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TABLE OF CONTENTS

Volume 29, No. 1 January 1965 Reviews 1–1110

MARCH

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Mathematical Reviews

TABLE OF CONTENTS

Volume 29, Number 2 February 1965 Reviews 1111–2139

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Mathematical Reviews

TABLE OF CONTENTS

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Vol. 29, No. 1 — January 1965 — Reviews 1–1110

GENERAL

See also 442, 622, 767.

Behnke, H.; Choquet, G.; Dieudonné, J.; Fenchel, W.; Freudenthal, H.; Hajós, G.; Pickert, G. 1

★Lectures on modern teaching of geometry and related topics.

Held at the ICMI-Seminar in Aarhus, May 30 to June 2, 1960. Elementaer Afdeling, Nr. 7.

Matematisk Institut, Aarhus Universitet, Aarhus, 1960. iv + 169 *pp.*

HISTORY AND BIOGRAPHY

See also 564.

Goldstein, Bernard R. 2

A medieval table for reckoning time from solar altitude.

Scripta Math. **27**, 61–66 (1964).

This article describes an anonymous Moslem manuscript—undated, but presumably going back to about the fourteenth century—now in the Leiden University Library. The manuscript includes a table in which rows and columns represent, respectively, meridian and observed altitudes of the sun; and the time since sunrise is to be computed as a function of these two variables. A graph of this relationship, based on values recomputed by Owen Gingerich and drawn by O. Neugebauer, is included in this paper, as is also a translation by E. S. Kennedy of the Persian introduction to the table.

C. B. Boyer (Brooklyn, N.Y.)

Simonov, R. A. 3

Numerical cryptography and Slavic mathematical education in the XIV–XVI centuries. (Bulgarian)

Fiz.-Mat. Spis. Bŭlgar. Akad. Nauk. **6 (39)** (1963), 199–206.

Biermann, Kurt-R. 4

Gotthold Eisenstein. Die wichtigsten Daten seines Lebens und Wirkens.

J. Reine Angew. Math. **214/215** (1964), 19–30.

In der Geschichte der elliptischen Funktionen und Integrale, und zumal in der Zahlentheorie nimmt Gotthold Eisenstein einen sehr ehrenvollen Platz ein. Bald nach seinem frühen Tode bemächtigte sich die Legende seiner Person und, so grosses Aufsehen seine Laufbahn einst erregt hatte, so schnell fiel er, wenn auch nicht seine Forschungsergebnisse, der Vergessenheit anheim. Der Verfasser hat in den letzten Jahren eine ganze Anzahl neuer Dokumente aufgefunden und veröffentlicht, wodurch besonders das Leben des Studenten und Dozenten so weit aufgehellt wurde, wie es die Quellenlage gegenwärtig gestattet. Gelegentlich des 65-ten Geburtstages von H. Hasse hat der Verfasser in dieser Arbeit tabellarisch-chronologisch in Kürze die wichtigsten Stationen des dramatischen Lebensweges Eisensteins (16.4.1823–11.10.1852) dargestellt. Insbesondere das freundschaftliche Verhältnis zum alten von Humboldt wird hervorgehoben. Der Verfasser betont auch die bekannte, schwererklärliche, hohe Meinung von Gauss für die Person und die Arbeiten Eisensteins, eine Quelle des Ärgers für den ehrgeizigen Jacobi, eine schwere Verdriesslichkeit für den bescheidenen, liebenswürdigen Lejeune-Dirichlet. (Nach Moritz Cantor soll Gauss gesagt haben: "Es hat in der Geschichte nur drei epochemachende Mathematiker gegeben: Archimedes, Newton und Eisenstein".) Auch hier findet diese rätselhafte Haltung des sonst "im Lobe kargen Mannes" (nach von Humboldt, 1846) keine Erklärung. Es wird weiter gezeigt, wie ausgezeichnet das Verhältnis Eisensteins mit den anderen namhaften Mathematikern in 1845 war, wie er damals auf Vorschlag Kummers (möglicherweise einer von Humboldt inspirierten Anregung Jacobis folgend) von der Universität Breslau als Student im dritten Semester zum Dr. phil. h.c. promoviert wurde, und wie er aber schon nach einem Jahre in Prioritätsstreitigkeiten mit Jacobi verwickelt wurde, der Eisenstein des Plagiats und falscher Darstellung bekannter Ergebnisse bezichtigte. Diese Arbeit schliesst mit einer vollständigen Anfgabe von Quellen und Literatur, nebst einem Verzeichnis der 38 in Crelles Journal erschienenen Arbeiten von Eisenstein.

Drei mathematische Genies sind in der ersten Hälfte des vorigen Jahrhunderts jung, zu früh, verstorben: Abel, Galois und Eisenstein. Die ersteren beiden sind unvergessen. Eisenstein verdient es, der Vergessenheit entrissen zu werden.

S. C. van Veen (Delft)

Ljunggren, W. 5

Thoralf Albert Skolem in memoriam.

Math. Scand. **13** (1963), 5–8.

A brief biography of Skolem. There is no bibliography, but reference is made to a bibliography of his works [Acta Math. **110** (1963), i–xi; addition, ibid. **110** (1963), 303; MR **27** #2389].

LOGIC AND FOUNDATIONS

See also 30, 45, 49, 54, 1106, 1108, 1109.

Boll, Marcel; Reinhart, Jacques 6

Sur une liaison triadique permutable, se particularisant en les huit liaisons de deux objets.

C. R. Acad. Sci. Paris **258** (1964), 3407–3409.

Curry, Haskell B. 7

Basic verifiability in the combinatory theory of restricted generality.

Essays on the foundations of mathematics, pp. 165–189. *Magnes Press, Hebrew Univ., Jerusalem*, 1961.

The author discusses a system V_2 of verifiability related to his system F_2 in much the same way that Gentzen's sequence calculi are related to formulations of predicate logic. The system F_2, based on the pure theory of combinators, contains statements that can be interpreted in the sense of restricted generality or formal implication. The system V_2 has two primitive predicates $\mathrm{Can}(M_1, \cdots, M_m, A)$ and $\mathrm{Ver}(M_1, \cdots, M_m, A)$ and governing rules that allow them to be interpreted, respectively, as the property of being a proposition and the property of being true. It is shown that an application in the proof of one of these statements of the only rule related to conversion can be postponed to the very end of the proof. An analogue of Gentzen's Hauptsatz is proved for V_2 in a final theorem. The paper is closely related to notions developed earlier by the author [Logique et Analyse **3** (1960), 15–39].

E. J. Cogan (Bronxville, N.Y.)

Takeuti, Gaisi 8

On the weak definability in set theory.

Proc. Japan Acad. **38** (1962), 43–46.

Consider the relational system R consisting of all sets with $\in$ as a binary relation and each of the ordinals as a special element. From the axioms of set theory other than the axiom of choice and the axiom that every set contains a set definable in R, the author proves: (a) every set is definable in R; (b) the axiom of choice holds. The idea for (a) is that if it were false, there would be a set consisting of the non-definable sets of lowest rank, which would contradict the assumption. The idea for (b) is that the definitions in R can be well-ordered. The proof is slightly obscured by a confusion between sets and the sentences defining them. The article ends with some comments on previous articles of the author.

J. R. Shoenfield (Durham, N.C.)

Lévy, A. 9

The interdependence of certain consequences of the axiom of choice.

Fund. Math. **54** (1964), 135–157.

Consider the following propositions.

$C^*(\alpha)$: For every function F on the ordinal α such that, for each $\beta<\alpha$, $F(\beta)$ is a non-empty set, there exists a function G on α such that $G(\beta)\in F(\beta)$ for $\beta<\alpha$. $C(\alpha)$: Let F be a function on α such that, for each $\beta<\alpha$, $F(\beta)$ is a non-empty set. If for every $\gamma<\alpha$ there exists a function H on γ such that $H(\beta)\in F(\beta)$ for $\beta<\gamma$, then there exists a function G on α such that $G(\beta)\in F(\beta)$ for $\beta<\alpha$. $D^*(\alpha)$: If R is a binary relation and A a set such that $(\forall\gamma<\alpha)(\forall f\in A^\gamma)(\exists x\in A)fRx$, then there is an R-admissible sequence in A^α. (A sequence h of length γ is called an R-admissible sequence if and only if, for every $\lambda<\gamma$, the restriction of h to λ has the relation R to $h(\lambda)$. $D(\alpha)$: If R is a binary relation and A a set such that $(\forall\gamma<\alpha)(\forall f\in A^\gamma)(\exists x\in A)fRx$ and such that if $\gamma<\alpha$ and f is an R-admissible sequence in A^γ, then for every $\gamma\leqq\delta<\alpha$ there is an R-admissible sequence g in A^δ which extends f, then there is an R-admissible sequence G in A^α. $H(\omega_\alpha)$: If A is a set such that $|A|>\omega_\beta$ for every $\beta<\alpha$ (and, for $\alpha=0$, $|A|>n$ for every $n\in\omega$), then $|A|\geqq\omega_\alpha$.

The following facts are proved easily (except for (VII), (IX), and (XI)): (I) $C^*(\alpha)\leftrightarrow(\forall\beta\leqq\alpha)C(\beta)$; (II) $C(0)$ & $C(\alpha+1)$; (III) $C(\alpha)\leftrightarrow C(\beta)$ if α and β are confinal with the same regular ordinal; (IV) $D^*(\alpha)\leftrightarrow(\forall\beta\leqq\alpha)D(\beta)$; (V) $D(0)$ & $D(\alpha+1)$; (VI) $D(\alpha)\leftrightarrow D(\beta)$ if α and β are confinal with the same regular ordinal; (VII) $D^*(\alpha)\leftrightarrow D(\alpha)$ if α is regular; (VIII) If $\beta\geqq\alpha$, then $H(\omega_{\beta+1})\to H(\omega_\alpha)$; (IX) If $\beta>\alpha$ and α and β are limit numbers confinal with the same regular ordinal, and if the generalized continuum hypothesis is assumed, then $H(\omega_\beta)\to H(\omega_\alpha)$; (X) $D(\alpha)\to C(\alpha)$ for every α; (XI) $D(\omega_\alpha)\to H(\omega_\alpha)$ for every α; (XII) $C(\omega_\alpha)\to H(\omega_\alpha)$ for every limit ordinal α; (XIII) $(\forall\alpha)D(\alpha)$ and $(\forall\alpha)H(\omega_\alpha)$ are equivalent to each other and to the axiom of choice.

Mostowski [Fund. Math. **32** (1939), 201–252] proved the independence of $D^*(\omega_0)\to C^*(\omega_1)$ in a system of set theory with urelements and without the axiom of choice. Using the Fraenkel-Mostowski modelling method, and working within the same axiom system, the author proves many independence results concerning $C(\alpha)$, $D(\beta)$, and $H(\gamma)$. For example, $H(\omega_{\alpha+1})$ is not provable even if we assume $(\forall\gamma)C(\omega_\gamma)$ & $(\forall\gamma\leqq\alpha)D(\omega_\gamma)$ & $(\forall\gamma)([\gamma\leqq\alpha$ or $\gamma=0$ or γ is a limit ordinal$]\to H(\omega_\gamma))$.

E. Mendelson (Flushing, N.Y.)

Lemmon, E. J. 10

A theory of attributes based on modal logic.

Acta Philos. Fenn. Fasc. 16 (1963), 95–122.

The author develops a formal system Σ of modal logic in which attributes (or properties) are "intensional" entities, to be distinguished from extensional classes. Classes are to be identified if they have the same members; attributes are to be identified if they necessarily apply to the same individuals. The result is a system which bears roughly the same relation to Church's paper "A formulation of the logic of sense and denotation" [in *Structure, method, and meaning* (edited by Henle, Kallen and Langer), pp. 3–24, Liberal Arts Press, New York, 1951], as is borne by von Neumann-Gödel set theory to Russell's type theory: Church takes account of the intensional-extensional distinction type-theoretically; the author does it set-theoretically.

A. R. Anderson (New Haven, Conn.)

Teh, H. H. 11

On 3-valued sentential calculus. An axiomatic approach.

Bull. Math. Soc. Nanyang Univ. **1963**, 1–37.

The paper deals with the sentential calculus with three values: 1 (true), -1 (false), 0 (neutral). Beyond the usual operators $\sim$, $\vee$, $\wedge$ and $\to$, the following are discussed: $\downarrow$ common-opinion, $\uparrow$ friendly common-opinion. $x\downarrow x=x\uparrow x=x$, if $x\neq y$ then $x\downarrow y=0$, $x\uparrow y=x+y$. Some basic logical laws are formulated with the help of the relation $\equiv$, which means the equality of values by every valuation. The author's main object is to introduce a method based on certain fundamental algebraic concepts like the model M, the free algebraic system S generated by the set of symbols and the collection of operators, homomorphisms of S into M, and the factor system $S/\equiv$. No reference to the literature is made, which makes it difficult to determine which results are new.

S. Jaśkowski (Toruń)

iu, Shih-Chao 12
Four types of general recursive well-orderings.
Notre Dame J. Formal Logic **3** (1962), 75–78.

eneral recursive well-orderings of all the natural umbers may, of course, have many non-constructive eatures. In order to require more constructive character f such an ordering, $\prec$, one may impose such conditions s (i) "x is a successor" and "x is a limit" are recursive, ii) there is a recursive σ such that $\sigma(n)$ is the successor of n if n is not last, or (iii) there is a partial recursive λ such hat if the recursive sequence $\{a_i\}$ with Gödel number e is ncreasing and bounded ($\prec$), then $\lambda(e)$ is the limit of $\{a_i\}$. f classes of recursive well-orderings are designated $I, \cdots, IV$ according as their members satisfy none of i), (ii) and (iii), only (i), only (i) and (ii), or all three onditions, respectively, then taken in order $I, \cdots, IV$ ontain orderings of increasingly constructive character. A fourth condition is given: (iv) there is a partial recursive μ such that if A is a non-empty recursive set with Gödel number e, then $\mu(e)$ is the first element of A. Orderings satisfying (iv) form class V, and $V \subset IV$ properly. Classes IV and V are considered as possibilities for the "class of onstructively given recursive well-orderings", a term used by the author informally elsewhere. Unhappily, orderings of these classes are all of order type $\leqq \omega\cdot 2$ and $<\omega$, espectively. The order types of the orderings of III, however, include all constructive ordinals.

P. Axt (University Park, Pa.)

Liu, Shih-Chao 13
Recursive linear orderings and hyperarithmetical functions.
Notre Dame J. Formal Logic **3** (1962), 129–132.

Spector [Fund. Math. **48** (1959/60), 313–320; MR **22** #10904] showed that for every recursive R there is a recursive S such that $(E\alpha)_{\alpha\in\mathrm{HA}}(x)S(\alpha, a, x) = (\alpha)(Ex)R(\alpha, a, x)$. Another proof of this result is given here through the use of the following two theorems. (1) For every recursive linear ordering $\prec$ which is not a well-ordering, if no HA function is strictly decreasing ($\prec$), then for every ordinal $\nu<\omega_1$, there is a segment of $\prec$ of order-type ν. (2) Given any recursive R, there is a recursive R' such that $(\alpha)(Ex)R(\alpha, a, x) \equiv (\alpha)(Ex)R'(\bar{\alpha}(x), a)$ and for no $\alpha \in \mathrm{HA}$, $(x)\bar{R}'(\bar{\alpha}(x), a)$. The presentation is marred by misprints.

P. Axt (University Park, Pa.)

Trahtenbrot, B. A. 14
On the frequency computability of functions. **(Russian)**
Algebra i Logika Sem. **2** (1963), *no.* 1, 25–32.

Let N stand for the set of natural numbers and let $m, n \in N$ satisfy $0<m\leqq n$. Let T be a function defined on N^n, $T(x_1, \cdots, x_n) = (y_1, \cdots, y_n)$. A function $f\colon N\to N$ is said to compute T with frequency m/n if for every n-tuple $(x_1, \cdots, x_n)$ such that $x_i \neq x_j$ for $i\neq j$ at least m of the n equations $f(x_1)=y_1, \cdots, f(x_n)=y_n$ are true. A function $f\colon N\to N$ is called frequency computable if there exists a recursive function which is computed by f with some frequency. Theorems: (1) For an arbitrary function T, the set of functions f which compute T with frequency $m/n>\frac{1}{2}$ is at most countable. (2) There exists a recursive function T_0 which is computed with frequency $m/n=\frac{1}{2}$ by uncountably many functions f. (3) Each function f which is computable with frequency $m/n>\frac{1}{2}$ is recursive. These results answer a question of Myhill and Rose presented by McNaughton [Advances in Computers, Vol. 2, pp. 379–421, Academic Press, New York, 1961; MR **24** #B2520]. Proofs depend in part on the introduction of a topology in the set of interpretations of a sequence of propositions, and the paper includes a constructive sharpening of the theorem of Löwenheim.

G. N. Raney (Storrs, Conn.)

Bodnarčuk, V. G. 15
Systems of equations in an algebra of events. **(Russian)**
Ž. Vyčisl. Mat. i Mat. Fiz. **3** (1963), 1077–1088.

The author considers the solution of certain types of equations in the algebra of events introduced by Kleene in connection with the characterisation of the behaviour of finite automata in terms of regular events. The elements of this algebra are events, i.e., sets of words on a given finite alphabet, and the operations are disjunction ($S_1 \vee S_2 =$ union of S_1, S_2), multiplication ($S_1S_2=$ set of all words of the form p_1p_2, where $p_1\in S_1$, $p_2\in S_2$), and iteration ($\{S\}$ consists of the empty word e and all words of the form $p_1\cdots p_k$ when $p_1, \cdots, p_k \in S$). The first equation considered is $X=XS\vee R$, and it is shown that if $e\notin S$, this has the unique solution $X=R\{S\}$; if $e\in S$, the solution takes the form $X=(R\vee T)\{S\}$, where T is arbitrary. Then the system of equations $X_i = X_1S_{1i} \vee X_2S_{2i} \vee \cdots \vee X_nS_{ni} \vee R_i$ $(i=1, \cdots, n)$ is treated by writing it in matrix form $\mathbf{X}=\mathbf{XS}\vee\mathbf{R}$. In matrix form the solution is the same here, except that the condition $e\in S$ is replaced by a condition asserting the existence of cycles in a graph associated with the matrix $\mathbf{S}$, and a condition is laid on $\mathbf{T}$ in the second case. It is also shown that when this system has a unique solution it can be found by a process of successive elimination of variables. Finally, several examples are given which illustrate how these methods may be used for the analysis of finite automata given by their state transition graphs.

J. C. Shepherdson (Bristol)

Red'ko, V. N. 16
On the algebra of commutative events. **(Russian. English summary)**
Ukrain. Mat. Ž. **16** (1964), 185–195.

Author's summary: "A certain complete system of identical relations of the algebra of commutative events is indicated in this paper. It is shown that the problem of identity in the algebra of regular commutative events is solvable algorithmically. It is proved that neither in the algebra of commutative events nor in the algebra of events in an arbitrary alphabet does there exist a finite complete system of identical relations."

Poljakov, E. A. 17
Algebras of recursive functions. **(Russian)**
Algebra i Logika Sem. **3** (1964), *no.* 1, 41–56.

Let A_p, A_{gr} and A_{pr} denote the set of all primitive-, general- and partial-recursive functions of one variable, respectively. By $+$, $*$, i and $^{-1}$ denote the operations of addition, composition, iteration and inversion: $(f+g)(x) = f(x)+g(x)$; $(f*g)(x)=f(g(x))$; $(if)(0)=0$, $(if)(x+1) = f((if)(x))$; and $f^{-1}(x)=\min_y(f(y)=x)$.

The author studies the structure of algebras $\mathfrak{A}_p =$

$\langle A_p;\ +,\ *,\ i\rangle$, $\mathfrak{A}_{gr}=\langle A_{gr};\ +,\ *,\ ^{-1}\rangle$ and $\mathfrak{A}_{pr}=\langle A_{pr};\ +,\ *,\ ^{-1}\rangle$ and some others; he proves many interesting results, a few of which will be mentioned here.

Every basis of these algebras necessarily contains functions of "almost high range", i.e., functions which take an infinity of different natural numbers as values and each one infinitely many times. The function f, defined by $f(2t)=t+1$ and $f(2t+1)=t \dot{-} [\sqrt{t}]^2$, is a basis for $\mathfrak{A}_p$. The function g, defined by $g(2t)=2k^2+3$ $(k>0)$ and $g(2t+1)=t\dot{-}[\sqrt{t}]^2$ is a basis for $\mathfrak{A}_{gr}$ and for $\mathfrak{A}_{pr}$. The algebras $\langle A_p;+,*\rangle$ and $\langle A_{gr};+,*\rangle$ are not finitely generated; both are simple. The only automorphism of $\mathfrak{A}_{pr}$ is the identity. *V. Vučković* (Notre Dame, Ind.)

Young, Paul R. 18

A note on pseudo-creative sets and cylinders.

Pacific J. Math. **14** (1964), 749–753.

A cylinder (in this paper) is a set of natural numbers which is recursively isomorphic to a Cartesian product $B \times N$, where B is a set of natural numbers and N is the set of all natural numbers. A recursively enumerable (r.e.) non-creative set A is pseudo-creative if for every r.e. subset B of A' there is an infinite r.e. subset of $A' \cap B'$. Myhill has observed that the cylinder of a non-recursive non-creative r.e. set is pseudo-simple. The author shows that if A is an infinite r.e. set which is not pseudo-creative, then $A \times A$ is many-one reducible to A if and only if $A \times A$ is a cylinder. Using a result of Fischer, it follows that there is a simple set S such that $S \times S$ is not a cylinder. The author shows that for S simple, $S \times S$ is pseudo-creative, thus obtaining a pseudo-creative set which is not a cylinder. A few related results are also proved. *J. R. Shoenfield* (Durham, N.C.)

Orevkov, V. P. 19

A constructive map of the square into itself, which moves every constructive point. (Russian)

Dokl. Akad. Nauk SSSR **152** (1963), 55–58.

The author shows that several classical theorems related to the Brouwer fixed-point theorem are false for the corresponding constructive spaces as defined by Šanin [Trudy Mat. Inst. Steklov. **67** (1962), 15–294; MR **28** #30]. For example, if one takes the constructive two-dimensional space formed by all pairs of constructive real numbers (i.e., the computable convergent sequence of rationals together with the modulus of convergence function), then he gives a constructive continuous retraction of a square onto its boundary, a constructive continuous map of a square of side 2 into itself which moves every point a distance of at least 1/8, a constructive uniformly continuous map of a square into itself without fixed point, and a constructive uniformly continuous tangential vector field on the 2-sphere which vanishes nowhere. These operators can all be taken to be infinitely differentiable. *J. C. Shepherdson* (Bristol)

Orevkov, V. P. 20

Constructive mappings of polyhedra. (Russian)

Dokl. Akad. Nauk SSSR **152** (1963), 278–281.

This is an extension of the work of Šanin on constructive (in the sense of recursive, computable) metric spaces [Šanin, Trudy Mat. Inst. Steklov. **67** (1962), 15–294; MR **28** #30] to simplicial complexes. Definitions, too long to reproduce here, are given of "constructive locally finite simplicial complex", "point of a complex", etc., and basic approximation and continuity theorems are proved on constructive operators from one complex to another, e.g., Theorem 2: Every constructive operator from one complex to another is continuous at each point at which it is defined. Theorem 3: Every constructive operator from a subcomplex L of a complex K into an arcwise connected complex R can be extended to a constructive operator from K into R. From these theorems follows the counter classical Lemma 2: If I is any complex of dimension >1, there exists a constructive operator from I onto the boundary of I leaving all points of the boundary fixed. A special case of this, a retraction of a square onto its boundary was given by the author [#19 above]. Theorem 4: Any two constructive operators defined on all points of a complex K and mapping it into an arcwise connected complex L are homotopic. Theorem 5: A complex is homotopically equivalent to a point if and only if it is arcwise connected. These last two theorems become false if "uniformly" is inserted before "homotopic". *J. C. Shepherdson* (Bristol)

Heyting, A. 21

Axiomatic method and intuitionism.

Essays on the foundations of mathematics, pp. 237–247. *Magnes Press, Hebrew Univ., Jerusalem,* 1961.

The article begins with a general discussion of axiom systems, concluding with the amazing statement that an axiomatic foundation of set theory is not conceivable. The argument seems to the reviewer to be vitiated by a common but incorrect assumption: that the notion of set naturally leads to a contradictory axiom system. This may be true of certain notions of set, but not (so far as we know) of the notion of set on which such axiom systems as Zermelo-Fraenkel set theory are built. The remainder of the paper lists some intuitionistic axiom systems for some well-known algebraic concepts. The most noticeable difference from classical axiom systems is the frequent use of the apartness relation, which is a sort of positive version of inequality (but is intuitionistically stronger than inequality). *J. R. Shoenfield* (Durham, N.C.)

Kušner, B. A. 22

Riemann integration in constructive analysis. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 255–257.

A constructive function in [0, 1] is called R-integrable if there is a recursive function δ such that two of its Riemann sums for nets with meshes less than $2^{-\delta(m)}$ differ by less than 2^{-m}. The following theorems are stated. (1) There is no algorithm which, given the description of an R-integrable function, bounded by 1, and its integral, yields the corresponding function δ. (2) There is no algorithm which leads from the description of an R-integrable function to its integral. (3) There is a bounded function on [0, 1] which is effectively non-integrable. (4) Neither the set of R-integrable functions, nor that of non-R-integrable functions is recursively enumerable. The theorem analogous to (2) holds for functions integrable in the sense of Zaslavskiĭ and Ceĭtin [Trudy Mat. Inst. Steklov. **67** (1962), 458–502; MR **27** #2408]. Moreover, for these functions there is no algorithm which, given an integrable

unction f and its integral u, yields a constructive real umber y such that $f(y)=u$. However, such a y exists lways as a constructive sequence of rationals for which it impossible that it is not convergent. Proofs are very riefly indicated. *A. Heyting* (Amsterdam)

Robinson, Abraham 23

On the construction of models.

Essays on the foundations of mathematics, pp. 207–217. *Magnes Press, Hebrew Univ., Jerusalem,* 1961.

By a partial valuation of a fixed set A we mean a mapping φ of a subset of A into the set $\{0, 1\}$. $D\varphi$ is the domain of definition of φ, and φ is called total if $D\varphi=A$. We write $\varphi|A'$ for the restriction of φ to a subset A' of A. Let N be an index set. J is said to be a net on the set N if J is a non-empty set of non-empty subsets of N such that if $J_1 \in J$ and $J_2 \in J$, then there exists a common subset of J_1 and J_2 in J. Fundamental lemma: Let $N=\{\nu\}$ be an index set for a set $\{\varphi_\nu\}$ of partial valuations of A, and let J be a net on N such that for every finite subset A' of A and for every element J' of J there exists a $\nu \in J'$ such that $A' \subseteq D\varphi_\nu$. Then there exists a total valuation ψ_0 of A such that for every finite subset A' of A and for every $J' \in J$ there exists a $\nu \in J'$ such that $A' \subseteq D\varphi_\nu$ and $\psi_0|A'=\varphi_\nu|A'$. The author expects that the fundamental lemma can play a central rôle in independent proofs of several theorems and constructions of models. The proofs of the following known theorems are given: If a set K of sentences in the lower predicate calculus is syntactically consistent, then K possesses a model. If every finite subset of a set of sentences K possesses a model, then K also possesses a model.

{Important misprints: p. 208, line 8, read $\varphi|A'$ instead of $\varphi|A$ and put φ' after "restriction"; p. 208, line 18, read φ_ν instead of φ_2; p. 210, line 12, read ψ_0 instead of Ψ_0; p. 210, line 13, read $A' \subseteq D\varphi_\nu$ instead of $A' \subseteq D_\nu$.}

S. Jaśkowski (Toruń)

Mullin, Albert A. 24

Models of the fundamental theorem of arithmetic.

Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 604–606.

Reichbach, Juliusz 25

Some characterizations of theses of the first-order functional calculus.

Math. Ann. **152** (1963), 195–207.

This is one of the author's many papers [e.g., Notre Dame J. Formal Logic **2** (1961), 1–15; errata, ibid. **2** (1961), 259; MR **25** #3815b; ibid. **2** (1961), 251–252; MR **25** #3815c; ibid. **3** (1962), 102–107; MR **26** #4897; Z. Math. Logik Grundlagen Math. **7** (1961), 175–184; MR **25** #3815a; ibid. **8** (1962), 267–276; MR **26** #3591] on modified notions of satisfiability. At the present time, these notions can be used only to obtain alternative proofs of known theorems. *G. F. Rose* (Pacific Palisades, Calif.)

Robinson, R. M. 26

Undecidability of the elementary theory of the field of rational functions of one variable with rational coefficients. (Russian)

Algebra i Logika Sem. **2** (1963), *no.* 4, 5–11.

This is not a paper by Robinson, but the translation, with minor modifications, of the summary of a communication which S. Gerig made in the seminar at Berkeley on a result of R. M. Robinson. On the parts of the curve $y^2=x^3-2$ where $|y| \geqq 5$ a dense set of rational points is constructed. From this fact and from the fact that the curve admits no rational parametric representation, a definition of the rationals in the theory mentioned in the title is obtained. The undecidability result then follows immediately. *A. Heyting* (Amsterdam)

Wang, Hao 27

Tag systems and lag systems.

Math. Ann. **152** (1963), 65–74.

A tag system T (first considered by Post in 1943) is defined by a finite alphabet $A=\{s_1, \cdots, s_\sigma\}$, a natural number β and a finite set of rules $T_i: s_i \to E_i$ $(i=1, \cdots, \sigma)$, $s_i \neq s_j$ for $i \neq j$, and each E_i is a word on the alphabet A. If W is a word on the alphabet A beginning with the letter s_i, then to apply the rules one deletes the first β letters of W and appends the string E_i at the end. The halting problem of T is that of deciding for an arbitrarily given word W whether, under the continued application of the rules, W leads to a word of length $<\beta$. The derivability problem is that of deciding for arbitrary W_1, W_2 whether W_1 leads to W_2. The existence of a tag system with undecidable halting (and hence derivability) problem was first proved by Minsky in 1961 [Ann. of Math. (2) **74** (1961), 437–455; MR **25** #3825] and the result was improved by Cocke and Minsky. The author gives a further improvement and obtains a "best possible" result in the sense that he shows all "simpler" tag systems are decidable. If ε_i is the length of E_i, then ε, ε^- are defined, respectively, as the greatest and least of the ε_i. The author shows all tag systems with $\beta=1$ are decidable, also those with $\beta \geqq \varepsilon$ or $\beta \leqq \varepsilon^-$ (i.e., where the rules always shorten or lengthen words), and he produces an undecidable one with $\beta=2$, $\varepsilon=3$, $\varepsilon^-=1$.

A lag system (introduced here for the first time) is a sort of opposite to the tag system. It consists of a set of σ^β rules $L_i: s_{i_1} \cdots s_{i_\beta} \to E_i$ such that if the first β symbols of a string are $s_{i_1} \cdots s_{i_\beta}$, the first symbol s_i, viz., s_{i_1}, is deleted and E_i is appended at the end of the string. For $\beta=1$ these are the same as tag systems. They are shown to be decidable also if $\varepsilon \leqq 1$ or $\varepsilon^- \geqq 1$, and an undecidable system with $\beta=2$, $\varepsilon=2$, $\varepsilon^-=0$ is constructed.

The proofs are elegant and quite short. The undecidable lag system is obtained by simulation of the behaviour of a universal machine of the type introduced by the reviewer and Sturgis [J. Assoc. Comput. Mach. **10** (1963), 217–255; MR **27** #1359] with instructions "print 0 (1) at the end of the word" and "scan and delete initial letter; if it is 0 go to next instruction, otherwise to instruction k". The undecidable tag system starts from a universal Turing machine. Finally, a simple proof is given of the known fact that there are monogenic normal systems with unsolvable halting problem. *J. C. Shepherdson* (Bristol)

Makarov, V. L. 28

Turing machines and finite automata. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 102–108.

Let $W(A)$ denote the set of words on the alphabet A. The author first shows that the class of mappings of $W(A) \to$

$W(A)$ realisable by Turing machines is not affected by the imposition of a series of restrictions. The end result is a Turing machine in which the head makes a series of passages from left to right through the word, modifying it or extending it as it goes, and each time is returned to the beginning in the same state, and without altering the word on this return (right to left) journey. Thus it behaves like a finite automaton with output which is successively applied to the input word, then to the resulting output word, and so on. The final theorem shows that another way of looking at it is an automaton which makes only one run through the word but which has an infinite number of states, but with primitive recursive output and transition functions. *J. C. Shepherdson* (Bristol)

Ljubič, Ju. I. 29

Estimates for the optimal determination of indeterminate autonomous automata. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 337–355.

An indeterminate autonomous automaton $\mathfrak{A}$ with the states $\sigma_1, \sigma_2, \cdots, \sigma_n$ is a multiple-valued mapping of the set S of states into itself:

$$(1) \qquad \sigma_i \to S_i, \qquad S_i \subset S \qquad (i = 1, \cdots, n).$$

The mapping (1) may be extended to the system Σ of all subsets of the set S:

$$(2) \qquad M \to M' = \bigcup_{\sigma_i \in M} S_i \qquad (M \in \Sigma).$$

The mapping (2) stands for a determinate autonomous automaton $\tilde{\mathfrak{A}}$ with the states Σ.

Suppose an initial state σ_1 of $\mathfrak{A}$ is selected; then by the "optimal determination of $\mathfrak{A}$", those states of $\tilde{\mathfrak{A}}$ which may not be reached starting from the set $\{\sigma_1\}$ are to be eliminated. The number of states of the automaton $\tilde{\mathfrak{A}}_0$ obtained after these eliminations is denoted by $d(\mathfrak{A})$.

Various estimates of $d(\mathfrak{A})$ are given. It is shown that $d(\mathfrak{A}) \leq f(n)$, where $f(n)$ is an arithmetical function with the property that

$$\lim_{n\to\infty} \frac{\ln f(n)}{n} = 0.$$

P. L. Ivănescu (Bucharest)

SET THEORY

See also 8, 9, 12, 45, 49.

Gavrilov, G. P. 30

Quasi-Peano behaviour of functions. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 1011–1013.

This paper is concerned with the class P of all functions (of any number of arguments) from a denumerable set into itself. By G we denote the set of all functions in G depending on not more than one variable, and we call a function quasi-Peanian if, together with G, it forms a complete class. {For terminology, cf. the author's paper [same Dokl. **128** (1959), 21–24; MR **21** #6327].} Various results, somewhat too involved to be stated here, are proved concerning quasi-Peanian functions and related notions. Included is a necessary and sufficient condition for quasi-Peanicity. *E. Mendelson* (Flushing, N.Y.)

COMBINATORIAL ANALYSIS

See also 53, 180.

Hurst, C. A. 31

Applicability of the Pfaffian method to combinatorial problems on a lattice.

J. Mathematical Phys. **5** (1964), 90–100.

For the solution of certain problems in statistical mechanics, a formulation in terms of the combinatorial problem of counting open or closed polygons drawn on a lattice has been found useful. The solution of such combinatorial problems has been given as a generating function which can be expressed as a Pfaffian. Necessary and sufficient conditions are given here to determine when such a Pfaffian representation is possible. If certain consistency conditions are satisfied, then the solution to the combinatorial problem can be written down immediately. The method makes some use of the ideas of emission and absorption operators for fermions.

G. Newell (Providence, R.I.)

Blakley, G. R. 32

Combinatorial remarks on partitions of a multipartite number.

Duke Math. J. **31** (1964), 335–340.

A partition of a multipartite number is regarded as specified by listing the number of appearances of each of its permitted parts; thus the set of lists is the set of non-negative integer-valued functions with domain the set of permitted parts. This prescription is used to obtain a recurrence relation for the number of partitions having r parts, zero (multipartite) parts permitted, or what is the same thing, the number with at most r parts. This recurrence, which with all its accompanying definitions is too long to quote, is used to prove E. M. Wright's result [Proc. London Math. Soc. (3) **11** (1961), 499–510; MR **24** #A2573] that for suitably restricted numbers partitioned, the number with at most r parts is a polynomial (in a special sense) in the number partitioned. Recurrences are also found for the total number of partitions with restricted and unrestricted parts. *J. Riordan* (Murray Hill, N.J.)

Blakley, G. R. 33

Algebra of formal power series.

Duke Math. J. **31** (1964), 341–345.

This paper contains a generalization to several variables of the explicit coefficient formulas of the algebra of power series. Specifically, if f is a multivariable function with a power series representation, the coefficients of f^k, k an integer, $g(f)$, g a power series, and $\log f$ are expressed in terms of the coefficients of f. The expression is in a compressed notation, formally elegant, but too long to quote.

J. Riordan (Murray Hill, N.J.)

Blakley, G. R. 34

Formal solution of nonlinear simultaneous equations: Reversion of series in several variables.

Duke Math. J. **31** (1964), 347–357.

Let $w_i = z_i f_i(z_1, \cdots, z_n)$, $i = 1(1)n$, in the notation of numerical tablemakers. The inverses of these are $z_i = w_i h_i(w_1, \cdots, w_n)$. Both the f_i and h_i are analytic and

nonzero at the origin. The author's main result is a theorem which gives the coefficients of the k_i in terms of the coefficients of the f_i. An intermediate result is a multivariable generalization of the formula for reversion of series. As in the companion papers above [#32; #33], a compressed notation is employed; a concluding section gives a detailed discussion of what would be required in digital computer applications. A set $H(n)$ appearing in this discussion is equinumerous with the fully labeled (free) trees, as the author notes and proves, and thus constitutes another description of these endemic entities.
J. Riordan (Murray Hill, N.J.)

Metcalf, Frederic T. 35
A theorem concerning rearrangements.
Amer. Math. Monthly **71** (1964), 172–174.

Let $0 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_n$ and $0 \leqq b_1 \leqq b_2 \leqq \cdots \leqq b_n$ be two sequences of real numbers. Let σ be a permutation of $1, 2, \cdots, n$, the image of i being denoted by σi. Let $m_\sigma = \min_{1 \leqq k \leqq n} a_k b_{\sigma k}$ and $M_\sigma = \max_{1 \leqq k \leqq n} a_k b_{\sigma k}$. The author proves the following equations, where it is understood that σ ranges over all permutations of $1, 2, \cdots, n$:

$$\max_\sigma m_\sigma = \min_{1 \leqq k \leqq n} a_k b_{n-k+1}, \quad \min_\sigma M_\sigma = \max_{1 \leqq k \leqq n} a_k b_{n-k+1},$$
$$\min_\sigma m_\sigma = a_1 b_1, \quad \max_\sigma M_\sigma = a_n b_n.$$

N. S. Mendelsohn (Winnipeg, Man.)

Fleischer, I.; Kooharian, A. 36
Optimization and recursion.
J. Soc. Indust. Appl. Math. **12** (1964), 186–188.

The authors state a set of sufficient conditions for an infimum operator to admit a recursive formulation.
D. S. Adorno (Wilton, Conn.)

Znám, Š. 37
On a combinatorical problem of K. Zarankiewicz.
Colloq. Math. **11** (1963), 81–84.

Let $k_j(n)$, $2 \leqq j < n$, be the smallest positive integer such that if A is an n-by-n matrix consisting of zeros and ones and if A contains $k_j(n)$ ones, then A has a minor of order j which consists exclusively of ones. The problem of Zarankiewicz is to determine $k_j(n)$.

In this paper it is shown that

$$k_j(n) < 1 + \left[\frac{j-1}{2} n + (j-1)^{1/j} n^{(2j-1)/j}\right].$$

This improves a previous result of Hyltén-Cavallius [same Colloq. **6** (1958), 59–65; MR **21** #1941]. For the special case $j = 2$, it is not as good as a result of Reiman [Acta Math. Acad. Sci. Hungar. **9** (1958), 269–273; MR **21** #63].
J. K. Goldhaber (College Park, Md.)

Tauber, S. 38
On N-numbers.
Elem. Math. **19** (1964), 57–62.

The N-numbers of the title are the coefficients of the relation

$$x^n + x^{-n} = \sum_{k=0}^{m} N^n_{n-2k}(x+x^{-1})^{n-2k}, \qquad m = [n/2],$$

with brackets indicating the largest integer function. As the author notes, they are also coefficients in an expansion of Chebyshev polynomials, namely,

$$t_n(x) = \sum_{k=0}^{m} N^n_{n-2k}(2x)^{n-2k}, \qquad m = [n/2],$$

with $t_0(x) = 1$, $t_n(x) = 2T_n(x)$, $n = 1, 2, \cdots$; $T_n(x) = \cos n\theta$, $\cos\theta = x$. Since

$$t_n(x) = \sum_{k=0}^{m} (-1)^k \frac{n}{n-k}\binom{n-k}{k}(2x)^{n-2k},$$

it follows at once that

$$N^n_{n-2k} = (-1)^k \frac{n}{n-k}\binom{n-k}{k},$$

a result curiously absent from the paper. Because the author deals separately with the cases n even and n odd in deriving orthogonal and recurrence relations, he omits the simplest recurrence, namely, $N^n_{n-2k} = N^{n-1}_{n-1-2k} - N^{n-2}_{n-2k}$. For the same reason his orthogonal relations are contained in that associated with the inverse pair

$$a_n = \sum \binom{n}{k} b_{n-2k}, \qquad b_n = \sum (-1)^k \frac{n}{n-k}\binom{n-k}{k} a_{n-2k},$$

both sums from zero to $[n/2]$, which is equation (17) of the reviewer's paper [Amer. Math. Monthly **71** (1964), 485–498].
J. Riordan (Murray Hill, N.J.)

Carlitz, L. 39
Some inversion formulas.
Rend. Circ. Mat. Palermo (2) **12** (1963), 183–199.

The chief inversion formula of this paper (which arose in the author's work on symmetric q-polynomials [Riv. Mat. Univ. Parma **10** (1959), 183–200; MR **27** #4969]) is as follows:

$$u_{mn} = u_{nm} = \sum_{r=0}^{M} \begin{bmatrix} m+n-2r \\ m-r \end{bmatrix} v_{m+n-2r,r}, \qquad M = \min(m, n),$$

is inverted by

$$v_{mn} = \sum_{s=0}^{n} (-1)^s q^{s(s-1)/2} \frac{1-q^{m+2s}}{1-q^{m+s}} \begin{bmatrix} m+s \\ s \end{bmatrix} u_{m+n+s,n-s}.$$

The bracketed expressions are q-binomial coefficients:

$$\begin{bmatrix} m \\ n \end{bmatrix} = \frac{(q)_m}{(q)_n (q)_{m-n}}, \qquad (q)_m = (1-q)(1-q^2)\cdots(1-q^m).$$

This has two companions which we do not quote. There are also two q-generalizations of inverse relations connected with Chebyshev polynomials, the first of which, in a slight modification of the author's notation, reads:

$$u_n = \sum_{2r \leqq n} \begin{bmatrix} n \\ r \end{bmatrix} v_{n-2r}$$

is inverted by

$$v_n = \sum_{2s \leqq n} (-1)^s q^{s(s-1)/2} \frac{1-q^n}{1-q^{n-s}} \begin{bmatrix} n-s \\ s \end{bmatrix} u_{n-2s}.$$

J. Riordan (Murray Hill, N.J.)

Barrucand, Pierre 40
Sur la somme des puissances des coefficients multinomiaux et les puissances successives d'une fonction de Bessel.
C. R. Acad. Sci. Paris **258** (1964), 5318–5320.

Put

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^\alpha}, \quad z = (f(x))^k = \sum_{n=0}^{\infty} \frac{C_{\alpha,k}(n)}{(n!)^\alpha} x^n.$$

The object of the paper is to obtain recurrences for the $C_{\alpha,k}(n)$. This is accomplished by making use of a differential equation satisfied by z. For example, when $\alpha=2$, $k=3$, z satisfies

$$\theta^4 z = x(10\theta^2+10\theta+3)-9x^2z,$$

where $\theta = x(d/dx)$. This leads to the recurrence

$$(1+n)^2C_{2,3}(1+n) = (10n^2+10n+3)C_{2,3}(n)-9n^2C_{2,3}(n-1).$$

As another simple example we cite

$$(1+n)^2C_{3,2}(1+n) = (7n^2+7n+2)C_{3,2}(n)+8n^2C_{3,2}(n-1).$$

In general, however, the recurrences are of higher order; for example, $C_{2,2k-1}$ and $C_{2,2k}$ satisfy recurrences of order k. *L. Carlitz* (Durham, N.C.)

Harper, L. H. 41

Optimal assignments of numbers to vertices.

J. Soc. Indust. Appl. Math. **12** (1964), 131–135.

Consider all possible ways of assigning the integers 1 to 2^n to the vertices of an n-cube. This paper characterizes by a simple algorithm all assignments which minimize the sum $\sum |i-j|$ over all neighboring pairs of vertices, and gives the value of the sum. The natural numbering is one of these. The assignments which maximize this sum are also characterized simply, and the maximum value is calculated. Finally, the average of the sum for all possible assignments is given. *W. W. Peterson* (Gainesville, Fla.)

de Bruijn, N. G. 42

Enumerative combinatorial problems concerning structures.

Nieuw Arch. Wisk. (3) **11** (1963), 142–161.

In a previous paper [Nederl. Akad. Wetensch. Proc. Ser. A **62** (1959), 59–69; MR **21** #4112] the author has generalized the well-known combinatorial theorem of Pólya. In the present paper he further broadens the range of enumerative techniques available to the combinatorial analyst. The paper is largely concerned with manipulation of sums of cycle-indexes, rather than of individual cycle-indexes (as in Pólya's theorem, for example); and although this is not an entirely new idea, the technique is used much more extensively than hitherto in the literature.

Several sets of combinatorial configurations ("structures", to use the author's terminology) are considered, with the object not merely of determining their number, but of computing the sum (the U-polynomial) of the cycle-indexes of their automorphism groups. From this U-polynomial it is possible to enumerate the structures obtained by imposing on the original structures some additional feature, such as, for example, colourings of their constituent elements. Thus from the sum of the cycle-indexes of the non-isomorphic graphs on a given number of nodes and edges, it is possible to count the number of such graphs in which each node is allotted one of a given set of colours.

Among the structures (defined over a set D) that the author considers are (a) the mappings of D into a fixed set R, (b) permutations of the elements of D, (c) cyclic permutations of the elements of D, (d) "nestings" of D, i.e., decreasing sequences of subsets of D, and (e) graphs with the elements of D as nodes. In these examples the group G which is associated with the set D, and which defines equivalent structures, is the symmetric group.

The author then defines "TH-structures", which are equivalence classes of mappings of a set T into D, the equivalence being defined by means of a group H of permutations of the elements of T. (Mappings f_1, f_2 are equivalent if $f_1 = f_2h$ for some h in H.) A formula is given for the U-polynomial of a TH-structure, and some specific and general results are given for colourings of the set D, and for "colourings with colour indifference", for which permutation of the colours is regarded as inessential.

The paper concludes with the consideration of pairs, triples, etc., of structures on one and the same set D.

It should be noted that the coloured graphs discussed as special cases of coloured structures are not subject to the restriction (usually tacitly implied) that adjacent nodes are not given the same colour. In particular, the bicoloured graphs enumerated in the paper are not the same as those enumerated by Harary [Pacific J. Math. **8** (1958), 743–755; MR **21** #2598], and the two enumerations are therefore essentially different.

{There are two misprints which might cause confusion. In equation (3.8) first line, the power of z should be $km/(k,m)$; and in (3.13) the exponent should be c_h, not c_n.} *R. C. Read* (Kingston)

ORDER, LATTICES

See also 13, 171.

Janowitz, M. F. 43

On the antitone mappings of a poset.

Proc. Amer. Math. Soc. **15** (1964), 529–533.

$(L, ')$ is an involution poset if L is a partially ordered set with largest element 1 and $'$ is a mapping (called an involution) of L into L such that $e''=e$ and $e \leqq f \Rightarrow f' \leqq e'$ for any $e, f \in L$. $(L, ')$ is called orthomodular if, in addition, for any $e, f \in L$, (i) $e \leqq f' \Rightarrow e \vee f$ exists, and (ii) $e \leqq f \Rightarrow f = e \vee (f' \vee e)'$. A mapping α is antitone if $e \leqq f \Rightarrow f\alpha \leqq e\alpha$. The main theorem gives necessary and sufficient conditions for an antitone mapping on an arbitrary poset L with 0 and 1 to turn L into an orthomodular lattice. The interest of this result stems from Mackey's use [*Lecture notes on the mathematical foundations of quantum mechanics*, Harvard Univ., Cambridge, Mass., 1960] of a certain orthomodular lattice for the "logic" of quantum mechanics. *E. Mendelson* (Flushing, N.Y.)

Jacobs, Eugene; Schwabauer, Robert 44

The lattice of equational classes of algebras with one unary operation.

Amer. Math. Monthly **71** (1964), 151–155.

The notion of the title is attributed to Hugo Ribeiro. The authors consider algebras $\langle A, {}^1\rangle$, where 1 is a unary operation on A which maps $x \in A$ onto $x^1 \in A$. For $n = 0, 1, 2, \cdots$, x^n is defined recursively by the equations: $x^0 = x$, $(x^n)^1 = x^{n+1}$. The equational class determined by an equation E is denoted by (E). Every equational class can be represented uniquely in one of the forms $(x^c = x^{c+d})$ or

$(x^c = y^c)$, where $c \geqq 0$, $d \geqq 0$. Under inclusion these classes form a lattice $\mathscr{L}$. The main result gives rules for computing the meet and the join of any two classes. It follows that $\mathscr{L}$ is distributive and has a universal upper bound $(x = x)$ and two atoms $(x^1 = y^1)$ and $(x = x^1)$.

A. Daigneault (Berkeley, Calif.)

Daigneault, Aubert 45

On automorphisms of polyadic algebras.

Trans. Amer. Math. Soc. **112** (1964), 84–130.

This paper is about (locally finite) polyadic algebras. The paper is divided into three chapters. In the first chapter the notions of homogeneous extensions and normal extensions are introduced. A series of results involving amalgamation properties, automorphisms, and extensions of polyadic algebras are proved, and extended to polyadic equality algebras when possible. Some of the theorems are related to the general Galois theory of Krasner [J. Math. Pures Appl. (9) **17** (1938), 367–385; ibid. (9) **18** (1939), 417–418; MR **1**, 198]. One main result is that every simple extension A_1 of a simple polyadic algebra A can be extended to a simple rich homogeneous normal extension B of A. A second main result is that the class of all polyadic algebras (and also certain subclasses) has a unique α^+-universal-homogeneous algebra for each infinite cardinal α; this result uses a theorem of Jónsson [Math. Scand. 8 (1960), 137–142; MR **23** #A2328] which depends on the generalized continuum hypothesis.

The second chapter is closely related to the reviewer's paper [Trans. Amer. Math. Soc. **107** (1963), 382–408; MR **26** #6054]. The notion of a generalized ultrapower is introduced and is shown to be equivalent to the notion of a limit ultrapower. The principal theorem concerns the representation of simple full polyadic algebras and generalized ultrapowers. When translated into the language of model theory, the theorem becomes a generalization of the reviewer's characterization of the class of all elementary extensions of a complete model by means of limit ultrapowers.

Chapter 3 contains a discussion of the connection between the algebraic results of the first two chapters and the theory of models of first-order logic. The results of Chapter 1 imply some known model-theoretic theorems of Beth [Nederl. Akad. Wetensch. Proc. Ser. A **56** (1953), 330–339; MR **15**, 385], Svenonius [Theoria (Lund) **25** (1959), 173–178; MR **25** #1986b], and Morley and Vaught [Math. Scand. **11** (1962), 37–57; MR **27** #37]. The results of Chapter 2 yield a characterization of elementary equivalence using generalized ultrapowers which is similar to the characterizations by the reviewer [loc. cit.] and by Kochen [Ann. of Math. (2) **74** (1961), 221–261; MR **25** #1992].

H. J. Keisler (Madison, Wis.)

GENERAL MATHEMATICAL SYSTEMS

See also 45.

O'Keefe, Edward S. 46

Independence in the small among universal algebras.

Math. Ann. **154** (1964), 273–284.

This paper is devoted to the proof of the following theorem: Let S be a finite set of constantive, primal in the small algebras, all of the same species, and let S contain no isomorphic pairs of algebras. If all the primitive operations of the algebras are surjective, then the set is independent in the small.

An algebra is constantive if every element can be represented by an expression. A set of algebras $\{\mathfrak{A}_1, \cdots, \mathfrak{A}_n\}$ is called independent in the small if for any set of functions $f_i(x_1, \cdots, x_r)$ and for every family of finite sets $N_i \subset A_i$ there exists an expression $E(x_1, \cdots, x_r)$, such that $\{a_1, \cdots, a_r\} \subset N_i$ implies $f(a_1, \cdots, a_r) = E(a_1, \cdots, a_r)$. If the set consists of only one algebra $\mathfrak{A}$, $\mathfrak{A}$ is called primal in the small.

The author first examines the consequences of the assumption that the operations are surjective, and then gives a condition which is equivalent to independence in the small. Let $\mathfrak{A}_i = \langle A_i, O_1, \cdots, O_m \rangle$ if for each family of finite sets $N_i \subset A_i$ there exists an expression $E(x_1, \cdots, x_r)$ such that for each i, $\{a_1, \cdots, a_r\} \in N_i$ implies $E(a_1, \cdots, a_r) = a_i$, then the algebras are independent in the small. He then proves that under his general assumptions this condition holds, first for two algebras and then for any finite set of algebras.

Literature: Foster [same Ann. **143** (1961), 29–58; MR **23** #A85; ibid. **148** (1962), 173–191; MR **27** #5714].

O. Pretzel (Berlin)

Sioson, Federico M. 47

Généralisation d'un théorème d'Hopkins-Brauer.

C. R. Acad. Sci. Paris **257** (1963), 1890–1892.

The author generalizes to algebras with a single m-ary operation R. H. Bruck's result: If S is a semigroup with zero which satisfies the minimal condition for nil ideals, then the nil radical of S is nilpotent [R. H. Bruck, *A survey of binary systems*, Theorem 8.1, Springer, Berlin, 1958; MR **20** #76].

G. B. Preston (Clayton)

Xu, Bao-lu [Hsu, Pao-Lu] 48

An association scheme $M_3(6)$ which is not an L_3-scheme.

Acta Math. Sinica **14** (1964), 177–178 (*Chinese*); *translated as Chinese Math.* **5** (1964), 197–198.

Soient un quasigroupe $Q = E(\ )$ et F l'ensemble produit EE. On définit une application de l'ensemble FF sur l'ensemble $(+, -)$ par les conditions suivantes: $\forall a, b, c, d \in E$, $a \neq b$, $c \neq d$, (i) $(ab, cd) = + \vee -$; (ii) $(ab, cd) = (cd, ab)$; (iii) $(ab, cd) = + \rightleftarrows b = d \vee a = c \vee ab = cd$; (iv) $(ab, cd) = - \rightleftarrows b \neq d \wedge a \neq c \wedge ab \neq cd$. L'auteur appelle cela un L_3-schéma d'association. Soit un ensemble F de s^2 éléments. On définit une application de FF sur l'ensemble $(+, -)$ de la façon suivante; (I) $\forall x \neq y \in F$, $xy = + \vee -$; (II) $xy = yx$; (III) $\forall x \in F$, l'ensemble des $y \neq x$ tels que $xy = +$ a pour puissance $3(s-1)$; (IV) Si $xy = +$, le nombre des éléments $z \neq x \neq y \in F$, tels que $xz = yz = +$ est s; (V) Si $xy = -$, le nombre des éléments $t \neq x$, $\neq y \in F$ tels que $xt = yt = +$ est 6. Cela s'appelle un M_3-schéma d'association. Tout L_3 est aussi un M_3. La réciproque n'est pas vraie: on donne un exemple de $M_3(6)$ qui n'est pas un L_3.

A. Sade (Marseille)

Feferman, Solomon 49

★The number systems. Foundations of algebra and analysis.

Addison-Wesley Publishing Co., Inc., Reading, Mass.-London, 1964. xii+418 *pp*. $8.75.

One of the major differences between American and Soviet

mathematics is that while many Soviet textbooks are written by prominent mathematicians (Vinogradov, Kolmogorov, Kurosh, Mal'cev, Novikov, to cite a few), very few prominent American mathematicians have ever written textbooks on the undergraduate level. Recently, the picture is changing in America, and the book under review is an excellent example of this new trend.

The contents of the book are the following (after the chapter we list in parentheses a few topics from that chapter). Chapter 1: The logical background; Chapter 2: The set-theoretical background (sets, relations, functions, mathematical systems); Chapter 3: The positive integers (Peano systems, recursive definitions, order, sums and products); Chapter 4: The integers and integral domains (construction and characterization of integers, elementary number theory, congruences); Chapter 5: Polynomials (polynomial functions and forms, transcendental extensions, fundamental theorem of symmetric polynomials); Chapter 6: The rational numbers and fields (ordered fields, fields of quotients, rational numbers, algebraic equations, factorization of polynomials); Chapter 7: Real numbers (continuously ordered fields, basic properties of real numbers, infinite series, roots of polynomials, algebraic and transcendental numbers); Chapter 8: The complex numbers (basic properties of polynomials and continuous functions, the fundamental theorem of algebra, roots of polynomials, cubic and fourth-degree equations); Chapter 9: Algebraic number fields and field extensions (simple transcendental and algebraic extensions, finite field extensions, applications to geometric construction problems); Appendix I: Some axioms for set theory; Appendix II: The analytical basis of the trigonometric functions.

Unfortunately, such an outline cannot show the best features of the book, and to illustrate, we cite a few examples. § 2 of Chapter 1: Logic. An excellent informal introduction to logic: mathematical statements and their structures; existence statements (constructive and nonconstructive); logical connectives. Some of the remarks in this section are interesting even for research mathematicians. § 1 of Chapter 2: Sets. Sets as abstractions from conditions. A very convincing argument why it is useful to introduce sets which are not defined by conditions. § 4 of Chapter 2. Mathematical systems of relations and functions. Such an early introduction of mathematical systems and isomorphism seems to be very useful.

Another interesting feature is that whenever a number system is extended, practical, algebraic and geometric motivations are given why is it necessary to do so (pp. 101, 103, 183, 184, 222, 224).

Several results are included which are usually not discussed on this level; the fundamental theorem of symmetric polynomials is a good example (though it should be mentioned that this is included in almost every European textbook of algebra which is on a comparable level). *G. Grätzer* (University Park, Pa.)

THEORY OF NUMBERS

See also 32, 34, 96, 97, 104, 109, 110, 281, 522.

Ehrhart, Eugène 50

Sur les points entiers d'un tétraèdre.

C. R. Acad. Sci. Paris **258** (1964), 4182–4183.

If a, b, c are integers relatively prime to each other, the author proves a formula for the number of integer points within the domain $X/a+Y/b+Z/c\leqq 1$. His argument is elementary and makes use of Euclid's algorithm. *S. Knapowski* (Marburg)

Dixon, John D. 51

Another proof of Lagrange's four square theorem.

Amer. Math. Monthly **71** (1964), 286–288.

A short new proof is given of the famous theorem that every positive integer can be written as the sum of four squares. *W. H. Mills* (Princeton, N.J.)

Yahya, Q. A. M. M. 52

An analogue of the Goldbach problem. (Spanish summary)

Rev. Mat. Hisp.-Amer. (4) **24** (1964), 66–72.

The author determines the number of solutions of the congruence $m\equiv x_1+\cdots+x_k \pmod n$ in $x_i \pmod n$, $(x_i, n)=1$, $i=1,\cdots,k$.

{Reviewer's remark: Reference is made to J. D. Dixon [Canad. Math. Bull. **3** (1960), 121–126; MR **23** #A844] and to D. Rearick [Amer. Math. Monthly **70** (1963), 837–840; MR **28** #2077].} *E. Cohen* (Knoxville, Tenn.)

Carlitz, L. 53

A basic analog of the multinomial theorem.

Scripta Math. **26**, 317–321 (1963).

The identities

$$\prod_{r=0}^{n-1}(1+p^r x) = \sum_{r=0}^{n} p^{r(r-1)/2}\begin{bmatrix} n\\ r\end{bmatrix}x^r,$$

$$\prod_{r=0}^{n-1}(1-p^r x)^{-1} = \sum_{r=0}^{\infty}\begin{bmatrix} n+r-1\\ r\end{bmatrix}x^r,$$

where

$$\begin{bmatrix} n\\ r\end{bmatrix} = \frac{(1-p^n)(1-p^{n-1})\cdots(1-p^{n-r+1})}{(1-p)(1-p^2)\cdots(1-p^r)},\qquad \begin{bmatrix} n\\ 0\end{bmatrix}=1,$$

are generalized to

$$\prod_{r=0}^{n-1}(1+p^r x+q^r y) = \sum_{r+s\leqq n} A_{rs}x^r y^s,$$

$$\prod_{r=0}^{n-1}(1-p^r x-q^r y)^{-1} = \sum_{r,s=0}^{\infty} B_{rs}x^r y^s,$$

by obtaining explicit expressions for A_{rs} and B_{rs}, including also the case when n tends to infinity. *M. S. Cheema* (Tucson, Ariz.)

Mullin, Albert A. 54

Elementary relations between the fundamental theorem of arithmetic, Schnirelmann's classical theorem, and Goldbach's conjecture.

Z. Math. Logik Grundlagen Math. **10** (1964), 199–201.

The fundamental theorem of arithmetic enables us to express any natural number $n>1$ in the form $n=p_1^{a_1}\cdots p_m^{a_m}$, where the p_i are primes and the exponents a_i are again natural numbers >1. If these exponents are again expressed in this fashion and this process is continued until it comes to an end, a collection of primes is obtained; the author denotes the sum of all these primes

by $\varphi^*(n)$. The results of the paper are expressed in terms of this function. It is easily seen, for example, that every natural number >1 is a sum of a finite number of primes (Schnirelmann's theorem) if and only if the range of $\varphi^*(n)$ is the set of all natural numbers >1. He also constructs the n-fold iterate of $\varphi^*(n)$ and shows that the value of the resulting function is always a prime.

H. W. Brinkmann (Swarthmore, Pa.)

Minc, Henryk; Sathre, Leroy 55

Some inequalities involving $(r!)^{1/r}$.

Proc. Edinburgh Math. Soc. (2) **14** (1964/65), 41–46.

Let $\phi(r)=(r!)^{1/r}$, $r>1$, r integer. The authors establish: (A) If $\psi(r)=r\phi(r+1)/\phi(r)$, then $\psi(r)-\psi(r-1)>1$; (B) $r<\psi(r)<r+1$; (C) $\sum[\phi(r_i)]^{-1}\leqq\prod\psi(r_i)/r_i$, if $r_i\geqq c+1$ and the sums taken over $i=1,\cdots,c$. {Numerical evaluation shows that apparently $\psi(r)-\psi(r-1)\downarrow 1$ monotonically; $\psi(10)-\psi(9)=1.01$. Hence (A) is remarkably exact.}

J. L. Brenner (Palo Alto, Calif.)

Stevens, Harlan 56

Linear homogeneous equations over finite rings.

Canad. J. Math. **16** (1964), 532–538.

The author first proves the following theorem. For any finite ring $\mathfrak{R}$ of q elements let $\{\mathfrak{H}_i\}$ be a collection of s subsets of $\mathfrak{R}$, each containing h_i $(i=1,2,\cdots,s)$ members, and let $D(\mathfrak{H}_i)$ denote the set of all differences $d'-d''$ with d' and d'' from $\mathfrak{H}_i$, including $d'=d''$. Furthermore, suppose that $\prod_{i=1}^{s}h_i>q^r$ $(r<s)$. Then the system of r linear equations

$$\sum_{i=1}^{s}a_{ji}x_i=0 \qquad (j=1,2,\cdots,r),$$

where $a_{ij}\in\mathfrak{R}$ or is a rational integer, has a nontrivial solution such that $x_i\in D(\mathfrak{H}_i)$ $(i=1,2,\cdots,s)$.

As an application of the theorem the author gives a theorem of Brauer and Reynolds on systems of linear congruences [same J. **3** (1951), 367–374; MR **14** #21], as well as a more general result in which both coefficients and unknowns are matrices. He also makes applications to the distribution of primitive roots in GF(p^k) and of nth power nonresidues.

L. Carlitz (Durham, N.C.)

Lochs, Gustav 57

Vergleich der Genauigkeit von Dezimalbruch und Kettenbruch.

Abh. Math. Sem. Univ. Hamburg **27** (1964), 142–144.

Using a result of A. Khintchine and P. Lévy the author proves the following theorem. Let m be the exact number of partial quotients of the regular continued fraction of a number $x\in[0,1)$ which can be obtained from the first n decimals of x. Then for almost all x

$$\lim_{n\to\infty}\frac{m}{n}=\frac{6\log 2\log 10}{\pi^2}=0.97027014\cdots.$$

J. E. Cigler (Vienna)

Salié, Hans 58

Arithmetische Eigenschaften der Koeffizienten einer speziellen Hurwitzschen Potenzreihe.

Wiss. Z. Karl-Marx-Univ. Leipzig Math.-Natur. Reihe **12** (1963), 617–618.

Put

$$\frac{\cosh z}{\cos z}=\sum_{n=0}^{\infty}S_{2n}\frac{z^{2n}}{(2n)!}.$$

The reviewer [Math. Mag. **32** (1958/59), 132, 136; MR **21** #2754] proved that S_{2n} is divisible by 2^n and that

$$S_{2n}'=2^{-n}S_{2n}\equiv(-1)^{n(n-1)/2} \pmod 4.$$

The present paper contains a refinement of this result. Put $\chi(n)=(-1)^{n(n-1)/2}$ and

$$\chi_k(n)=0 \quad (n\equiv 0,1,\cdots,2^k-1 \pmod{2^{k+1}}),$$
$$=1 \quad (n\equiv 2^k,\cdots,2^{k+1}-1 \pmod{2^{k+1}}).$$

Then it is proved that

$$S_{2n}'\equiv\chi(n)+4\chi_1(n)+8\chi_2(n) \pmod{16}.$$

L. Carlitz (Durham, N.C.)

Rauzy, Gérard 59

Répartition modulo 1 pour des suites partielles d'entiers. Développements en série de Taylor donnés sur des suites partielles.

C. R. Acad. Sci. Paris **258** (1964), 4881–4884.

Let $\mathcal{J}$ be a set of positive integers. The least upper bound of the set of real numbers $A\geqq1$ satisfying $\mathcal{J}\supseteq\{n: x\leqq n<Ax\}$ for infinitely many x is called the frequency of $\mathcal{J}$. The main theorem is the following: Let $\vartheta>1$ be a real algebraic number not belonging to the class T of algebraic integers all of whose conjugates have modulus $\leqq1$. Then there exist two constants $A(\vartheta)>1$, $\varepsilon(\vartheta)>0$ such that, for every set $\mathcal{J}$ of frequency $A>A(\vartheta)$,

$$\lim_{n\in\mathcal{J}\to\infty}\|\lambda\vartheta^n\|\geqq\varepsilon(\vartheta)$$

for all real $\lambda\neq0$ ($\|X\|=\min_n|X-n|$).

J. E. Cigler (Vienna)

Turyn, Richard J. 60

The linear generation of the Legendre sequence.

J. Soc. Indust. Appl. Math. **12** (1964), 115–116.

Let p be a prime and $\chi(h)=\pm1$ be the Legendre symbol of quadratic residuacity modulo p. The question raised by the author is how to generate $\chi(h)$, or its equivalent $(\chi(h)+1)/2$, for $h=1(1)p-1$. He answers the question of how not to do this by a linear recurring series with constant coefficients mod 2. Why anyone would attempt to do this is not made clear. In any case the recurrence is shown to be of degree $(p-1)/2$. {The reviewer has generated such sequences for p up to 10^6 by a severely practical method using second differences and bit registration in a straightforward manner.}

D. H. Lehmer (Berkeley, Calif.)

Rotkiewicz, André 61

Sur les nombres pseudopremiers de la forme $ax+b$.

C. R. Acad. Sci. Paris **257** (1963), 2601–2604.

n is a pseudo-prime if $n|(2^n-2)$. The author proves that every arithmetic progression $ax+b$ $(x=0,1,2,3,\cdots)$, where a and b are relatively prime positive integers, contains an infinity of pseudo-primes. The proof uses the Brun-Titchmarsh estimate of the number of primes $\equiv l \pmod k$ and not exceeding x.

S. Chowla (University Park, Pa.)

Rotkiewicz, A. [Rotkiewicz, André] 62

Sur les nombres composés n de la forme $cx+d$ pour lesquels $n|a^{n-1}-b^{n-1}$.

Bull. Soc. Roy. Sci. Liège **32** (1963), 823–829.

Le théorème suivant est démontré: quels que soient les nombres naturels a et b, toute progression arithmétique $cx+d$, où c et d sont des nombres naturels premiers entre eux, contient une infinité des nombres composés n tels que $n|a^{n-1}-b^{n-1}$.

La démonstration est similaire à la démonstration de l'auteur pour le cas $a=2$, $b=1$ [see #61 above] mais les détails sont plus compliqués.

A. Schinzel (Columbus, Ohio)

Graham, R. L. 63

Complete sequences of polynomial values.

Duke Math. J. **31** (1964), 275–285.

Let $S=(s_1, s_2, \cdots)$ be a sequence of real numbers and $P(S)$ the set of sums of the form $\sum_{k=1}^{\infty} \varepsilon_k s_k$, where ε_k is 0 or 1 and all but a finite number of ε_k are 0. If all sufficiently large integers belong to $P(S)$, then S is said to be complete. It is proved in the paper in a simple and elementary manner that if $f(x)$ is a polynomial with real coefficients,

$$f(x) = \alpha_0+\alpha_1\binom{x}{1}+\cdots+\alpha_n\binom{x}{n}, \qquad \alpha_n \neq 0,$$

$$\binom{x}{r} = \frac{x(x-1)\cdots(x-r+1)}{r!},$$

then the sequence $S(f)=(f(1), f(2), \cdots)$ is complete if and only if (1) $\alpha_k=p_k/q_k$ for some integers p_k and q_k with $(p_k, q_k)=1$ and $q_k\neq 0$ for $0\leqq k\leqq n$; (2) $\alpha_n>0$; (3) g.c.d.$(p_0, p_1, \cdots, p_n)=1$.

M. J. Wonenburger (Toronto, Ont.)

Graham, R. L. 64

A theorem on partitions.

J. Austral. Math. Soc. **3** (1963), 435–441.

It is first shown that every integer greater than 77 can be partitioned into distinct positive integers whose reciprocals add to 1. This depends on a table of such partitions for the integers between 78 and 333, together with a pair of transformations by which this table may be extended recursively. Using this result and a further lemma, it is then shown that if α and β are positive rational numbers, there is an $r=r(\alpha, \beta)$ such that if $n>r$, there is a partition of n into integers larger than β whose reciprocals add to α.

W. J. LeVeque (Ann Arbor, Mich.)

Mordell, L. J. 65

The Diophantine equation $y^2=Dx^4+1$.

J. London Math. Soc. **39** (1964), 161–164.

W. Ljunggren [Skr. Norske Vid.-Akad. Oslo I Mat.-Naturv. Kl. **1936**, no. 12] has shown that the equation of the title has at most two solutions when D is a positive non-square integer, and that these can be found explicitly. In the present paper, the author obtains, by elementary methods, two sets of sufficient conditions that the equation have no solution. One is that $D\not\equiv 0, -1, 3, 8 \pmod{16}$, and that D have no factorization $D=pq$, where $p>1$, p is odd, $(p, q)=1$, and such that $p\equiv \pm 1$, $p-q\equiv \pm 1$, $p-4q\equiv \pm 1 \pmod{16}$. The other is similar.

W. J. LeVeque (Ann Arbor, Mich.)

Chawla, L. M.; Shafaat 66

On certain classes of diophantine equations defining equivalence relations over integers.

J. Natur. Sci. and Math. **3** (1963), 163–173.

This paper continues the study of equivalence relations on subsets S of the integers, determined by solutions of particular types of Diophantine equations of the general form (*) $F(x, y, z)=0$ [L. M. Chawla, same J. **3** (1963), 83–98; MR **27** #5725]. Given x and y in S, if (*) holds for some integral z, then $x\sim y$. This will be an equivalence relation for certain S and F.

The following types of equivalence relation and corresponding Diophantine equation are considered: (1) $x-y\equiv a \pmod{m}$ for some a in S. These are associated with equations of the form $f(ax+by)=mz$; (2) $x\equiv ay \pmod{p}$ for some a in S, with p an odd prime. These are associated with Diophantine equations of the form:

$$(x-a_1y)\cdots(x-a_ky) = pz,$$

$$((xy)^n-a_1)\cdots((xy)^n-a_k) = pz;$$

(3) $f(x)\sim f(y)$, where $f(x)$ is an integral polynomial, and $x\sim y$ is an equivalence relation of one of the previous types.

Necessary and sufficient conditions for each of these cases are obtained. The results are illustrated by examples.

O. Frink (University Park, Pa.)

Chawla, L. M. 67

On a determinantal diophantine equation.

J. Natur. Sci. and Math. **3** (1963), 175–178.

The author solves a generalization of the problem of finding all triangles with integral vertices whose areas are perfect squares; that of finding all r-dimensional simplexes whose vertices have integral coordinates, and with hypervolume a perfect rth power. More generally, he finds all integral solutions of the equation $D=m$, where D is an $(r+1)$-rowed determinant with 1 everywhere in the last column, the other elements being unknown integers; m is a given integer. The solutions are found explicitly, but are too complicated to list here. *O. Frink* (University Park, Pa.)

Ferentinou-Nicolacopoulou, Jeanne 68

Une propriété des diviseurs du nombre $r^{r^m}+1$. Applications au dernier théorème de Fermat.

Bull. Soc. Math. Grèce (N.S.) **4** (1963), *no.* 1, 121–126.

Let r, m, n be integers, $r\geqq 2$, $m\geqq 0$, $n\geqq 2$, let n^k be the largest power of n which divides $r^{r^m}+1$. The author proves that if r is even, then $r^{n^k-1}\not\equiv \pm 1 \pmod{n^{k+1}}$, while if r is odd, $r^{n^k-1}\not\equiv(-1)^{n+1} \pmod{n^{k+1}}$. The proofs are straightforward. Corollaries involving Fermat's Last Theorem are derived from the result (with n an odd prime and $k=1$) in an obvious fashion by using the well-known results of Morishima, Rosser and Furtwangler.

S. L. Segal (Rochester, N.Y.)

Kapferer, Heinrich 69

Verifizierung des symmetrischen Teils der Fermatschen Vermutung für unendlich viele paarweise teilerfremde Exponenten E.

J. Reine Angew. Math. **214/215** (1964), 360–372.

The author proves the following theorem: Let E and n be positive rational integers, and let $(E, 6)=1$. If x^E+y^E+

$z^E = 0$ has nonzero rational integral solutions x, y, z such that $(x, y) = (y, z) = 1$, and, for a divisor $a > 0$ of x, $a^n - 1 = EF$ with $(E, F) = 1$, where F is a rational integer, then x is divisible by E. Now, let $K_n(a) = \prod_{t|n} (a^{n/t} - 1)^{\mu(t)}$, $n = 1, 2, 3, \cdots$, where the notation indicates a product over all the divisors of n, $\mu(t)$ is the Möbius function and a is a positive integer. Then $a^n - 1$ is divisible by $K_n(a)$. Using $K_{n_1}(a)$ or $K_{n_2}(a)/p$ with a prime p, the author proves that there exist infinitely many E and that any two of these E are prime to each other. Further, using the above theorem, the author proves the following result: Let g be any even positive integer and prime to 3, and let π_t be a prime such that $\pi_t > 3^{\varphi(g)}$ and $\pi_t - 1 = E_t g$ with $(E_t, g) = 1$, where $\varphi(g)$ is the Euler function. If $x^{E_t} + y^{E_t} + z^{E_t} = 0$, $(x, y) = (y, z) = 1$ has nonzero rational integral solutions, then one of x, y, z is divisible by E_t. There exist infinitely many such E_t and any two of these E_t are relatively prime.

T. Morishima (Tokyo)

Lang, Serge 70

Transzendente Zahlen.

Bonn. Math. Schr. No. 21 (1963), viii + 22 *pp.*

The main text contains an elegant and elementary exposition of a theorem and proof due in substance to T. Schneider [*Einführung in die transzendenten Zahlen*, Satz 13, Springer, Berlin, 1957; MR **19**, 252], concerning the algebraic dependence of sets of functions satisfying suitable analytic and arithmetic conditions. In the foreword, various unsolved problems are discussed, one of them being to find circumstances under which the following assertion is true: Let G be an algebraic group variety of dimension n, defined over an algebraic number field. Let T be its tangent space at the unit element, with its structure as algebraic vector space. Let $\alpha \in T$ be a nonvanishing tangent vector, and let the curve $\exp(t\alpha)$ be of dimension n over the field of rational functions in t. Then the transcendence degree of the point $(\alpha, \exp(\alpha))$ is at least n.

W. J. LeVeque (Ann Arbor, Mich.)

Hooley, Christopher 71

On the representation of a number as the sum of two h-th powers.

Math. Z. **84** (1964), 126–136.

For any positive integer n let $r_h(n)$ denote the number of solutions of $n = x^h + y^h$ in non-negative integers x and y, where the order of the terms is relevant; put

$$N_h(x) = \sum_{\substack{r_h(n) > 0 \\ n \leq x}} 1.$$

If h is any prime $\geqq 7$, the author proves that

$$\sum_{n \leq x} r_h(n)^2 = \frac{1}{h} \frac{\Gamma(1/h)^2}{\Gamma(2/h)} x^{2/h} + O\left(\frac{x^{2/h} \log\log x}{(\log x)^{\Delta_h}}\right),$$

where $\Delta_h = 1 - 2(h-1)^{-1/2}$, and

$$N_h(x) = \frac{1}{4h} \frac{\Gamma(1/h)^2}{\Gamma(2/h)} x^{2/h} + O\left(\frac{x^{2/h} \log\log x}{(\log x)^{\Delta_h}}\right).$$

The same formulae hold also for $h = 3$ and $h = 5$ with Δ_h replaced by $2 - 2(h-1)^{-1/2}$, and they have been proved for $h = 3$ by the author [same Z. **82** (1963), 259–266; MR **27** #5742], and follow by the same method also for $h = 5$.

K. Mahler (Canberra)

Mordell, L. J. 72

On Lerch's class number formulae for binary quadratic forms.

Ark. Mat. **5**, 97–100 (1964).

In this paper, the author gives new proofs of the following two theorems of Lerch. Theorem 1: Let $D > 0$ and $-\Delta < 0$ be two fundamental discriminants with $(D, \Delta) = 1$. Then

$$h(-\Delta D) = 2 \sum_{a=1}^{\Delta/2} \left(\frac{-\Delta}{a}\right) \sum_{n=1}^{aD/\Delta} \left(\frac{D}{n}\right),$$

$$-h(-\Delta D) = 2 \sum_{a=1}^{D/2} \left(\frac{D}{a}\right) \sum_{n=1}^{a\Delta/D} \left(\frac{-\Delta}{n}\right).$$

Theorem 2: Let $D_1, \cdots, D_r$ denote fundamental discriminants relatively prime in pairs with absolute values $\Delta_1, \cdots, \Delta_r$ and with $2s+1$ of the D negative. Then

$$h(D_1, \cdots, D_r) = (-1)^s \sum \left(\frac{D_1}{n_1}\right) \cdots \left(\frac{D_r}{n_r}\right) \left[\frac{n_1}{\Delta_1} + \cdots + \frac{n_r}{\Delta_r}\right],$$

where the square bracket denotes the integer part and the summation is taken over $0 < n_1 < \Delta_1, \cdots, 0 < n_r < \Delta_r$. The author's proof is very simple and elementary, being deduced from the well-known formula

$$h(-\Delta) = -\frac{1}{\Delta} \sum_{n=1}^{\Delta} \left(\frac{-\Delta}{n}\right) n,$$

where $-\Delta$ is a fundamental discriminant. He first gives a proof of Theorem 2; then Theorem 1 is deduced from Theorem 2 by using a simple transformation. One might object to his use of notation (for example, he writes $\prod (D_1/n_1)$ for $(D_1/n_1) \cdots (D_r/n_r)$); however, this hardly causes any confusion.

T. Tamagawa (New Haven, Conn.)

Lenz, Hanfried 73

Einige Ungleichungen aus der Algebra der quadratischen Formen.

Arch. Math. **14** (1963), 373–382.

In this paper the author improves some inequalities of Tsuzuku [J. Math. Soc. Japan **6** (1954), 325–331; MR **16**, 669]. Let φ denote a quadratic form in n variables, of non-zero determinant over a non-formally real field K of characteristic different from 2. Let s denote the least number r such that -1 is the sum of r squares in K and t is determined by $2^t \leqq s < 2^{t+1}$. The author denotes by φ^k the k-fold direct sum of φ with itself, $E(\varphi)$ the vector space determined by φ and $G(\varphi)$ the set of forms φ for which $\varphi(x) \neq 0$ with x in $E(\varphi)$. $Q(\varphi)$ denotes the set of non-zero square classes represented by φ, $q(\varphi)$ is the weight of $Q(\varphi)$ and $G_k = ((1)^k)$.

Two of the theorems proved by the author are the following. Theorem 2: Let k be one of the numbers 1, 2, 4, 8 and $a \notin G_k$, $-a \in G_s$ and $s > k$. Then either $(1, a)^k$ is a null form or $q((1, a)^k) \geqq 4q_k$, where q_k is the weight of $Q((1)^k)$. Theorem 3: Let e be one of the numbers 0, 1, 2, 3, $f = 2^e$ and $a \in G_f$. Then $q(1, a) \geqq 2^{t-e}$.

Burton W. Jones (Boulder, Colo.)

Cohen, Eckford 74

Arithmetical notes. II. An estimate of Erdős and Szekeres.

Scripta Math. **26**, 353–356 (1963).

Part I appeared in Proc. Amer. Math. Soc. **12** (1961),

214-217 [MR 24 #A1329]. The problem of counting the number of abelian groups of a given order has been raised by Erdős and Szekeres [Acta Litt. Sci. Szeged 7 (1934), 95-102] who actually solved a somewhat more general number-theoretic problem. This has been generalized and refined further by Richert [Math. Z. 56 (1952), 21-32; MR 14, 349] and by Bateman and the reviewer [Illinois J. Math. 2 (1958), 88-98; MR 20 #3205]. The present author generalizes the corresponding number-theoretic problem as follows. Let $n=p_1^{e_1}p_2^{e_2}\cdots p_t^{e_t}$, where $e_i=ak$ or $e_i=ak+b$ ($k\geq 0$, $a>0$, $b>0$ are all integers; $i=1, 2, \cdots, t$). The number of such $n\leq x$ is denoted by $R_{a,b}(x)$. If $\tau_{a,b}(n)$ is the number of decompositions of n in the form $n=d^a\delta^b$, and $T_{a,b}(x)=\sum_{n\leq x}\tau_{a,b}(n)$, then

$$R_{a,b}(x) = \sum_{n\leq x^{1/2b}} \mu(n)T_{a,b}(x/n^{2b}).$$

Using different estimates for $T_{a,b}(x)$, the author obtains corresponding results for $R_{a,b}(x)$. The best error term is obtained by using Richert's [loc. cit.] estimate for $T_{a,b}(x)$ and is given by $R_{a,b}(x)=\alpha^* x^{1/a}+\beta^* x^{1/b}+E(x)$, where $\alpha^*=\zeta(b/a)/\zeta(2b/a)$, $\beta^*=\zeta(a/b)/\zeta(2)$ and

$$\begin{aligned} E(x) &= O(x^{1/2b}) \quad \text{if} \quad 2b<5a,\\ &= O(x^{1/2b}\log x) \quad \text{if} \quad 2b=5a,\\ &= O(x^{2/(2b+5a)}) \quad \text{if} \quad 2b>5a. \end{aligned}$$

E. Grosswald (Philadelphia, Pa.)

Knapowski, S.; Turán, P. 75

Further developments in the comparative prime-number theory. I.

Acta Arith. 9 (1964), 23-40.

This is Part I of a new series on this subject, the earlier series of eight papers having appeared in Acta Math. Acad. Sci. Hungar. 13 (1962), 299-314 [MR 26 #3682a]; ibid. 13 (1962), 315-342 [MR 26 #3682b]; ibid. 13 (1962), 343-364 [MR 26 #3682c]; ibid. 14 (1963), 31-42 [MR 26 #3683a]; ibid. 14 (1963), 43-63 [MR 26 #3683b]; ibid. 14 (1963), 65-78 [MR 26 #3683c]; ibid. 14 (1963), 241-250 [MR 28 #70a]; ibid. 14 (1963), 251-268 [MR 28 #70b]. The present paper begins by posing 44 problems dealing with irregularities in the distribution of primes in the various residue classes modulo k; these are essentially amplifications of those posed in Part I of the first series. It is intended that the first few parts of the present series deal with what the authors call "accumulation problems". In this part the following theorem is proved: If k is a modulus for which the so-called Haselgrove condition holds, then there exists $T_0(k)$ such that for $T>T_0$ and for all $(l,k)=1$ there are suitable U_1, U_2 with $T\exp(-\log^{11/12}T)\leq U_1<U_2\leq T$ so that

$$\sum_{\substack{n\equiv 1(k)\\ U_1\leq n\leq U_2}} \Lambda(n) - \sum_{\substack{n\equiv l(k)\\ U_1\leq n\leq U_2}} \Lambda(n) > T^{1/2}\exp(-\log^{11/12}T)$$

holds; a symmetrical upper bound holds also, not necessarily for the same U_i. Here $\Lambda(n)$ has its usual meaning. The Haselgrove condition requires the explicit value of an $E=E(k)$ such that no $L(s,\chi)$ belonging to the modulus k vanishes for $0<\sigma<1$, $|t|\leq E$; it is to be noted that it is satisfied for $3\leq k\leq 10$. The methods used in the proof are similar to those used in the first series.

H. W. Brinkmann (Swarthmore, Pa.)

Gerl, Peter 76

Einige metrische Sätze in der Theorie der Gleichverteilung mod 1.

J. Reine Angew. Math. 216 (1964), 50-66.

Generalizing methods and results of J. F. Koksma [Composito Math. 2 (1935), 250-258] and W. J. LeVeque [Michigan Math. J. 1 (1953), 139-162; MR 14, 1067], the author proves some metrical theorems on the uniform distribution mod 1 of sequences in r-dimensional space. The first part of the paper is concerned with the usual concept of uniform distribution and the second one with uniform distribution with respect to certain weighted means. There are some applications to sequences of matrices, among which the following is typical: The sequence $\begin{pmatrix} a & b\\ c & d\end{pmatrix}^n$ is uniformly distributed mod 1 for almost all a, b, c, d in the domain $a\geq 0$, $b\geq 0$, $a+d\geq 1$, $ad-bc<0$.

J. E. Cigler (Vienna)

Chowla, S.; Hawkins, D. 77

Asymptotic expansions of some series involving the Riemann zeta function.

J. Indian Math. Soc. (*N.S.*) 26 (1962), 115-124.

Hardy and Littlewood (stimulated by a conjecture of Ramanujan) proved that the truth of

$$\sum_{n=0}^{\infty} \frac{(-x)^n}{n!\zeta(2n+1)} = O(x^{-1/4+\varepsilon})$$

is a necessary and sufficient condition for the truth of the Riemann hypothesis. {Remark: Hardy and Littlewood have proved the equivalence with the Riemann hypothesis if $\varepsilon=0$ [cf. Hardy and Littlewood, Acta Math. 41 (1917), 119-196; Titchmarsh, *The theory of the Riemann zeta-function*, p. 328, Clarendon, Oxford, 1951; MR 13, 741].} The authors of the present paper obtain the following result which has a superficial resemblance with the above, namely,

$$\text{(A)} \quad \sum_{n=2}^{\infty} \frac{(-x)^n}{n!}\zeta(n) = x\log x+(2C-1)x+\tfrac{1}{2}+O(e^{-c\sqrt{x}})$$

(x is large and positive; c is a certain positive constant). It is presumed that the error term cannot be essentially sharpened, but the authors assert that they are unable to prove this. Furthermore, they obtain the similar result

$$\text{(B)} \quad \sum_{n=2}^{\infty} (-1)^n\binom{x}{n}\zeta(n) = x\log x+(2C-1)x+o(1).$$

The proof of this formula is based upon the study of the ratio

$$g(x,s) = \frac{\sum_{n\leq x}(x-n)^s\left\{\frac{x}{n}-\left[\frac{x}{n}\right]\right\}}{\sum_{n\leq x}(x-n)^s}$$

for a fixed natural number s, as $x\to\infty$. It is shown that $\lim_{x\to\infty} g(x,s)=h(s)$ exists, and furthermore that $\lim_{s\to\infty} h(s)=\frac{1}{2}$. The paper ends with the proof of an exact expansion of the term $O(\exp(-c\sqrt{x}))$ in (A), viz.,

$$O(\exp(-c\sqrt{x})) = \sum_{n=1}^{\infty} \frac{K(2\pi ny)}{n}$$

with

$$K(y) = \frac{1}{4\pi i}\int_{\sigma=1/2} \frac{\Gamma(s)}{\Gamma(-s)}\frac{y^{-s}}{\cos\frac{1}{2}s\pi}\,ds.$$

The authors end with the remark that it is not hard to show that $K(y)$ is expressible in terms of "almost" Bessel functions, but it is not quite clear which kind of functions they mean.

{Remark: A part of the proof is obscured by the fact that on pp. 116, 118 in the same formula the letter m is used simultaneously as a variable index of summation, and as an arbitrary positive constant integer. It seems, however, that the results are right.}

S. C. van Veen (Delft)

Chowla, S.; Hawkins, D. 78

Asymptotic expansions of some series involving the Riemann zeta function. I, II.

Norske Vid. Selsk. Forh. (Trondheim) **35** (1962), 99–103; *ibid.* **35** (1962), 104–110.

The content of both these papers is identical with the preceding paper [#77]. *S. C. van Veen* (Delft)

Kesava Menon, P. 79

On a function of Ramanujan.

J. Indian Math. Soc. (N.S.) **25** (1961), 109–119 (1962).

Ramanujan [*Collected papers*, Cambridge Univ. Press, London, 1927] obtained a formula for

$$(1)\quad \Phi(2s+1,x) = \sum_{n=0}^{\infty}\{(x+n+1)^{1/2}-(x+n)^{1/2}\}^{2s+1} = \sum_{n=0}^{\infty}\{(x+n+1)^{1/2}+(x+n)^{1/2}\}^{-(2s+1)}\quad (s=1,2,3,\cdots)$$

in terms of the generalised zeta-function $\zeta(s,a) = \sum_{n=0}^{\infty} 1/(a+n)^s$ (Re $s>1$). He proved by induction that

$$(2)\quad \Phi(2s+1,x)+\tfrac{1}{2}[\{x^{1/2}+(x-1)^{1/2}\}^{2s+1} + \{x^{1/2}-(x-1)^{1/2}\}^{2s+1}]+(2s+1)\sum_{m=0}^{[(s-1)/2]}\frac{2^{2(s-2m-1)}}{s-m} \times\binom{2s-2m}{2m+1}\zeta(2m+\tfrac{1}{2}-s,x) = 0.$$

The author claims to give a direct proof of this result. He starts from a formula (which he calls "known") which ought to be, as it seems,

$$(3)\quad \frac{1}{2s+1}(\sqrt{y}-\sqrt{(y+1)})^{2s+1} = \sum_{m=0}^{s}\frac{2^{2(s-m)}}{2s-m+1}\binom{2s-m+1}{m}\{y^{s-m+1/2}-(-1)^m(y+1)^{s-m+1/2}\}.$$

He attains his aim indeed by a method which seems to be sound, but his development is quite unintelligible, as it is mutilated by an enormous quantity of grave errors and misprints (e.g., (1), (2) and (3) contain already six serious errors). Ramanujan's result (2) is expressed in a different form by means of Hurwitz's well-known formula for $\zeta(s,a)$ if Re $s<0$ (cf. Whittaker and Watson, *A course in modern analysis*, 4th ed., 13.15, Cambridge Univ. Press, Cambridge, 1927). By differentiation of (2) the author obtains Ramanujan's evaluation of

$$\Psi(2s+1,x) = \sum_{n=0}^{\infty}\frac{\{(x+n)^{1/2}+(x+n+1)^{1/2}\}^{-(2s+1)}}{\{(x+n)(x+n+1)\}^{1/2}}\quad (s=1,2,3,\cdots).$$

From his result the author tries to find simple transformations for certain sums, e.g.,

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{\{(x+n)^{1/2}+(x+n+1)^{1/2}\}^{2s+1}}.$$

He states as an example

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(\sqrt{n}+\sqrt{(n+1)})^3} = \frac{2\sqrt{2}}{3\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^{3/2}},$$

but an easy verification gives for the sum in the left-hand member a number between 0.93 and 0.96, but for the right-hand member gives $0.51\cdots$. *S. C. van Veen* (Delft)

Kesava Menon, P. 80

Series associated with Ramanujan's function $\tau(n)$.

J. Indian Math. Soc. (N.S.) **27** (1963), 57–65.

The series

$$F(s) = \sum_{1}^{\infty}\frac{\tau(n)}{n^s},\qquad f(s) = \sum_{1}^{\infty}\frac{\tau^2(n)}{n^s}$$

satisfy known functional equations. The former has an Euler product conjectured by Ramanujan and proved by Mordell. The author introduces a few other functions involving $\tau(n)$ and establishes identities between them.

S. Chowla (University Park, Pa.)

Levinson, N. 81

The prime number theorem from $\log n!$.

Proc. Amer. Math. Soc. **15** (1964), 480–485.

The author gives another proof of the prime number theorem, this time by applying the Wiener Tauberian theory with the kernel $K(s)=e^{-s}R(e^s)$, $s>0$; $K(s)=0$, $s<0$, where

$$R(x) = 1+2r(x)-\frac{4}{x}\int_1^x r(\xi)\,d\xi$$

and $r(x)=[x]-x+\frac{1}{2}$. Starting from the formula

$$\int_1^x\left[\frac{n}{y}\right]d\psi(y) = \log n! = n\log n - n + O(\log n),$$

where as usual

$$\psi(x) = \sum_{p^m\le x}\log p = \sum_{p\le x}\left[\frac{\log x}{\log p}\right]\log p,$$

he obtains the prime number theorem in the form $\psi(x)\sim x$ as $x\to\infty$.

In an addendum the author notes that an earlier proof based on $\log n!$ had been given by Ingham [J. London Math. Soc. **20** (1945), 171–180; MR **8**, 147]. It should be noted however that Ingham's proof is a corollary of more general Tauberian theorems of some interest (in fact, all the familiar equivalent forms of the prime number theorem follow easily and directly from Ingham's analysis), whereas the author confines himself to $\psi(x)\sim x$ alone.

S. L. Segal (Rochester, N.Y.)

Pan, Cheng-dong [Pan, Cheng-Tung] 82

A note on the large sieve method and its applications.

Acta Math. Sinica **13** (1963), 262–268 (*Chinese*); *translated as Chinese Math.* **4** (1963), 283–290.

The author proves the following. Theorem 1: Suppose q

is square-free, and let $k = \log q/\log A + 1$. If $k \leqq \log^3 A$, then for all primes p such that $(p, q) = 1$ and $A < p \leqq 2A$, except for at most $A^{1-\varepsilon}$ $(\varepsilon > 0)$ L-functions belonging to the modulus $D = pq$, when $\chi_D(n)$ is primitive with respect to p, $L(s, \chi_D)$ is not zero in the following region:

$$\sigma > 1 - \frac{\frac{2}{9}-\varepsilon}{k} \frac{\log D}{4 \log D + 2 \log(|t|+1)}, \qquad |t| \leqq T.$$

With regard to this result the author says: "The important difference between the results we have here obtained and those of A. Rényi is that here T may be arbitrarily large and is not limited to $|T| \leqq \log^3 D$. This point is useful in the theory of the distribution of primes" [see A. Rényi, Izv. Akad. Nauk SSSR Ser. Mat. **12** (1948), 57–78; MR **9**, 413; Amer. Math. Soc. Transl. (2) **19** (1962), 299–321; MR **24** #A1264]. From Theorem 1 he can deduce Theorem 2: Let $N_{\min}(p, k)$ represent the smallest positive non-residue of order k (mod p), where $A < p \leqq 2A$; then with the exception of at most $A^{1-\varepsilon}$ exceptional primes p, we always have

$$N_{\min}(p, k) < C_1(\log A)^{18+\varepsilon}.$$

{A sharper form of Theorem 2 when $k = p - 1$ is due to N. C. Ankeny, however, under the assumption of the "extended Riemann hypothesis".}

S. Chowla (University Park, Pa.)

Rieger, G. J. 83

Über die Summe von zwei n-ten Primzahlpotenzen.

Math. Z. **84** (1964), 137–142.

Erdős and Mahler [J. London Math. Soc. **13** (1938), 134–139] proved that if $A(u)$ is the number of solutions of

$$x_1^n + x_2^n = k, \qquad 0 < x_1, x_2 \leqq u^{1/n},$$

then, for $n \geqq 3$,

$$\liminf \frac{A(u)}{u^{2/n}} > 0 \quad \text{as } u \to \infty.$$

They also deduced from a theorem of Siegel that

$$\limsup \frac{A(u)}{u^{2/n}} < \infty.$$

The author proves that if $B(u)$ is the number of solutions of

$$p_1^n + p_2^n = k, \qquad 0 < p_1, p_2 \leqq u^{1/n},$$

where p_1, p_2 are primes, then

$$\text{(1)} \qquad \liminf \frac{B(u)}{u^{2/n}/\log^2 u} > 0.$$

Since, from the prime number theorem,

$$\limsup \frac{B(u)}{u^{2/n}/\log^2 u} < \infty,$$

it follows that (1) cannot be improved.

S. Chowla (University Park, Pa.)

Rieger, G. J. 84

Über die Darstellbarkeit quadratfreier Zahlen als Summe einer Primzahl und des Quadrates einer quadratfreien Zahl.

Acta Arith. **10** (1964/65), 1–7.

The author first proves the following generalization of Romanov's theorem: The sequence of square-free numbers which can be represented as a sum of a prime and the square of a natural number has a positive density. He is then able to replace the word "natural number" by "square-free number" in the above generalization. Finally it is shown that the density of the sequence of square-free numbers which can be written as a sum of a prime number from a fixed prime residue class modulo k and the square of a square-free number is bigger than $C/\{\phi(k)\}^2$, where C is a certain absolute positive constant and ϕ is Euler's totient function.

H. Gupta (Chandigarh)

Rieger, G. J. 85

Über ein lineares Gleichungssystem von Prachar mit Primzahlen.

J. Reine Angew. Math. **213** (1963), 103–107.

In this paper the author considers what he calls a Prachar system, viz., the system $p_1 + p_2 = m$, $p_2 + p_3 = n$, where p_1, p_2, p_3 are primes. Let $A\{\ \}$ denote the number of elements in the set $\{\ \}$. Let $f(m, n) = A\{p_1, p_2, p_3; p_1 + p_2 = m, p_2 + p_3 = n, p_1, p_2, p_3 \text{ primes}\}$. Theorem: There exists an absolute constant c such that, for $x > 8$,

$$A(m, n; n < m < x, f(m, n) > 0) > cx^2.$$

Various generalizations are considered.

R. Ayoub (University Park, Pa.)

Rieger, G. J. 86

Anwendung der Siebmethode auf einige Fragen der additiven Zahlentheorie. I.

J. Reine Angew. Math. **214/215** (1964), 373–385.

In this paper the author applies the sieve method and thereby obtains generalizations of classical results. Let $A\{\ \}$ denote the number of elements in the set $\{\ \}$. Theorems of the following type are proved. Let $F(x)$ be a positive monotone increasing function tending to infinity with x, $F(x) \leqq \sqrt{\log x}$. Then there exists a constant $C(F, k)$ depending only on F and k and a positive constant $C(k)$ depending on k such that for $x > C(F, k)$ we have

$$A\{m; m \leqq x, m = x_1^2 + x_2^2 + x_3^k, x_3 \leqq F(x)\} < \frac{C(k)x^{F(x)}}{\sqrt{\log x}}.$$

Let $F(x)$ be a positive monotone increasing function tending to infinity with x, $F(x)/\log F(x) \leqq \log x$. Then there exists a constant $C(F)$ depending only on F and an absolute constant C such that for $x > C(F)$

$$A\{n; n \leqq x, n = p_1 + p_2, p_1, p_2 \text{ primes}, p_2 \leqq F(x)\} > C \frac{x}{\log x} \frac{F(x)}{\log F(x)}.$$

R. Ayoub (University Park, Pa.)

Rieger, G. J. 87

Some theorems on prime ideals in algebraic number fields.

Pacific J. Math. **13** (1963), 687–692.

In this paper the author obtains several theorems on prime ideals in algebraic number fields, theorems which use "elementary" methods and which generalize known results for the field of rational numbers as well as known

results for algebraic number fields. For example, the following is proved: Let $\mathfrak{f}$ be an ideal, and consider the classes of ideals mod $\mathfrak{f}$. Let H_0 be the principal class and H any one of the $h(\mathfrak{f})$ classes. If

$$\sum_{\substack{N(\mathscr{P})\leq x\\ \mathscr{P}\in H_0}} \frac{\log N(\mathscr{P})}{N(\mathscr{P})} \to \infty,$$

then for every H,

$$\frac{1}{x}\sum_{\substack{N(\mathscr{P})\leq x\\ \mathscr{P}\in H}} \log^2 N(\mathscr{P}) \to \infty.$$

The proof is based on generalizations of Selberg's lemma.

R. Ayoub (University Park, Pa.)

Philipp, Walter 88

Über einen Satz von Davenport-Erdős-LeVeque.

Monatsh. Math. **68** (1964), 52–58.

In the paper of the title [Michigan Math. J. **10** (1963), 311–314; MR **27** #3618], it was shown that if $\{f_k(x)\}$ is a sequence of bounded integrable functions on $[a, b]$, and if $s_n(x) = |n^{-1}\sum_1^n \exp(2\pi i m f_k(x))|^2$ and $I(n) = \int_a^b s_n(x)\,dx$, then $\{f_k(x)\}$ is uniformly distributed (mod 1) for almost all $x \in [a, b]$ if $\sum n^{-1} I(n)$ converges for all integral $m \neq 0$. In the present paper, the author extends this as follows. Let X be a compact space with countable basis, and let $\mathfrak{W}(X)$ be the set of all positive normalized measures on X. Let $\mathfrak{C}(X)$ be the normed (by $\|f\| = \sup_{x\in X}|f(x)|$) linear space of all complex continuous functionals on X, and let $H \subset \mathfrak{C}(X)$ be a countable set spanning a space dense in $\mathfrak{C}(X)$. Let $A = (a_{nk})$ be a complex matrix with $\lim_n \sum_{k=1}^\infty a_{nk} = 1$ and $\sup_n \sum_{k=1}^\infty |a_{nk}| < \infty$. If $\omega = \{x_k\}$ is an infinite sequence of elements of X, put $\lambda_n(\omega, A, f) = \sum_{k=1}^\infty a_{nk} f(x_k)$. The author calls ω strongly A-equidistributed to the measure $\mu \in \mathfrak{W}(X)$ if for each $f \in H$, $\lim s_n = 0$ and $\sum |(s_n - s_{n+1})s_n| < \infty$, where $s_n = |\lambda_n - \mu(f)|^2$. Satz 2: The sequence ω is strongly A-equidistributed to the measure μ if and only if, for each $f \in H$, there is a divergent series $\sum c_n$ with $c_n \geq 0$, $\sum c_n s_n < \infty$, and $|s_n - s_{n+1}| \leq c_n$ for $n = 1, 2, \cdots$. Satz 3: Let $\{T_k\}$ be a sequence of measurable operators in X. If to each $f \in H$ there corresponds a divergent series $\sum c_n$ with $c_n \geq 0$, $\sum c_n \mu(s_n) < \infty$ and $|s_n - s_{n+1}| \leq c_n$, then the sequence $\{T_k x\}$ is strongly A-equidistributed to the measure μ for μ-almost all $x \in X$. Applications are made to distribution on a compact group with countable basis.

W. J. LeVeque (Ann Arbor, Mich.)

Kuipers, L. 89

Some aspects of the theory of asymptotic distribution modulo 1.

Bull. Soc. Math. Belg. **15** (1963), 379–388.

The first part of this paper has an expository character. It gives the definitions of uniform distribution in the discrete and continuous cases and states the general results obtained by H. Weyl and J. G. van der Corput. Furthermore, some Fejér type theorems are mentioned. Then the author uses the notion of independence in the sense of Kac and Steinhaus to derive results on the uniform distribution of certain sums such as $\alpha t + \sin t$.

J. E. Cigler (Vienna)

Fröhlich, A. 90

A remark on the classfield tower of the field $P(\sqrt[l]{m})$.

J. London Math. Soc. **37** (1962), 193–194.

Let l be an odd prime, $m \neq \pm 1$, $m \not\equiv 0 \pmod{l^k}$ for every $k \geqq 1$. Let P denote the field of rational numbers and $K = P(\sqrt[l]{m})$. Let K_1 be the absolute class field of K and K_2 the absolute class field of K_1. K is said to have a non-trivial class field tower of length 2 if $K_2 \neq K_1$. Let p_j $(j = 1, 2, \cdots, s)$ be the distinct prime factors of m with $p_j \equiv 1 \pmod{l}$. For each j, let χ_j be a primitive lth power residue character mod p_j. If $j \neq k$, let $\chi_j(p_k) = e^{(2\pi i/l) r_{jk}}$, where r_{jk} are regarded as elements of $GF(l)$. Theorem: K has a non-trivial class field tower of length 2 except in the following cases: (i) $s \leqq 1$; (ii) $s = 2$ and $r_{12} \neq 0$ or $r_{21} \neq 0$; (iii) $s = 3$ and $r_{12}r_{23}r_{31} - r_{21}r_{32}r_{13} \neq 0$.

R. Ayoub (University Park, Pa.)

Schanuel, S. 91

On heights in number fields.

Bull. Amer. Math. Soc. **70** (1964), 262–263.

The object of this note is to announce a proof of a generalization of the classical result due to Dedekind and Weber. A corollary is formulated as follows: K is a number field of degree N over Q. S_∞ is the set of Archimedean absolute values of K normalized to extend the absolute value of Q. For $x \in K^*$, $\nu \in S_\infty$, let $\|X\|_\nu = |X|_\nu^{N_\nu}$, where $N_\nu = [K_\nu : Q_\nu]$ so that $N_\nu = 1$ or 2. Let $X = (X_1, \cdots, X_m) \in K^m$ and $\|X\|_\nu = \sup \|X_i\|$, $H_\infty(X) = \prod_{\nu\in S_\infty} \|X\|_\nu$. Let $[X]$ denote the fractional ideal generated by $X_1, \cdots, X_m$. Define the height of X to be $N[X]^{-1} H_\infty(X)$. Finally, the class of a point in projective space $P^{n-1}(K)$ is the class modulo principal ideals of the fractional ideal generated by homogeneous coordinates for X. Theorem: The number of points in $P^{m-1}(K)$ of a given class with height at most B is

$$\frac{k_m B^m}{\zeta_K(m)} + O(B^{(m-1)/N}),$$

where

$$k_m = \left(\frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{d}}\right)^m \frac{R}{w} m^r.$$

R. Ayoub (University Park, Pa.)

FIELDS AND POLYNOMIALS

See also 63, 92, 127.

Hodges, John H. 92

Simultaneous pairs of linear and quadratic matrix equations over a finite field.

Math. Z. **84** (1964), 38–44.

Let GF(q) denote the finite field of odd order q. Let A be a nonsingular symmetric matrix of order m, B and D symmetric matrices of order t, C an arbitrary $m \times t$ matrix, all with elements in GF(q). The author seeks the number of $m \times t$ matrices X over GF(q) which satisfy the system of equations

$$(*) \qquad X'AX = B, \quad X'C + C'X = D,$$

where the prime denotes transpose. It is shown that the number of solutions of (*) can be expressed in terms of

certain generalized Kloosterman sums previously introduced by the author [same Z. **66** (1956), 13–24; MR **18**, 643]. In the case $C = D = 0$ the number of solutions of (*) is the number of solutions of $X'AX = B$, which has been found by the reviewer [Duke Math. J. **21** (1954), 123–137; MR **15**, 604]. For $A = B = 0$, (*) reduces to $X'C + C'X = D$, which was discussed by the author [Ann. Mat. Pura Appl. (4) **44** (1957), 245–250; MR **20** #1684].

The number of symmetric matrices F of order t such that $CF = 0$ is also obtained; this is simply $q^{(t-s)(t-s+1)/2}$, where s is the rank of C. *L. Carlitz* (Durham, N.C.)

Davenport, H.; Schinzel, A. 93
Two problems concerning polynomials.
J. Reine Angew. Math. **214/215** (1964), 386–391.

The theorems proved in this paper are the following. Theorem 1: If $f(x, y, z)$ is an irreducible polynomial over the complex number field C with the property that there are infinitely many values of z for which f is reducible as a polynomial in x and y, then there exists an algebraic extension $C(z, \theta)$ of $C(z)$ such that $Q(z)f(x, y, z)$ is the norm from $C(z, \theta)$ to $C(z)$ of an element of the form $\sum_{i=0}^{n-1} u_i(x, y, z)\theta^i$, where $Q, u_0, \cdots, u_{n-1}$ are polynomials. Theorem 2 asserts that over a field K of characteristic 0 a polynomial $f(x_1, \cdots, x_r) + g(y_1, \cdots, y_s)$ is reducible if and only if $f(x_1, \cdots, x_r) = F(R(x_1, \cdots, x_r)), g(y_1, \cdots, y_s) = G(S(y_1, \cdots, y_s))$, where F, G, R, S are polynomials with coefficients in K and $F(u) + G(v)$ is reducible in K.

{The formula given at the end of p. 390 is incorrect, for in this formula, as well as in the rest of the argument, $R(x_1, x_2, \cdots, x_r, 0, \cdots, 0)$ should be used instead of $R(x_1, x_2, 0, \cdots, 0)$.} *M. J. Wonenburger* (Toronto, Ont.)

Hinrichs, Lowell A.; Niven, Ivan; Vanden Eynden, C. L. 94
Fields defined by polynomials.
Pacific J. Math. **14** (1964), 537–545.

If F is an infinite field, let us define two operations $a \oplus b = P(a, b)$ and $a \odot b = Q(a, b)$, where P, Q are polynomials. Then F is a field under the addition $\oplus$ and multiplication $\odot$ if and only if $a \oplus b = P(a, b) = f^{-1}(f(a) + f(b))$ and $a \odot b = Q(a, b) = f^{-1}(f(a) \cdot f(b))$, where $f(c) = (u - z)c + z$, $u \neq z$. Hence $(F, \oplus, \odot)$ is isomorphic to $(F, +, \cdot)$. For finite fields the theorem holds if and only if the field contains less than 5 elements.

Let R be the real field and define two operations on the set $R \times R$ of pairs of real numbers by

$$(a, b) \oplus (c, d) = (P_1(a, b, c, d), P_2(a, b, c, d)),$$

$$(a, b) \odot (c, d) = (Q_1(a, b, c, d), Q_2(a, b, c, d)).$$

The problem of finding all polynomials P_1, P_2, Q_1, Q_2 such that $R \times R$ is a field under the addition $\oplus$ and the multiplication $\odot$ is solved only under the assumption that P_1, P_2, Q_1, Q_2 are linear in each variable or that $(a, b) \odot (c, 0) = (ac, bc)$. It is shown that there exist solutions in which the polynomials are not linear and their degrees can be arbitrarily high, but in any case the field obtained is the complex field. The last assertion is a particular case of a more general theorem derived from Pontrjagin's theorem about continuous fields.
M. J. Wonenburger (Toronto, Ont.)

Cassels, J. W. S. 95
On the representation of rational functions as sums of squares.
Acta Arith. **9** (1964), 79–82.

The author proves the following theorems. Theorem 1: Let k be any field and denote by $k(x)$ and $k[x]$, respectively, the field of rational functions and the ring of polynomials in a single variable x having coefficients in k. Then any $f \in k[x]$ which is the sum of squares of elements of $k(x)$ is the sum of the same number of squares of elements of $k[x]$. The new part of the theorem is the statement that the same number of squares suffices. Theorem 2: Let $d \in k$ and suppose that the characteristic of k is not 2. A necessary and sufficient condition that $x^2 + d$ be the sum of $n > 1$ squares in $k(x)$ is either that -1 is the sum of $n-1$ squares of k or that d is the sum of $n-1$ squares of k. Theorem 3: Let R denote the field of real numbers and let $x_1, x_2, \cdots, x_n$ be independent variables over R. Then $x_1^2 + \cdots + x_n^2$ is not the sum of $n-i$ squares of elements of $R(x_1, \cdots, x_n)$. *J. W. Andrushkiw* (S. Orange, N.J.)

Nagell, Trygve 96
Contributions à la théorie des corps et des polynomes cyclotomiques.
Ark. Mat. **5**, 153–192 (1964).

This paper contains both new results and classical results with non-classical proofs. Let K_n be the cyclotomic field generated by a root of the nth cyclotomic polynomial $F_n(x)$. The theorem about the discriminant and the integral basis of K_n is proved following an earlier paper of the author [Norsk Mat. Tidsskr. **1** (1919), 99–101]. All irrationalities of the form $\sqrt[q]{a}$ (a a rational integer) contained in some K_n are determined, and in particular it is shown that if Δ is a square-free rational integer $\neq 1$, then $\sqrt{\Delta}$ is contained in K_n if and only if $\Delta \equiv 1 \pmod 4$ and $n \equiv 0 \pmod \Delta$ or $\Delta \equiv 2$ or $3 \pmod 4$ and $n \equiv 0 \pmod{4\Delta}$. If these conditions are satisfied, there exists a factorization $F_n(x) = A(x)B(x)$, where A and B are polynomials with integral coefficients in $K(\sqrt{\Delta})$ and conjugate in this field. It is proved that if $\varepsilon = e^{2\pi i/n}$ is a root of $A(x)$, then all its roots are ε^a, where $1 \leq a < n$, $(a, n) = 1$ and a satisfies the conditions: $(a/|\Delta|) = 1$ if $\Delta \equiv 1 \pmod 4$, $(a/|\Delta|) = (-1)^{(a-1)/2}$ if $\Delta \equiv 3 \pmod 4$, $(\Delta/a) = 1$ if $\Delta \equiv 2 \pmod 4$. A part of this result is contained in a slightly different notation in a paper of the reviewer [Proc. Cambridge Philos. Soc. **58** (1962), 555–562, Lemma 1 and Theorem 1; MR **26** #1280]. The present proofs are different.
A. Schinzel (Columbus, Ohio)

Kazandzidis, G. S. 97
On the cyclotomic polynomial: Coefficients.
Bull. Soc. Math. Grèce (N.S.) **4** (1963), *no.* 1, 1–11.

In this paper, which is a continuation of a preceding one [same Bull. **4** (1963), no. 1, 50–73; MR **28** #5060], the author's aim is to obtain a general formula for the coefficients of the cyclotomic polynomial $\phi_N(x) = \prod_{d\delta=N} (1 - x^d)^{\mu(\delta)} = 1 + a_1 x + \cdots + a_M x^M + \cdots$. Utilizing the results of E. Gagliardo [Boll. Un. Mat. Ital. (3) **8** (1953), 269–273; MR **15**, 291] the author proves that

$$a_M = \sum \left(\frac{-s_1}{1}\right)^{\alpha_1} \left(\frac{-s_2}{2}\right)^{\alpha_2} \cdots \left(\frac{-s_N}{N}\right)^{\alpha_N},$$

where the summation is taken over all the non-negative

integral solutions of the Diophantine equation $\alpha_1+2\alpha_2+\cdots+N\alpha_N=M$ and s_n stands for the sum of the nth powers of the roots of $\phi_N(x)$. The author also finds a second general formula for a_M and considers the case $N=pq$.

J. W. Andrushkiw (S. Orange, N.J.)

Gerstenhaber, Murray 98

On the Galois theory of inseparable extensions.

Bull. Amer. Math. Soc. **70** (1964), 561–566.

Jacobson [Trans. Amer. Math. Soc. **42** (1937), 206–224; Amer. J. Math. **66** (1944), 645–648; MR **6**, 115] has given a Galois theory for purely inseparable extensions K/F of exponent one. The fundamental correspondence relates the intermediate fields to restricted (i.e., closed under pth powers, where p is the characteristic) Lie algebras D of F-derivations of K, which are closed under multiplication by elements of K. The author shows that the condition of Lie closure is redundant in the presence of the remaining conditions (including restrictedness) on D. In fact, each such set of derivations is shown to be the K-span of the iterated pth powers of one of its members [see also Jacobson, *Lectures in abstract algebra*, Vol. III, Chap. IV, § 8, especially Exercise 4, Van Nostrand, Princeton, N.J., 1964].

G. B. Seligman (New Haven, Conn.)

Bellman, Richard 99

A note on the solution of polynomial congruences.

Boll. Un. Mat. Ital. (3) **19** (1964), 60–63.

As is known, the number N of different solutions in GF(p) of $f(x)\equiv 0\ (p)$, where $f(x)=x^n+a_1x^{n-1}+\cdots+a_n$, may be expressed in the form

$$N=\frac{1}{p}\sum_{t,x}e^{2\pi i t f(x)/p},$$

where t and x run independently through GF(p). The paper sketches the proof of an analogous result, nowhere precisely stated, that turns out to be:

$$(1)\qquad N=\frac{1}{p^{n-1}(p-1)}\sum_{(t,x)\equiv 0(p)}e^{2\pi i(t,Ax)/p}-\frac{p}{p-1},$$
$$t,x\in V_n[\mathrm{GF}(p)],$$

where A is any matrix similar to the companion matrix of f, i.e., has minimal polynomial equal to its characteristic polynomial equal to f. Two final equations refer to the case in which A is symmetric but the author slips in asserting them for the case $p=2$. An example is provided but it is simply a translation of the result (1) and does not compute N. {There are two unfortunate typographical errors.}

J. J. Stone (Waltham, Mass.)

Douady, Adrien 100

Détermination d'un groupe de Galois.

C. R. Acad. Sci. Paris **258** (1964), 5305–5308.

For any set X, let $L(X)$ denote the natural projective limit of the family of finite groups F/N, where F is the free group generated by X, and where N ranges over all normal subgroups of finite indices in F, containing almost all elements of X. Clearly $L(X)$ is the projective limit of the groups $L(E)$ when E ranges over all finite subsets of X. The author proves that if, conversely, a compact group G is the projective limit of the $L(E)$ in a certain natural way, then G is isomorphic to $L(X)$.

Now, let C be an algebraically closed field of characteristic 0, K the field of rational functions of one variable over C, and Ω an algebraic closure of K. Then the main theorem of the paper states that the Galois group of Ω/K is isomorphic to the group $L(C)$. An outline of the proof is as follows: First, a lemma of Grothendieck reduces the proof to the case where C is the complex field. For any finite subset E of the complex field C, let $\Omega(E)$ denote the composite of all finite extensions of K which are unramified on $C-E$. Then the classical function theory shows that the Galois group of $\Omega(E)/K$ is isomorphic to $L(E)$ in a certain nice manner. Since the Galois group of Ω/K is the projective limit of the Galois groups of $\Omega(E)/K$ when E ranges over all finite subsets of C, the theorem follows from the group-theoretical result stated above.

K. Iwasawa (Cambridge, Mass.)

ABSTRACT ALGEBRAIC GEOMETRY

See also 70.

Mumford, D. B. 101

Some aspects of the problem of moduli. (Japanese)

Sûgaku **15** (1963/64), 155–157.

An invited address given at the Annual Meeting of the Mathematical Society of Japan in May, 1963.

Mattuck, A.; Mayer, A. 102

The Riemann-Roch theorem for algebraic curves.

Ann. Scuola Norm. Sup. Pisa (3) **17** (1963), 223–237.

Es sei C eine irreduzible, singularitätenfreie projektive algebraische Kurve vom Geschlecht g, definiert über einem algebraisch abgeschlossenen Körper k. Für eine natürliche Zahl $n>0$ bezeichne $C(n)$ das n-fache symmetrische Produkt von C mit sich selbst; dies ist eine singularitätenfreie n-dimensionale Mannigfaltigkeit, deren Punkte umkehrbar eindeutig den ganzen Divisoren $\mathfrak{a}$ vom Grade n auf C entsprechen. Es sei $\alpha_n: C(n)\to J$ die (auf einen festen Punkt $\mathfrak{p}_0\in C$ bezogene) kanonische Abbildung von $C(n)$ auf die Jacobische Mannigfaltigkeit J von C. Durchläuft x die Punkte von J, so durchläuft $\alpha_n^{-1}(x)$ die sämtlichen linearen Vollscharen von Divisoren n-ten Grades auf C. Sei nun $\mathfrak{a}\in C(n)$, $\alpha_n(\mathfrak{a})=x$, $F=\alpha_n^{-1}(x)$; es werde angenommen, dass $\mathfrak{a}$ ein einfacher Punkt von F ist. $i: F\to C(n)$ sei die Inklusionsabbildung. Die Sequenz

$$F\underset{i}{\to}C(n)\underset{\alpha}{\to}J$$

induziert eine entsprechende Sequenz für die Tangentialräume in den betreffenden Punkten:

$$T_{\mathfrak{a},F}\underset{di}{\to}T_{\mathfrak{a},C(n)}\underset{d\alpha}{\to}T_{x,J}.$$

Satz: Diese Sequenz ist exakt. Es folgt, dass auch die durch Dualisierung entstehende Sequenz von Abbildungen der zugehörigen Differentialräume exakt ist, und dadurch ergibt sich durch Dimensionsvergleich der Riemann-Rochsche Satz für $\mathfrak{a}$. (Der Riemann-Rochsche Satz für beliebige, nicht notwendig positive Divisoren kann leicht daraus hergeleitet werden.) Bei diesem Beweis des Riemann-Rochschen Satzes ergibt sich der Spezialitätsindex $i(\mathfrak{a})$ eines Divisors $\mathfrak{a}$ gleich zu Beginn als die

Dimension des Raumes der durch $\mathfrak{a}$ teilbaren Differentiale auf C; hierin sehen die Verfasser den entscheidenden Vorteil ihres Beweises gegenüber den bekannten Beweisen des Riemann-Rochschen Satzes. Vorausgesetzt wird die Kenntnis der Chowschen Konstruktion der Jacobischen Mannigfaltigkeit, wobei entscheidend ist, dass bei dieser Konstruktion nicht der Riemann-Rochsche Satz verwendet wird, sondern nur der sogenannte Riemannsche Satz: $\dim(\mathfrak{a}) = \deg(\mathfrak{a}) - g$, falls der Grad $\deg(\mathfrak{a})$ hinreichend gross ist. Aus dem Riemannschen Satz folgt, dass der oben formulierte Satz richtig ist, wenn n hinreichend gross ist; der Beweis des Satzes für $m < n$ ergibt sich dann durch Diskussion der (auf einen festen Punkt $\mathfrak{p}_0 \in C$ bezogenen) natürlichen Einbettung $C(m) \subset C(n)$.

Literatur: W. L. Chow [Amer. J. Math. **76** (1954), 453–476; MR **15**, 823]. *P. Roquette* (Tübingen)

Baily, Walter L., Jr. 103

On the theory of θ-functions, the moduli of abelian varieties, and the moduli of curves.

Ann. of Math. (2) **75** (1962), 342–381.

The familiar Torelli theorem establishes a natural one-to-one correspondence between the conformal equivalence classes of compact Riemann surfaces and the points of the set $\mathscr{E}$ of equivalence classes of normalized period matrices of the curves [Andreotti, Amer. J. Math. **80** (1958), 801–828; MR **21** #1309; Weil, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. IIa **1957**, 33–53; MR **19**, 683]; the author demonstrated in an important previous paper [Ann. of Math. (2) **71** (1960), 303–314; MR **22** #1583] that the set $\mathscr{E}$ is a Zariski-open subset of an irreducible algebraic variety $\mathscr{E}'$. The present paper contains a more purely algebraic proof of the main result of the preceding paper; and this result is also extended and in a sense completed in the present paper as follows. It is shown that the set $\mathscr{E}'$ is defined over the rational field Q, and that $\mathscr{E}$ is Q-open on $\mathscr{E}'$. Further, the field $Q(e)$ generated over Q by the coordinates of a point $e \in \mathscr{E}$ is precisely the field of moduli $F(S_e)$ of the corresponding Riemann surface S_e; this follows from the corresponding result for polarized abelian varieties. Still further, there is a projective variety $\mathscr{F}'$ defined over Q, and a regular map $\lambda: \mathscr{F}' \to \mathscr{E}'$, also defined over Q, such that $\lambda^{-1}(e)$ is the quotient of S_e by the group of all conformal automorphisms of S_e for any point $e \in \mathscr{E}$. The proofs rest heavily on the classical theory of theta functions; a convenient summary of the results required from that theory is also included.

R. C. Gunning (Princeton, N.J.)

Cassels, J. W. S. 104

Arithmetic on curves of genus 1. VI. The Tate-Šafarevič group can be arbitrarily large.

J. Reine Angew. Math. **214/215** (1964), 65–70.

Part V appeared in J. London Math. Soc. **38** (1963), 244–248 [MR **26** #6171]. Let $\mathbf{C}$ be the 1-dimensional abelian variety consisting of the curve

(1) $$x^3 + y^3 + dz^3 = 0$$

with an addition in which $(1, -1, 0)$ is the zero point. Let us consider the set of homogeneous spaces over $\mathbf{C}$ defined by the curves

(2) $$mx^3 + m^{-1}y^3 + dz^3 = 0,$$

where m is an element of the multiplicative group Q^* of rationals. Equation (2) defines a map $\omega: Q^* \to WC$, where WC is the group of classes of homogeneous spaces over $\mathbf{C}$. The Tate-Šafarevič group Ш consists of those elements of WC which have points in every p-adic field and in the reals, but no rational points. The author proves that if $m | P^2$, $m > 1$, where $P = \prod_{1 \le j \le T} p_j$ and the p_j are distinct positive rational primes of the form $9n + 8$, then $\omega(m) \in$ Ш. Hence, the order of Ш is at least 3^T.

P. Abellanas (Madrid)

Manin, Ju. I. 105

On branched coverings of algebraic curves. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **25** (1961), 789–796.

Es sei k ein algebraischer Funktionenkörper einer Variablen über einem algebraisch abgeschlossenen Konstantenkörper k_0 der Charakteristik $p \geqq 0$. Das "Verzweigungsdatum" einer endlich-algebraischen Erweiterung K von k besteht aus: (i) dem Grad n von K/k; (ii) den in K verzweigten Primdivisoren $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ von k; (iii) für jedes $\mathfrak{p}_i$ die Verzweigungsexponenten $e_{i1}, \cdots, e_{im_i}$ der über $\mathfrak{p}_i$ liegenden Primdivisoren $\mathfrak{P}_{i1}, \cdots, \mathfrak{P}_{im_i}$ von K, dabei sind die $e_{ij} > 0$ und $\sum_{1 \le j \le m_i} e_{ij} = n$. Das Verzweigungsdatum heisst "regulär", wenn alle e_{ij} zu p teilerfremd sind (im Falle $p = 0$ ist jedes Verzweigungsdatum regulär). Satz: Zu vorgegebenem regulärem Verzweigungsdatum gibt es nur endlich viele Erweiterungen K von k. Dieser Satz wurde vor einigen Jahren von Abhyankar bewiesen [Amer. J. Math. **79** (1957), 825–856; MR **20** #872; errata, MR **20**, p. 1372]. In der vorliegenden Arbeit wird für diesen Satz ein neuer Beweis gegeben, der die vom Autor in einer früheren Arbeit schon benutzte Methode der Konstantenreduktion mit Hilfe abstrakter Differentiationen verwendet [der Verfasser, Dokl. Akad. Nauk SSSR **125** (1959), 488–491; MR **21** #4150].

P. Roquette (Tübingen)

Manin, Ju. I. 106

On the classification of formal Abelian groups. (Russian)

Dokl. Akad. Nauk SSSR **144** (1962), 490–492.

Let K be the class of formal abelian groups G over an algebraically closed field of characteristic $p > 0$ for which the homomorphism pI (multiplication by p) is an isogeny of $(pI)G$. There exists an integer H, depending only on G, such that for any two groups G', G'' isogenic to G, there exists an isogeny $G' \to G''$ or $G'' \to G'$ of height $\leqq H$.

L. Fuchs (Budapest)

Becken, Sigrid 107

Eine Kennzeichnung der orthogonalen Gruppen über Körpern der Charakteristik $\neq 2$.

Abh. Math. Sem. Univ. Hamburg **26** (1963/64), 211–229.

In this paper, the author gives an axiomatic characterization of orthogonal groups of quadratic forms of index > 0 and dimension $\geqq 3$ in the case where the characteristic of the ground field is not equal to 2. Similar theory on orthogonal groups of index 0 is treated by Ahrens [Arch. Math. **11** (1960), 116–126; MR **22** #8424]. In this paper the author treats the case of index > 0. Ahrens' method cannot be applicable in this case. Instead the author takes advantage of the existence of a hyperbolic

plane, and gives an axiomatic discussion of the hyperbolic plane P and abelian subgroups H and H' attached to singular vectors in P. The author imposes seven axioms on the set $\mathcal{S}$ of all "symmetries", and planes (in the author's term "Büschel", which is the set of all symmetries defined by nonsingular vectors in a plane). The axioms are somewhat involved; however, everything is stated in purely geometric terms, and the author gives an explicit way of introducing a vector space structure into H, and of representing the whole group as the orthogonal group of a quadratic form. *T. Tamagawa* (Evanston, Ill.)

Satake, Ichiro 108

Algebraic groups over p-adic number fields. (Japanese)
Sûgaku **12** (1960/61), 195–202.

This is a survey of the theory of algebraic groups defined over perfect fields, especially p-adic fields. In § 1–§ 3, a proof of A. Borel's theorem on conjugacy of maximal trigonalizable subgroups is given. In § 4, the author shows that if the ground field is a p-adic field, the homogeneous space of an algebraic group modulo a maximal trigonalizable subgroup is compact. In § 5, the author gives a few remarks on maximal compact subgroups of G_k where k is a p-adic field. In § 6, groups of compact type, i.e., groups with trivial maximal trigonalizable subgroups, are discussed. The last section is a brief summary of the author's work on spherical functions on p-adic algebraic group. The details of the theory are given in the author's paper [Inst. Hautes Études Sci. Publ. Math. No. 18 (1963), 5–69]. Because of the age of this paper, many results on p-adic algebraic groups have been obtained by several researchers in the past few years which are not covered here.

T. Tamagawa (Evanston, Ill.)

Doi, Koji 109

On the field of moduli of an abelian variety with complex multiplication.
J. Math. Soc. Japan **15** (1963), 237–243.

The theory of complex multiplication as developed by Shimura and Taniyama [*Complex multiplication of abelian varieties and its applications to number theory*, Math. Soc. Japan, Tokyo, 1961; MR **23** #A2419] is concerned in its main part with simple abelian varieties. The author proposes to consider also composite varieties. As a first step in this direction, he considers in this paper an abelian variety A which is a direct product $B_1 \times \cdots \times B_h$ of simple abelian varieties of the same type, where the "type" of a simple abelian variety is defined as in Shimura and Taniyama's book [loc. cit.]. The main result of the author implies the following theorem: The field of moduli of A (with respect to any polarization of A) is contained in the class field of the field of moduli of the factors B_i. See also Shimura [Osaka Math. J. **14** (1962), 33–44, Proposition 1]. *P. Roquette* (Tübingen)

Šafarevič, I. R. 110

Principal homogeneous spaces defined over a function field. (Russian)
Trudy Mat. Inst. Steklov. **64** (1961), 316–346.

Es sei A eine über dem Körper k definierte abelsche Mannigfaltigkeit der Dimension d. Gegenstand der Untersuchung ist die erste Kohomologiegruppe $H^1(G, A_K)$, wobei K die maximale separabel-algebraische Erweiterung von k bezeichnet, A_K die Gruppe der in K rationalen Punkte von A, und G die Galois-Gruppe von K/k. Zur Abkürzung werde $H^1(G, A_K) = H^1(k, A)$ gesetzt. Die Elemente dieser Gruppe sind die prinzipalen homogenen Räume über A; sie entsprechen grob gesprochen denjenigen über k definierten Mannigfaltigkeiten, die über K zu A biregulär äquivalent werden. Jedes Element von $H^1(k, A)$ besitzt endliche Ordnung. Es wird vornehmlich der Fall untersucht, in dem k ein algebraischer Funktionenkörper in einer Variablen über einem algebraisch abgeschlossenen Konstantenkörper k_0 ist. Die angewendete Betrachtungsweise ist die in der Zahlentheorie übliche Methode des Übergangs vom Lokalen zum Globalen. Zuerst wird die lokale Seite des Problems untersucht, d.h. als Grundkörper wird die komplette Hülle $k_\mathfrak{p}$ des Körpers k nach einem seiner Primdivisoren $\mathfrak{p}$ genommen. Es bedeute $\hat{A}$ die zu A duale Mannigfaltigkeit (Picardsche Mannigfaltigkeit), und $\hat{A}_{k_\mathfrak{p}}{}^0$ die Gruppe derjenigen in $k_\mathfrak{p}$ rationalen Punkte von $\hat{A}$, die durch sämtliche zur Charakteristik p von k_0 primen Zahlen teilbar sind. Ferner bedeute B_n die Gruppe der Elemente der Periode n der abelschen Gruppe B. Das Hauptresultat im lokalen Falle ist der Dualitätssatz: Ist n prim zu p, so sind die Gruppen $H^1(k_\mathfrak{p}, A)_n$ und $(\hat{A}_{k_\mathfrak{p}}{}^0)_n$ dual zueinander, d.h. jede ist isomorph zur Charaktergruppe der anderen. Diese Dualität wird vermittelt durch die von A. Weil für abelsche Mannigfaltigkeiten über beliebigen Körpern angegebene Paarung $e_n(x, \xi)$, $x \in A_n$, $\xi \in \hat{A}_n$ [vgl. S. Lang, *Abelian varieties*, insbesondere S.189, Interscience, New York, 1959; MR **21** #4959]. Wenn A eine reguläre Reduktion modulo $\mathfrak{p}$ besitzt, so ergibt sich aus allgemeinen Sätzen über abelsche Mannigfaltigkeiten, dass $(\hat{A}_{k_\mathfrak{p}}{}^0)_n$ direktes Produkt von $2d$ zyklischen Gruppen der Ordnung n ist; dasselbe gilt daher nach dem Dualitätssatz auch für $H^1(k_\mathfrak{p}, A)_n$. Im allgemeinen Falle ist $H^1(k_\mathfrak{p}, A)_n$ direktes Produkt von $d' + 2d''$ zyklischen Gruppen der Ordnung n, wobei sich die Invarianten d' und d'' aus dem Verhalten von A bei Reduktion modulo $\mathfrak{p}$ bestimmen; es ist $d' + d'' \leq d$.

Bei der Behandlung des globalen Problems wird die folgende Zusatzvoraussetzung gemacht: Alle Punkte von zu p primer Ordnung von A sind rational über der maximalen Erweiterung K' von k, die keine höhere Verzweigung über k besitzt. (Für $p = 0$ bedeutet dies keine Einschränkung.) Diese Zusatzvoraussetzung ist anscheinend oft erfüllt. Sie ist notwendig, weil die Galois-Gruppe G' von K'/k nach Grothendieck [Séminaire Bourbaki, 1958/59, Exp. 182, Secrétariat mathématique, Paris, 1959; MR **28** #1091] wohl bekannt ist: sie besitzt nämlich dieselbe Struktur wie im klassischen Falle, wo k_0 der Körper der komplexen Zahlen ist. Es sei $\varphi_\mathfrak{p}$ der durch die Einbettung $k \subset k_\mathfrak{p}$ definierte Homomorphismus $\varphi_\mathfrak{p}$: $H^1(k, A) \to H^1(k_\mathfrak{p}, A)$. Bei gegebenem $a \in H^1(k, A)$ ist $\varphi_\mathfrak{p}(a) = 0$ für alle Primdivisoren $\mathfrak{p}$ von k bis auf endlich viele. Also definieren die $\varphi_\mathfrak{p}$ durch Zusammensetzung einen Homomorphismus φ: $H^1(k, A) \to \sum_\mathfrak{p} H^1(k_\mathfrak{p}, A)$, wobei $\sum$ die Bildung der direkten Summe andeutet. Gegenstand der Untersuchung ist Bild und Kern dieses Homomorphismus, d.h. die Beantwortung der folgenden Fragen: (1) Welche Systeme lokaler Invarianten können über k realisiert werden? (2) Wie sind diejenigen Elemente aus $H^1(k, A)$ beschaffen, für welche alle lokalen Invarianten

verschwinden? Zu (1): Es sei n prim zur Charakteristik p. Nach Satz 1 haben wir einen Homomorphismus

$$\varphi_n : H^1(k, A)_n \to \sum_{\mathfrak{p}} H^1(k_{\mathfrak{p}}, A)_n = \sum_{\mathfrak{p}} \mathrm{Char}(\hat{A}_{k_{\mathfrak{p}}}{}^0)_n = \mathrm{Char} \prod_{\mathfrak{p}} (\hat{A}_{k_{\mathfrak{p}}}{}^0)_n,$$

wobei die Charaktergruppe und das Produkt $\prod$ im topologischen Sinne zu verstehen sind. Es sei $\mathfrak{t}$ die Gruppe derjenigen Elemente aus $\hat{A}_k$, die endliche, zu p prime Ordnung besitzen und die für jedes $\mathfrak{p}$ vermöge $\varphi_{\mathfrak{p}}$ in $\hat{A}_{k_{\mathfrak{p}}}{}^0$ abgebildet werden. Durch die Einbettungen $k \subset k_{\mathfrak{p}}$ wird ein Homomorphismus $\psi_n : \mathfrak{t}_n \to \prod_{\mathfrak{p}} (\hat{A}_{k_{\mathfrak{p}}}{}^0)_n$ definiert, aus dem durch Dualisierung ein Homomorphismus $\psi_n{}^*$ der entsprechenden Charaktergruppen in umgekehrter Richtung entsteht. Satz 2: Die Sequenz

$$H^1(k, A)_n \xrightarrow[\varphi_n]{} \sum_{\mathfrak{p}} H^1(k_{\mathfrak{p}}, A)_n \xrightarrow[\psi_n{}^*]{} \mathrm{Char}(\mathfrak{t}) \to 0$$

ist exakt.

Zu (2): Es sei $\mathfrak{h}$ der Kern von φ. Satz 3: Wenn $\mathfrak{t} = 0$, so ist $\mathfrak{h}$ unbeschränkt teilbar; und zwar ist $\mathfrak{h}$ dann isomorph zum e-fachen Produkt der Gruppe der Einheitswurzeln von zu p primer Ordnung mit sich selbst, wobei $e = 2d(m + 2g - 2) - \sum_{\mathfrak{p}} e_{\mathfrak{p}} - r$. Hierbei bedeuten: g das Geschlecht von k; r den Rang von A über k, wobei $(A_k : nA_k) = n^r \cdot ((A_k)_n : 1)$ [Lang und Néron, Amer. J. Math. **81** (1959), 95–118; MR **21** #1311]; m die Anzahl derjenigen $\mathfrak{p}$, für die A keine reguläre Reduktion modulo $\mathfrak{p}$ besitzt; für ein solches $\mathfrak{p}$ bedeutet $e_{\mathfrak{p}}$ eine gewisse Invariante, die von dem Verhalten von A bei Reduktion modulo $\mathfrak{p}$ abhängt, und zwar ist $e_{\mathfrak{p}}$ der Rang der ersten Homotopiegruppe $\pi_1(A_{k_{\mathfrak{p}}})$ im Sinne von Serre [Inst. Hautes Études Sci. Publ. Math. No. 7 (1960); MR **22** #9493], wobei $A_{k_{\mathfrak{p}}}$ als proalgebraische Gruppe über k_0 aufzufassen ist ($e_{\mathfrak{p}} = 0$, wenn A eine reguläre Reduktion modulo $\mathfrak{p}$ besitzt).

Es wird noch die Gruppe $\mathfrak{t}$ eingehender untersucht und insbesondere ein Kriterium dafür angegeben, dass $\mathfrak{t} = 0$. Eine abelsche Mannigfaltigkeit A/k heisst unverzweigt, wenn sie eine reguläre Reduktion nach jedem Primdivisor $\mathfrak{p}$ von k besitzt. Die maximale unverzweigte Teilmannigfaltigkeit A_0 von A heisst die unverzweigte Spur von A. Die Existenz der unverzweigten Spur wird durch die gleichen Überlegungen bewiesen, mit denen die Existenz der Chowschen k/k_0-Spur A^* von A bewiesen wird [vgl. Lang, loc. cit.]; es gilt $A^* \subset A_0$. Satz 4: $\mathfrak{t}$ ist isomorph zur Gruppe der Punkte mit endlicher, nicht durch p teilbarer Ordnung von $(A_0)_k$. Satz 5: Wenn $k = k_0(t)$ der rationale Funktionenkörper ist, so ist $A_0 = A^*$, d.h. A_0 ist schon über k definiert. Insbesondere ist dann $\mathfrak{t}$ und also auch $\mathfrak{h}$ unbeschrankt teilbar. Wenn k/k_0 nicht rational ist, so ist die Situation verwickelter; Verfasser zeigt, dass A_0 im Falle der Dimension 1 in gewissem Sinne den Typus eines Faserraumes besitzt, während das im höherdimensionalen Falle nicht notwendig richtig ist.

Der Verfasser betrachtet noch genauer den Fall, in dem $A = A^*$ über k_0 definiert ist; in diesem Falle vereinfachen sich die obigen Ergebnisse sehr. Auch kann dann die Theorie unter der allgemeineren Annahme entwickelt werden, dass k der Funktionenkörper einer singularitätenfreien Mannigfaltigkeit X beliebiger Dimension über k_0 ist. In dem ersten, der lokalen Theorie gewidmeten Teil der Arbeit findet sich ein Beispiel dafür, dass Index und Exponent eines Elements $a \in H^1(k_{\mathfrak{p}}, A)$ nicht notwendig zusammenfallen, wenn $d > 1$, und zwar kann der Exponent $n \not\equiv 0 \bmod p$ beliebig vorgegeben werden. Dagegen ist für $d = 1$ und $k_{\mathfrak{p}}$ als Grundkörper stets Index = Exponent. (Vgl. hierzu auch: Lang und Tate [Amer. J. Math. **80** (1958), 659–684; MR **21** #4960].) Die Hauptresultate dieser Arbeit wurden unabhängig davon auch von Ogg veröffentlicht [Ann. of Math. (2) **76** (1962), 185–212; MR **27** #5758]. *P. Roquette* (Tübingen)

LINEAR ALGEBRA

See also 56, 92, 99, 165.

Mostowski, Andrzej; Stark, Marceli 111

★Elementy algebry wyższej [The elements of higher algebra].

Second edition, corrected and supplemented. Biblioteka Matematyczna, Tom 16.
Państwowe Wydawnictwo Naukowe, Warsaw, 1963. 391 *pp.* *zl* 38.00.

This is the second edition of a previously reviewed textbook [1958; MR **21** #1927]. It contains some additional material on quadratic and hermitian forms and on Descartes' rule of signs. *H. Minc* (Santa Barbara, Calif.)

Marcus, Marvin; Minc, Henryk 112

★A survey of matrix theory and matrix inequalities.

Allyn and Bacon, Inc., Boston, Mass., 1964. xvi+180 *pp.* $12.95.

From the preface: "In this book we have tried within relatively few pages to provide a survey of a substantial part of the field. The first chapter starts with the assumption that the reader has never seen a matrix before, and then leads within a short space to many ideas which are of current research interest. In order to accomplish this within limited space, certain proofs that can be found in any of a very considerable list of books have been left out. However, we have included many of the proofs of classical theorems as well as a substantial number of proofs of results that are in the current research literature." This book is divided into three chapters with a bibliography at the end of each chapter.

In the first chapter, the elementary theory of matrices and vectors is introduced, the matrices being over the real numbers, complex numbers, integers or polynomials. The ordinary operations are discussed, as are direct sums, block multiplication and the Kronecker product. Properties are developed of the trace, permanent, characteristic polynomials, characteristic roots and the companion matrix. There is also a treatment of different normal forms, the solution of linear equations, determinantal divisors, unitary matrices, elementary inequalities, normal matrices, positive definite matrices.

The second chapter is on convexity and begins with convex sets and polyhedrons. There follow the Birkhoff theorem, simplexes, convex functions, symmetric functions, Hölder inequality, Minkowski inequality and the Kantorovich inequality. There is a discussion of the Hadamard determinant theorem, the Minkowski determinant theorem and several inequalities for characteristic roots. The chapter concludes with a section on non-negative matrices, primitive matrices, and doubly stochastic matrices.

The last chapter is on the localization of characteristic roots and includes Bendixson's theorem, Hirsch's theorem, Schur's inequality, and Perron's theorem. Several theorems on regions of location of the characteristic roots are given, and a section is devoted to applying these estimates to different special types. The chapter concludes with a section on the spread of a matrix.

A wide variety of topics is touched upon, only some of which have been mentioned here. The second and third chapters, in particular, contain much recent research, including that of the authors. The bibliographical material is extensive and seems complete. There are many important illustrative examples worked out in the text. The reviewer found few typographical errors, and as a whole the book seems well thought out. The relation between vectors and matrices is, for example, a well-known one; the treatment here is fresh and interesting. The first chapter would be adequate for a third or fourth year text and the book should prove valuable to anyone who has a question about matrices or matrix inequalities.

G. M. Petersen (Swansea)

Wilker, Peter 113

Über die maximale Zerlegung linearer Abbildungen von endlich-dimensionalen Vektorräumen.

Arch. Math. **14** (1963), 361–366.

By extending the base field k of a finite-dimensional vector space to a splitting field of the minimal polynomial $\mu(x)$ of a linear transformation A, one arrives at a maximal decomposition into invariant subspaces. Let m_r be the number of these subspaces which have dimension r. The author shows how to determine m_r by rational operations in $k[x]$. In the easier case of characteristic zero, $m_r = \rho_{r-1} - 2\rho_r + \rho_{r+1}$, where ρ_r is the rank of $[\mu^*(A)]^r$ with $\mu^* = \mu/(\mu, \mu')$ {a type of formula encountered more generally in the theory of the Wehr and Segre numbers}. In case the characteristic is $p \neq 0$, he uses a rational manipulation of the invariant factors $\{q_i\}$ to overcome multiplicities divisible by p. Among remarks on resultants, it is proved that $\det \mu'(A) = \prod_i R(q_i, \mu/q_i)R(q_i, q_i')$.

B. Vinograde (Ames, Iowa)

Wigner, Eugene P.; Yanase, Mutsuo M. 114

On the positive semidefinite nature of a certain matrix expression.

Canad. J. Math. **16** (1964), 397–406.

The paper is devoted to proving the following theorem on hermitian matrices: If S is positive definite, N self-adjoint, and T is the (unique, positive semidefinite) solution of $ST + TS = N^2$, then $Q = S \times \bar{T} + T \times \bar{S} - N \times \bar{N}$ is positive semidefinite ($\times$ denoting the cross product and bars, complex conjugates).

This remarkable and deep theorem is rooted in a problem in information theory and expresses "the fact that the information content of a mixture $\rho = \alpha\rho_1 + (1-\alpha)\rho_2$ of two ensembles ρ_1 and ρ_2 is a convex function of the mixing parameter α".

The proof of the theorem treats the hermitian form associated with Q, shown to be given by $(\psi, Q\psi) = \mathrm{Trace}(P^*SPT + P^*TPS - \frac{1}{2}P^*N^*PN - \frac{1}{2}P^*NPN^*)$, where P is the "matrix" of the n^2-dimensional vector ψ under a suitable ordering of indices, as a hermitian form in the elements of P and N. After an invariantive transformation $N \to S^{1/2}PS^{1/2} = N'$, $P \to S^{-1/2}NS^{-1/2} = P'$, where $S^{1/2}$ is the positive definite square root of S, it is shown that if the transformed hermitian form, constrained by suitable normalizations of P' and N', were to have a negative minimum, then the established positive semidefinite character of P' would be violated.

A variant of the main theorem is also proved, in which S is relaxed to be merely positive semidefinite but T is required to be positive semidefinite.

The machinery developed for the proof of the main theorem includes a number of lemmas on nth-order square matrices which are interesting in their own right. Lemma 3: Let B be positive definite hermitian. Then for a given matrix C there is one and only one matrix A such that $AB + BA = C$. If C is self-adjoint, or positive definite, or positive semidefinite, so is A. Lemma 5: If A and B are hermitian positive definite (semidefinite), so is $A \times B$. Lemma 7: If $A > A'$ (i.e., A and A' are hermitian and $A - A'$ is positive definite), $B > B'$, and $A + A'$, as well as $B + B'$, is positive definite, then $A \times B > A' \times B'$. Lemma 8: Any characteristic vector of Q can be assumed to have such a form that the corresponding matrix C is self-adjoint. Lemma 9: $C = I$ is the matrix of a characteristic vector of Q; its characteristic value is zero. Lemma 10: The trace of the matrix C of a non-zero characteristic value of Q vanishes; no such matrix can be definite or semidefinite.

R. F. Rinehart (Cleveland, Ohio)

Gomory, R. E.; Hoffman, A. J.; Hsu, N. C. 115

Some properties of the rank and invariant factors of matrices.

Canad. Math. Bull. **7** (1964), 85–96.

This paper is concerned with doubly stochastic (0, 1) matrices. Two types of problem are considered: (1) properties of the elementary divisors in the ring of integers and (2) embedding properties.

Let $\mathfrak{A}(n, k, p)$ represent the class of all doubly stochastic $n \times n$ matrices whose entries are 0 and 1, whose row and column sums are k and whose rank is p. Let $B < \mathfrak{A}(n, k, p)$ mean that B is a non-singular $p \times p$ submatrix of some matrix $A \in \mathfrak{A}(n, k, p)$. Let J stand for the matrix all of whose entries are 1, and if A is a matrix, let $\tilde{A} = J - A$. With this notation the authors prove the following theorems. Theorem 1: If $B < \mathfrak{A}(n, k, p)$, then the sum of the entries of B^{-1} is n/k. Theorem 2: Let A be a matrix of order n with integral entries and each row and column sum equal k with $0 \neq k \neq n$. Then $\pi l_i / \pi \tilde{l}_i = k/(n-k)$, where l_i, $\tilde{l}_i$ are the non-zero invariant factors of A, $\tilde{A}$. Theorem 3: The theorem gives necessary and sufficient conditions that a matrix with non-negative real entries be a submatrix of a doubly stochastic matrix of order n and rank p, where $p < n \leqq 2p - 1$. Theorem 4: If B is a non-singular matrix of order p with integral elements such that B^{-1} has entries whose sum is n/k, $n > p$, then B is contained in a matrix of order n and rank p with integral entries and all of whose row and column sums are k.

N. S. Mendelsohn (Winnipeg, Man.)

Dulmage, A. L.; Mendelsohn, N. S. 116

The characteristic equation of an imprimitive matrix.

J. Soc. Indust. Appl. Math. **11** (1963), 1034–1045.

The authors make a systematic study of matrices which are block cyclic of index k. If A is such a matrix, then A^k

has a block diagonal form with matrices $A_1, \cdots, A_k$ on the diagonal. The authors show that the matrices A_i have the same nonzero proper values with the same multiplicities. If α is a proper value of A_1 and w a corresponding proper vector of A_1, then a formula is given for a proper vector of A corresponding to each kth root of α. The second part uses graph-theoretical methods to obtain the connection between irreducibility of A and $A_1, \cdots, A_k$. Also, if φ [see the reviewer, Czechoslovak Math. J. **8** (**83**) (1958), 487–495; MR **22** #1520a] is irreducible with index of imprimitivity d, then φ^m is irreducible if and only if d and m are relatively prime. An application to Markov chains is given. *V. Pták* (Prague)

Dimovski, Ivan 117
On an infinite system of linear equations. (Bulgarian)
Fiz.-Mat. Spis. Bŭlgar. Akad. Nauk. **6** (**39**) (1963), 136–139.

Consider the infinite system of linear equations

$$\sum_{n=1}^{\infty} \chi(kn) = a(k), \qquad k = 1, 2, \cdots,$$

where the infinite sequence $\{a(k)\}$ is bounded. The author shows that, with respect to the unknowns $\chi(n), n = 1, 2, \cdots$, there is at most one solution $\chi(n) = \sum_{m=1}^{\infty} \mu(m)a(mn)$, where $\mu(n)$ denotes the Möbius arithmetical function, the infinite sequence $\{\chi(n)\}$ being also bounded.
T. P. Anđelić (Belgrade)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 56, 147.

Petro, J. W. 118
Some results on the asymptotic completion of an ideal.
Proc. Amer. Math. Soc. **15** (1964), 519–524.

Let $\mathfrak{o}$ be a commutative ring with identity and let $\mathfrak{a}$ be an arbitrary ideal of $\mathfrak{o}$. A semi-prime operation is defined to be a map $\mathfrak{a} \to \mathfrak{a}_p$ of ideals into ideals which satisfies the following conditions: (i) $\mathfrak{a} \subseteq \mathfrak{a}_p$, (ii) $\mathfrak{a} \subseteq \mathfrak{b} \Rightarrow \mathfrak{a}_p \subseteq \mathfrak{b}_p$, (iii) $(\mathfrak{a}_p)_p = \mathfrak{a}_p$ and (iv) $\mathfrak{a}_p\mathfrak{b}_p \subseteq (\mathfrak{a}\mathfrak{b})_p$. The a-operation $\mathfrak{a} \to \mathfrak{a}_a$ is defined by $\mathfrak{a}_a = \{x \in \mathfrak{o} : x$ is integral over $\mathfrak{a}\}$. When $\mathfrak{o}$ is a normal ring, it is known that the a-operation is a prime operation, which satisfies one more condition than above and was introduced by Krull [Math. Z. **41** (1936), 545–577]. One of the main results of this article is that the a-operation is in general a semi-prime operation. For $x \in \mathfrak{o}$, we define $v_\mathfrak{a}(x) = \sup n$ for n such that $x \in \mathfrak{a}^n$, and $\bar{v}_\mathfrak{a}(x) = \lim v_\mathfrak{a}(x^n)/n$. $\bar{v}_\mathfrak{a}$ is well-defined as was proved by Rees [Proc. London Math. Soc. (3) **5** (1955), 107–128; MR **16**, 669]. The author defines the asymptotic completion of $\mathfrak{a}$ to be $\mathfrak{a}_s = \{x \in \mathfrak{o} ; \bar{v}_\mathfrak{a}(x) \geqq 1\}$. (If $\mathfrak{o}$ is Noetherian, then $\mathfrak{a}_a = \mathfrak{a}_s$. In the general case, we have $\mathfrak{a}_a \subseteq \mathfrak{a}_s$ only.) The other of the main results is the following: The s-operation $\mathfrak{a} \to \mathfrak{a}_s$ is a semi-prime operation and satisfies "$(\mathfrak{a}\mathfrak{c})_s \subseteq (\mathfrak{b}\mathfrak{c})_s$ and $\mathfrak{a} \subseteq$ radical of $\mathfrak{c} \Rightarrow \mathfrak{a}_s \subseteq \mathfrak{b}_s$". Furthermore, $\mathfrak{a}_s = \mathfrak{b}_s$ if and only if $\bar{v}_\mathfrak{a} = \bar{v}_\mathfrak{b}$. *M. Nagata* (Kyoto)

Mott, Joe Leonard 119
Equivalent conditions for a ring to be a multiplication ring.
Canad. J. Math. **16** (1964), 429–434.

If "ring" is taken to mean a commutative ring with identity and a multiplication ring is a "ring" in which, when A and B are ideals with $A \subset B$, there is an ideal C such that $A = BC$, then it is shown that the following statements are equivalent. (I) R is a multiplication ring; (II) if P is a prime ideal of R containing the ideal A, then there is an ideal C such that $A = PC$; (III) R is a ring in which the following three conditions are valid: (a) every ideal is equal to the intersection of its isolated primary components; (b) every primary ideal is a power of its radical; (c) if P is a minimal prime of B and n is the least positive integer such that P^n is an isolated primary component of B, and if $P^n \neq P^{n+1}$, then P does not contain the intersection of the remaining isolated primary components of B. (Here an isolated P-primary component of A is the intersection of all P-primary ideals that contain A.) *H. T. Muhly* (Iowa City, Iowa)

Gilmer, Robert W., Jr. 120
Extension of results concerning rings in which semi-primary ideals are primary.
Duke Math. J. **31** (1964), 73–78.

Let R be a commutative ring not necessarily having an identity element. R is said to satisfy condition (*) if primary ideals are the only ideals whose radicals are prime. R is called a u-ring if the radical of every proper ideal is also proper. A complete characterization of rings satisfying (*) in terms of u-rings is given: R satisfies (*) if and only if R is one of the following types: (A) a nilring, (B) an integral domain having at most two prime ideals, (C) a zero-dimensional u-ring, (D) a one-dimensional u-ring with the property that if P and M are proper prime ideals such that $P \subset M$, and if $p \in P$, then $p = pm$ for some $m \in M$. Besides, it is shown that every one-dimensional u-domain contains an identity element and satisfies (*), while a one-dimensional u-ring without identity element need not satisfy (*). The following is a conjecture of the author: An indecomposable one-dimensional u-ring satisfying (*) is an integral domain. *G. Azumaya* (Sapporo)

Ayoub, Raymond; Ayoub, Christine 121
On the commutativity of rings.
Amer. Math. Monthly **71** (1964), 267–271.

This is an expository article in which the authors give elementary proofs (i.e., proofs that do not make use of Zorn's lemma) of special cases of Jacobson's theorem that a ring R is commutative if there is an integer n such that $a^n = a$ for all $a \in R$. The cases treated are $n = 2, 3, 4, 5, 7$ and $n = p + 1$, where p is a prime of the form $2^k + 2^m - 1$.
H. T. Muhly (Iowa City, Iowa)

Nöbauer, Wilfried 122
Bemerkungen über die Darstellung von Abbildungen durch Polynome und rationale Funktionen.
Monatsh. Math. **68** (1964), 138–142.

If R is a commutative ring with unity and R^n is the n-fold Cartesian product $R \times R \times \cdots \times R$, a mapping α of R^n into R^n, $\alpha(u) = (g_1(u), \cdots, g_n(u))$ (where $u = (u_1, u_2, \cdots, u_n)$) is said to be representable by polynomials or rational functions according as $g_1, \cdots, g_n$ are polynomial or rational functions. The sets P and R of mappings

that are so representable form subsemigroups of the semigroup H of all mappings of R^n into itself with the usual function composition. It is shown that each of the four conditions $R=H$, $P=H$, $\bar{R}=S$, $\bar{P}=S$ holds if and only if R is a Galois field. (Here S is the group of all permutations of R^n and $\bar{R}=R\cap S$, $\bar{P}=P\cap S$.)

H. T. Muhly (Iowa City, Iowa)

Scheja, Günter 123

Über die Bettizahlen lokaler Ringe.

Math. Ann. **155** (1964), 155–172.

Let R be a (Noetherian) local ring with maximal ideal $\mathfrak{m}$. For each non-negative integer p, the pth Betti number $b_p(R)$ of R is defined by the length of $\mathrm{Tor}_p{}^R(k,k)$, where $k=R/\mathfrak{m}$ (or, equivalently, the smallest number of generators for $\mathrm{Syz}^p\, k$). The Betti series of R is defined by $\sum_{p=0}^{\infty} b_p Z^p$, where Z is an indeterminate and $b_p=b_p(R)$. Known facts on the Betti series, say $\mathcal{R}$, of R are: (1) R is regular if and only if $\mathcal{R}$ is a polynomial, and in this case $\mathcal{R}=(1+Z)^n$ with $n=b_1=$ Krull dimension of R [Serre, Proc. Internat. Sympos. Algebraic Number Theory (Tokyo and Nikko, 1955), pp. 175–189, Sci. Council of Japan, Tokyo, 1956; MR **19**, 119] and (2) if R is a complete intersection, then, with $n=b_1(R)$ and $d=$ Krull dimension of R, $\mathcal{R}=(1+Z)^n/(1-Z^2)^{n-d}$ [Tate, Illinois J. Math. **1** (1957), 14–27; MR **19**, 119]. The author gives in this article an explicit expression for $\mathcal{R}$ in the case where the homological codimension of R is not less than $b_1(R)-2$ (hence the typical case is the case where R is a Macaulay ring and is a homomorphic image of a regular local ring by an ideal of height 2). Some of the other results are: (i) If $x\in\mathfrak{m}$, $x\notin\mathfrak{m}^2$, and if x is not a zero-divisor, then $\mathcal{R}=(1+Z)\bar{\mathcal{R}}$, where $\bar{\mathcal{R}}$ is the Betti series of R/xR, and (ii) the completion of R is a complete intersection if and only if $b_1(R)=b_2(R)+$ Krull dimension of $R-\binom{n}{2}$, where $n=b_1(R)$.

M. Nagata (Kyoto)

Laplaza, Miguel L. 124

Some properties of immersion with respect to a unit. (Spanish)

Rev. Mat. Hisp.-Amer. (4) **23** (1963), 242–252.

Let R be a commutative ring and Z be the integers. Using the canonical structure on $R\times Z=\bar{R}$, R is then embedded in a ring with identity. The author studies a number of connections between R and $\bar{R}$, typical of which are the following. (1) $\bar{R}$ is noetherian if and only if R is noetherian; (2) If $P\neq R$ is a prime ideal in R, then there is a unique prime ideal $\bar{P}$ in $\bar{R}$ such that $\bar{P}\cap R=P$; (3) Every ideal of $\bar{R}$ which contains R is a finite intersection of primary ideals.

Not all of the results obtained are new, but in the case of known results the author has supplied new details in his proofs.

E. H. Batho (Durham, N.H.)

Patterson, E. M. 125

Commutation problems involving rings of infinite matrices.

Proc. Edinburgh Math. Soc. (2) **14** (1964/65), 55–60.

The author continues investigations published in two previous papers [Proc. Roy. Soc. Edinburgh Sect. A **65** (1960), 263–271; MR **24** #A746; ibid. **66** (1961/62), 42–46; MR **26** #151]. Besides the rings $M_\rho(R)$ and $M_\rho{}^*(R)$ considered in these papers the author introduces another ring containing $M_\rho{}^*(R)$ and contained in $M_\rho(R)$. If the matrix $A=(a_{ik})$ is row-finite, then for each integer i there exists a finite set $N(i)$ of $n(i)$ integers such that $a_{ik}=0$ whenever $k\notin N(i)$. If there exists a finite upper bound for all $n(i)$, the matrix A is called uniformly row-finite. The set of all uniformly row-finite matrices is a subring $M_\rho{}'(R)$ of $M_\rho(R)$. Theorem 1: The Jacobson-Perlis radical Γ satisfies $\Gamma(M_\rho{}'(R))=M_\rho{}'(\Gamma(R))$ if and only if $\Gamma(R)$ is nilpotent. Let R^α ($\alpha\geqq 2$) be the set of all elements of the form $\sum x_1x_2\cdots x_\alpha$ ($x_i\in R$) in which the summation is over a finite number of terms. Theorem 2: $\{M_\rho(R)\}^\alpha=M_\rho(R^\alpha)$ for any ring R. Let $R_q{}^\alpha$ denote the set of all members of R^α which are sums of at most q terms of the form $x_1x_2\cdots x_\alpha$ ($x_i\in R$). Theorem 3: For any ring R, we have $\{M_\rho{}'(R)\}^\alpha=\bigcup_q M_\rho{}'(R_q{}^\alpha)$, q ranging over all positive integers. Theorem 4: $\{M_\rho{}'(R)\}^\alpha=M_\rho{}'(R^\alpha)$ if and only if there exists an integer q such that $R_q{}^\alpha=R_{q+1}^\alpha$. Theorem 5 is analogous to Theorem 4 with $M_\rho{}^*$ instead of $M_\rho{}'$.

R. Kochendörffer (Rostock)

Kupisch, Herbert 126

Zu einem Satz von Nakayama und Ikeda.

Arch. Math. **15** (1964), 144–145.

The author gives an alternative proof of the theorem of Nakayama and Ikeda [Osaka Math. J. **2** (1950), 7–12, Theorem 2; MR **12**, 797], assuming Theorem 1 in their paper.

H. Nagao (Osaka)

Flamant, Maurice 127

L'algèbre de Clifford associée à une extension cyclique.

C. R. Acad. Sci. Paris **257** (1963), 2381–2383.

Author's summary: "Nous nous proposons d'étudier dans cette Note quelques propriétés de l'algèbre de Clifford associée à une extension cyclique, à savoir: étant donné un corps K de caractéristique $p\neq 2$ et une extension cyclique E de K, de degré n sur K, soit B la forme bilinéaire sur E définie par $B(x,y)=\mathrm{Tr}_{E/K}(xy)$ et T la forme quadratique associée; nous déterminons dans certaines conditions la structure de l'algèbre de Clifford $C(T)$ de la forme quadratique T."

E. H. Batho (Durham, N.H.)

Mount, K. R. 128

Some remarks on Fitting's invariants.

Pacific J. Math. **13** (1963), 1353–1357.

Let M be a finitely generated module over a commutative ring R and let $0\to K\to F\to M\to 0$ be exact with F a free module on the basis $e(1),\cdots,e(n)$. If $k(i)=\sum x(ij)e(j)$ generate K, then define $f(j:M)$ to be the ideal of R generated by all $(n-j)\times(n-j)$ determinants of the form $(x(uv))$; further, let $F(j:M)$ be the R-module $R/f(j:M)$ and let $F(M)$ be the direct sum of the $F(j:M)$. The author establishes a number of connections between M and $F(M)$, the most important of which are the following. Theorem 1: If R is a not-necessarily noetherian local ring, then M is free if and only if $F(M)$ is free. If M is free and I is the maximal ideal of R, then $\dim_{R/I}(R/I\otimes_R M)=\mathrm{rank}\,F(M)=\mathrm{rank}\,M$. Theorem 2: M is flat if and only if $F(M)$ is flat if and only if $F(j:M)$ is flat for each j. Theorem 3: M is projective if and only if $F(M)$ is projective.

E. H. Batho (Durham, N.H.)

Goldman, Oscar 129
Quasi-equality in maximal orders.
J. Math. Soc. Japan **13** (1961), 371–376.

Let R be an integrally closed noetherian domain with field of quotients K, Σ a separable K-algebra, and Λ a maximal order in Σ. The author then considers the set of all finitely generated reflexive R-submodules of Σ which span Σ over K and which are closed under both left and right multiplication by Λ, which he makes into an abelian group by defining $A \circ B$ to be $(AB)^{**}$ where $(\ )^*$ is Hom(, R). This group $G(\Lambda)$ is shown to have the following properties. If Γ is another maximal order in Σ, then $G(\Lambda)$ is naturally isomorphic to $G(\Gamma)$. There is a natural isomorphism $G(\Lambda) \approx \sum_{\mathfrak{p}} G(\Lambda R_{\mathfrak{p}})$, where $\mathfrak{p}$ runs through all minimal prime ideals of R, which gives the result that if Σ is a central simple K-algebra, then $G(\Lambda)$ is a free abelian group whose generators are in natural one-to-one correspondence with the minimal prime ideals in R.

M. Auslander (Waltham, Mass.)

Jouanolou, Jean-Pierre 130
Sur la profondeur des modules de type fini.
C. R. Acad. Sci. Paris **258** (1964), 2245–2247.

Let A be a noetherian local ring with the maximal ideal $\mathfrak{m}$. For an A-module M, $\text{prof}_A(M)$ can be interpreted as the smallest non-negative integer q such that $\text{Ext}_A{}^q(A/\mathfrak{m}, M) \neq 0$. The following statements are announced without proofs. (1) Let A be a Macaulay local ring regular of codimension 2. If M, N are reflexive modules of finite type such that $\text{Hom}_A(M, N)$ is free, then $\text{Ext}_A{}^1(M, N) = 0$. (2) Let A be an integrally closed noetherian domain and M a torsion-free A-module of finite type. If there exists an integer r (≥ 2) such that $\text{prof}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq 2$ for all $\mathfrak{p}$ whose height is $\geq r$, then $M = \bigcap_{h(\mathfrak{p}) = r-1} M_{\mathfrak{p}}$. (3) Let A be a noetherian domain and let $0 \to M \to N \to P \to 0$ be an exact sequence of A-modules of finite type. If N is torsion-free and M is reflexive, then the associated primes to P are of height $\leq r - 1$. (4) Let $B \supset A$ be irreducible localities over a field k whose quotient field extension is separably generated. If A is regular and B is a normal complete intersection over k, then the associated primes to the module of differentials $D_A(B)$ are of height ≤ 1.

D. S. Rim (Waltham, Mass.)

Jans, J. P. 131
Module classes of finite type.
Pacific J. Math. **13** (1963), 603–609.

Let R be a ring with minimum condition on both left and right ideals. All R-modules considered are finitely generated. A class C of R-modules is said to be of finite type if for each integer n there are only a finite number of non-isomorphic modules in C of composition length less than n. Various classes of modules are listed, and then it is shown that if some of these classes are of finite type, then certain others must be also. For instance, the class of submodules of left projectives is of finite type if and only if the class of submodules of right projectives is of finite type. Also the class of reflexive left R-modules is of finite type if and only if the class of torsionless left R-modules A such that $\text{Ext}_R{}^1(A, R) = 0$ is of finite type.

E. Matlis (Evanston, Ill.)

Natarajan, N. S. 132
Ideal theory for semirings.
J. Madras Univ. B **32** (1962), 161–168 (1963).

Author's summary: "Semirings arise in a natural way, as the totality of endomorphisms of a commutative semigroup with the usual operations on the endomorphisms. Our definitions of two-sided ideals, prime ideals, primary ideals, etc., are such that when the semiring is a ring, they reduce to the usual notions. The ideal theory developed here for a semiring in all generality is thus a generalisation of the ideal theory of a ring as formulated by Krull (1935), McCoy (1949) and Murdoch (1952)."

H.-J. Hoehnke (Berlin)

NON-ASSOCIATIVE ALGEBRA

See also 98, 506.

Frank, Marguerite 133
Two new classes of simple Lie algebras.
Trans. Amer. Math. Soc. **112** (1964), 456–482.

From the author's introduction: "In this paper we define a Lie algebra R of dimension p^{2n+1} over a field F of characteristic $p > 2$, which we show to be central simple if $n + 2$ is not divisible by p. From the fact that all derivations of R are inner we conclude that R is not isomorphic to known algebras of identical dimension (except, possibly, if $n = \frac{1}{2}(p^{2r} + 2r - 1)$, $r \geqq 1$). If $n + 2 \equiv 0 \pmod p$, it is shown that the [derived] algebra R^2, of dimension $p^{2n+1} - 1$ over F, is central simple and also new. We finally perform a Cartan decomposition of both algebras."

The algebra R is defined, in a straightforward way, as a certain subalgebra of the generalized Witt algebra of Jacobson [Duke Math. J. **10** (1943), 107–121; MR **4**, 187]. Details of the definition are too complicated to be given here.

Rimhak Ree (Vancouver, B.C.)

Morikawa, Hisasi 134
On the exponential maps and the triangular 2-cohomology of graded Lie rings of length three.
Nagoya Math. J. **23** (1963), 5–13.

A normal series $H = H_1 \supset H_2 \supset \cdots \supset H_{n+1} = \{1\}$ in a group H is called an N-series if $(H_i, H_j) \subset H_{i+j}$ for all i and j. To such a series corresponds a graded Lie ring $L = A_1 \oplus \cdots \oplus A_n$, where $A_i = H_i/H_{i+1}$, such that $[A_i, A_j] \subset A_{i+j}$ for all i and j [see M. Hall, Jr., *Theory of groups*, p. 329, Macmillan, New York, 1959; MR **21** #1996]. Now let $L = A_1 + A_2 + A_3$ be a graded Lie ring of length 3. The author defines a triangular 2-cocycle of L to be a triple $\gamma = (\gamma_{21}, \gamma_{31}, \gamma_{32})$, where each γ_{ji} is a 2-cochain of A_i with coefficients in A_j, and the γ_{ij} satisfy certain identities. For each triangular 2-cocycle γ, a group $\text{Exp}_\gamma(L)$ is defined such that $\text{Exp}_\gamma(L)$ has an N-series of length 3 whose associated graded Lie ring is L. Conversely, for each N-series of length 3, $H = H_1 \supset H_2 \supset H_3 \supset \{1\}$, it is possible to define a triangular 2-cocycle γ of the associated graded Lie ring $L = (H_1/H_2) + (H_2/H_3) + H_3$ such that there is an isomorphism of $\text{Exp}_\gamma(L)$ onto H which carries the natural N-series of $\text{Exp}_\gamma(L)$ onto the given N-series of H.

C. W. Curtis (London)

HOMOLOGICAL ALGEBRA

See also 123, 128, 131.

Maranda, J. M. 135

Some remarks on limits in categories.

Canad. Math. Bull. **5** (1962), 133–146.

This paper is concerned with the existence of direct limits (= right roots) of functors $\mathscr{U}\to\mathscr{X}$, $\mathscr{U}$ being a category with ob $\mathscr{U}$ a set. Among the special cases are direct sums ($\mathscr{U}$ has only identity maps) and cokernels ($\mathscr{U}$ the diagram-scheme $A\rightrightarrows B$). The classical case occurs when $\mathscr{U}$ is a directed set with maps $(A, B): A\to B$ for $A\leq B$.

By an easy generalization of the usual argument the author shows that if $\mathscr{X}$ admits direct sums and cokernels, then all functors into $\mathscr{X}$ have direct limits, that a functor $\mathscr{X}\to\mathscr{Y}$ which preserves sums and cokernels preserves all direct limits, and that a cokernel $\mathscr{U}$ of $\mathscr{S}\rightrightarrows\mathscr{T}$ in the category of functors inherits the direct-limit-preserving properties of $\mathscr{S}$ and $\mathscr{T}$. Of course all these results dualize.

Using Grothendieck's axiom for the category of modules he also shows easily that if $\mathscr{Y}$ is a category of modules and $0\to\mathscr{S}\to\mathscr{T}\to\mathscr{U}$ is exact in the category of functors $\mathscr{X}\to\mathscr{Y}$, then $\mathscr{S}$ inherits the classical direct-limit-preserving properties of $\mathscr{T}$ and $\mathscr{U}$. This last result might easily be generalized. *A. Heller* (Urbana, Ill.)

Dedecker, Paul; Mersch, Jacques 136

Précatégories et relations d'équivalence dans les catégories.

C. R. Acad. Sci. Paris **256** (1963), 4811–4814.

This note consists of definitions and theorems without proofs. If $\mathfrak{A}$ is a precategory, then there exists a category $\tilde{\mathfrak{A}}$ and a monomorphism $i: \mathfrak{A}\to\tilde{\mathfrak{A}}$ such that if $\mathscr{C}$ is a category and $\phi: \mathfrak{A}\to\mathscr{C}$ is a morphism, then ϕ can be factored through $\tilde{\mathfrak{A}}$ by means of a unique functor. If R is an admissible equivalence relation in a category $\mathscr{C}$, then $\mathfrak{A}=\mathscr{C}/R$ is a precategory and there is associated to R a functor $\tau_R: \mathscr{C}\to\tilde{\mathfrak{A}}$. Conversely, if $T: \mathscr{C}\to\mathscr{C}'$ is a functor and $\mathscr{C}$, $\mathscr{C}'$ categories, and if R_T is the equivalence relation in $\mathscr{C}$ defined by $a\sim a'$ if and only if $T(a)=T(a')$, then R_T is an admissible equivalence relation in $\mathscr{C}$. T admits a unique factorization $T: \mathscr{C}\xrightarrow{j}\mathfrak{A}\xrightarrow{k}\mathfrak{A}'\xrightarrow{i}\mathscr{C}'$, where $\mathfrak{A}$ is the precategory $\mathscr{C}/R_T$, $\mathfrak{A}'$ is the precategory $T(\mathscr{C})$, j is the canonical epimorphism, i the canonical monomorphism and k is an isomorphism. T is compatible with an equivalence relation R if and only if T admits a factorization $T: \mathscr{C}\xrightarrow{\tau_R}\tilde{\mathfrak{A}}\xrightarrow{\tilde{T}}\mathscr{C}'$. The case of groupoids is considered. *E. Matlis* (Evanston, Ill.)

Shih, Weishu 137

Sur les extensions successives.

C. R. Acad. Sci. Paris **249** (1959), 607–609.

Shih, Weishu 138

Une remarque sur le système projectif de suites spectrales.

C. R. Acad. Sci. Paris **251** (1960), 20–21.

Inasaridze, H. N. 139

Extensions of semigroups. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **33** (1964), 263–269.

This paper contains some applications of the author's previous results on extensions in general categories of a suitable type [same Soobšč. **30** (1963), 537–544; MR **28** #3074]. Let "semigroup" mean always a semigroup with unit element. Extensions in the following categories are investigated: (1) the category of all semigroups, (2) of all left Λ-semimodules (Λ being a semiring), (3) of all commutative semigroups with cancellation, (4) of all commutative generalized groups, (5) of all compact commutative semigroups. Typical example concerning the case (1): For a group G and a semigroup D central extensions $G\to X\to D$ are considered. The set $E^1(D, G)$ of their equivalence classes is an abelian group under the Baer sum which is isomorphic to the group $S^1\text{Hom}(D, G)$. In cases (2)–(5) there are similar results. Extensions have to be understood in the sense of the author's previous paper mentioned above. *K. Drbohlav* (Prague)

Oort, F. 140

Yoneda extensions in abelian categories.

Math. Ann. **153** (1964), 227–235.

Let $\mathfrak{C}$ be an abelian category, $\mathfrak{C}'$ the category of contravariant left-exact functors $\mathfrak{C}\to\mathfrak{AG}$ ($\mathfrak{AG}$ is the category of abelian groups). The properties of $\mathfrak{C}'$ are similar to those of $\mathfrak{AG}$; in particular, it has enough injectives. Moreover, $\varphi_{\mathfrak{C}}: \mathfrak{C}\to\mathfrak{C}'$, $\varphi_{\mathfrak{C}}X=\text{Hom}_{\mathfrak{C}}(-, X)$, is a full embedding. This makes $\mathfrak{C}'$ a valuable tool. The author proves that $\varphi_{\mathfrak{C}}$ induces isomorphisms $\varphi_{\mathfrak{C}}^*: \text{Ext}_{\mathfrak{C}}^n(A, B)\cong\text{Ext}_{\mathfrak{C}'}^n(\varphi A, \varphi B)$ between the groups of Yoneda extensions, provided $\mathfrak{C}$ is a noetherian category ("noetherian" means that every object satisfies the ascending chain condition). Dually for artinian categories (defined by descending chain conditions). *A. Dold* (Heidelberg)

GROUP THEORY AND GENERALIZATIONS

See also 47, 48, 106, 107, 134, 139, 192, 564, 603.

Lipschutz, Seymour 141

On square roots in eighth-groups.

Comm. Pure Appl. Math. **15** (1962), 39–43.

Sufficient conditions are given for an element of an eighth-group (i.e., a group for which non-inverse cyclic permutations of defining relators or their inverses cancel less than one-eighth of each other) to have no [respectively, a unique] square root. *M. Greendlinger* (Ivanovo)

Lyndon, R. C.; Schützenberger, M. P. 142

The equation $a^M=b^Nc^P$ in a free group.

Michigan Math. J. **9** (1962), 289–298.

If $a^M=b^Nc^P$ in a free group, where M, N and P are integers greater than one, then a, b and c are all powers of a common element. *M. Greendlinger* (Ivanovo)

Macdonald, I. D. 143

On cyclic commutator subgroups.

J. London Math. Soc. **38** (1963), 419–422.

The following theorem is proved: If the commutator group G' of a group G is cyclic and either G is nilpotent

or G' is infinite, then G' is generated by a suitable commutator. For any given positive integer n, however, there is a group G in which G' is cyclic and is generated by no set of less than n commutators.

K. W. Gruenberg (London)

Suzuki, Michio 144

On a class of doubly transitive groups. II.

Ann. of Math. (2) **79** (1964), 514–589.

A (ZT)-group is a doubly transitive permutation group of odd degree containing no regular normal subgroup such that no non-identity element leaves more than three letters invariant. The author has previously classified all (ZT)-groups [same Ann. (2) **75** (1962), 105–145; MR **25** #112].

The purpose of this paper is to prove the following result. Theorem: Let G be a permutation group on the finite set Ω. Assume that (*) If $a \in \Omega$, the subgroup H of G consisting of all the elements leaving a invariant is a proper subgroup of G and contains a normal subgroup Q which is regular on $\Omega - \{a\}$. (**) The degree is odd and the group H/Q is a solvable group of odd order. Then one of the following must occur: (i) There is a near field F and a group Σ of automorphisms of F so that G is isomorphic to the totality of semi-linear transformations $y = ax^\sigma + b$, $a \neq 0$, $a, b \in F$, $\sigma \in \Sigma$; (ii) G contains a normal subgroup G_1 which is either a (ZT)-group or isomorphic to $HO(3, 2^m)$ [$= PU_3(2^m)$] and G is isomorphic to a subgroup of the group of all automorphisms of the group G_1 in such a way that Q corresponds to a Sylow 2-group of G_1.

The proof of this theorem is long and complicated and makes use of many ideas. Let G be a group satisfying the hypotheses of the theorem. Choose $b \in \Omega$, $b \neq a$. There exists an involution t interchanging a and b. Let K be the subgroup of G consisting of all elements leaving both a and b fixed, let $V = K \cap C_G(t)$, let $U = \{x\colon\ x \in K, t^{-1}xt = x^{-1}\}$, and let $W = V \cap C_G(U)$. It is shown that there exists a unique involution $s \in Q$ such that $t^{-1}st = r^{-1}tr$ with $r \in Q$. s is called the distinguished involution of Q. The author calls this a structure identity of G. If s is the distinguished involution, let $\chi(G)$ be the order of the element st. It is shown that $\chi(G)$ is always a prime. The proof of the theorem is divided into three major cases. If $\chi(G) > 5$, then case (i) of the theorem holds. If $\chi(G) = 5$, then either case (i) holds or G is isomorphic to a subgroup of the automorphism group of a Suzuki group. The proof in this case relies heavily on a previous paper of the author (see reference above). The main difficulty comes in handling the case that $\chi(G) = 3$. Here it is necessary to consider several cases depending on various possibilities for V and W. It is shown eventually that a group G which satisfies the assumptions of the theorem and also one of several sets of conditions is uniquely determined. Furthermore, the groups mentioned in the conclusion of the theorem each satisfy one of these sets of conditions, thus completing the proof of the theorem.

W. Feit (Ithaca, N.Y.)

Suzuki, Michio 145

Finite groups of even order in which Sylow 2-groups are independent.

Ann. of Math. (2) **80** (1964), 58–77.

A finite group G is said to satisfy the condition (T.I.) if any two distinct Sylow 2-groups have only the identity in common. The purpose of this paper is to prove the following result. Theorem 2: Let G be a group of even order satisfying the condition (T.I.). Then one of the following three statements is true: (i) A Sylow 2-group of G is a normal subgroup; (ii) A Sylow 2-group of G is either a cyclic group or a generalized quaternion group; or (iii) G contains normal subgroups G_1 and G_2 such that $G \supseteq G_1 \supseteq G_2 \supseteq \{1\}$, where both G/G_1 and G_2 are of odd order and G_1/G_2 is isomorphic to one of the groups $L_2(q)$, $U_3(q)$ or $G(q)$. Here q is a power of 2, $G(q)$ denotes the Suzuki group [the author, Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 868–870; MR **22** #11038], and $L_2(q)$, $U_3(q)$ denote the projective linear group and projective unitary group, respectively. Since groups of odd order are solvable [the reviewer and J. G. Thompson, Pacific J. Math. **13** (1963), 775–1029] and a generalized quaternion group cannot be a Sylow 2-group of a simple group [R. Brauer and the author, Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 1757–1759; MR **22** #731], this theorem immediately implies Theorem 1: A non-abelian simple group satisfying the condition (T.I.) is isomorphic to one of the groups $L_2(q)$, $U_3(q)$ or $G(q)$ for a power q of 2.

This elegant theorem is the end result of a long series of papers by various authors, and the proof is far from easy. However, most of the hard work has been done. Of special importance are two recent papers by the author in which he characterized (ZT)-groups and the groups $U_3(q)$ [Ann. of Math. (2) **75** (1962), 105–145; MR **25** #112; #144 above]. If the results of these two papers are assumed, the proof of Theorem 2 is relatively short. Theorem 2 is proved by induction. Let S be a Sylow 2-group of G. Let $H = N_G(S)$. Let j be an involution in S and let $J = C_G(j)$. After some preliminary reductions it may be assumed that G satisfies neither condition (i) nor (ii) of the conclusion of Theorem 2. Three cases are then considered. Case 1: For any element $x \neq 1$ in J, $C_G(x) \subseteq H$. Case 2: There exists $x \neq 1$ in J for which $C_G(x) \nsubseteq H$ and for any such element $C_G(x)$ has a Sylow 2-group which is either cyclic or generalized quaternion. Case 3: Neither Case 1 nor Case 2 holds. In Case 1 it is shown that G is a (ZT)-group and the proof is completed by appealing to a previous result (see #144 above). In Case 2 it is shown that G contains a normal subgroup G_1 which is a (ZT)-group and $|G : G_1|$ is odd. The proof is completed as in Case 1. The proof in Case 3 is somewhat more difficult and it is necessary here to look at a structure equation of G (see #144 above). The proof is then completed by appealing to the author's characterization of $U_3(q)$.

W. Feit (Ithaca, N.Y.)

Ishii, Ikuma; Morita, Tohru 146

On Φ_1- and Φ_2-groups of a finite group. (Japanese)

Sûgaku **14** (1962/63), 169–171.

Let G be a finite group and $\Phi_1(G) = \bigcap_i G_i^*$, where G_i^* ranges over the non-normal maximal subgroups of G, if G has any non-normal maximal subgroup, and $\Phi_1(G) = G$ if G has no maximal subgroup which is not normal. The authors prove (among other things) the following: (1) $\Phi_1(G)$ is nilpotent; (2) If G has a maximal subgroup G^* of prime index and $G^* \cap D(G) \subseteq \Phi_1(G)$, then G is supersolvable. To prove (2) they use Huppert's theorem [Math. Z. **60** (1954), 409–434; MR **16**, 332]. *H. Nagao* (Osaka)

Green, J. A. 147

A transfer theorem for modular representations.

J. Algebra **1** (1964), 73–84.

Let G be a finite group, and k a complete local ring which is a principal ideal domain. Let m be the maximal ideal of k, and p the characteristic of k/m. A kG-module is a finitely generated k-module on which G acts as a group of right operators. The Krull-Schmidt theorem holds for kG-modules. The representation ring $a(G)$ of G over k is the additive abelian group generated by the isomorphism classes (U) of kG-modules U, subject to the relations $(U)+(U')=(U\oplus U')$. Multiplication in $a(G)$ is defined by $(U)(U')=(U\otimes U')$, where $x\in G$ acts on $u\otimes u'\in U\otimes U'$ according to the rule $(u\otimes u')x=ux\otimes u'x$. Then $a(G)$ is a commutative associative ring with the classes $\{(U),\ U$ indecomposable$\}$ as an additive basis. If H is a subgroup of G, the maps $(U)\to(U_H)$, U a kG-module, and $(V)\to(V^G)$, V a kH-module, define additive maps $r_{GH}: a(G)\to a(H)$ (restriction) and $t_{HG}: a(H)\to a(G)$ (induction). Restriction r_{GH} is also a multiplicative homomorphism. For kG-modules U and U', $U'|U$ means that there exists a kG-module U'' such that $U\cong U'\oplus U''$. A kG-module U is called H-projective, for a subgroup H of G, if there exists a kH-module V such that $U|V^G$. The additive subgroup of $a(G)$ generated by the classes (U), with U an H-projective kG-module, is denoted by $a_H(G)$. Then $a_H(G)$ is an ideal in $a(G)$. The author's first theorem deals with the following situation. Let D and H be subgroups of G such that H contains the normalizer $N(D)$ of D. Let $\mathfrak{X}$ be the family of subgroups $\{D^x\cap D: x\in G-H\}$, and $\mathfrak{Y}$ the family $\{D^x\cap H: x\in G-H\}$. If $\mathfrak{Z}$ is any set of subgroups of G, we write $a_{\mathfrak{Z}}(G)=\sum_{Z\in\mathfrak{Z}}a_Z(G)$. Then $a_{\mathfrak{X}}(H)\subset a_D(H)$ and $a_{\mathfrak{X}}(H)\subset a_{\mathfrak{Y}}(G)$. The map t_{HG} induces an additive homomorphism, called the transfer, $t: a_D(H)/a_{\mathfrak{X}}(H)\to a_D(G)/a_{\mathfrak{X}}(G)$. The first theorem asserts that t is an isomorphism of rings, and if $\mathfrak{Z}$ is any family of subgroups of D with $\mathfrak{Z}\supset\mathfrak{X}$, then $t(a_{\mathfrak{Z}}(H))=a_{\mathfrak{Z}}(G)$, and t induces an isomorphism: $a_D(H)/a_{\mathfrak{Z}}(H)\to a_D(G)/a_{\mathfrak{Z}}(G)$. More specifically, $t^{-1}=rs$, where r is the map $a_D(G)/a_{\mathfrak{X}}(G)\to a_D(H)+a_{\mathfrak{Y}}(H)/a_{\mathfrak{Y}}(H)$ induced by the restriction map r_{GH}, and s is the natural isomorphism $a_D(H)+a_{\mathfrak{Y}}(H)/a_{\mathfrak{Y}}(H)\to a_D(H)/a_{\mathfrak{X}}(H)$.

The proof of this theorem is based on the following correspondence between indecomposable modules. Let $\mathfrak{Z}$ be a set of subgroups of G, and write $U=O(\mathfrak{Z})$ if $(U)\in a_{\mathfrak{Z}}(G)$. Let H, D be subgroups such that $H\supset N(D)$, and let $\mathfrak{X}$, $\mathfrak{Y}$ be the families of subgroups defined above. Let $\mathfrak{A}$ be the family of subgroups D^* of D which are not conjugate in G to subgroups of members of $\mathfrak{X}$. Then there exists a one-to-one correspondence between the isomorphism classes of indecomposable kG-modules U which have vertices in $\mathfrak{A}$ [see the author, Math. Z. **70** (1958/59), 430–445; MR **24** #A1304], and the isomorphism classes of indecomposable kH-modules V which have vertices in $\mathfrak{A}$, such that U and V correspond if and only if either $V|U_H$ or $U|V^G$. Corresponding modules have the same vertex and satisfy $U_H=V+O(\mathfrak{Y})$, $V^G=U+O(\mathfrak{X})$.

Next let D be a subgroup of G, and $\mathfrak{Z}$ the set of all proper subgroups of D. Define $w_D(G)=a_D(G)/a_{\mathfrak{Z}}(G)$; then $w_D(G)$ is a ring. Let $A(G)$ and $W_D(G)$ be the algebras over the complex field C obtained by tensoring $a(G)$ and $w_D(G)$ with C. It is not known in general whether $A(G)$ is a semi-simple algebra. The author proves, however, that $A(G)$ is semi-simple if $W_D(G)$ is semi-simple for all subgroups D of G. As a corollary, it follows that $A(G)$ is semi-simple if $W_D(N(D))$ is semi-simple for each p-subgroup D of G. Using a theorem of M. F. O'Reilly [Illinois J. Math., to appear], it follows that $A(G)$ is semi-simple if G has cyclic p-Sylow subgroups. *C. W. Curtis* (London)

Frame, J. Sutherland; Tamaschke, Olaf 148

Über die Ordnungen der Zentralisatoren der Elemente in endlichen Gruppen.

Math. Z. **83** (1964), 41–45.

The second of the authors recently obtained a precise theory of unitary central S-rings over a finite group [same Z. **80** (1962/63), 328–354; MR **27** #3696; ibid. **80** (1962/63), 443–465; MR **27** #3697]. Now using results there the authors prove the following remarkable theorem. Let C_i $(i=1,\cdots,t)$ be the classes of conjugate cyclic subgroups of a finite group H. Let H_i be a representative of C_i and c_i be the order of the centralizer of H_i in H $(i=1,\cdots,t)$. Then $\prod_{i=1}^{t}c_i=2^ez^2$ holds, where z is an integer and $e=0$ or 1. Furthermore, they prove that $e=0$ either if H has an odd order or if every irreducible character of H is rational, and conjecture that $e=0$ always (actually they make a little stronger conjecture). *N. Ito* (Nagoya)

Hall, P. 149

Some constructions for locally finite groups.

J. London Math. Soc. **34** (1959), 305–319.

Two main results, together with many subsidiary results, are proved in this paper. The first main theorem is that there is a group C which is determined to within isomorphism by the following properties: (1) C is countable and locally finite (that is, every finite subset generates a finite subgroup); (2) C contains an isomorphic copy of every finite group; (3) any two isomorphic finite subgroups of C are conjugate in C. Moreover, C has the property (3′), on the face of it stronger than (3), that every isomorphism between finite subgroups of C is induced by an inner automorphism of C; moreover, (4) to every countably infinite locally finite group, C contains continuously many distinct isomorphic copies. The classes of conjugate elements of C are just the sets S_m consisting of all elements of order m (this is immediate from (3′)); and $C=S_mS_m$ for every $m>1$. It follows that C is simple. Moreover, C is "verbally complete", that is to say, given any non-trivial word $f(x_1,x_2,\cdots,x_n)$, its values as the variables range over C also range over the whole of C. The second main theorem states that to every countable abelian p-group Z there exists a countably infinite locally finite verbally complete p-group L whose centre is isomorphic to Z. The construction of the group C of the first theorem is by a straightforward direct limit process; but the proof of its essential uniqueness is delicate, and so is the proof of (4). The second theorem also proceeds by a direct limit construction which leans heavily on the iterated wreath product of permutation groups. The verbal completeness of the group L constructed by these methods is shown by means of a third theorem which contains a sufficient condition, too complicated for statement here, for verbal completeness of a countable direct limit of finite groups. The paper concludes with an unsolved problem, namely, how many mutually non-isomorphic, verbally complete, countably infinite, locally finite p-groups with trivial centre there are for any given prime number p.

B. H. Neumann (Zbl **88**, 23)

Garbe, Dietmar; Mennicke, Jens L. 150
Some remarks on the Mathieu groups.
Canad. Math. Bull. **7** (1964), 201–212.

The remarks in question consist of: (i) A characterisation of the 2-Sylow subgroups of M_{11}, M_{12}, M_{22} and M_{24}. The 2-Sylow subgroup of M_{24} is stated to be the holomorph of the elementary Abelian group of type $(1, 1, 1, 1)$; presumably the 2-Sylow subgroup of this holomorph is intended. (ii) A representation of M_{12} of degree 10 over $GF(3)$. (iii) Abstract definitions for M_{11}, M_{12} and M_{22}.
J. A. Todd (Cambridge, England)

Chen, Zhong-mu [Chen, Chung-mu] 151
An extension of Burnside's theorem.
Acta Math. Sinica **14** (1964), 75–77 (*Chinese*); *translated as Chinese Math.* **5** (1964), 81–84.

Generalizing a theorem of Burnside, the author proves: Suppose the order of a finite group G is $p^\alpha n$, $(p, n) = 1$, and P is a Sylow p-subgroup of G. If every element of order prime to p in the normalizer $N_{P'}$ of every subgroup P' of P containing the center of P commutes with elements of P', then G has a normal subgroup of order n. He proves also a certain generalization of Frobenius's theorem, and investigates finite groups whose Sylow p-subgroups are cyclic.
K. Honda (Tokyo)

Honda, Kin'ya 152
Realism in the theory of abelian groups. III.
Comment. Math. Univ. St. Paul. **12** (1964), 75–111.

For Part II see same Comment. **9** (1961), 11–28 [MR **26** #199]. In the first chapter the socles S_0 of reduced abelian p-groups G are studied. To the descending chain $S_0 \supseteq S_1 \supseteq \cdots \supseteq S_\alpha \supseteq \cdots \supseteq S_\lambda = 0$, where $S_\alpha = S_0 \cap p^\alpha G$, there corresponds an ascending chain $0 = P_0 \subseteq P_1 \subseteq \cdots \subseteq P_\alpha \subseteq \cdots \subseteq P_\lambda = S_0$ of subgroups such that $S_0 = P_\alpha \oplus S_\alpha$. If $C_\alpha = P_{\alpha+1} \cap S_\alpha$, then the direct decompositions $S_0 = P_\alpha \oplus C_\alpha \oplus S_{\alpha+1}$ give rise to a homomorphism $u \to \bar{u} = \langle c_0, c_1, \cdots, c_\alpha, \cdots \rangle$ of S_0 into the complete direct sum V of the C_α, where $c_\alpha \in C_\alpha$ is the component of u in the above decomposition. A diagonal vector $\mathrm{u} = \langle c_{10}, \cdots, c_{\alpha+1,\alpha}, \cdots, 0, 0, \cdots \rangle$ of $u \in p^\gamma G$ is defined by selecting first $u_\alpha \in p^\alpha G$ such that $pu_\alpha = u$ and then taking $c_{\alpha+1,\alpha}$ from $u_{\alpha+1} - u_0 = \langle c_{\alpha+1,0}, \cdots \rangle$, and finally using 0 in the γth place on. By means of the u's a homomorphism of $p^\gamma G$ with kernel $p^{\gamma+1}G$ is obtained into V modulo the image of P_γ in V.

The second chapter is devoted to principal p-groups which are defined by the property $S_0 = \sum_\alpha C_\alpha$ (direct sum). Necessary and sufficient conditions are established on the Ulm invariants of principal p-groups.
L. Fuchs (Budapest)

Khabbaz, S.; Walker, E. A. 153
The number of basic subgroups of primary groups.
Acta Math. Acad. Sci. Hungar. **15** (1964), 153–155.

As a solution of Problem 8 in the reviewer's book [*Abelian groups*, Publ. House Hungar. Acad. Sci., Budapest, 1958; MR **21** #5672] the cardinality $b(G)$ of the set of different basic subgroups B of an abelian p-group G is determined. If G is of bounded order or is divisible, then $b(G) = 1$. If G is the direct sum of m quasicyclic groups and cyclic groups of orders $p^{n_1}, \cdots, p^{n_k}$, then $b(G) = \prod_{i=1}^{k} p^{mn_i}$. In all other cases, $b(G) = |G|^{|B|}$, where bars indicate cardinality. The generalized continuum hypothesis is assumed.
L. Fuchs (Budapest)

Husain, F. 154
Characterization of abelian groups.
Nieuw Arch. Wisk. (3) **12** (1964), 1–4.

The following two axioms characterize an abelian group in terms of a binary operation $x \circ y$ defined in a set G: (I) $x \circ y \in G$ for $x, y \in G$, (II) $y = x \circ \{(x \circ z) \circ (y \circ z)\}$. The proof is reduced to the characterization by J. V. Whittaker [Amer. Math. Monthly **62** (1955), 636–640; MR **18**, 279; erratum, MR **18**, 1118].
H. Nagao (Osaka)

Sergeev, M. I. 155
Completely FN-factorizable groups. (Russian)
Dokl. Akad. Nauk SSSR **155** (1964), 532–534.

A subgroup A of a group G is called F-complemented in G if $AB = G$ with some subgroup B of G such that $A \cap B$ is finite. In case $A \cap B$ is contained in a finite normal subgroup of B, A is FN-complemented, and if $A \cap B = 1$, A is complemented. If each subgroup of G is F-complemented, etc., then G is said to be completely F-factorizable, etc. A group with FN-complemented abelian subgroups is completely FN-factorizable. G is completely FN-factorizable if and only if it contains a finite normal subgroup N such that G/N is a semidirect product of a completely factorizable group and a normal subgroup which is the direct product of a completely factorizable group and a divisible abelian group with minimum condition on subgroups and whose abelian subgroups are FN-complemented in G/N. A completely FN-factorizable group is the product of a finite normal subgroup, which is identical with its commutator subgroup, and a solvable group. Among the completely F-factorizable groups the nilpotent and ZA-groups are described. No proofs are given.
L. Fuchs (Budapest)

Viljoen, Gerhardus 156
★A contribution to the extensions of abelian groups.
Ter Verkrijging van de Graad van Doctor in de Technische Wetenschap aan de Technische Hogeschool te Delft.
Drukkerij "Luctor et Emergo", Leiden, 1963. viii + 42 *pp.*

A rather explicit description is given for all abelian factor sets of an abelian group L into another one K in case L is a cyclic or quasi-cyclic group or else the additive group of rational numbers. It is shown that if $\mathrm{Ext}(L, C(\infty)) = 0$ holds for some L, where $C(\infty)$ is the infinite cyclic group, then L is $\aleph_1$-free and a subdirect sum of $C(\infty)$; moreover, every element of L is contained in a cyclic direct summand. Furthermore, if L has an infinite cardinality $\mathfrak{m}$ satifying $\mathfrak{m}^{\aleph_0} = \mathfrak{m}$, then $\mathrm{Hom}(L, C(\infty))$ is of cardinality $2^{\mathfrak{m}}$, and L/pL and L are of the same rank for every prime p. Groups L satisfying $\mathrm{Ext}(L, T) = 0$ for every torsion group T are also considered. A simple proof is given that the complete direct sum of countably many $C(\infty)$ does not have this property. If the cardinal number $\mathfrak{m}$ of L satisfies $\mathfrak{m}^{\aleph_0} = \mathfrak{m}$, then the ranks of L/pL and L are the same [cf. J. Rotman, Acta Math. Acad. Sci. Hungar. **12** (1961), 245–254].
L. Fuchs (Budapest)

Simon, Hermann 157

Noethersche Gruppen mit endlicher Hyperzentrumsfaktorgruppe.

Illinois J. Math. 8 (1964), 231–240.

Un gruppo è detto noetheriano quando ogni suo sottogruppo ammette un sistema finito di generatori. Questo lavoro e il seguente sono dedicati al problema di fornire condizioni necessarie e sufficienti affinchè i gruppi con un sistema finito di generatori e verificanti una data proprietà E risultino noetheriani. In questo lavoro il problema è risolto quando $E=$"il gruppo fattoriale G/H, con $H=$ ipercentro di G, è finito". (Se G è un gruppo, si ponga $Z_0(G)=1$; per ogni ordinale k, che non sia un numero limite, sia $Z_k(G)$ il sottogruppo di G tale che $Z_k(G)/Z_{k-1}(G)$ $=$ centro di $G/Z_{k-1}(G)$; per ogni numero limite σ si ponga $Z_\sigma(G)=\bigcup_{\nu<\sigma} Z_\nu(G)$; allora se λ è il più piccolo numero ordinale tale che $Z(G/Z_\lambda(G))=1$, $Z_\lambda(G)$ è detto ipercentro di G.)

Un gruppo G è detto t-semirisolubile se ogni suo gruppo fattoriale infinito ha un sottogruppo abeliano normale privo di torsione, ed è detto pressochè-risolubile ("fastauflösbar") quando ogni suo gruppo fattoriale $F\neq 1$ ha un sottogruppo normale $\neq 1$ a derivato finito. Un gruppo G è detto quasiomogeneo quando esiste un insieme infinito Φ di numeri primi, tale che ogni fattore finito di G induca in ogni suo p-sottogruppo normale P, con $p\in\Phi$, un gruppo di automorfismi il cui ordine è potenza di p (Fattore = gruppo fattoriale di un sottogruppo).

Il teorema principale asserisce che per un gruppo G le seguenti condizioni sono equivalenti: (I) G è noetheriano e il fattoriale G/H, con H ipercentro di G, è finito; (II) G è noetheriano, pressochè-risolubile e quasiomogeneo; (III) G ha un sistema finito di generatori, è quasiomogeneo e ha un sottogruppo t-semirisolubile di indice finito.

In particolare, si ha che per un gruppo G privo di torsione, le seguenti proprietà sono equivalenti: (I) G è noetheriano e nilpotente (di classe finita); (II) G ha un sistema finito di generatori, è quasiomogeneo, e ha un sottogruppo t-semirisolubile di indice finito.

G. Zappa (Florence)

Širšov, A. I. 158

On certain near-Engel groups. (Russian)

Algebra i Logika Sem. 2 (1963), *no.* 5, 5–18.

The author defines inductively $[a, b, 1]=[a, b]=aba^{-1}b^{-1}$; $[a, b, k]=[[a, b, k-1], b]$, and $f^{(1)}(x, y)=xy$, $\varphi^{(1)}(x, y)=yx$;

$$f^{(k)}(x, y) = f^{(k-1)}(x, y)\varphi^{(k-1)}(x, y),$$
$$\varphi^{(k)}(x, y) = \varphi^{(k-1)}(x, y)f^{(k-1)}(x, y).$$

The group is an Engel group if $[a, b, k]=1$ for all a, b in G and a fixed integer k; likewise G is a ν-group if $f^{(k)}(x, y)=\varphi^{(k)}(x, y)$ for all x and y and a fixed integer k. He shows (Theorem 1) G is three-Engel if and only if (i) G is a ν_3-group, and (ii) $xy^2xyxyx^2y=yx^2yyxxy^2x$; conditions (i) and (ii) are independent. (Theorem 2) If G is nilpotent by nilpotent satisfying an identity of the form $x^{2^k}=1$, then G is a ν-group. (Theorem 4) Finite ν-groups are extensions of nilpotent groups of odd order by two-groups. (Theorem 5) A three-Engel element of G is a two-Engel element in any subgroup generated by itself and by one of its conjugates under G.

H. Heineken (Frankfurt a.M.)

Baumslag, Gilbert 159

On a problem of Plotkin concerning locally nilpotent groups.

Math. Z. 83 (1964), 25–26.

A rational series in a group is an ascending series with factors that are isomorphic to subgroups of the additive group of the rational numbers. A group in which extraction of nth roots is possible for every n is here called an $\mathscr{E}$-group. V. S. Čarin [Dokl. Akad. Nauk SSSR **66** (1949), 809–811; MR **10**, 677] has proved that an $\mathscr{E}$-group with a finite rational series is nilpotent. This gave rise to the question whether an $\mathscr{E}$-group with a (non-finite) rational series is necessarily locally nilpotent. The present paper answers this question in the negative. The author takes a free metabelian group M of rank 2 and forms the cartesian product P of the Mal'cev completions of all torsion-free nilpotent factor groups of M. It is known that the natural homomorphism of M into P is a monomorphism. Hence P is not locally nilpotent. But P both is an $\mathscr{E}$-group and has a rational series. *K. W. Gruenberg* (London)

Čan Van Hao [Ch'ang Wang Hao] 160

Nilgroups of finite rank. (Russian)

Sibirsk. Mat. Ž. 5 (1964), 459–464.

The nilgroups in the title of this paper are to be understood in the sense of Baer [Math. Z. **62** (1955), 402–437; MR **17**, 124]: every cyclic subgroup is subnormal (finitely accessible). The paper begins with the observation (Theorem 1) that nilgroups can also be characterized by the fact that every element is a left-Engel element (for this see also K. W. Gruenberg [Illinois J. Math. **3** (1959), 151–168; MR **21** #3483]). The author then considers nilgroups subject to certain finiteness conditions. We recall that in the classification of Mal'cev [Dokl. Akad. Nauk SSSR **67** (1949), 23–35; MR **11**, 78; Mat. Sb. (N.S.) **28 (70)** (1951), 567–588; MR **13**, 203] an abelian group is of type A_2 if it has finite special rank and of type A_3 if, in addition, its periodic part satisfies the minimal condition for subgroups. Theorem 2 states that for a p-group G the following properties are equivalent: (1) G is a nilgroup in which all abelian subgroups are of type A_2; (2) G is nilpotent with minimal condition for subgroups [see also K. W. Gruenberg, Illinois J. Math. **5** (1961), 436–466; MR **25** #122]; (3) G is a finite central extension of a direct product of a finite number of groups of type p^∞; (4) G is locally normal and of finite special rank. The proof utilizes previous results of Baer [loc. cit.], Mjagkova [Izv. Akad. Nauk SSSR Ser. Mat. **13** (1949), 495–512; MR **11**, 321] and Černikov [Mat. Sb. (N.S.) **27 (69)** (1950), 185–200; MR **12**, 477]. Finally, (Theorem 3) if in a nilgroup G all abelian subgroups are of type A_3, then G is nilpotent of finite rank. The corresponding result for ZA-groups or locally nilpotent groups is known to be false. The paper ends with a clever example of a nilgroup of finite special rank that is not nilpotent.

K. A. Hirsch (Madison, Wis.)

Šmel'kin, A. L. 161

Free polynilpotent groups. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. 28 (1964), 91–122.

This paper presents in detail statements and proofs of the results announced earlier [Dokl. Akad. Nauk SSSR **151** (1963), 73–75; MR **27** #5820]. Although the review

quoted describes the content of this paper adequately, it may be added that the main result of § 7 is that a free polynilpotent group may be embedded in a complete polynilpotent group (for the same finite sequence of integers), "complete" meaning that every equation $x^a = g$ has a unique solution x. *O. H. Kegel* (Frankfurt a.M.)

Grün, Otto 162
Beiträge zur Gruppentheorie. XI.
J. Reine Angew. Math. **214/215** (1964), 71–75.
A finite group $\mathfrak{G}$ is called a Cp-group, for a fixed prime p dividing its order $[\mathfrak{G}]$, if all its elements S whose orders are divisible by p (p-irregular elements) are elements of p-power order (p-singular elements); that is, if the centralizer of each p-irregular element is a p-group. $\mathfrak{G}$ is a Ct-group if it is a Cp_i-group for all prime divisors p_i of $\mathfrak{G}$. Each Ct-group is a CN-group, in which the centralizer of each element $\neq 1$ is nilpotent. A number of properties of these groups are proved in this paper, including the following. Factor groups and subgroups of a Cp-group are also Cp-groups if their orders are divisible by p. Every nilpotent Cp-group is a p-group. Centralizers of two distinct p-Sylow subgroups of a Cp-group have only the identity in common. Two elements of the centralizer $\mathfrak{Z}(\mathfrak{P})$ of a p-Sylow subgroup $\mathfrak{P}$ of a Cp-group $\mathfrak{G}$ that are conjugate in $\mathfrak{G}$ are conjugate in the common normalizer $\mathfrak{N}(\mathfrak{P})$ of $\mathfrak{P}$ and $\mathfrak{Z}(\mathfrak{P})$. If two p-Sylow groups of $\mathfrak{G}$ have an intersection > 1, this contains the centralizer of at least one p-Sylow group of $\mathfrak{G}$. The maximal nilpotent factor group of $\mathfrak{G}$ is either 1, p-regular, or a p-group which is its own normalizer in $\mathfrak{G}$. *J. S. Frame* (E. Lansing, Mich.)

Berkovič, Ja. G. 163
Quasi-nilpotent groups. (Russian)
Sibirsk. Mat. Ž. **5** (1964), 249–252.
Let G be a finite group. Two subgroups H, F of G are defined as forming a nilpotent pair if one of the following conditions is satisfied: (a) $H \leq F$, (b) $F \leq H$, (c) $N_H(H \cap F) > H \cap F$ and $N_F(H \cap F) > H \cap F$. G itself is defined to be quasi-nilpotent if any two subgroups form a nilpotent pair. The author proves that a non-nilpotent group G which is quasi-nilpotent is solvable of order $p^a q^b$, where p, q are distinct primes. Additional details on the Sylow structure of G are also proved.
P. Fong (Berkeley, Calif.)

Berkovič, Ja. G. 164
The existence of subgroups of a finite non-solvable group. (Russian)
Dokl. Akad. Nauk SSSR **156** (1964), 1255–1257.
Various results are announced concerning the question of what can be said about a non-soluble finite group if we suppose something about its (soluble) subgroups. The results illustrate the fact that the structure of certain subgroups affects the structure of the group.
Z. Janko (Canberra)

Ram, Siya 165
On the completion of rectangular matrices to unimodular matrices.
Gaṇita **13** (1962), 37–39.
Let R be a Dedekind ring, that is, a ring in which "classical" ideal theory is valid. Consider matrices with entries in R. A square matrix is called unimodular if its determinant is a unit in R. Now let A be an $r \times n$ matrix, where $r \leqq n$, and define A to be primitive if its $r \times r$ minors generate the unit ideal of R. On the other hand, call A completable if there exists an $n \times n$ unimodular matrix whose first r rows form the matrix A.
It follows from results of Steinitz [Math. Ann. **71** (1911), 328–354] that a matrix is completable if and only if it is primitive. For the special case where $r = 1$, a concise proof of this fact was given by the reviewer [Amer. Math. Monthly **63** (1956), 246–247; MR **17**, 938]. In the present note, the author attempts to give a simple proof of Steinitz' Theorem for arbitrary r. {There are numerous misprints in this note, but even allowing for these, the author's proof is incorrect.} It should be remarked, however, that the result for arbitrary r follows easily from the special case where $r = 1$.
I. Reiner (Urbana, Ill.)

Newman, M.; Smart, J. R. 166
Symplectic modulary groups.
Acta Arith. **9** (1964), 83–89.
Let I_t denote the $t \times t$ unit matrix, M' the transpose of a matrix M, and set

$$J = \begin{pmatrix} 0 & I_t \\ -I_t & 0 \end{pmatrix}.$$

Consider matrices whose entries are rational integers, and write $M \equiv N \pmod{n}$ to indicate elementwise congruence.
The symplectic modular group Γ consists of all $2t \times 2t$ matrices M (with rational integral entries) such that $MJM' = J$. Its principal congruence subgroup $\Gamma(n)$ consists of all $M \in \Gamma$ such that $M \equiv I_{2t} \pmod{n}$. If a divides b, define the symplectic modulary group $\mathfrak{M}(a, b) = \Gamma(a)/\Gamma(b)$.
The authors prove various results about the structure of $\mathfrak{M}(a, b)$. A typical theorem is as follows: Let p^u divide m, where p is prime. Then $\mathfrak{M}(m, mp^u)$ is an abelian group of type $(p^u, \cdots, p^u)$, the number of factors being $2t^2 + t$.
Of independent interest is the preliminary theorem which the authors establish: If $NJN' \equiv J \pmod{n}$, then there exists an $M \in \Gamma$ such that $M \equiv N \pmod{n}$.
I. Reiner (Urbana, Ill.)

Newman, Morris 167
Normal congruence subgroups of the modular group.
Amer. J. Math. **85** (1963), 419–427; *errata, ibid.* **85** (1963), 753; *addendum, ibid.* **86** (1964), 465.
Let Γ be the inhomogeneous 2×2 modular group, the quotient group $SL(2, \mathbf{Z})/(\pm I)$; and let $\Gamma(n)$ be the subgroup of Γ consisting of all elements of Γ congruent to a scalar matrix modulo n. The principal theorem of the paper is the following: If G is a normal subgroup of Γ such that $\Gamma(mn) \subset G \subset \Gamma(n)$, where $(m, 6) = 1$, then $G = \Gamma(nd)$ for some divisor d of m. Calling a subgroup $G \subset \Gamma$ a "congruence subgroup of level m" if m is the least integer such that $\Gamma(m) \subset G$, a corollary of the above theorem is that for $(m, 6) = 1$, $\Gamma(m)$ is the only normal congruence subgroup of level m. The restriction $(m, 6) = 1$ is essential, as the author shows by examples.
R. C. Gunning (Princeton, N.J.)

Vislavskiĭ, M. N. 168

Absolutely irreducible representations of certain groups. I. (Russian)

Vesci Akad. Navuk BSSR Ser. Fiz.-Tèhn. Navuk **1963**, *no.* 4, 25–33.

Let G be a group with a normal subgroup H of finite index. Then every (finite-dimensional linear) representation of G induces one of H. Two such representations of G are said to be related (relative to H) if they coincide on H. In this paper the author considers the case that G/H is abelian and either cyclic or elementary abelian of order p^2. A complete survey is given of the classes of similar (relative to H) absolutely irreducible representations of G. Two representations are called similar if they are related relative to H and differ on a fixed basis of G/H only by suitable roots of unity as multiplicative factors.

O. H. Kegel (Frankfurt a.M.)

Gabriel, John 169

New methods for reduction of group representations using an extension of Schur's lemma.

J. Mathematical Phys. **5** (1964), 494–504.

The author's approach is fundamentally physical in the sense that actual matrices in reduced form are his goal. Though he gives many references, he does not seem to realize that the ideas he is describing are to be found in Schur's thesis and papers, where the relation between the representation theory of $GL(n)$ and that of the commuting algebra S_m is described in detail. Specific references to this and much subsequent work are to be found in the reviewer's book [*Representation theory of the symmetric group*, Univ. Toronto Press, Toronto, Ont., 1961; MR **23** #A3182].

G. de B. Robinson (Toronto, Ont.)

Glauberman, George 170

Fixed points in groups with operator groups.

Math. Z. **84** (1964), 120–125.

Let A be an operator group which acts faithfully on the group G. If B is a subgroup of A and H is a subgroup of G fixed by B, define X to be their semi-direct product. The author investigates consequences of one or both of the following conditions. (S) For every subgroup W of X such that $WH=X$, W splits over $W\cap H$. (Z) Every complement of H in X is conjugate to B. The results in the paper are generalizations of results which are known in case $(|A|, |G|)=1$, where both conditions (S) and (Z) are satisfied.

W. Feit (Ithaca, N.Y.)

Narasimha Swamy, K. L. 171

Autometrized lattice ordered groups. I.

Math. Ann. **154** (1964), 406–412.

This paper studies the geometry of lattice-ordered abelian groups qua autometrized algebras, the generalized distance being given by the usual absolute value: $d(a,b)=|a-b|=(a\cup b)-(a\cap b)$. It is shown, for example, that the geometric concept of "betweenness" (x is between a and b if $d(a,b)=d(a,x)+d(x,b)$) is equivalent to $a\cap b\leqq x\leqq a\cup b$, which is the usual lattice-theoretical notion of betweenness. The concepts of B-linearity and D-linearity introduced by Blumenthal [Rend. Circ. Mat. Palermo (2) **1** (1952), 343–360; MR **15** #247] are shown to be equivalent in lattice-ordered abelian groups. It is proved that there are no equilateral triangles in the geometry of lattice-ordered abelian groups, and that segments are congruent if and only if they have the same length. {Reviewer's remark: Many of the results of this paper become more transparent if account is taken of the fact that every abelian l-group is a subdirect product of ordered groups. For example, it is obvious that there are no equilateral triangles in an ordered group, and this fact immediately carries over to abelian l-groups in general.}

R. S. Pierce (Seattle, Wash.)

Šik, František 172

Über additive und isotone Funktionale auf geordneten Gruppen. (Russian summary)

Czechoslovak Math. J. **12** (**87**) (1962), 611–621.

Let (G, P) denote an additive abelian group G in which a partial order has been introduced with the positive cone P. The author obtains various results about the set $\hat{}(G,P)$ of all isotone homomorphisms of (G,P) into the ordered additive group of real numbers, including the following theorems: (1) $\hat{}(G,P)$ contains a non-zero homomorphism if and only if an extension $S\neq\{0\}$ of the partial order P of G and an element $a\in G$ can be found such that, for every $x\in G$, there exists an integer m satisfying $ma-x\in S-\{0\}$. (2) For a subgroup H of G, a non-zero homomorphism $f\in \hat{}(H, H\cap P)$ is the restriction of an element of $\hat{}(G,P)$ to H if and only if there exists an extension Q of the partial order P of G such that $f\in\hat{}(H, H\cap Q)$ and, for every $x\in G$, there exists an $a\in H$ satisfying $a-x\in Q-\{0\}$.

A. Jaeger (Cincinnati, Ohio)

Jakubíková, Mária 173

On some subgroups of l-groups. (Russian. German summary)

Mat.-Fyz. Časopis Sloven. Akad. Vied **12** (1962), 97–107.

The sets $\mathscr{K}$ and $\mathscr{K}'$ of all convex subgroups and all convex subgroup-sublattices of a lattice-ordered group G are studied. They are complete lattices under inclusion, where intersection is to be taken in the set-theoretical sense. $\mathscr{K}'$ satisfies the infinite distributive law $A\wedge(\bigvee A_i)=\bigvee(A\wedge A_i)$ $(A, A_i\in\mathscr{K}')$; its dual is satisfied in complete lattice-ordered groups G. $\mathscr{K}$ is a modular (distributive) lattice if and only if G is linearly ordered.

L. Fuchs (Budapest)

Jakubík, Ján 174

The interval topology of an l-group. (Russian summary)

Mat.-Fyz. Časopis Sloven. Akad. Vied **12** (1962), 209–211.

In a lattice-ordered group G an element $a>0$ is called Archimedean if for each $b\in G$ there is a positive integer n such that $na\not\geqq b$. It is shown that if G contains disjoint Archimedean elements, then it is not a topological group in its interval topology.

L. Fuchs (Budapest)

Lesohin, M. M. 175

Systems with exterior multiplication with periodic and complete components. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 2 (39), 94–99.

For terminology see the author [same Izv. **1963**, no. 4 (35), 89–91; MR **28** #148; ibid. **1963**, no. 5 (36), 59–62; MR **28** #149]. Let (A, B, C) be a system with exterior multiplication. Under various conditions concerning the semigroups A, B, C the regularity and completeness of the system with respect to the components are studied. The conditions imposed on the semigroups A, B, C are, e.g., periodicity, completeness or regularity of some of the components. There are ten theorems corresponding to various assumptions concerning A, B, C.

Št. Schwarz (Bratislava)

Munn, W. D. 176

A certain sublattice of the lattice of congruences on a regular semigroup.

Proc. Cambridge Philos. Soc. **60** (1964), 385–391.

A semigroup S is called regular if and only if $a \in aSa$ for every element a of S. The set Λ of all congruences on a given semigroup is a lattice. This paper reports that a certain sublattice of congruences on a regular semigroup is a modular lattice. Let $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{H}$ be the equivalences defined as follows: $a\mathscr{L}b$ if and only if $S^1a=S^1b$; $a\mathscr{R}b$ if and only if $aS^1=bS^1$; $\mathscr{H}=\mathscr{L}\cap\mathscr{R}$. Let $\Sigma(\mathscr{H})=\{\rho\in\Lambda:\rho\subseteq\mathscr{H}\}$. The main theorem is this: If S is a regular semigroup, then $\Sigma(\mathscr{H})$ is a modular sublattice with a greatest and a least element. To prove this, the following lemma is used: $\rho\circ\sigma=\sigma\circ\rho$ if $\rho, \sigma\in\Sigma(\mathscr{H})$ where $\rho\circ\sigma=\{(a,c)\in S\times S: (a,b)\in\rho, (b,c)\in\sigma$ for some $b\}$. The author introduces idempotent-separating congruences: A congruence ρ is idempotent-separating if and only if each congruence class contains at most one idempotent. Clearly if $\rho\in\Sigma(\mathscr{H})$, then ρ is idempotent-separating. The main theorem is applied to the following special cases: (1) S is a regular semigroup in which all the idempotents form a subsemigroup; (2) S is a semigroup that is a union of groups; (3) S is a regular semigroup which satisfies the minimal condition on principal left ideals. In each case, all the idempotent-separating congruences on S are contained in $\mathscr{H}$ and hence they form a modular sublattice with a greatest and a least element.

T. Tamura (Davis, Calif.)

Saitô, Tôru 177

On some equivalence relations in semigroups.

Bull. Tokyo Gakugei Univ. **14** (1963), 1–5.

The author discusses the equivalence relation $\mathfrak{r}'$, defined by $a\equiv b\,(\mathfrak{r}')$ if and only if $aS=bS$, on a right reductive semigroup S, i.e., on a semigroup such that $as=bs$ for all $s\in S$ implies $a=b$. Let $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$ and $\mathscr{D}$ denote Green's equivalences on S: $\mathscr{L}=\{(a,b): Sa\cup a=Sb\cup b\}$, $\mathscr{R}$ is the left-right dual of $\mathscr{L}$, $\mathscr{H}=\mathscr{L}\cap\mathscr{R}$ and $\mathscr{D}=\mathscr{L}\circ\mathscr{R}=\mathscr{R}\circ\mathscr{L}$. It is shown that $\mathscr{L}\circ\mathfrak{r}'=\mathfrak{r}'\circ\mathscr{L}$. Write $\mathscr{D}'=\mathscr{L}\circ\mathfrak{r}'$ and $\mathscr{H}'=\mathscr{L}\cap\mathfrak{r}'$. Then $\mathscr{D}'$ is an equivalence on S and each $\mathscr{D}'$-class is a "rectangle" of $\mathscr{H}'$-classes. It is shown that a multiplication may be defined on a $\mathscr{D}'$-class D' in such a manner that D' becomes a completely simple semigroup with the $\mathscr{L}$-classes and $\mathfrak{r}'$-classes in D' as the minimal left and minimal right ideals, respectively, of this completely simple semigroup. The $\mathscr{H}'$-classes of the semigroup D' then coincide with the $\mathscr{H}'$-classes of the $\mathscr{D}'$-class D'. If D' contains an idempotent (of S), then, within D', $\mathfrak{r}'$ coincides with $\mathscr{R}$. Hence, in this event, D' is a $\mathscr{D}$-class, and the new multiplication introduced turns a $\mathscr{D}$-class into a completely simple semigroup.

G. B. Preston (Clayton)

Graham, Nancy 178

Note on M-groupoids.

Proc. Amer. Math. Soc. **15** (1964), 525–527.

A groupoid is a system with a binary operation. A groupoid S is called an M-groupoid if and only if S satisfies the following conditions: (I) Existence of left identities e: $ex=x$ for all $x\in S$; (II) (weakened associativity) $x(yz)=(xy)z$ for all $x\in S$, if y or z is a left identity; (III) For any $x\in S$, there is a unique left identity e of S such that $xe=x$. In an earlier paper of Merkel, Latimer and the reviewer [same Proc. **14** (1963), 118–123; MR **26** #2534], the following question was raised: If a groupoid S satisfies (I), (II) above and (III′) below, is S an M-groupoid? (III′) (Right invertibility): For every $a, b\in S$, there is $x\in S$ such that $ax=b$. In this note the author gives a negative answer to this question by showing an elegant example. Clearly such an example has to be infinite. Also the author proves a theorem: A groupoid S is an M-groupoid if S satisfies (I) and (II) and S is uniquely right invertible, that is, for every $a, b\in S$, there is a unique $x\in S$ such that $ax=b$. The reviewer adds that a groupoid S satisfies (I), (II) and unique right invertibility if and only if S is the direct product of a right zero semigroup and a uniquely right invertible groupoid with a two-sided identity.

T. Tamura (Davis, Calif.)

Tagamlicki, Ja. 179

On the separability principle in associative Abelian spaces. (Bulgarian. Russian and German summaries)

Bŭlgar. Akad. Nauk. Izv. Mat. Inst. **7** (1963), 169–183.

Soit K un ensemble arbitraire et (K,σ) un multigroupoïde (si $a\in K$, $b\in K$, on a $a\sigma b\equiv ab\subset K$). L'auteur appelle σ convexité dans K et emploit pour un multigroupoïde (K,σ) associatif et commutatif la dénomination d'espace associatif abélien. On dit que l'ensemble A $(\subset K)$ est convexe si $a\in A$, $b\in A$ entraînent $ab\subset A$. Les convexités σ_1 et σ_2 dans le même ensemble sont dites équivalentes si elles déterminent les mêmes ensembles convexes. On dit que les éléments a et b de K sont dans la relation d'incidence (i) si l'intersection de leurs variétés convexes est non-vide. $A\subset K$ est appelé demi-espace, si les espaces A et $K\setminus A$ sont non-vides et convexes. On dit que (K,σ) remplit la condition de séparabilité (s) si pour tous deux sous-ensembles A et B de K non-vides convexes et disjoints il existe un demi-espace qui contient A et n'a pas de point commun avec B.

L'auteur démontre qu'un multigroupoïde (K,σ) associatif et commutatif remplit la condition (s) si et seulement si la relation (i) est transitive et s'il existe une convexité σ_1 dans K équivalente à σ et telle que l'on ait pour elle

$$(1)\qquad a\frac{b}{c}\subset\frac{ab}{c}\qquad\left(\frac{a}{b}=_{\text{def}}\{x\mid b\in ax\}\right).$$

L'article contient en outre quatre théorèmes dont l'un établit la condition suffisante de (s) pour un multigroupoïde associatif, commutatif et remplissant la

condition (1), sous la forme $abb \subset ab$, et les derniers trois sont relatifs aux multigroupoïdes munis de structure topologique.
S. B. Prešić (Belgrade)

Dénes, József; Pásztor, Endréné 180
Some problems on quasigroups. (Hungarian)
Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **13** (1963), 109–118.

To every $n \geqq 4$ there exists a quasigroup of order n containing a subquasigroup of order 2. It follows that to every $n \neq 3$ there exist two Latin squares differing in four places from each other, but Latin squares are identical if they differ in at most three places. A conjecture of D. W. Wall [Pacific J. Math. **7** (1957), 1711–1714; MR **19**, 1159] is disproved by showing that to every $m > 2$ there exists a quasigroup which is the union of m pairwise disjoint subquasigroups of equal orders.
L. Fuchs (Budapest)

Wilker, Peter 181
Double loops and ternary rings.
Bull. Amer. Math. Soc. **70** (1964), 543–547.

Let T be a ternary ring with the ternary operation $T(a, b, c)$ and the distinguished elements 0, 1 [see Pickert, *Projektive Ebenen*, Springer, Berlin, 1955; MR **17**, 399]. On T, two loop structures can be defined by means of the binary operations $a + b = T(a, 1, b)$ and $ab = T(a, b, 0)$. The resulting loops are called the additive and the multiplicative loops of T, respectively. Together with $a0 = 0a = 0$ they define the structure of a double loop on T which satisfies condition (1): For every a, b in T with $b \neq 1$, the equation $x + a = xb$ is uniquely solvable. It is shown (Theorem 1) that any infinite double loop satisfying (1) can be given canonically the structure of a ternary ring; moreover, that any infinite loop can be the additive or multiplicative loop of a double loop satisfying (1) and hence of a ternary ring (Theorems 2m and 2a).

A ternary ring is called linear if $T(a, b, c) = ab + c$ (identically). In this case, the double loop of T satisfies two further conditions ((2) and (3) in the paper) and any double loop satisfying (2) and (3) can be made into a linear ternary ring. Theorem 3 gives sufficient conditions for an infinite loop to be the additive loop of a double loop satisfying (2) and (3). These conditions are satisfied by all infinite groups. On the other hand, Theorem 4 presents a construction yielding infinite loops of any cardinality which cannot be additive loops of linear ternary rings. This confirms and extends a conjecture of D. R. Hughes [Proc. Amer. Math. Soc. **6** (1955), 973–980; MR **17**, 451].

Detailed proofs are to appear later.
L. G. Kovács (Canberra)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 215, 423, 424, 425, 426, 483, 486, 506, 567, 572, 610, 631, 976, 979, 980.

Bochner, S. 182
Analytic measures on compact Bohr groups.
Scripta Math. **27**, 5–26 (1964).

For $k \in \mathbf{N}$, let G_k be a Bohr compactification of $\mathbf{R}_k$, and let μ be a Borel measure on G_k. Decomposing μ into its (Haar) absolutely continuous and singular parts: $\mu = \mu_a + \mu_s$, the author obtains interesting generalizations of the Helson-Lowdenslager techniques, particularly as they relate to the classical F. and M. Riesz theorem on measures having non-negative spectrum.

If μ is a Borel measure on G_k, let $\hat{\mu}(\lambda)$ be its Fourier transform confined to $\mathbf{R}_k$, and let $S_p\{\mu\} = \{\lambda \mid \hat{\mu}(\lambda) \neq 0\}$. Theorem: If $S_p\{\mu\} \subset \{\lambda \mid (\lambda, c) \equiv \sum_{i=1}^k \lambda_i c_i \geqq \Lambda\} \equiv H_\Lambda$, then $S_p\{\mu_a\} \subset H_\Lambda$ and $S_p\{\mu_s\} \subset H_\Lambda$. Theorem: If $S_p\{\mu\}$ is countable and if, for some c, $\sum_{(\lambda, c) \leq \Lambda} |\hat{\mu}(\lambda)| < \infty$ holds for all Λ, then μ is absolutely continuous.

If $k = 1$, μ is called analytic if $S_p\{\mu\} \subset \mathbf{R}^+$. Theorem: If μ is analytic, then $S_p\{\mu_a\} \cup S_p\{\mu_s\} \subset S_p\{\mu\}$ and $S_p\{\mu_s\}$ is two-sidedly dense in itself. The F. and M. Riesz theorem is a corollary to this conclusion.

Other more detailed results are to be found but are omitted here.
B. R. Gelbaum (Irvine, Calif.)

Varopoulos, Nicholas Th. 183
Sur la continuité des fonctions définies sur un groupe localement compact.
C. R. Acad. Sci. Paris **258** (1964), 2716–2719.

A function $f(x)$ defined on a topological space T is called a function (S) if $\lim x_n = x$ implies $\lim f(x_n) = f(x)$. The space T is said to have property (Σ) if every function (S) on T is continuous. For a topological group G, let $g(G)$ be the smallest cardinal such that there exists a base at the identity whose cardinality is $g(G)$. Theorem: If G is a locally compact group such that $g(G)$ is smaller than the first inaccessible cardinal, then G has property (Σ).
Deane Montgomery (Princeton, N.J.)

Varopoulos, N. Th. [Varopoulos, Nicholas Th.] 184
A theorem on the continuity of homomorphisms of locally compact groups.
Proc. Cambridge Philos. Soc. **60** (1964), 449–463.

The theorem mentioned in the title is this: Let G and H be two locally compact groups and $f: G \to H$ an algebraic homomorphism. Suppose that the following conditions are satisfied: (a) f is sequentially continuous. (b) The cardinality of G is such that there is no ultrafilter on G with the countable intersection property unless it is the collection of sets containing a point (i.e., card G is "bornological"). Then f is continuous. This interesting theorem is proved by a reduction to essentially the case that G is a direct product of metric compact groups and H a compact Lie group. For that purpose a lifting lemma for sequences and several structure theorems are proved. The gist of the whole idea may best be indicated by considering the following example: Let C and H be compact groups, I a set and $\mathfrak{f}$ an ultrafilter on I with the countable intersection property which is not trivial (i.e., is not the collection of all sets containing a point). Let $G = C^I$ (the group of all functions $g: I \to C$ with the pointwise topology). Suppose $\varphi: C \to H$ is a nontrivial continuous homomorphism. For each $g \in G$ let $\bar{\varphi}(g) = \lim \varphi(g(\mathfrak{f}))$; this limit exists since $\mathfrak{f}$ is an ultrafilter and since H is compact. Then $\bar{\varphi}: G \to H$ is an algebraic homomorphism; if G' denotes the subgroup of all $g: I \to C$ which take a value $\neq 1$ only for finitely many points, then $\bar{\varphi}(G') = \{1\}$. But if $g(i) = c$ for all $i \in I$ and for some $c \in C$ with $\varphi(c) \neq c$, then

$\bar\varphi(g)=\varphi(c)\neq 1$. Since G' is dense in G, this shows that $\bar\varphi$ is not continuous. On the other hand, $\bar\varphi$ is sequentially continuous if H is metric for then given any g there is an $F\in\mathfrak{f}$ such that $\bar\varphi(g)=\varphi(g(F))$ and if g_n is a sequence in G, then there is even an $F\in\mathfrak{f}$ such that $\bar\varphi(g_n)=\varphi(g_n(F))$ for all n which establishes sequential continuity of $\bar\varphi$. The cardinality assumption on G rules this sort of pathology out and the proof actually leads to a point where a situation similar to the one in the example has to be considered. *K. H. Hofmann* (New Orleans, La.)

Kuipers, L.; Scheelbeek, P. A. J. 185
Gleichverteilung in kompakten, topologischen Gruppen.
Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 262–265.

Let G be a compact Hausdorff group, and let $\{a_1,a_2,\cdots,a_k\}$ be a subset of G such that $(a_1{}^n)_{n=1}^{\infty}$ is uniformly distributed in G. {This obviously implies that G is Abelian.} For $i=1,\ 2,\ 3,\ \cdots$, let $\Psi(i)$ be the finite sequence $(a_1{}^i, a_1{}^{i-1}a_2, \cdots, a_1{}^{i-1}a_k, a_1{}^{i-2}a_2{}^2, a_1{}^{i-2}a_2a_3, \cdots, a_2{}^i, \cdots, a_{k-1}^i, \cdots, a_{k-1}^{i-1}a_k, a_{k-1}a_k{}^{i-1}, a_k{}^i)$. Theorem: The sequence $\Psi(1), \Psi(2), \Psi(3), \cdots$ is uniformly distributed in G.
Edwin Hewitt (Seattle, Wash.)

Helmberg, Gilbert 186
On a convolution of sequences in a compact group.
Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 266–274.

Let G be a compact Hausdorff group. Let ν be a non-negative linear functional on $\mathfrak{C}(G)$ such that $\nu(1)=1$, and let μ be the Haar functional on $\mathfrak{C}(G)$. A sequence $(x_n)_{n=0}^{\infty}$ with values in G [and $x_0=e$] is ν-summable if $\lim_{N\to\infty}[(N+1)^{-1}\sum_{n=0}^{N}f(x_n)]=\nu(f)$ for all $f\in\mathfrak{C}(G)$. This generalizes the class of equidistributed sequences, which are just the μ-summable ones. The convolution $\nu_1*\nu_2$ of functionals is defined as usual. Let $(x_n)\star(y_n)$ be the sequence $(e, x_1, y_1, x_2, x_1y_1, y_2, x_3, x_2y_1, x_1y_2, y_3, \cdots)$. Main theorem: Let $(x_n{}^{(j)})_{n=0}^{\infty}$ be ν_j-summable ($j=1,\ 2$). Then $(x_n{}^{(1)})\star(x_n{}^{(2)})$ is $\nu_1*\nu_2$-summable. This generalizes one case of a theorem of Kuipers and Scheelbeek [see preceding review #185] and also generalizes a theorem of Eckmann [Comment. Math. Helv. **16** (1944), 249–263; MR **6**, 146].
Edwin Hewitt (Seattle, Wash.)

Čarin, V. S. 187
Some classes of locally bicompact groups with ascending chain condition for Abelian subgroups. (Russian)
Sibirsk. Mat. Ž. **5** (1964), 438–458.

In this review, G always denotes a locally compact group. The author establishes certain classes of groups for which the ascending chain condition for closed abelian subgroups (ACCA) implies the ascending chain condition for closed subgroups (ACC). The group G is called periodic if every element belongs to a compact subgroup of G; G is called a p-group if for every $g\in G$ and neighborhood U of the identity, all but a finite number of the elements g^{p^k} ($k=0, 1, 2, \cdots$) belong to U. Theorems: Suppose that G is periodic and solvable, or that G is periodic and locally nilpotent, or that G is a locally nilpotent p-group. If G satisfies the ACCA, then G is compact and satisfies the ACC. Both theorems in the author's earlier paper [Dokl. Akad. Nauk SSSR **145** (1962), 1010–1011; MR **25** #4023] are implied by this statement. In the negative direction there exists a nilpotent group G satisfying the ACCA but not the ACC.

The author gives a number of interesting auxiliary results. For example, if G is locally nilpotent and satisfies the ACCA, then G contains a compact nilpotent normal subgroup B satisfying the ACC for which G/B is discrete, locally nilpotent, and torsion-free. If G is a p-group and G has hypercenters for all natural numbers $1=Z_0\subset Z_1\subset\cdots\subset Z_n\subset\cdots$, then G does not satisfy the ACCA. If G is a locally nilpotent periodic group with a nontrivial compact normal subgroup satisfying the ACC, then G has nontrivial center. If G is locally nilpotent and has a compact maximal abelian subgroup satisfying the ACC then G has nontrivial center.
K. A. Ross (Rochester, N.Y.)

Conner, P. E.; Floyd, E. E. 188
A note on the action of SO(3).
Proc. Amer. Math. Soc. **10** (1959), 616–620.

In this note the authors prove that if the 3-dimensional rotational group SO(3) acts on a closed n-cell, then the orbit space is acyclic over the integers.
M. Mahowald (Zbl **89**, 254)

Connell, E. H.; Montgomery, D.; Yang, C. T. 189
Compact groups in E^n.
Ann. of Math. (2) **80** (1964), 94–103.

The principal result is that if a compact group acts differentiably on Euclidean space with a fixed-point set diffeomorphic to a hyperplane of codimension at least 3 and if all the non-trivial orbits are of the same type and of codimension at least 5, then the action of the group is differentiably equivalent to a linear one. To prove this one (a) straightens out the fixed-point hyperplane (b) adjusts the action to be linear near this hyperplane (c) applies techniques of the engulfing theorem to the orbit space, which show that outside the fixed hyperplane the orbit space is a product manifold, and this enables one to push the linearity near the hyperplane out to the entire space.

There seems to be an error with regard to the concept "weakly p-connected"; for example, Lemma 3, page 96 has the counter-example: Let N be a contractible compact 5-manifold with non-simply-connected boundary M the interior of N; U a small 5-cell; C empty; V the intersection of M and a tubular neighborhood of the boundary of N. The difficulty is that, according to the authors' definition, if M is contractible and V arbitrary then (M, V) is weakly p-connected for all p.

The mistake can be corrected perhaps along these lines: If $A\subset B\subset X$, then that A is "p-nice" in B in X might mean that every polyhedron P of dimension $\leqq p$ in X is deformable into B in such a way that the deformation does not move $P\cap A$. Then Lemma 2, the engulfing theorem, could be stated in terms of a sequence (of uncertain length) $C\subset U_1\subset\cdots\subset U_k\subset M$ for which U_i-C is p-nice in $U_{i+1}-C$ in $M-C$. Then Proposition 2, page 100, is strong enough to imply a modified version of Proposition 3, which could then be used on the following page in the proof of Theorem B.
John Stallings (Princeton, N.J.)

Krensbein, J.; Welton, T. A. 190
Note on parametrization of the four-dimensional real symplectic group.
J. London Math. Soc. **39** (1964), 291–295.
The "parametrization" of the group in the title is in the form of products of certain subsets of lower dimension. The proof given seems inadequate, amounting to only a heuristic dimension-counting argument.
R. Hermann (Evanston, Ill.)

Vilenkin, N. Ja. 191
Special functions associated with class 1 representations of the motion groups of spaces of constant curvature. (Russian)
Trudy Moskov. Mat. Obšč. **12** (1963), 185–257.
Special functions such as the Legendre and Jacobi polynomials, the Bessel functions, etc., can be generated by matrix representations of the Euclidean rotational groups. They may appear as the elements of a basis, as matrix elements or as the traces of the matrices. In this expository paper this technique is systematically explored, which eventually leads to the description of special functions of a rather general kind not studied so far. These functions arise in connection with the irreducible unitary representations of certain motion groups. There are three chapters, each consisting of four sections. A brief indication of the contents is given below. (I) concerns the group $SO(n)$ of the rotations of the $(n-1)$-sphere. (1) Contains fundamental notions such as the Eulerian angles. (2) A description is given of class 1 representations of $SO(n)$ with respect to a subgroup $SO(n-1)$. (3) The matrix elements of the representations of the previous section are determined in the form of explicit integral expressions. By this a class of generalized Jacobi polynomials $P^{nl}_{kmj}(\cos\theta)$ is defined. (4) This section contains properties of the functions $P^{nl}_{kmj}(\cos\theta)$ such as Fourier expansions, orthogonality relations, etc. (II) concerns the group of hyperbolic rotations in the n-dimensional Lobachevsky space. The four sections are the hyperbolic analogues of those of the preceding chapter. (III) concerns the class 1 representations of the group of rotations in the n-dimensional Euclidean space. In this chapter a certain type of generalized Bessel function $j^{n}_{kmj}(x)$ is carefully studied. Using representation theory, a number of interesting relations between the various types of generalized special functions are derived.

This paper also contains generalizations of the well-known integral transforms such as the Fourier and Hankel transform. *H. A. Lauwerier* (Amsterdam)

Delsarte, Jean 192
Sur une condition pour qu'une suite de formes soit une base de l'anneau des invariants d'un groupe.
C. R. Acad. Sci. Paris **258** (1964), 4895.
Let G be a group of linear transformations of C_n and let $u_1, u_2, \cdots, u_k$ be a type of base of the invariant polynomials. Then a set of differential operators is associated with this base and a condition on them is satisfied. There is a partial converse. *Deane Montgomery* (Princeton, N.J.)

Bogaevskiĭ, A. N. 193
Harmonic functions on GL(2). (Russian)
Dokl. Akad. Nauk SSSR **153** (1963), 751–753.
A Laplace operator is defined for some class of functions on GL(2), and harmonic functions relative to this operator are defined as usual. Properties of a certain class of functionals on a certain class of harmonic functions are stated. A telegraphic style, an absence of proofs, and a putative misprint on line 7, p. 751, make this note unintelligible to the reviewer.
Edwin Hewitt (Seattle, Wash.)

Orihara, Akio 194
Bessel functions and the Euclidean motion group.
Tôhoku Math. J. (2) **13** (1961), 66–74.
Let $G=SO(n)$ and $H=SO(n-1)$. Suppose $g\to U_g$ is an irreducible unitary representation of G on the space $\mathscr{H}$ with the property that there exists an essentially unique vector f_0, $f_0\in\mathscr{H}$, so that $U_hf_0=f_0$, $h\in H$, $\|f_0\|=1$; then the function $\varphi(g)=(U_gf_0, f_0)$ is called a zonal harmonic. The author begins by determining these zonal harmonics by their differential equations as functions on the sphere S^{n-1}, and shows that in polar coordinates they are (up to a constant multiple) the Gegenbauer polynomials $C_m{}^{\nu}(\cos\theta)$, $\nu=\frac{1}{2}(n-2)$, $m=0, 1, 2, \cdots$.

Next, the author turns to the similar situation where now G is the group of n-dimensional Euclidean motions and H is the subgroup of rotations. The zonal harmonics are now Bessel functions; more particularly, they are (up to a constant multiple) $J_\nu(Rr)/r^\nu$, where R is a positive number parameterizing the appropriate irreducible representations. In the case of two dimensions this was obtained by N. Ja. Vilenkin [Uspehi Mat. Nauk **11** (1956), no. 3 (69), 69–112; MR **19**, 153]. By the use of such considerations the author rederives Gegenbauer's identity

$$e^{iz\cos\theta} = 2^\nu\Gamma(\nu)\sum_{m=0}^{\infty}(\nu+m)i^m\frac{J_{m+\nu}(z)}{z^\nu}C_m{}^{\nu}(\cos\theta).$$

Finally considering the case $G=SO(n)$, $H=SO(p)\times SO(q)$ $(p+q=n)$, the author is led to a derivation of Sonine's integral formula. However this approach seems, in reality, very close to the one given in Section 12.12 of Watson's book on Bessel functions [*A treatise on the theory of Bessel Functions*, Cambridge Univ. Press, Cambridge, England, 1944; MR **6**, 64].
E. M. Stein (Princeton, N.J.)

Kohari, Akihiro 195
Harmonic analysis on the group of linear transformations of the straight line.
Proc. Japan Acad. **37** (1961), 250–254.
Let G be the group of affine linear transformations $\{g: g(\xi)=a\xi+b,\ 0<a,\ -\infty<b<\infty\}$ of the straight line, and H the subgroup of homogeneous transformations. Let $U_g: f(x)\to a^{1/2}e^{ibx}f(ax)$ be the indicated unitary representation of G on $L^2(-\infty,\infty)$. As is known [see Gel'fand and Naĭmark, Dokl. Akad. Nauk SSSR **55** (1947), 571–574; MR **8**, 563], this representation splits into two irreducible parts, which yields all the irreducible infinite-dimensional representations of G up to equivalence. In the context of these representations and the subgroup H the author discusses spherical functions, invariant operators, and canonical bases. As is seen, these notions have simple interpretations in terms of the usual Fourier or Mellin transforms. *E. M. Stein* (Princeton, N.J.)

Kobayashi, Shoshichi; Nagano, Tadashi 196

A theorem on filtered Lie algebras and its applications.

Bull. Amer. Math. Soc. **70** (1964), 401–403.

La liste des groupes linéaires irréductibles dont le groupe déduit est non trivial a été donnée par E. Cartan dans le cas complexe [Ann. Sci. École Norm. Sup. (3) **26** (1906), 93–161] et par Y. Matsushima dans le cas réel [Colloq. de Topologie de Strasbourg, 1954–1955, Inst. Math., Univ. Strasbourg; MR **19**, 431]. Cette note annonce, sans démonstration, qu'il convient d'ajouter de nouveaux groupes à ces listes, liés notamment aux espaces hermitiens symétriques exceptionnels. Les auteurs ont repris ces problèmes de classification en utilisant un lemme sur les algèbres de Lie filtrées; ils indiquent d'autres applications de ce lemme aux espaces symétriques.

J. L. Koszul (Grenoble)

FUNCTIONS OF REAL VARIABLES

See also 211.

Staniszewska, J. 197

Sur la classe de Baire des dérivées de Dini.

Fund. Math. **47** (1959), 215–217.

O. Hájek [Fund. Math. **44** (1957), 238–240; MR **20** #1734] has proved that the derivative $\bar{f}'(x)$ of a finite-valued function belongs to Baire's second class. The author proves that there is a function φ, belonging to the Lipschitz class, such that $\bar{\varphi}'(x)$ belongs to Baire's second class but not to the first class. For the proof, the author uses Z. Zahorski's results [Ann. Soc. Polon. Math. **19** (1946), 66–105; MR **9**, 89].

S. Izumi (Tokyo)

Kulbacka, M. 198

Sur certaines propriétés des dérivées approximatives.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 17–20.

Z. Zahorski a démontré que la dérivée d'une fonction continue appartient à la classe $\mathscr{M}_2$ [Trans. Amer. Math. Soc. **69** (1950), 1–54; MR **12**, 247]. L'auteur généralise ce fait, en démontrant que la dérivée approximative d'une fonction approximativement continue appartient à la classe $\mathscr{M}_2$. A l'aide de ce résultat, on trouve que la dérivée approximative d'une fonction approximativement continue jouit de la propriété de Darboux.

Soit Q une certaine classe de fonctions réelles de variable réelle. Un ensemble réel E est dit déterminant pour la classe Q si pour $f\in Q$, $g\in Q$ on a $f=g$ dès que $f(x)=g(x)$ pour $x\in E$. Le référent a posé le problème de trouver tous les ensemble déterminants pour la classe P des dérivées approximatives des fonctions approximativement continues [C. R. Acad. Sci. Paris **255** (1962), 1685–1687; MR **26** #1402]. L'auteur montre qu'un ensemble E est déterminant pour la classe P si et seulement si le complémentaire de E est de mesure intérieure nulle.

Comme généralisation d'un théorème du référent [Monatsh. Math. **66** (1962), 435–440; MR **27** #1543] l'auteur établit le théorème suivant: Si f est une dérivée approximative d'une fonction approximativement continue, alors pour deux nombres réels arbitraires α et β l'ensemble $\{x:\alpha<f(x)<\beta\}$ est vide ou de mesure positive.

S. Marcus (Bucharest)

Oeconomidis, Nicolas 199

Sur les nombres dérivés d'une suite de fonctions réelles.

C. R. Acad. Sci. Paris **256** (1963), 3229–3232.

Soit f_n une suite convergente de fonctions réelles définies dans un ensemble de nombres réels E. L'auteur donne des conditions nécessaires et suffisantes pour que les formules

$$\lim_{n\to\infty}\liminf_{x\to x_0+} f_n(x)=\liminf_{x\to x_0+} f(x),$$

$$\lim_{n\to\infty}\limsup_{x\to x_0+} f_n(x)=\limsup_{x\to x_0+} f(x),$$

respectivement,

$$\lim_{n\to\infty}\liminf_{x\to x_0-} f_n(x)=\liminf_{x\to x_0-} f(x),$$

$$\lim_{n\to\infty}\limsup_{x\to x_0-} f_n(x)=\limsup_{x\to x_0-} f(x),$$

et $\lim_{n\to\infty} D_+f_n(x)=D_+f(x)$, $\lim_{n\to\infty} D^+f_n(x)=D^+f(x)$ soient valables.

{Remarque: A la page 3230, dans la première ligne du paragraphe 2, on doit remplacer n par h.}

S. Marcus (Bucharest)

Oeconomidis, Nicolas 200

Sur les nombres dérivés d'une suite de fonctions réelles.

C. R. Acad. Sci. Paris **256** (1963), 4346–4349.

Cette note est une reproduction, mot à mot, d'une note déjà publiée par l'auteur [voir #199 ci-dessus].

S. Marcus (Bucharest)

Laatsch, Richard G. 201

Extensions of subadditive functions.

Pacific J. Math. **14** (1964), 209–215.

The author obtains a number of results on extensions of subadditive functions from the set of non-negative integers or the interval $[0, a]$ to the set of non-negative reals, and the like. Questions of monotonicity, convexity, and evenness arise. If f is a subadditive function on $[0, a]$, then Sf is defined for each $x\in[0,\infty)$ by $Sf(x)=\inf\sum f(x_i)$, where the infimum is taken over all finite collections $\{x_1,\cdots,x_n\}$ such that $x_i\in[0,a]$ and $\sum x_i=x$. Sf is called the maximal subadditive extension of f to $[0,\infty)$. Various results on Sf analogous to those of Bruckner [same J. **10** (1960), 1155–1162; MR **23** #A275] are obtained without assumptions of continuity. Two theorems on convergence and uniform convergence of a sequence of subadditive functions are stated. A function defined on the complement of the Cantor "middle-third" set is shown to be subadditive.

R. A. Rosenbaum (Middletown, Conn.)

Conlan, James 202

A generalization of Gronwall's inequality.

Contributions to Differential Equations **3** (1964), 37–42.

Gronwall's inequality, that if $g(x)$ is continuous and satisfies $0\leqq g(x)\leqq\gamma+L\int_0^x g(x)\,dx$ for $0\leqq x\leqq A$, where γ and L are non-negative constants, then $g(x)\leqq\gamma e^{Lx}$, is here generalized to bounded integrable functions $g(x_1,\cdots,x_n)$. The present generalization is similar to the one previously given by the author and Diaz [Contributions to Differential Equations **2** (1963), 277–289; MR **27** #5046] but is more general and gives a sharper bound.

E. F. Beckenbach (Los Angeles, Calif.)

Besicovitch, A. S. 203
On power series with non-negative coefficients.
J. London Math. Soc. **39** (1964), 13–14.

The following two problems are treated in this paper: Given a function $f(x)$, increasing and positive for $0 \leqq x < 1$ and such that $f(1-) = \infty$. (a) Does there exist a power series $P(x) = \sum a_n x^n$ such that $P(x) > f(x)$ for $0 \leqq x < 1$? (b) If, in addition to the above hypotheses, $\int_0^1 f(x)\,dx$ is assumed to converge, does there exist a power series $P(x) > f(x)$ such that $\int_0^1 P(x)\,dx$ also converges? Both questions are answered in the affirmative. The corresponding problems with the inequality between $P(x)$ and $f(x)$ reversed were treated by the same author in an earlier paper [same J. **38** (1963), 223–225; MR **26** #6337].
F. Herzog (E. Lansing, Mich.)

MEASURE AND INTEGRATION

See also 22, 88, 182, 231, 262, 584, 649, 656, 1008.

Fréchet, Maurice 204
Sur la raréfaction d'un ensemble de mesure nulle.
Rend. Circ. Mat. Palermo (2) **12** (1963), 229–238.

Suppose that $a_n \to 0$ and $b_n \to 0$; to say that $\{a_n\}$ converges faster than $\{b_n\}$ means that $a_n/b_n \to 0$. The same phrase for infinite series is to be interpreted as applying to their tails.

If E is a set of Lebesgue measure zero on the line, then there exists a sequence of distinct intervals (a "majorizing sequence" of E) such that (a) the sum of their lengths is finite, and (b) each point of E belongs to infinitely many of them. Slightly modifying a concept introduced by E. Borel, the author says of two sets E and F of measure zero that E is more rarefied than F in case E has a majorizing sequence the series of whose lengths converges faster than the corresponding series for every majorizing sequence of F. The purpose of the paper is to state (without proofs) some properties of this concept. An easy assertion is that no set is more rarefied than a countable set; i.e., among the possible "orders of rarefaction" there is a maximum, attained by all countable sets. More generally no set is more rarefied than one that can be covered by a countable union of closed sets of measure zero.

There is no explicit statement about whether or not two distinct orders of rarefaction have yet been observed in nature. Reference is made to two related works [Moszner, C. R. Acad. Sci. Paris **256** (1963), 3556–3559; MR **27** #255; the author, Ann. Sci. École Norm. Sup. (3) **80** (1963), 139–172; MR **28** #172].
P. R. Halmos (Ann Arbor, Mich.)

Grabar', M. I. 205
Indecomposable measures in dynamical systems. (Russian)
Dokl. Akad. Nauk SSSR **148** (1963), 1009–1012.

For a continuous parameter group $\{S_t\}$ of homeomorphisms of a compact space R onto itself, invariant and indecomposable normed measures m for the system $\{S_t\}$ and μ for the fixed S_τ, $\tau \neq 0$, are studied. For each m there is a μ with $m(A) = (1/\tau)\int_0^\tau \mu(S_t A)\,dt$ for all Borel A, and each integral average of a μ is an m. Characteristic functions and frequencies are related to S_τ, $\{S_t\}$, and m indecomposable for S_τ. *J. H. Abbott* (Albuquerque, N.M.)

Goffman, Casper; Serrin, James 206
Sublinear functions of measures and variational integrals.
Duke Math. J. **31** (1964), 159–178.

An analysis of general set functions and previous results of J. Serrin [Trans. Amer. Math. Soc. **101** (1961), 139–167; MR **25** #1466] lead the authors to a generalization of Tonelli's theorem on Lebesgue area of nonparametric surfaces and of a corresponding result of I. Verchenko to certain nonparametric integrals. Here μ denotes a countably additive set function on a σ-ring R of subsets of a given set S with values on a Banach space $\mathfrak{X}$ (in applications $\mathfrak{X}$ may be R^1 or R^n). On the other hand, $\mathfrak{F}: \mathfrak{X} \to R^1$ is a given function such that $\mathfrak{F}(p+q) \leqq \mathfrak{F}(p) + \mathfrak{F}(q)$, $\mathfrak{F}(\lambda p) = \lambda\mathfrak{F}(p)$, $\mathfrak{F}(p) \leqq C|p|$ for all $p, q \in \mathfrak{X}$, $\lambda \geqq 0$, C a given constant. Then the set function is defined: $\mathfrak{F}\mu(E) = \operatorname{Sup} \sum_i \mathfrak{F}(\mu(E_i))$, $E \in R$, where Sup is taken over all finite partitions $E = \bigcup E_i$, $E_i \in R$. If $\mathfrak{F}(p) = |p|$, then $\mathfrak{F}\mu(E)$ is the usual total variation and denoted by $|\mu|(E)$. Two main properties of $\mathfrak{F}\mu(E)$ are proved. One is a decomposition theorem: If $\mu = \alpha + \beta$, α a ν-integral, β singular, then $\mathfrak{F}\mu(E) = \int \mathfrak{F}(a)\,d\nu + \mathfrak{F}\beta(E)$. The other is a lower-semicontinuity theorem: If $\mathfrak{X} = R^n$, if $[\mu_k]$ is a sequence of measures weakly convergent to μ, then $\mathfrak{F}\mu(G) \leqq \liminf \mathfrak{F}\mu_k(G)$ on every B-set G. These theorems are then extended to an analogous functional $\mathfrak{I}[u, E]$, depending on a function $u: S \to \mathfrak{X}$. Nonparametric variational integrals of the form $\int f(u_x)\,dx$ are particular cases of $\mathfrak{I}$. The first theorem for $\mathfrak{I}$ essentially extends Tonelli's theorem for area. The authors then prove for $\mathfrak{I}$ a theorem of the Verchenko type. If u, v are weakly differentiable and continuous (weaker hypotheses are also given), if the derivative measures of u and v coincide on a B-set E, then $\mathfrak{I}[u, E] = \mathfrak{I}[v, E]$. *L. Cesari* (Ann Arbor, Mich.)

Chatterji, S. D. 207
Certain induced measures on the unit interval.
J. London Math. Soc. **38** (1963), 325–331; *addendum, ibid.* **39** (1964), 320.

The author gives an interesting proof that general product measures on the space of infinite binary sequences yield induced measures on the unit interval which are purely atomic, purely singular non-atomic, or purely absolutely continuous relative to Lebesgue measure. Characterizing equivalent conditions on the set of coordinate measures separate the cases.

In the addendum the author cites priorities, but shows that his results go a bit farther.
J. H. Abbott (Albuquerque, N.M.)

Rätz, Jürg 208
Invariante Funktionen über teilgeordneten algebraischen Halbstrukturen.
Comment. Math. Helv. **38** (1963), 121–162.

An abstract structure is defined, consisting of a set $\mathfrak{M}$, a binary operation, a commutative group of one-to-one transformations of $\mathfrak{M}$, a partial ordering of a certain set of mappings of $\mathfrak{M}$ into the non-negative integers, and a distinguished element $E \in \mathfrak{M}$. The element E is required

to satisfy a certain "boundedness condition". The aim is to develop a theory of invariant "fields" $\mathfrak{F} \subset \mathfrak{M}$ and invariant functions φ on $\mathfrak{F}$ with $\varphi(E)=1$, which will include as special cases basic results concerning invariant content functions on the one hand, and invariant integrals on the other. Main results are (A) a necessary and sufficient condition for the existence of an invariant system $(\mathfrak{F}, \varphi)$, (B) determination of the maximal field of elements of $\mathfrak{M}$ for which any two invariant functions are equal, assuming (A) satisfied, and (C) the existence of an extension of any invariant system $(\mathfrak{F}, \varphi)$ to the whole of $\mathfrak{M}$. In the content-theoretic interpretation these reduce to known theorems of Tarski and Banach. In other interpretations the theory includes results concerning limitable sequences, rotation-invariant functions on the class of convex subsets of the plane, and a characterization of the function $\log x$.

J. C. Oxtoby (Bryn Mawr, Pa.)

McShane, E. J. 209

Integrals devised for special purposes.

Bull. Amer. Math. Soc. **69** (1963), 597–627.

This is a thorough survey of developments in integration theory and probabilistic functional analysis in the last thirty years. The topics treated or cited are (in order of appearance): the order Riemann integrals of Cogburn; the averaging operators of Moy and the reviewer; the Dunford-Schwartz finitely additive integrals; work by Coleman, the author, Cameron, Martin and Hatfield on the Wiener integral; Schwartz's definition of an integral from the Gauss-Wiener measure on Hilbert space; the Friedrichs-Shapiro extension of the Gauss-Wiener measure by embedding Hilbert space into a larger space; Gross's topology; Segal's theory of weak distributions and Fourier-Wiener transforms; Segal's non-commutative integration; stochastic integrals, with the contributions of Doob, Itô, the author, Wiener, and others; the use of function-space integration for the solution of partial differential equations; Feynman integrals, with recent contributions by Feldman and Nelson.

The exposition is concise, and the main ideas of each of these "integration" theories are described in remarkably clear detail in the space of a few lines. This paper should be required reading for any student of functional analysis.

G.-C. Rota (Cambridge, Mass.)

Foguel, S. R. 210

On order preserving contractions.

Israel J. Math. **1** (1963), 54–59.

Let (Ω, Σ, μ) be a measure space and let P be an operator on $L_2(\Omega, \Sigma, \mu)$ with $\|P\| \leqq 1$, $Pf \geqq 0$ a.e. whenever $f \geqq 0$. If the subspace K is defined by

$$K = \{x \mid \|P^n x\| = \|P^{*n} x\| = \|x\|, \quad n = 1, 2, \cdots\},$$

then $K = L_2(\Omega, \Sigma_1, \mu)$, where $\Sigma_1 \subset \Sigma$, and on K the operator P is "essentially" a measure-preserving transformation. Thus the eigenvalues of P of modulus one form a group under multiplication.

This last result was proved by the reviewer [Bull. Amer. Math. Soc. **67** (1961), 556–558; addendum, ibid. **68** (1962), 49; MR **24** #A1621] for finite μ (here finiteness is not assumed) and is a generalization of a theorem of Frobenius and Perron on positive matrices.

G.-C. Rota (Cambridge, Mass.)

Vituškin, A. G. 211

Some properties of linear superpositions of smooth functions. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 1003–1006.

Cette note représente la continuation de la recherche concernant la ε-entropie de certaines classes de fonctions. {Pour la définition de ε-entropie voir Kolmogorov [mêmes Dokl. **108** (1956), 385–388; MR **18**, 324] et, sur le même sujet, V. M. Tihomirov [ibid. **117** (1957), 191–194; MR **20** #3749], l'auteur [ibid. **117** (1957), 745–747; MR **20** #3750] et le livre de l'auteur, *Estimation of the complexity of the tabulation problem* (Russian) [Fizmatgiz, Moscow, 1959; MR **22** #8265].} Soit $C(G_n)$ l'espace des fonctions continues et réelles définies dans R^n. Deux fonctions $f_1(x)$, $f_2(x) \in C(G_n)$ sont (ε, δ)-différentes s'il existe une sphère S_δ telle que $\min|f_1(x)-f_2(x)| \geqq \varepsilon$, $x \in S_\delta$. Soit $N_{\varepsilon,\delta}(E)$ le nombre de (ε, δ)-différents éléments d'un sous-ensemble $E \subset C(G_n)$ et soit F un compact dans $C(G_n)$ contenant avec f et λf, $\rho(f)$ une fonctionnelle positive définie dans F et telle que si $f \in F$, $\rho(\lambda f) \to 0$, $\lambda \to 0$, $F_{\varepsilon,r}$ l'intersection de deux ensembles

$$\{f: \max|f(x)| \leqq \varepsilon; f(x) \in C(G_n)\}, \qquad \{f: \rho(f) \leqq r, f \in F\},$$

alors, quel que soit r,

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \left[\frac{-\log_2 N_{\varepsilon,\delta}(F_{\varepsilon,r})}{\log \delta}\right] = n.$$

On applique ce théorème pour démontrer qu'il existe des fonctions analytiques qui ne sont pas représentables par une superposition de la forme

$$\sum_{k=1}^{\nu} p_k f_k(q_k{}^1, q_k{}^2, \cdots, q_k{}^{(n-1)}),$$

où $\{f_k\}$ sont des fonctions arbitraires de $n-1$ variables, $q_k{}^i(x_1, x_2, \cdots, x_n)$ les fonctions données de n variables continûment différentiables et p_k les fonctions continues de n variables données par avance.

M. Tomić (Belgrade)

Metivier, Michel 212

Limites projectives de mesures. Martingales. Applications.

Ann. Mat. Pura Appl. (4) **63** (1963), 225–352.

Some of the results of this paper have been announced in a sequence of notes [C. R. Acad. Sci. Paris **252** (1961), 491–493; MR **26** #279; ibid. **252** (1961), 654–656; MR **26** #279; ibid. **254** (1962), 207–209; MR **25** #165; ibid. **256** (1963), 1062–1065; MR **26** #5114; ibid. **256** (1963), 2993–2995; ibid. **256** (1963), 3802–3805; MR **27** #2834]. Although these notes have been previously reviewed, an attempt will be made to summarize the entire contents of this paper.

Chapter I is preliminary in nature and develops the necessary terminology for treating measures, regular measures, and contents in situations more general than the usual topological situation. The terminology pertaining to measure-theoretic concepts used here is that of the author and is more general than that customarily used. The principal result in this chapter is a theorem which gives sufficient conditions for a content function defined on a class of compact sets to generate a regular measure.

Chapter II treats the theorem and problem of Bochner concerning projective limits of measure spaces [*Harmonic analysis and the theory of probability*, Ch. 5, Univ. California Press, Berkeley, Calif., 1955; MR **17**, 273] and

a corresponding theorem of J. R. Choksi [Proc. London Math. Soc. (3) 8 (1958), 321–342; MR 20 #3251]. The problem can be stated as follows. Consider a projective system of measures $(X_\alpha, \mathfrak{Z}_\alpha, \mu_\alpha, \pi_{\alpha\beta})_{\alpha\in I}$, where I is a directed set, X_α is a space, $\mathfrak{Z}_\alpha$ is a σ-ring of subsets of X_α, μ_α is a measure on $\mathfrak{Z}_\alpha$, and $\pi_{\alpha\beta}$ is a mapping from X_β into X_α, with certain consistency conditions holding for the maps, σ-rings, and measures. Let X denote the inverse projective limit of the X_α's and π_α the projection of X into X_α. The problem of Bochner is that of showing that there is a measure **m** defined on the σ-ring generated by $\bigcup_{\alpha\in I}\pi_\alpha^{-1}(\mathfrak{Z}_\alpha)$ with the property that $\mu_\alpha = \mathbf{m}\circ\pi_\alpha^{-1}$. Bochner's theorem states the existence of such a measure under topological conditions on the X_α, regularity conditions on the μ_α, and a sequential maximality condition on the projective system. Bochner raised the question of whether or not the measure **m** inherits the regularity properties of the μ_α's. Choksi showed that **m** does inherit the properties of the μ_α's provided these properties include additional conditions on the class of compact sets. The author gives an alternative solution to Bochner's problem in the case where the X_α's are topological. The author also points out in this chapter the essential equivalence of a projective system of measures and a martingale in the sense of Bochner (i.e., a directed family of measures defined on an associated directed family of σ-rings and satisfying the usual equations of martingale theory).

Chapter III improves upon Choksi's work by showing that one of Choksi's compactness conditions can be eliminated. The results of Chapter III are extended in Chapter IV to projective systems of measures of arbitrary sign by decomposing such systems into the difference of two projective systems of positive measures.

Chapter V treats set functions taking on values in an arbitrary locally convex space **V** in duality with another such space **V**′. This chapter includes a theorem which states that a content defined on a Boolean ring $\mathscr{A}$ with values in **V** has a unique extension to a measure defined on the σ-ring generated by $\mathscr{A}$ with values in **V**″, the topological dual of **V**. The results of the preceding chapters are generalized to the vector-valued case with, of course, the projective limit measure taking on values in **V**″.

Various generalizations of Doob's pointwise martingale convergence theorem and theorems on the adjunction of a last element to the martingale are given in Chapter VI. The author first treats projective systems of measures taking on values in a locally convex space **V** in duality with **V**′ and defined by densities. Under uniform integrability conditions on the densities, the author shows that such systems have a projective limit measure taking on values in **V**″. If **V** is semi-reflexive, then the projective limit measure takes on values in **V** rather than **V**″. The author next treats martingales $(\mathfrak{Z}_\alpha, \mathbf{m}_\alpha)_{\alpha\in I}$, where I is a directed set, $\mathfrak{Z}_\alpha$ is an increasing family of σ-algebras of subsets of a space X, and the $\mathbf{m}_\alpha$'s are measures on $\mathfrak{Z}_\alpha$ taking on values in a locally convex space **V** in duality with **V**′, defined by densities h_α relative to a base ν, and satisfying the usual martingale equations. The author gives conditions under which there is an h_ω such that the system $(\mathfrak{Z}_\alpha, \mathbf{m}_\alpha)_{\alpha\in\bar I}$, where $\bar I$ is the directed set obtained by adjoining ω as largest element and $\mathbf{m}_\omega$ is the indefinite integral of h_ω, is again a martingale. Under the same conditions it is shown that for every $\nu' \in \mathbf{V}'$, the families $\langle \nu', h_\alpha\rangle$ converge in the mean to $\langle \nu', h_\omega\rangle$ (relative to the measure ν) and that it is possible to extract a sequence $\{h_{\alpha_n}\}$ which converges a.e. (ν) to h_ω. The chapter concludes with a treatment of the vectorial form of the Radon-Nikodym theorem along the lines of Dieudonné's work [Ann. Univ. Grenoble Sect. Sci. Math. Phys. (N.S.) 23 (1948), 25–53; MR 10, 519; Canad. J. Math. 3 (1951), 129–139; MR 13, 448].

The paper concludes with a chapter on the problem of disintegration of measures which corresponds to the problem of the existence of regular versions of conditional probability distributions in probability theory. Since the last chapter is adequately described in detail by a previous review [MR 26 #5114], this chapter will not be discussed.

L. L. Helms (Urbana, Ill.)

Timan, A. F. 213

The order of growth of ε-entropy of spaces of real continuous functionals defined on a connected compactum. (Russian)

Uspehi Mat. Nauk **19** (1964), *no.* 1 (115), 173–177.

For a subset A of a metric space R let $H_\varepsilon(A)$ denote its ε-entropy. An exposition of this concept and calculation of the growth rate of the function $H_\varepsilon(A)$ as $\varepsilon\to 0$ for several examples of compact subsets A in function spaces can be found in A. N. Kolmogorov and V. M. Tihomirov [Uspehi Mat. Nauk **14** (1959), no. 2 (86), 3–86; MR **22** #2890; Amer. Math. Soc. Transl. (2) **17** (1961), 277–364; MR **23** #A2031]. In describing growth rates of functions as $\varepsilon\to 0$ the following notation is used: $f(\varepsilon)\asymp g(\varepsilon)$ if $f(\varepsilon) = O(g(\varepsilon))$ and $g(\varepsilon) = O(f(\varepsilon))$; $f(\varepsilon)\prec\prec g(\varepsilon)$ if $f(\varepsilon) = o(g(\varepsilon))$. Considered in the above-mentioned work is the set of $D_\omega^A(C)$ of real-valued functions defined on A which are bounded in absolute value by the constant C and which satisfy Hölder's condition with a modulus of continuity $\omega=\omega(t)$. The set $D_\omega^A(C)$ is a metric space with metric $\rho(F_1, F_2) = \sup_{x\in A}|F_1(x) - F_2(x)|$ and its ε-entropy can be discussed. Kolmogorov and Tihomirov relate the ε-entropy of A with the ε-entropy of $D_\omega^A(C)$ by the theorem: If A is totally bounded and $\log_2\log_2 1/\varepsilon \prec\prec H_\varepsilon(A)\asymp H_{\varepsilon k}(A)$ for arbitrary $k>0$, then $\log_2 H_\varepsilon(D_\omega^A(C))\asymp H_{\omega^{-1}(\varepsilon)}(A)$. The author points out in the present article that they actually do not obtain this result unless the condition $\omega^{-1}(2\varepsilon)\asymp\omega^{-1}(\varepsilon)$ is also assumed. The main purpose of this paper is to relate the ε-entropy of A and the ε-entropy of $D_\omega^A(C)$ without assuming this condition. The author obtains the following theorem. If A is a connected compact set such that $\log_2\log_2 1/\varepsilon\prec\prec H_\varepsilon(A)\asymp H_{\varepsilon k}(A)$ for arbitrary $k>0$, then $\log_2 H_\varepsilon(D_\omega^A(C))\asymp H_{\omega^{-1}(2\varepsilon)}(A)$.

R. L. Adler (Yorktown Heights, N.Y.)

Tihomirov, V. M. 214

The works of A. N. Kolmogorov on ε-entropy of function classes and superpositions of functions. (Russian)

Uspehi Mat. Nauk **18** (1963), *no.* 5 (113), 55–92.

This expository paper contains a review of the results of Kolmogorov (from the period 1953–1958) which concern probabilistic and metric entropy and superpositions of functions. There are sketches of proofs and some interesting biographical remarks.

G. G. Lorentz (Syracuse, N.Y.)

Foiaş, Ciprian 215

Sur les mesures spectrales qui interviennent dans la théorie ergodique.

J. Math. Mech. **13** (1964), 639–658.

The author studies spectral measures E on a locally compact abelian group G, with values that are projections on L^2 over some measure space. If f and g are in L^2, he considers the vector-valued measures Ef and Eg and defines for them a kind of convolution $Ef * Eg$. This convolution is not itself a vector-valued measure, but it behaves like one on the Fourier transforms of finite complex measures on G. The principal result is that E is the spectral measure of a weakly continuous unitary representation of G over L^2 if and only if E is multiplicative, in the sense that if f, g, and $f \cdot g$ are in L^2, then $Ef * Eg = E(f \cdot g)$. The result has several consequences of interest; some special cases of some of them were known before. Here are two samples. (1) If G has a countable base, and if the unitary representation U of G is ergodic, then the support of the spectral measure of U is a closed subgroup of G. (2) If U is the unitary operator induced by an invertible aperiodic measure-preserving transformation, then the spectrum of U is the entire unit circle.

P. R. Halmos (Ann Arbor, Mich.)

Besicovitch, A. S. 216

On one-sided densities of arcs of positive two-dimensional measure.

Proc. Cambridge Philos. Soc. **60** (1964), 517–524.

The author considers plane simple arcs which have positive two-dimensional Lebesgue measure. If x is a point of such an arc AB, a point $x' \neq x$ is said to be to the left of x [right of x] if it belongs to the arc Ax [xB]. If $C(x, r)$ denotes the closed disk of radius r centered at x, then the upper limit of $[m_2\{AxC(x, r)\}]/\pi r^2$ as $r \to 0$ is called the upper left density at x. Lower left density is defined similarly.

The author establishes, in a straightforward way, that the one-sided lower density of AB is $\leq \frac{1}{2}$ at m_2 almost all points of AB. Surprisingly enough, he then shows that this result is best possible by constructing an arc at almost all points of which the one-sided density exists and is equal to $\frac{1}{2}$. Finally, he considers the familiar Peano curve which fills the square, and shows that its one-sided lower density equals 1/12 at almost all of its points.

W. P. Ziemer (Bloomington, Ind.)

FUNCTIONS OF A COMPLEX VARIABLE

See also 103, 167, 203, 261, 264, 281, 302, 356, 403, 428.

Heins, Maurice 217

★Selected topics in the classical theory of functions of a complex variable.

Athena Series: Selected Topics in Mathematics.

Holt, Rinehart and Winston, New York, 1962. xi + 160 *pp.* $3.50.

This is an excellent book on the geometric theory of functions. This text is intended for mathematics students who have had or are completing a first course in the theory of a complex variable. The object is to present to such students a number of important topics, such as the big Picard theorem, the Riemann mapping theorem and the Fatou radial-limit theorem from the classical theory of functions. However, the author's methods are fresh and systematic, and expositions are self-contained and rigorous. In this text, a knowledge of the Lebesgue theory of integration is not presupposed, although the author points out that this theory plays an indispensable role in the study of many important classes of functions. What is needed from real-variable theory is developed in this text. The author adopts a fresh interesting method of exposition which presents certain topics via (finite) sequences of exercises rendered accessible by adequate indications of essential difficulties. The author states modestly that the *Aufgaben und Lehrsätze aus der Analysis* [J. Springer, Berlin, 1925] of Pólya and Szegö is a model for this method. The reviewer notes that the author has made many important contributions to complex analysis by applying the theory of subharmonic functions systematically, and a large number of theorems and methods in this text were obtained by the author himself.

This book consists of six chapters and an appendix. In Chapter 1, Preliminaries, the reviewer is very much interested in nice explanations of the notion of simple connectivity, analytic logarithm and the relation between Fourier series and analytic functions, together with important problems. In Chapter 2, covering properties of meromorphic functions, argument principle, local analysis of an analytic function, lower semicontinuity of the valence function, boundary values of a meromorphic function, boundary-preserving maps and the Riemann mapping theorem are discussed; here the use of sequences of exercises is made in the proof of the Riemann mapping theorem based on Koebe's method. In Chapter 3, the Picard Theorem, the Bloch Theorem, the theorem of Schottky and the proof of the big Picard Theorem are stated. In Chapter 4, harmonic and subharmonic functions, and Chapter 5, applications, properties of harmonic functions, the Poisson-Stieltjes integral, the Fatou theorem, properties of subharmonic functions, Perron families, $\beta \geq \frac{1}{4}\sqrt{3}$, where β denotes the Bloch constant, the Dirichlet problem, Green's functions, the Lindelöf principle, the inequality of Milloux-Schmidt, the Phragmén-Lindelöf theorem, Wiman's theorem and a method of Carleman are discussed. As to the Bloch constant β, the reviewer refers to the author's recent paper [Nagoya Math. J. **21** (1962), 1–60; MR **26** #1451] in which the author proves that β is actually greater than $\frac{1}{4}\sqrt{3}$. In Chapter 6, The boundary behavior of the Riemann mapping function for simply-connected Jordan regions, we have a nice exposition of Beckenbach's stronger form of the Cauchy integral theorem. The Appendix is devoted to the Riesz representation theorem for positive additive functionals, Lebesgue's theorem on the derivative of a monotone function and the Jordan curve theorem. The reviewer notes that throughout this text the author tries to point out the important fact that complex function theory is organically interrelated with many branches of mathematics. To make this review more complete, it seems necessary to add the author's correction of some minor slips and misprints: p. 13, line 5 from the bottom, "spherical" is to be replaced by "chordal"; p. 19, lines 5, 13, 21, replace "1" by "3"; p. 33, line 14, replace the period at the end of the line by ", $|f| < 1$."; p. 47, line 13, replace "Ex. 1" by "Ex. 2"; p. 51, line 4 from the bottom, replace the whole line by "$|\log|F_\rho(iy)|| \leq B_{m+1}$, $0 \leq y \leq (m+1)/2$, where B_{m+1} is independent of admitted ρ."; p. 81, lines 7, 6 from the bottom, replace "is convex to $\{\inf u < x < \sup u\}$." by "to $\{\inf u < x < \sup u\}$ is convex.";

p. 82, line 11 from the bottom, replace "the theorem" by "that of the second paragraph"; p. 82, line 2 from the bottom, delete final "and"; p. 82, line 1 from the bottom, delete "harmonic" and insert before the first comma "and harmonic on $\{R_1 < |z| < R_2\}$"; p. 111, line 12, replace "$\pi/2$" by "π"; p. 111, lines 15, 16, 17, replace sentence "If f is ..." by "A theorem of similar type is: If f is analytic on K but not a polynomial, then

$$\lim_{r\to\infty} \log M(r; f)/\log r = +\infty\text{''}.$$

K. Noshiro (Nagoya)

Boas, Ralph P., Jr.; Buck, R. Creighton 218

★Polynomial expansions of analytic functions.

Second printing, corrected. Ergebnisse der Mathematik und ihrer Grenzgebiete, N. F., Bd. 19.

Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin, 1964. viii+77 *pp.* $4.00.

Except for the correction of misprints, this second printing is identical to the first [Springer, Berlin, 1958; MR **20** #984].

Tanzi Cattabianchi, Luigi 219

Alcuni criteri di non prolungabilità per le serie di potenze.

Riv. Mat. Univ. Parma. (2) **3** (1962), 169–191.

Des conditions compliquées, généralisant la lacunarité, sont données pour que le cercle de convergence soit une coupure. *S. Mandelbrojt* (Paris)

Bertrandias, Françoise 220

Séries de Taylor à coefficients rationnels.

C. R. Acad. Sci. Paris **257** (1963), 3793–3795.

Let $f(z)$, which is given by $\sum_0^\infty a_n z^{-n-1}$ in the neighborhood of $z=\infty$, be holomorphic and uniform in the complement of a compact set K_f of transfinite diameter τ_f and be meromorphic with a finite number of poles in $|z| > \rho_f$. It is well known that if the a_n are rational and $\tau_f < 1$, then $f(z)$ is rational. Now let $a_n \in Q$, the rational field. Denote by Q_p the field of p-adic numbers, by K_p a finite algebraic extension of Q_p and by $|\alpha|_p$ the p-adic absolute value on K_p. Then the author states without proof a number of theorems that generalize the above result. A typical theorem follows. Suppose there exists a positive constant A such that for all primes p and for all $n > n_0$, $|a_n|_p \leqq An$. Then if $\rho_f < \exp(-A)$, $f(z)$ is rational. With the same hypothesis if $\tau_f < \exp((-3/2)A)$, then $f(z)$ is rational. *V. F. Cowling* (New Brunswick, N.J.)

Whiteman, Russell A. 221

A converse form of Daïovitch's theorem.

Duke Math. J. **31** (1964), 321–324.

Daïovitch (Dajović) [Bull. Soc. Math. Phys. Serbie **8** (1956), 23–28; MR **20** #5275] proved that if $f(z)$ and $g(z)$ are analytic in the open unit disk and $f \in H_p$, $1 < p \leqq 2$, and $g \in H_q$ $(1/p + 1/q = 1)$, then the Hadamard composition function $h(z) = f(z) \circ g(z)$ is a bounded function in $|z| < 1$. The present author proves the following converse form of Daïovitch's theorem. Consider the subclass of analytic functions in $|z| < 1$ for which each function has Maclaurin series coefficients that satisfy a Gauss condition G. Let $1 < p \leqq 2$; if for all $f \in H_p$ there is a g such that the Hadamard composition $h = f \circ g$ is bounded in $|z| < 1$, then $g \in H_q$, $1/p + 1/q = 1$. Furthermore, p and q are interchangeable.

The Gauss condition G is defined as follows: let the sequence $\{a_n\}$ be a sequence of Maclaurin coefficients. The sequence $\{a_n\}$ satisfies the condition G if a_n is a positive real-valued function of n and $a_{n+1}/a_n = 1 - h/n - A(n)/n^\lambda$, where $\lambda > 1$, $A(n)$ is bounded and h is a real number. The method of proof is classical, using the theorems of Young, Hausdorff, and Hardy and Littlewood on the Hardy and Lebesgue classes. *W. C. Royster* (Lexington, Ky.)

Bombieri, Enrico 222

Sopra un teorema di H. Bohr e G. Ricci sulle funzioni maggioranti delle serie di potenze.

Boll. Un. Mat. Ital. (3) **17** (1962), 276–282.

Désignons par $F_h(\lambda)$ la famille de fonctions $f(z)$, holomorphes dans $|z| < 1$, telles que $|f(z)| \leqq 1$ dans ce cercle, et telles que $f(z) = \lambda z^h + a_{h+1} z^{h+1} + \cdots$, $0 \leqq \lambda < 1$. Soit $\mathfrak{M}(f; r) = \lambda r^h + |a_{h+1}| r^{h+1} + \cdots$, et soit $B_h(\lambda)$ la borne supérieure des r pour lesquelles $\mathfrak{M}(f; B_h(\lambda)) \leqq 1$ quelle que soit la fonction f de F_h. On démontre que pour $\frac{1}{2} \leqq \lambda < 1$, on a $B_0(\lambda) = 1/(1+2\lambda)$, et les fonctions extrémales (pour lesquelles $\mathfrak{M}(f; B_h(\lambda)) = 1$) sont alors $(\lambda - e^{i\mu} z)/(1 - \lambda e^{i\mu} z)$, μ réel. On a $B_1 = \inf B_1(\lambda) = 1/\sqrt{2}$, $B_0(\lambda) \leqq 1/\sqrt{2}$. *S. Mandelbrojt* (Paris)

Kahane, J.-P. 223

Lacunary Taylor and Fourier series.

Bull. Amer. Math. Soc. **70** (1964), 199–213.

Exposé général des théorèmes lacunaires récents concernant les séries de Taylor avec des lacunes "à la Hadamard", d'une part, et les séries lacunaires de Fourier, d'autre part. Dans la première partie on considère les séries comme étant des séries de fonctions presqu'indépendantes. Dans la seconde partie on donne des réponses à un problème général (concernant les séries de Fourier) posé par le rapporteur. *S. Mandelbrojt* (Paris)

Ahmedov, T. K. 224

On the analytic continuation of certain series of Taylor-Dirichlet type. (Russian. Azerbaijani summary)

Akad. Nauk Azerbaĭdžan. SSR Dokl. **19** (1963), *no.* 9, 3–7.

In this paper it is proved that the difference between the functions represented by the Taylor-Dirichlet series $\sum_1^\infty \phi(n) s^{\psi(n)} e^{-sf(n)}$ and the corresponding integral $\int_0^\infty \phi(x) s^{\psi(x)} e^{-sf(x)}\,dx$ is a function that is analytic on the Riemann surface for $\log s$. Here ϕ, ψ and f are functions that are analytic in a suitable domain and satisfy certain growth conditions. There are several minor misprints in the paper; in some cases different fonts of a letter have been used interchangeably, and in at least one case the significance of a symbol has to be deduced from the context. However, the proof is quite straightforward. *J. H. Williamson* (Cambridge, England)

Beurling, Arne 225

Notes on Dirichlet series.

J. Indian Math. Soc. (N.S.) **27** (1963), 19–26.

Let $\varphi(s)=\sum_1^\infty c_n n^{-s}$ in some half-plane, and assume that $\varphi(s)$ is analytic and single-valued in a neighborhood of ∞ of the finite plane. Assume further that (*) $|\varphi(s)|<\Gamma(1+|s|)(h/2\pi)^{|s|}$ holds for sufficiently large $|s|$ and some finite h. Let k_1 [respectively, k_2] denote the infimum of h for which (*) holds for sufficiently large odd [even] negative integers and put $k_0=\max(k_1,k_2)$. The following theorems are proved: (I) If $k_0<1$, then φ is entire and $\limsup_{n\to\infty} n^{-1}\log|c_n|\leqq -\pi(4/k_0^2-1)^{1/2}$. (II) If $k_1<2$ and $k_2<1$, then φ has no finite singularities other than simple poles located at odd integral values of s. (III) If φ is entire, $k_1<2$ and (**) $\limsup_{n\to\infty}|\varphi(-2n)|^{1/2n}<1$, then $\varphi\equiv 0$. (IV) If $k_1<2$ and (**), then φ is a linear combination of the functions $\zeta(s+2p)$, $p=0,\pm 1,\pm 2,\cdots$.

A. Dvoretzky (Jerusalem)

Gašimov, G. M. 226

A uniqueness theorem for Dirichlet series. (Russian. Azerbaijani summary)

Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Mat. Tehn. Nauk **1963**, *no.* 5, 25–36.

This paper contains detailed proofs of two theorems previously announced (with outline proofs) by the author [Dokl. Akad. Nauk SSSR **150** (1963), 722–725; MR **27** #1569].

J. H. Williamson (Cambridge, England)

Leont'ev, A. F. 227

Some uniqueness theorems for Dirichlet series. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **27** (1963), 1251–1280.

L'auteur démontre le théorème suivant. Supposons que la suite $\{\lambda_n\}$, $0<\lambda_n\uparrow\infty$, puisse être partagée en deux suites $\{\lambda_n'\}$ et $\{\lambda_n''\}$ avec les propriétés suivantes: Les λ_n' sont les zéros d'une fonction entière $L(z)\not\equiv 0$ de type exponentiel avec $|L(re^{\pm(\pi i/2)})|\leqq Be^{Ar}$, la suite $\{\lambda_n''\}$ étant soit vide, soit telle que $\sum(\lambda_n'')^{-1}<\infty$. Soit $F(s)=\lim_{k\to\infty}\sum_{\lambda_n<\rho_k}c_ne^{\lambda_n s}$ ($\rho_k\uparrow+\infty$), la convergence étant uniforme dans chaque compact. Si dans $|t|\leqq A$ ($s=\sigma+it$), $|F(s)|<e^{\mu\sigma}$, $\mu<\lambda_1$, $\sigma>\sigma_0$, alors $F(s)\equiv 0$. On démontre un théorème indiquant que les hypothèses du théorème précédent ne sont pas loin d'être les meilleures possible pour tirer la conclusion. On démontre aussi des théorèmes plus fins, où la décroissance de $F(s)$ est précisée surtout sur une droite (l'axe réel).

S. Mandelbrojt (Paris)

Leont'ev, A. F. 228

On the completeness of the system $\{e^{\lambda_n z}\}$ in a closed strip. (Russian)

Dokl. Akad. Nauk SSSR **152** (1963), 266–268.

Voici un des théorèmes démontrés par l'auteur. Les λ_n satisfaisant à la même condition que dans le travail analysé plus haut [voir #227], soit $P_n(z)$ une combinaison linéaire du système $\{e^{\lambda_n z}\}$. Si $P_n(z)$ converge uniformément sur tout compact de la bande $|y|\leqq A$, ces combinaisons convergent uniformément sur tout compact.

S. Mandelbrojt (Paris)

Roux, Delfina 229

Sulla composizione secondo Hurwitz-Pincherle di due serie di Dirichlet.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **34** (1963), 364–368.

Une extension d'un théorème du rapporteur concernant la composition "à la Hurwitz-Pincherle" de deux séries de Dirichlet.

S. Mandelbrojt (Paris)

Gupta, J. S. 230

On the order and type of integral functions defined by Dirichlet series. (Turkish summary)

İstanbul Üniv. Fen Fak. Mec. Ser. A **26** (1961), 37–43.

It is supposed that the functions are representable by Dirichlet series $f(s)=\sum_{n=1}^\infty a_ne^{s\lambda_n}$, with $\lambda_{n+1}>\lambda_n$; $\lambda_1\geqq 0$, $\lim_{n\to\infty}\lambda_n=\infty$, $s=\sigma+it$ and $\limsup_{n\to\infty}((\log n)/\lambda_n)=D<\infty$, and furthermore that they are integral. $M(\sigma)$ is the least upper bound of $|f(\sigma+it)|$ $(-\infty<t<+\infty)$,

$$\rho=\limsup_{\sigma\to\infty}\frac{\log\log M(\sigma)}{\sigma}\quad\text{(order)},$$

$$\lambda=\liminf_{\sigma\to\infty}\frac{\log\log M(\sigma)}{\sigma}\quad\text{(lower order)},$$

$$\tau=\limsup_{\sigma\to\infty}\frac{\log M(\sigma)}{e^{\rho\sigma}}\quad\text{(type)},$$

$$\nu=\liminf_{\sigma\to\infty}\frac{\log M(\sigma)}{e^{\rho\sigma}}\quad\text{(lower type)}.$$

The author proves Theorem 1: If $\Phi(s)$, $\Psi(s)$ are two functions of finite orders ρ_1,ρ_2, lower orders λ_1,λ_2 and of least upper bounds $M_1(\sigma)$, $M_2(\sigma)$, then, if

$$\log\log M(\sigma)\sim\log\{\log M_1(\sigma)\log M_2(\sigma)\},$$

the order ρ and the lower order λ of $f(s)$ of the least upper bound $M(\sigma)$ are such that $\lambda_1+\lambda_2\leqq\lambda\leqq\rho\leqq\rho_1+\rho_2$, and if

$$\log\log M(\sigma)\sim|\surd(\log\log M_1(\sigma)\log\log M_2(\sigma))|,$$

then $\surd(\lambda_1\lambda_2)\leqq\lambda\leqq\rho\leqq\surd(\rho_1\rho_2)$. The result of this theorem can be extended to m functions. In Theorem II he proves that if $\Phi(s)$, $\Psi(s)$ have the same finite order ρ, finite types τ_1,τ_2, lower types ν_1,ν_2 and least upper bounds $M_1(\sigma)$, $M_2(\sigma)$, then if

$$\log\log M(\sigma)\sim\log\{\log M_1(\sigma)\log M_2(\sigma)\},$$

$$\nu_1+\nu_2\leqq\nu\leqq\tau\leqq\tau_1+\tau_2,$$

and if

$$\log\log M(\sigma)\sim|\surd(\log\log M_1(\sigma)\log\log M_2(\sigma))|,$$

$$\surd(\nu_1\nu_2)\leqq\nu\leqq\tau\leqq\surd(\tau_1\tau_2).$$

The result of this theorem can also be extended to m functions.

S. C. van Veen (Delft)

Boas, R. P., Jr. 231

Indefinite integration by residues.

Amer. Math. Monthly **71** (1964), 298–300.

The author shows how the integral $\int_\alpha^\beta R(\cos\theta,\sin\theta)\,d\theta$, $0\leqq\alpha<\beta\leqq 2\pi$, can be evaluated by the method of residues, not only in the special case $\alpha=0$, $\beta=2\pi$, which is usually treated in textbooks. In the paper the integrand is assumed

to be a rational function without poles on the arc $\alpha \leqq \theta \leqq \beta$ of the unit circle, but the author has informed the reviewer that the method may fail if the integrand has poles on the arc complementary to $\alpha \leqq \theta \leqq \beta$. As an example of the method, the author evaluates the integral $\int_0^\pi (d\theta/(a+b\sin\theta))$, where $-a<b<a$. For another paper on the subject of indefinite integration by means of residues, see E. H. Neville [Math. Student **13** (1945), 16–25; MR **7**, 193]. *F. Herzog* (E. Lansing, Mich.)

Huckemann, Friedrich 232

On decompositions of a rectangle into two quadrilaterals.

Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 202–204.

Let Q be the rectangle $0<\Re z<M$, $0<\Im z<1$. When a continuum C separates the vertical sides of Q, then two quadrilaterals Q_1, Q_2 are produced whose conformal moduli M_1, M_2, satisfy $M_1+M_2 \leqq M$. The author studies the converse problem: Given M_1, M_2, M, what can one deduce about the horizontal projection of C? He finds best possible functions $x'(M_1, M_2)$, $x''(M_1, M_2)$ of the two variables M_1, M_2 such that C must lie in the strip $x'(M_1, M_2) \leqq \Re z \leqq x''(M_1, M_2)$. Indications of proof are given and involve the introduction of an auxiliary hexagon. The results should be of interest in problems of quasiconformal mapping. *E. Reich* (Stanford, Calif.)

Dziubiński, Izydor 233

Sur le minimum de diamètre du domaine doublement connexe.

Bull. Soc. Sci. Lettres Łódź **14** (1963), *no.* 6, 9 *pp.*

Let F be the family of functions $f(z)=\sum_{-\infty}^{\infty} a_n z^n$ holomorphic in the annulus A, $0<r<|z|<1$, that are characterized by the property $|a_1|=|a|\neq 0$ for every member $f\in F$. Consider the functional $\delta(f)=\sup|f(z_2)-f(z_1)|$, where the supremum is taken over all z_1 and z_2 in A. It is proven in this paper that min $\delta(f)$ for all $f\in F$ is attained by the function $f(z)=a_0+az$. If, instead of fixing the modulus of the coefficient of z, one fixes the modulus of the coefficient of z^{-1}, that is, $|a_1|=|b|\neq 0$, then min $\delta(f)$ is attained by the function $f(z)=bz^{-1}+a_0$.

Consider a doubly connected region D, each of whose boundaries contains at least two distinct points. Let $w=f^*(z)$ map this region conformally onto the annulus A^*, $1<|z|<R$, and let $z=h^*(w)$ be the inverse function of f^*. If F_c is the family of functions analytic in D with each member $F(h^*(w))=\sum_{-\infty}^{\infty} a_n w^n$, $w\in A$, satisfying the condition $|a_1|=c\neq 0$, then min $\delta(F)$ is attained by the function $w=a_0+a_1 f^*(z)$, $|a_1|=c\neq 0$, and only by this function. *W. C. Royster* (Lexington, Ky.)

Goodman, A. W. 234

A note on bivalent functions.

J. London Math. Soc. **39** (1964), 215–219.

The author's purpose is to obtain an example of an analytic function $f(z)$ bivalent in UH, the upper half-plane $\Im z>0$, for which $f'(z)$ vanishes infinitely often in UH and, furthermore, to exhibit an explicit formula for this function.

Using a generalized Schwarz-Christoffel transformation, the author maps UH by $f(z)$ onto a surface S obtained by placing over the half-plane $v>0$ infinitely many non-overlapping half-strips: $(2n+1)\delta+nW<u<(2n+1)\delta+(n+1)W$, $v>0$, where $\delta>0$, $W>0$, $n=0, \pm 1, \pm 2, \cdots$, and joining each of these strips to the base half-plane with a simple branch point. $f'(z)$ is then obtained in closed form.

The formula for the limit function $f_0(z)$ is obtained by letting one of the parameters tend to zero. In this case the width W of the upper strips tends to zero and $f_0(z)$ maps UH onto a comb-like region obtained by deleting from the upper half-plane an infinite number of equally spaced vertical slits, each of the same length.

M. S. Robertson (New Brunswick, N.J.)

McGregor, M. T. 235

On three classes of univalent functions with real coefficients.

J. London Math. Soc. **39** (1964), 43–50.

This paper deals with some classes of univalent functions $f(z)=z+a_2z^2+\cdots$ which are regular in the unit disc and have real coefficients. The author shows, e.g., that each f belonging to the class S_c of convex univalent functions with real coefficients takes every value situated inside the curve

$$u(t) = \pm \frac{1}{2t}\left(\frac{\sin \pi t}{\pi t} - 1\right),$$

$$v(t) = \pm \frac{1}{2t^2\pi}(1-\cos \pi t), \qquad 0 \leqq t \leqq 1.$$

Thus the values of each $f\in S_c$ cover the ellipse

$$\frac{u^2}{1/4}+\frac{v^2}{\pi^2/16} < 1.$$

Analogous results for classes S_* (functions starlike with respect to the origin with real coefficients) and S_I (functions convex with respect to the imaginary axis with real coefficients) are obtained. *Z. Lewandowski* (Lublin)

Suita, Nobuyuki 236

On the coefficient theorem of Jenkins. (Japanese)

Sûgaku **14** (1962/63), 129–137.

This is an expository paper mainly on the Jenkins general coefficient theorem in Jenkins's monograph [*Univalent functions and conformal mapping*, Springer, Berlin, 1958; MR **20** #3288] and results in his paper [*Analytic functions*, pp. 159–194, Princeton Univ. Press, Princeton, N.J., 1960; MR **22** #8126a]. *Y. Kusunoki* (New York)

Krzyż, Jan; Reade, Maxwell O. 237

The radius of univalence of certain analytic functions.

Michigan Math. J. **11** (1964), 157–159.

Suppose that $f(z)=z+a_2z^2+\cdots$ and $g(z)=z+b_2z^2+\cdots$ are analytic in the disc $D=\{z|\ |z|<1\}$. T. H. MacGregor [Proc. Amer. Math. Soc. **14** (1963), 514–520; MR **26** #6388] has shown that if $g(z)$ is univalent and starlike in D and if $\Re[f(z)/g(z)]>0$ in D, then $f(z)$ is univalent and starlike for $|z|<2-\sqrt{3}$, and the result is sharp. The authors show that the condition that $g(z)$ be starlike in D is unnecessary to obtain the same conclusions since, when $g(z)$ is merely univalent in D, the sharp inequality (used in MacGregor's proof)

$$(1) \qquad \Re\frac{zg'(z)}{g(z)} \geqq \frac{1-|z|}{1+|z|}$$

holds for $|z|<\tanh\frac{1}{2}=0.46212\cdots$ $(>2-\sqrt{3})$.

A second theorem of MacGregor [ibid. **14** (1963), 521–524; MR **26** #6389] states that if $g(z)$ is univalent and starlike in D and if $|f(z)/g(z)-1|<1$ in D, then $f(z)$ is univalent and starlike for $|z|<1/3$ [not in D as appears in the quotation in the paper reviewed here], and the result is sharp. By further use of (1) the authors show that again the condition that $g(z)$ be starlike in D is unnecessary to obtain MacGregor's second theorem.

M. S. Robertson (New Brunswick, N.J.)

Das, K. M. 238

A note on measure of curvature of level curves and orthogonal trajectories of functions finite-valent in the unit circle.

Gaṇita **13** (1962), 107–109.

Let $p(z)$ be regular in $|z|<1$. If for all z in $r<|z|<1$, $0<r<1$, $|p(z)|\leq(1-|z|^2)^{-2}$, then the equation (*) $y''(z)+p(z)y(z)=0$ is non-oscillatory in $|z|<1$. The author finds expressions for the curvature of the level curves and orthogonal trajectories of the finite-valent function $f(z)=u(z)[v(z)]^{-1}$, where u and v are linearly independent solutions of (*). These formulae correct certain formulae in a paper by Kulshretha (Gaṇita **9** (1958), 1–4; MR **21** #2744]. This paper contains several misprints.

W. C. Royster (Lexington, Ky.)

Weyl, Hermann 239

★Die Idee der Riemannschen Fläche.

Vierte Auflage. Unveränderter Nachdruck der dritten, vollständig umgearbeiteten Auflage.

B. G. Teubner Verlagsgesellschaft, Stuttgart, 1964. viii + 162 *pp.* *DM* 25.80.

An unaltered reprint of the third edition [1955; MR **16**, 1097].

Rodin, Burton 240

Extremal length of weak homology classes on Riemann surfaces.

Proc. Amer. Math. Soc. **15** (1964), 369–372.

On an open Riemann surface let $\{c\}$ denote the class of all cycles weakly homologous to a given cycle c. Let $\lambda\{c\}$ denote the extremal length of this family of curves. Let $\psi(c)$ be the semi-exact square integrable harmonic differential such that for all semi-exact ω, $\int_c \omega=(\omega,\psi(c))$. (For definitions see L. V. Ahlfors and L. Sario [*Riemann surfaces*, Chapter V, Princeton Univ. Press, Princeton, N.J., 1960; MR **22** #5729].) The author proves that the square of the Dirichlet norm of $\psi(c)$ equals $\lambda\{c\}$.

The results and methods are similar to those used by C. Blatter [C. R. Acad. Sci. Paris **250** (1960), 1167; MR **22** #9599] and the reviewer [Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 540–543; MR **22** #9598] in proving a similar result for strong homology classes of curves.

R. Accola (Providence, R.I.)

Oikawa, Kôtaro 241

A remark to the construction of Riemann surfaces by welding.

J. Sci. Hiroshima Univ. Ser. A-I Math. **27** (1963), 213–216.

Let $f(x)$ be a monotone increasing continuous function on $-1\leq x\leq 1$ such that $f(-1)=-1$, $f(1)=1$. Let $D^+=\{z;|z|<1, \operatorname{Im} z\geq 0\}$ and $D^-=\{z;|z|<1, \operatorname{Im} z\leq 0\}$. If there exist conformal mappings $\zeta=\phi^+(z)$ and $\zeta=\phi^-(z)$ of the interiors of D^+ and D^-, respectively, which are homeomorphic on D^+ and D^-, and are such that $\phi^+\circ f(x)=\phi^-(x)$ on $-1\leq x\leq 1$, then it is said that D^+ and D^- are welded together by f.

Assume that f is real-analytic on $-1\leq x<0$ and $0<x\leq 1$. The author considers the possibility of welding throughout $-1\leq x\leq 1$. Consider a pair of curves γ^+ and γ^- with the following properties: they are rectifiable, $\gamma^+\subset D^+-\{f(0)\}$, $\gamma^-\subset D^--\{0\}$, their endpoints are on $[-1,0)\cup(0,1]$, f (endpoint of γ^-) = (initial point of γ^+), f(initial point of γ^-) = (endpoint of γ^+). Let $\gamma=\gamma^+\cup\gamma^-$, and consider the family $\{\gamma\}$ consisting of all the γ. Denote its extremal length by $\lambda\{\gamma\}$.

The author shows that the condition $\lambda\{\gamma\}=0$ is sufficient but not necessary for the possibility of welding. This answers a question raised by Pfluger [J. Indian Math. Soc. (N.S.) **24** (1960), 401–412; MR **24** #A2663].

R. Accola (Providence, R.I.)

Tienari, Martti 242

Fortsetzung einer quasikonformen Abbildung über einen Jordanbogen.

Ann. Acad. Sci. Fenn. Ser. A I No. 321 (1962), 32 *pp.*

Let γ be a Jordan curve which divides the extended z-plane into two disjoint domains G_1 and G_2. Then γ is said to be a quasiconformal curve in the large if every quasiconformal mapping from G_1 onto the upper half w-plane can be extended to a quasiconformal mapping of the entire plane. The author proves that γ is quasiconformal in the large if and only if γ is the image of a circle under a quasiconformal mapping of the whole plane. Next, let γ be either an open Jordan arc or a closed Jordan curve. γ is said to be quasiconformal at a point z in γ if there exists an open subarc γ_z of γ and a quasiconformal mapping of a domain G such that $\gamma_z\subset G$, and the image of γ_z is a line segment. γ is called quasiconformal if it is quasiconformal at each point. The author shows that a quasiconformal Jordan curve is quasiconformal in the large.

Let G_1 and G_2 be two disjoint, simply connected domains which have an open, quasiconformal, free boundary arc γ in common. Let G_1', G_2', and γ' satisfy the same conditions. If f is a quasiconformal mapping of G_1 onto G_1' which maps γ onto γ', then for each compact subarc γ_0 of γ, f can be extended to a quasiconformal mapping of $G_1\cup\gamma_0\cup G_2$ onto $G_1'\cup\gamma_0'\cup G_2'$. The author also shows the quasiconformality of certain classes of arcs, e.g., arcs with continuous curvature.

J. A. Kelingos (Durham, N.C.)

Väisälä, Jussi 243

Remarks on a paper of Tienari concerning quasiconformal continuation.

Ann. Acad. Sci. Fenn. Ser. A I No. 324 (1962), 6 *pp.*

The author shows that the graph of a real-valued function satisfying a Lipschitz condition is quasiconformal. He also shows that an arc having a continuous tangent is quasiconformal. Finally, he generalizes Tienari's extension theorem [see the preceding review, #242] to multiply connected domains.

J. A. Kelingos (Durham, N.C.)

Biluta, P. A. 244

Some extremal problems in the class of mean quasiconformal mappings. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 503–505.

The author considers homeomorphisms whose dilatation $p(z)$ satisfies restrictions of the type $\iint\Phi[p(z)]\,d\sigma_z\leqq M$. He announces the solution of a number of extremal problems by use of the variational method of Belinskiĭ [same Dokl. **121** (1958), 199–201; MR **24** #A3285]. An example is the following (Theorem 1): Let $w=f(z)$ map $r\leqq|z|\leqq 1$ quasiconformally onto $\rho\leqq|w|\leqq 1$, with $\iint_{r\leq|z|\leq 1}p(z)\,d\sigma_z\leqq M$. (1) If $M\leqq 2\pi(1-r)$, then $\rho\leqq c^{-1}\exp(cr-1)$, where $c^{-1}=r+\sqrt{(\pi^{-1}M+r^2-1)}$. (2) If $M>2\pi(1-r)$, then $\rho\leqq\exp[2\pi M^{-1}(r-1)]$. Mappings for which ρ achieves the stated upper bound are listed. *E. Reich* (Stanford, Calif.)

Li, Chung 245

On the existence of homeomorphic solutions of a system of quasilinear partial differential equations of elliptic type.

Acta Math. Sinica **13** (1963), 454–461 (*Chinese*); *translated as Chinese Math.* **4** (1964), 493–500.

From the author's introduction: "This paper is devoted to the discussion of the homeomorphisms formed by the solutions of the system of elliptic partial differential equations

$$-v_y+A_{11}u_x+A_{12}u_y = 0,$$
$$v_x+A_{21}u_x+A_{22}u_y = 0,$$
$$A_{11}>0,\quad 4A_{11}A_{22}-(A_{12}+A_{21})^2\geqq\delta_0>0.$$

Our aim is to establish the existence of solutions which map a general finitely-connected domain onto some canonical domain. The result obtained is equivalent to solving the existence question for quasiconformal mappings for multiply connected domains."

Terpigoreva, V. M. 246

Extremal problems for Orlicz classes of analytic functions in the unit circle. (Russian. Latvian and English summaries)

Litovsk. Mat. Sb. **3** (1963), *no.* 1, 237–269.

In this interesting work the author gives detailed proofs of results published earlier [Dokl. Akad. Nauk SSSR **142** (1962), 38–41; MR **25** #5174]. *D. Mitrović* (Zagreb)

Anselone, P. M.; Boas, R. P., Jr. 247

The atypical zeros of a class of entire functions.

J. Math. Anal. Appl. **8** (1964), 278–281.

The entire functions concerned are characteristic functions of certain difference-differential-integral equations. A typical function is

$$\Psi(z) = e^{-z}-e^{-2z}-\int_0^1 e^{-zs}K(s)\,ds.$$

If $K(s)$ is absolutely continuous, $K(0)\neq 0$, $K'(s)$ essentially bounded, then $\Psi(z)$ is shown to have two separate families of zeros $2\pi in+O(1)$ and $2\pi in+\log|n|+O(1)$. In more general cases conditions which assume the existence of zeros of arbitrarily large real part are given.

A. J. Macintyre (Cincinnati, Ohio)

Wang, Zhen-yu [Wang, Chen-yu] 248

On an interpolation problem of entire functions.

Acta Math. Sinica **13** (1963), 1–16 (*Chinese*); *translated as Chinese Math.* **4** (1963), 1–17.

Mainly the following sort of interpolation problem for entire functions of finite order is discussed. Given a sequence $\{\lambda_n\}$ of distinct complex numbers such that $r_n=|\lambda_n|\uparrow\infty$, and such that

$$(*)\qquad \limsup_{n\to\infty}\frac{n^2}{r_n^{\rho}}=\tau<\infty,\qquad \tau\geqq 0,\ \rho>0;$$

let $\{a_n\}$ be another sequence of complex numbers satisfying

$$(**)\qquad \limsup_{n\to\infty}\frac{\log|a_n|}{r_n^{\rho}}=\nu<\infty.$$

Under what conditions does there exist an entire function $w(z)$ of order not exceeding ρ and such that $w^{(n)}(\lambda_n)=a_n$, $n=1,2,\cdots$; if such an entire function exists, of what form is it? The following theorem of the author is the answer to the above problem, but with an additional requirement on the sequence $\{\lambda_n\}$. Theorem: Let $\{\lambda_n\}$ and $\{a_n\}$ be the given sequences satisfying (*) and (**) and such that

$$(***)\qquad \limsup_{n\to\infty}\frac{1}{r_n^{\rho}}\log\left|\frac{1}{F^{(n+1)}(\lambda_n)}\right|=\delta<\infty,$$

where

$$F(z)=\prod_{n=1}^{\infty}{}'\left(1-\frac{z^m}{\lambda_n^m}\right)^{n+1},$$

and m is an integer greater than ρ. (The prime after the product sign means that the product can have only those factors for which the λ_n^m are distinct.) Then the above interpolation problem has a solution, and the order of the entire function is ρ. In another theorem, (***) is replaced by a simpler condition. Further, it is also remarked that the function $F(z)$ appearing in (***) does not have to be of that form. It can be any entire function of order ρ, satisfying (***) and having zeros of $(n+1)$st order at λ_n, $n=1,2,\cdots$.

To that effect another theorem is proved. Corresponding results for functions of infinite order are stated. Finally, general interpolation problems, similar to those of Hermite and Abel-Gončarov, are discussed. The paper ends with the construction of interpolating polynomials.

Hari Shankar (Athens, Ohio)

Wittich, H. 249

Über eine Klasse holomorpher Funktionen.

J. Reine Angew. Math. **214/215** (1964), 321–327.

Consider the class $\{f(s)\}$ of functions $f(s)$ holomorphic in $G=\{s|s\neq s_1,\cdots,s_n\}$ and with $f(\infty)=0$, and also the class $\{F(z)\}$ of entire functions $F(z)=\sum_1^n e^{s_jz}F_j(z)$, where $F_j(z)$ is at most of minimal type of order 1. The author shows that every $f(s)$ can be written as the Laplace transform of an $F(z)$ and vice versa, and $f(s)$ is rational if and only if all $F_j(z)$ are polynomials. From this a result due to H. Selberg [Avh. Norske Vid.-Akad. Oslo Mat. Natur. Kl. **1931**, no. 10] follows: If $F_1(z)$ and $F_2(z)$ are exponential sums of the form $\sum_1^n c_je^{s_jz}$ and F_1/F_2 is entire, then it is itself an exponential sum. The function $f(s)$ can be written $f(s)=\sum_1^n f_j(s)$, where $f_j(s)$ is entire in $1/(s-s_j)$. For the functions $f_j(s)$ and $F_j(z)$ the order can then be defined

in the usual way as $l(f_j) = \limsup \ln\ln M(r, f_j)/\ln r$, $l(F_j) = \limsup \ln\ln M(r, F_j)/\ln r$, where r is $1/|s-s_j|$ and $|z|$, respectively, and where M denotes the maximum on the r-circle. Clearly, $l(f_j)$ coincides with the local order $l(f, s_j)$ of f in the neighborhood of s_j. Now we may exhaust G by means of a Selberg function $u(s) = \sum_1^n \mu_j \ln(|s-s_j|^{-1})$ with $\mu_j > 0$ and $\sum_1^n \mu_j = 1$, and define the order $L(f) = \limsup \ln\ln M(\lambda, f)/\lambda$ of f, where $M(\lambda, f)$ is the maximum of $|f|$ on the level $u(s) = \lambda$. Here $L(f)$ depends on the choice of the masses μ_j, but the lower order $L_0(f) = \inf_\mu L(f)$ is unique [cf. the reviewer, Ann. Acad. Sci. Fenn. Ser. A I No. 193 (1955); MR **18**, 27]. The following formulae are derived: $L(f) = \max_{j=1}^n l(f_j)/\mu_j$, $l(F_j) = l(f_j)/(1+l(f_j))$ with $l(F_j) = 1$ if and only if $l(f_j) = \infty$, $L_0(f) = \sum_1^n l(f, s_j) = \sum_1^n l(F_j)/(1-l(F_j))$. Further results: The function $f(s)$ is rational if and only if $\liminf \ln M(\lambda, f)/\lambda < \infty$. The equation $f(s) = a \neq \infty$ has an infinity of roots, unless $L_0(f)$ is an integer. A Denjoy-Carleman-Ahlfors theorem is valid: $f(s)$ cannot have more than $2L_0(f)$ finite asymptotic values. *G. af Hällström* (Åbo)

Lappan, Peter 250

Some sequential properties of normal and non-normal functions with applications to automorphic functions.

Comment. Math. Univ. St. Paul. **12** (1964), 41–57.

Using the classical results (and standard notations) of the theory of normal families of functions of a complex variable, the author establishes a sufficient condition that a meromorphic function f defined in the unit disk be a normal function [O. Lehto and K. I. Virtanen, Acta Math. **97** (1957), 47–65; MR **19**, 403]. In addition, a necessary and sufficient condition for non-normality is proven. The author also shows that even though the sum of two normal functions need not be normal, the sum of two uniformly normal functions is again uniformly normal. A function f defined in $|z| < 1$ is uniformly normal if for each $M > 0$ there exists a finite $K > 0$ such that for each z_0, $|z_0| < 1$, $\rho(z, z_0) < M$ implies $|f(z) - f(z_0)| < K$, where ρ is the non-Euclidean hyperbolic distance between z_0 and z.

Various other results on the normality and non-normality of automorphic functions are also given.

J. A. Cima (Stanford, Calif.)

Trofimenko, V. A. 251

A refined theorem of Lindelöf for topological mappings of class $\tilde{B}L$. (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. **169** (1963), 13–17.

A theorem of Lindelöf asserts that if a holomorphic function f maps the unit disk D univalently onto a plane domain G, and Γ is a Stolz path in D, then the cluster set of f on Γ consists of the principal points of the corresponding prime end of G. The theorem has been extended by Collingwood and the reviewer [Arch. Math. **10** (1959), 379–386; MR **26** #2599]: If $\{s_n\}$ is any side-chain of cross-cuts of G, then the image $f(\Gamma)$ meets at most finitely many of the s_n. The author shows that the extended theorem holds for the functions of the larger class that he calls $\tilde{B}L$. These are the topological mappings $f(z) = u(x, y) + iv(x, y)$ for which u and v possess generalized partial derivatives in the sense of Sobolev, and whose spherical Dirichlet integral over D is finite. *G. Piranian* (Ann Arbor, Mich.)

Dunducenko, L. E. [Dundučenko, L. E.]; Kasianiuk, S. A. [Kas′janjuk, S. A.] 252

Functions of bounded type in a circular annulus. I. (Romanian)

Acad. R. P. Romîne An. Romîno-Soviet. Ser. Mat.-Fiz. (3) **16** (1962), *no.* 2 (41), 63–81.

Simonenko, I. B. 253

The Riemann boundary-value problem for n pairs of functions with measurable coefficients and its application to the study of singular integrals in L_p spaces with weights. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 277–306.

The author considers the Riemann boundary problem (1) $\phi^+(t) = G(t)\phi^-(t) + g(t)$, $t \in C$, where C is a finite set of simple, closed, non-intersecting Liapunov curves bounding the interior domain D^+, the domain complementary to $D^+ + C$ with respect to the extended plane being denoted by D^-. The problem requires determination of n-vectors $\phi^+(z)$ and $\phi^-(z)$, analytic in D^+ and D^-, respectively, whose limiting values on C satisfy (1), where $g(t)$ is a specified n-vector and $G(t)$ is a given n-square matrix. The cases $n = 1$ and $n > 1$ are considered separately.

The case $n = 1$ was first solved by F. D. Gahov assuming $G(t)$ and $g(t)$ Hölder-continuous on C. This was extended by B. Hvedelidze who assumed $g(t) \in L_p(C)$, $p > 1$ [F. D. Gahov, *Boundary problems* (Russian), Fizmatgiz, Moscow, 1958; MR **21** #2879]. Further weakening of the requirements on G and g were made by the author, S. G. Mihlin, I. Gohberg, V. Ivanov, I. Daniljuk, H. Widom, et al. Using methods of functional analysis the author now shows that the basic Noether theorems associated with (1) continue to hold for $g \in L_p(C)$ and with $G(t)$ bounded, measurable and such that for each t_0 on C there is a neighborhood of t_0 and a $\delta > 0$ for which $G(t)$ is contained in a sector with vertex at the origin and of opening $(2\pi - \delta)/\max(p, p/p-1)$.

For the case $n > 1$, the author assumes that the components of the n-vector g belong to $L_2(C)$ and the n-square matrix $G(t)$ has the following properties: (i) its elements are bounded and measurable, (ii) it may be written as the product $G_1(t)G_2(t)G_3(t)$ of matrices wherein $G_1(t)$, $G_3(t)$ are continuous and non-singular and $G_2(t)$ is such that (2) $\operatorname{Re} G_2(t) = \frac{1}{2}[G(t) + G^*(t)] \geqq \nu > 0$ (ν is independent of t and $\geqq$ refers to a comparison of Hermitian matrices). The number $(1/2\pi) \arg \Delta(G_1G_3)_C$ is invariant with respect to the representation (2) and defines the index of G on C.

J. F. Heyda (King of Prussia, Pa.)

Čerepanov, G. P. 254

The Riemann-Hilbert problem for the region exterior to a set of cuts along a line or a circle. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 275–277.

The author considers the complex z-plane with a number of cuts $L_j(a_j, b_j)$ $(j = 1, 2, \cdots, n)$, $a_1 < b_1 < a_2 < \cdots$, along the real axis. The problem is to determine a sectionally holomorphic function $\varphi(z)$ bounded at infinity and making jumps at the cuts satisfying the boundary conditions

$$\operatorname{Re}\{(a^+ + ib^+)\varphi^+\} = f^+(x),$$

$$\operatorname{Re}\{(a^- + ib^-)\varphi^-\} = f^-(x),$$

where $a^\pm$, $b^\pm$, $f^\pm$ are real and piecewise continuous functions satisfying Hölder conditions at the intervals of

continuity. An explicit solution is derived using the theory developed by the author in a previous paper [Prikl. Mat. Meh. **26** (1962), 907–912; MR **26** #5785].

H. A. Lauwerier (Amsterdam)

Gusev, V. A. 255

On certain classes of quaternionic monogenic functions. (Russian)

Mat. Sb. (*N.S.*) **63 (105)** (1964), 321–328.

Let Q_R and Q_C denote, respectively, the algebras of real and of complex quaternions. Let 1, e_1, e_2, e_3 be a common basis for the algebras, and let $f=f^0+f^1e_1+f^2e_2+f^3e_3$ be a function of class C^1 with domain $\Delta\subset Q_R$ and range in Q_C (i.e., the f^i are complex-valued functions of four real variables of class C^1). Let $p_1=ix_0+e_1x_1+e_2x_2-e_3x_3$, $p_2=ix_0-e_1x_1+e_2x_2+e_3x_3$, $p_3=ix_0+e_1x_1-e_2x_2+e_3x_3$, $p_4=ix_0-e_1x_1-e_2x_2-e_3x_3$. The (formal) derivatives of f, $\partial f/\partial p_s$, are defined to be the (unique) solutions φ_s, $s=1,\cdots,4$, of the system $f_{x_h}=\sum_{s=1}^4(p_s)_{x_h}\varphi_s$, $h=0,1,2,3$.

The following classes of functions f are defined:

$$F_1(\Delta)=\{f:\varphi_4=0\},\qquad F_1{}^*(\Delta)=\{f:\varphi_4=\varphi_3=0\},$$

$$F_1{}^{**}\Delta=\{f:\varphi_4=\varphi_3=\varphi_2=0\}.$$

These classes of functions comprise, respectively, the functions f which are monogenic in the sense of Fedorov [same Sb. (N.S.) **50 (92)** (1960), 101–108; MR **22** #3796] with respect to p_1, p_2, p_3, with respect to p_1, p_2, and with respect to p_1. The author gives characterizations of these classes.

$F_1(\Delta)$ coincides with the class of functions monogenic in the sense of Moisil [Bull. Sci. Math. (2) **55** (1931), 168–174]; the solution of the nonhomogeneous Maxwell equations is developed in terms of such functions (of class C^2). In case two component functions, say f^1 and f^3, are prescribed, then the general function of $F_1(\Delta)$ is a function of explicitly developed special form with domain and range in, respectively, the algebras of the real and complex bicomplex numbers.

The class $F_1{}^*(\Delta)$ consists of those functions whose component functions f^i are expressible as special linear combinations of two functions of the arguments x_1-ix_3 and x_0-x_2, and two functions of the arguments x_1+ix_3 and x_0+x_2, the functions being analytic in their first argument and continuous in their second.

The class $F_1{}^{**}(\Delta)$ is shown to comprise those functions which are "q-monogenic" in the sense of Krylov [Dokl. Akad. Nauk SSSR **55** (1947), 799–800; MR **9**, 233].

R. F. Rinehart (Cleveland, Ohio)

Gusev, V. A. 256

On certain matrix differential operators. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 2 (39), 65–73.

Der Autor führt zunächst den Begriff einer monogenen Matrizenfunktion ein. Zwei Matrizenfunktionen p und q vom Typ (N,N) der Klasse $C_N(\Delta)$ bilden ein Funktionenpaar $(p,q;\Delta,N)$ im Gebiet Δ, falls (1) die Matrizen p, q, p_x, q_x, p_y, q_y paarweise vertauschbar sind und (2) die Matrix $\delta=p_xq_y-p_yq_x$ nicht singulär ist. Mit Hilfe eines solchen Funktionenpaares werden für Funktionen $f(x,y)$ der Klasse $C_N{}^k(\Delta)$ die folgenden Differentialoperatoren eingeführt:

$$\frac{\partial f}{\partial p}=\delta^{-1}(q_yf_x-q_xf_y);\qquad \frac{\partial f}{\partial q}=\delta^{-1}(p_xf_y-p_yf_x);$$

$$\frac{\partial^2 f}{\partial p^2}=\frac{\partial}{\partial p}\left(\frac{\partial f}{\partial p}\right)\quad \text{u.s.w.}$$

Eine Funktion $f(x,y)\in C_N{}^1(\Delta)$ heißt monogen bezüglich $\varphi\in C_N{}^1(\Delta)$ $(f=f[\varphi;\Delta])$ im Gebiet Δ, falls eine eindeutige Matrizenfunktion $\theta(x,y)$ vom Typ (N,N) existiert, sodaß in Δ die folgenden Beziehungen gelten: $f_x=\varphi_x\theta$; $f_y=\varphi_y\theta$. θ heißt dann die Ableitung von f nach φ.

Anschließend werden einige grundlegende Eigenschaften der monogenen Matrizenfunktionen untersucht und der Begriff der Stammfunktion einer monogenen Matrizenfunktion definiert. Mit Hilfe dieser Eigenschaften läßt sich schließlich eine Verallgemeinerung des Theodorescuschen Satzes angeben: Falls für ein vorgegebenes Funktionenpaar $(p,q;\Delta,N)$ die Beziehung $\partial^{n+1}f/\partial q^{n+1}=0$, $(x,y)\in\Delta$, gilt, so läßt sich $f(x,y)$ in der Form

$$f=\sum_{k=0}^{n}q^kh_k(x,y),\qquad h_k=h_k[p;\Delta],\qquad (x,y)\in\Delta,$$

darstellen und umgekehrt.

Abschließend wird eine partielle Matrizendifferentialgleichung betrachtet. Es wird gezeigt, daß sich die Lösungen der Differentialgleichung

$$\text{(I)}\qquad \frac{\partial^2\varphi}{\partial q^2}+2a(x,y)\frac{\partial\varphi}{\partial q}+b(x,y)\varphi=0$$

für ein vorgegebenes Funktionenpaar $(p,q;\Delta,N)$ stets in der Form

$$\varphi(x,y)=e^{-aq}(\varphi_0F[p;\Delta]+\varphi_1\Phi[p;\Delta])$$

darstellen laßen, wobei F und Φ beliebige bezüglich p monogene Funktionen bezeichnen, und die Funktionen φ_0 bzw. φ_1 durch die folgenden Beziehungen definiert sind:

$$\varphi_0=\sum_{n=0}^{\infty}\frac{\alpha^nq^{2n}}{(2n)!},\qquad \varphi_1=\sum_{n=0}^{\infty}\frac{\alpha^nq^{2n+1}}{(2n+1)!},\qquad \alpha=a^2-b.$$

Dieses Ergebnis wird schließlich zur Lösung der Matrizendifferentialgleichung

$$A\frac{\partial^2 f}{\partial x^2}+2B\frac{\partial^2 f}{\partial x\partial y}+C\frac{\partial^2 f}{\partial y^2}=0$$

benutzt, die sich mit Hilfe der Substitution $f=u(x,y)\varphi$ (u bezeichnet eine skalare Funktion) auf die Form (I) zurückführen läßt.

W. Schmidt (Simferopol)

POTENTIAL THEORY

See also 350, 460.

Weinberger, H. F. 257

On bounding harmonic functions by linear interpolation.

Bull. Amer. Math. Soc. **70** (1964), 525–529.

Let u be a function harmonic inside and on a circle, and P a point inside the circle. Let $A(\theta)$ be the value obtained by linearly interpolating the boundary values of u at the ends of the chord through P making an angle θ with some fixed direction. It follows from Poisson's formula that

$u(P)$ is equal to the average of $A(\theta)$ and thus that $u(P) \leq \operatorname{Max} A(\theta)$. Attempts have been made to prove this inequality for convex or star-shaped regions. The author proves that no such inequality, even allowing a large multiplicative factor on the right, is true for star-shaped regions, and that it holds at all points of a convex region if and only if the region is a circle.

John W. Green (Los Angeles, Calif.)

Ninomiya, Nobuyuki 258

Sur un principe du maximum pour le potentiel de Riesz-Frostman.

J. Math. Osaka City Univ. **13** (1962), 57–62.

Dans un travail récent [J. Math. Osaka City Univ. **12** (1961), 115–138; MR **26** #2632; ibid. **12** (1961), 139–143; MR **26** #2633] l'auteur a introduit la notion de principe du maximum pour un noyau par rapport à un autre noyau, et établi que le noyau de Riesz d'ordre α $(m-2 \leq \alpha < m)$ satisfait au principe du maximum par rapport au noyau d'ordre β $(0 < \beta \leq m-2)$. Ce résultat est étendu au cas $0 < \beta \leq \alpha$.

J. Deny (Orsay)

Rubel, L. A. 259

On separation by harmonic functions.

Acta Math. Acad. Sci. Hungar. **15** (1964), 175–176.

The author here investigates the following question: For which values of n is it true that if f is a superharmonic function on Euclidean space E^n, h a subharmonic function there, and $f \leq h$ everywhere, then there exists a harmonic function g such that $f \leq g \leq h$ everywhere? It is true for $n=1$, since a concave function f that lies below a convex function h clearly can be separated from h by a linear function g. The author shows by example, however, that it is false for all integers $n > 1$.

E. F. Beckenbach (Los Angeles, Calif.)

Meyer, Paul-André 260

Brelot's axiomatic theory of the Dirichlet problem and Hunt's theory.

Ann. Inst. Fourier (Grenoble) **13** (1963), *fasc.* 2, 357–372.

Let E be a locally compact space with a countable base on which harmonic functions are defined satisfying the axioms 1, 2, 3 of M. Brelot's theory [*Lectures on potential theory*, Tata Inst. Fundamental Res., Bombay, 1960; MR **22** #9749]. One assumes furthermore the existence of a potential >0 and the harmonicity of the constant functions. The author then proves the existence of a submarkovian semi-group $(P_t)_{t \geq 0}$ of kernels on E having the following properties: For each continuous function $g: E \to \mathbf{R}$ with compact support, the function $P_t g$ is continuous and $\lim_{t \to 0} P_t g = g$ holds uniformly on compact subsets. The finite superharmonic functions ≥ 0 on E coincide with the finite excessive functions with respect to (P_t). Starting with a finite potential f the author reproves first R.-M. Hervé's result [Ann. Inst. Fourier (Grenoble) **12** (1962), 415–571; MR **25** #3186] on the existence of a positive kernel V which is uniquely determined by the following properties: $V(1)=f$; Vg is superharmonic and harmonic in the complement of the closed support of g for each bounded Borel-measurable $g: E \to \mathbf{R}_+$. By assuming f to be a continuous strict potential in the reviewer's sense [Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **1** (1962/63), 197–229; MR **27** #5926] the semi-group (P_t) is then constructed in such a way that $V = \int_0^\infty P_t\, dt$. The author's methods apply with some modifications also to the reviewer's axiomatic theory of harmonic functions [Math. Ann. **146** (1962), 1–59; MR **26** #1612].

H. Bauer (Hamburg)

Boboc, N.; Constantinescu, C.; Cornea, A. 261

On the Dirichlet problem in the axiomatic theory of harmonic functions.

Nagoya Math. J. **23** (1963), 73–96.

A new axiomatic theory of the Dirichlet problem is developed which generalizes both the theory of M. Brelot [*Lectures on potential theory*, Tata Inst. Fundamental Res., Bombay, 1960; MR **22** #9749] and of the reviewer [Math. Ann. **146** (1962), 1–59; MR **26** #1612; Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **1** (1962/63), 197–229; MR **27** #5926]. The new theory includes, e.g., the treatment of compact Riemann surfaces. The axioms are derived from those of the reviewer's theory by replacing his separation axiom by one of its consequences, namely, the minimum principle for hyperharmonic functions. More precisely: An open set $U \neq \emptyset$ is called an M.P.-set if U is either compact and all hyperharmonic functions on U are ≥ 0 or if U is non-compact and $\lim \inf_{x \to \omega} s(x) \geq 0$ implies $s \geq 0$ for all hyperharmonic functions s on U (ω is the point at infinity of U). The axiom replacing the separation axiom states then the existence of a covering of the underlying locally compact space by M.P.-sets. Starting with these axioms the Dirichlet problem for certain open subsets U is treated by means of the Perron-Wiener-Brelot method. A precise study of the local behavior of potentials and boundary points leads to a generalization of the resolutivity theorem of M. Brelot in the case where the boundary of U is σ-compact. The main results have been announced already [C. R. Acad. Sci. Paris **256** (1963), 2754–2757; MR **27** #337].

H. Bauer (Hamburg)

Šafeev, M. N. 262

Boundary properties of the S_p integral. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 2 (39), 165–175.

The author considers some boundary-behavior properties of a double Poisson-Stieltjes integral

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{1-r_1^2}{1+r_1^2-2r_1\cos(\varphi_1-\alpha_1)} \times \frac{1-r_2^2}{1+r_2^2-2r_2\cos(\varphi_2-\alpha_2)}\, d\phi(\Delta)$$

considered as a function of a point in the bicylinder $\{(\varphi_1, r_1, \varphi_2, r_2): r_1, r_2 < 1\}$. The integration extends over the boundary of the bicylinder with respect to some function of an interval of bounded variation. The following two results are generalizations of those due to Privalov concerning the ordinary Poisson-Stieltjes integral: (1) at almost every boundary point $(\alpha_1, 1, \alpha_2, 1)$ the angular limit of the integral (i.e., the limit as $r_1, r_2 \to 1$, $1/k \leq (\varphi_1-\alpha_1)/(\varphi_2-\alpha_2) \leq k$ for some $k>0$) exists with the value equal to the ordinary derivative of $\phi(\Delta)$ at (α_1, α_2); (2) In case $\phi(\Delta)$ satisfies certain additional regularity conditions, the existence of the radial limit (i.e., $r_1, r_2 \to 1$; $\varphi_1 = \alpha_1$, $\varphi_2 = \alpha_2$) at a particular boundary point is ensured with the

value equal to the strong derivative of $\phi(\Delta)$ at this point. (For the concepts of ordinary and strong derivative, cf. S. Saks [*Theory of the integral*, 2nd rev. ed., p. 106, Stechert, New York, 1937].) *H. Fast* (Notre Dame, Ind.)

Vekilov, Š. I. 263

The first boundary-value problem for the Laplace equation in a composite region with corners. (Russian. Azerbaijani summary)

Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Tehn. Mat. Nauk **1964**, *no.* 1, 7–12.

Let D be a domain in 3-space bounded by a piecewise smooth closed surface S. Let C denote a closed Lyapunov surface in D. The problem considered by the author is to determine a continuous function U on $D \cup S$ which is harmonic in $D-C$ and satisfies the following conditions: (1) $U=f$ on S, (2) $K_1(\partial U/\partial n)_i = K_2(\partial U/\partial n)_i + hU = g$ on C, where f is a continuous function prescribed on S, g, $h \geqq 0$ are continuous functions prescribed on C, and $K_1 \neq K_2$ are given positive constants. By representing U as a sum of the potential due to a single layer on C and the potential due to a double layer on S, the problem is reduced to integral equations for the densities of these layers. For sufficiently smooth S a similar problem was treated earlier by A. I. Guseĭnov [Prikl. Mat. Meh. **12** (1948), 103–108; MR **9**, 438] and the author [Izv. Akad. Nauk Azerbaidžan. SSR Ser. Fiz.-Tehn. Mat. Nauk **1953**, no. 4, 3–25]; now S is allowed to have corners along a piecewise smooth curve without cuspidal points. *J. Král* (Prague)

Ivanov, L. D. 264

On the analytic capacity of linear sets. (Russian)

Uspehi Mat. Nauk **17** (1962), *no.* 6 (108), 143–144.

For definitions and some similar results see Pommerenke [Arch. Mat. **11** (1960), 270–277; MR **22** #5740c]. After discussing sets of intervals and closed sets, an inner analytic capacity of a linear set [cf. Ahlfors and Beurling, Acta Math. **83** (1950), 101–129; MR **12**, 171] is shown to be precisely one quarter of its inner measure.

A. J. Macintyre (Cincinnati, Ohio)

Ogawa, Shirô 265

Sur un théorème du balayage.

J. Math. Osaka City Univ. **13** (1962), 135–141.

Démonstration du théorème du balayage sur tout ensemble fermé non compact pour un noyau-fonction symétrique satisfaisant aux principe de domination et du maximum dilaté, et s'annulant convenablement à l'infini.

J. Deny (Orsay)

Ogawa, Shirô 266

Sur le balayage pour des ensembles analytiques.

J. Math. Osaka City Univ. **14** (1963), 35–44.

Démonstration de l'identité des balayages intérieur et extérieur sur tout ensemble K-analytique pour des noyaux-fonctions symétriques satisfaisant au principe de domination et qui sont parfaits au sens de Fuglede [Acta Math. **103** (1960), 139–215; MR **22** #8232]. L'auteur utilise le fait que le potentiel balayé d'un potentiel fixe est une fonction alternée d'ordre 2 du compact sur lequel on balaye; il est facile de voir que c'est même une fonction alternée d'ordre infini. *J. Deny* (Orsay)

Kishi, Masanori 267

Note on balayage and maximum principles.

Proc. Japan Acad. **39** (1963), 415–418.

Enoncé sans démonstrations de divers résultats concernant les principes fondamentaux de la théorie du potentiel par rapport à un noyau-fonction G semi-continu inférieurement, non nécessairement symétrique, sur un espace localement compact séparable. Exemples: supposons que l'adjoint de G soit régulier; alors (1) la conjonction des principes de domination et du maximum entraîne le principe complet du maximum; (2) le principe fort du maximum entraîne le principe complet. Ces résultats apportent une réponse positive à des conjectures du reviewer, posées, il est vrai, dans un cadre plus général que celui des noyaux-fonctions. *J. Deny* (Orsay)

Kishi, Masanori 268

Maximum principles in the potential theory.

Nagoya Math. J. **23** (1963), 165–187.

Let Ω be a locally compact space and let $G: \Omega \times \Omega \to \bar{\mathbf{R}}_+$ be a lower semi-continuous kernel. One defines the potential and the energy of a measure $\mu \geqq 0$ on Ω with compact support $S\mu$ with respect to G or the adjoint kernel $\check{G}$ in the usual way. A property is said to hold G-p.p.p. on a set $X \subset \Omega$ if it holds almost everywhere on X with respect to each measure $\mu \geqq 0$ on Ω with compact support and finite energy. The following existence theorem is the central result of the paper: Let $K \subset \Omega$ be compact and separable and let u be a finite upper semi-continuous function $\geqq 0$ on K; suppose that $\check{G}$ satisfies the usual continuity principle. Then there exists a measure $\mu \geqq 0$ on K such that $G\mu \geqq u$ G-p.p.p. on K and $G\mu \leqq u$ on $S\mu$. This generalizes a result of N. Ninomiya [J. Inst. Polytech. Osaka City Univ. Ser. A **8** (1957), 147–179; MR **21** #776] for continuous symmetric G. The theorem is proved first for compact spaces with a finite number of points and then extended to the general case. Using the existence theorem and assuming that both G and $\check{G}$ satisfy the continuity principle, the author shows that the balayage [equilibrium] principle for G is equivalent to the domination [maximum] principle for G. For this part of the paper one assumes in addition that each compact subset of Ω is separable and that G is locally bounded outside of the diagonal of $\Omega \times \Omega$. Other maximum principles, in particular the complete maximum principle, are also studied. The results of the paper were previously announced [see #267 above].

H. Bauer (Hamburg)

Nakai, Mitsuru 269

On the fundamental existence theorem of Kishi.

Nagoya Math. J. **23** (1963), 189–198.

There are two aspects to this paper. The author proves the theorem of the title [M. Kishi, Theorem I.1; see #268 above] without assuming separability of the compact K. His device is to introduce a pseudo-metric d relative to the kernel G by

$$d(y, y') = \sup_{z \in K} |G(y, z) - G(y', z)|,$$

with respect to which K has a countable dense subset even if K is non-separable in its relative topology. The author also simplifies Kishi's proof of Theorem I.3 [loc. cit.]

through the application of a fixed-point theorem due to S. Kakutani [Duke Math. J. **8** (1941), 457–459; MR **3**, 60].

E. C. Schlesinger (New London, Conn.)

SEVERAL COMPLEX VARIABLES

Bose, S. K.; Sharma, Devendra 270

Integral functions of two complex variables.

Compositio Math. **15**, 210–226 (1963).

Nach einigen Abschätzungen für die Betragsmaxima und die Maxima des Realteils ganzer Funktionen von zwei komplexen Veränderlichen verallgemeinern die Autoren Methoden von G. Valiron [*Lectures on the general theory of integral functions*, E. Privat, Librairie de l'Univ., Toulouse, 1923] auf ganze Funktionen von zwei Veränderlichen, indem sie zu einer solchen Funktion

$$f(z_1, z_2) = \sum_{m_1, m_2=0}^{\infty} a_{m_1 m_2} z_1^{m_1} z_2^{m_2}$$

mit Hilfe der Punkte mit den Koordinaten $(m_1, m_2, -\log|a_{m_1 m_2}|)$ einen Polyeder $S(f)$, genannt "Newton's polyhedron", konstruieren, der eine Verallgemeinerung der Newtonschen Polygone zu ganzen Funktionen einer Veränderlichen nach G. Valiron darstellt.

Sie definieren $\mu(r_1, r_2)$ als den Maximalterm aus der Matrix der Beträge $|a_{m_1 m_2}|r_1^{m_1} r_2^{m_2}$ der Summanden der Doppelreihe von $f(z_1, z_2)$ zu vorgegebenem r_1 und r_2 und $\nu(r_1, r_2)$ als den zugehörigen Grad von $\mu(r_1, r_2)$, außerdem $\rho = \lim\sup_{r_1, r_2\to\infty} \log \nu(r_1, r_2)/\log(r_1, r_2)$ als die Ordnung der ganzen Funktion $f(z_1, z_2)$. Über die Ordnung einer ganzen Funktion beweisen sie folgenden Satz: Die ganze Funktion $f(z_1, z_2)$ besitzt genau dann eine endliche Ordnung, wenn $\mu = \lim\sup_{m_1, m_2\to\infty} \log(m_1^{m_1} m_2^{m_2})/\log(1/|a_{m_1 m_2}|)$ endlich ist. Die Ordnung ρ ist dann gleich μ. Für $0<\rho<\infty$ setzen sie $\alpha = \lim\sup_{m_1, m_2\to\infty} (m_1^{m_1} m_2^{m_2}|a_{m_1 m_2}|^{\rho})^{1/(m_1+m_2)}$, bezeichnen $\tau = \lim\sup_{r_1, r_2\to\infty} \log M(r_1, r_2)/(r_1^{\rho} + r_2^{\rho})$, $M(r_1, r_2) = \mathrm{Max}\,|f(z_1, z_2)|$ für $|z_i| \leqq r_i$, als den Typ von $f(z_1, z_2)$ und folgern das Ergebnis, daß unter der Bedingung $0<\alpha<\infty$ für die Ordnung ρ und den Typ τ einer ganzen Funktion die Beziehung $\alpha = e\tau\rho$ gilt.

R. Epe (Münster)

Michiwaki, Yoshimasa 271

The kernel functions and associated minimum problems, Riemannian curvature in an analytic direction.

Res. Rep. Nagaoka Tech. College **1** (1962), 11–23.

Let $W(Z) = (w_1(Z), \cdots, w_m(Z))'$ be a column vector of single-valued analytic functions, regular in a bounded domain D in the n-complex-dimensional space and belonging to the class $L^2(D)$. Let $[W_1, \overline{W}_2] = \int_D W_1 W_2{}^* dv_z$, $[W] = [W, \overline{W}]$, where W^* is the conjugate transpose of W. $[W]$ is a positive definite matrix. The author introduces the partial order $[W_1] \geqq [W_2]$ if $[W_1] - [W_2]$ is a positive semidefinite matrix. Generalizing J. M. Stark's results [Pacific J. Math. **6** (1956), 565–582; MR **18**, 474; ibid. **10** (1960), 1021–1038; MR **22** #12237] he develops Bergman's method of the minimum integral and, in terms of the Bergman kernel function of D, he expresses the minimum of $[W]$ in the sense of the above order relation and the minimizing vector-valued function when, e.g., $W(Z)$ and its first and second derivatives satisfy certain conditions at a point t of D. The results are then applied to characterizing some pseudo-conformal mappings of circular domains bounded by analytic hyperplanes onto similar domains, and to expressing the Riemann curvature of the Bergman metric in an analytic direction in terms of the above mentioned minima (when $m=1$, $n=2$). Some of the proofs are incomplete, and the author informs the reviewer that complete proofs will be published elsewhere. The author points to an error in the second paper of J. M. Stark, referred to above, concerning the expression for the curvature in an analytic direction. Actually, Stark's error resulted from a misprint in S. Bergman [*Sur les fonctions orthogonales du plusieurs variables complexes avec les applications à la théorie des fonctions analytiques*, Mémor. Sci. Math., No. 106, Gauthier-Villars, Paris, 1947; MR **11**, 344], where formula (16) should read: $\lambda^{(3)} = 1/[K(2-R)\times(\sum T_{\alpha\bar\beta} u_\alpha \bar u_\beta)^2]$. The correct expression for the curvature is already contained in B. Fuchs [Mat. Sb. (N.S.) **2** (**44**) (1937), 567–594].

M. Maschler (Jerusalem)

Geče, F. I. 272

Systems of entire functions of several variables and their applications to the theory of differential equations. (Russian. Armenian summary)

Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk **17** (1964), *no.* 2, 17–46.

The author studies analytic mappings of C^n into C^m. A point $Z\in C^n$ is written as a matrix with a single column and Z^k is used as a notation for a column matrix with all numbers

$$(k!\alpha_1!^{-1}\cdots\alpha_n!^{-1})^{1/2} z_1^{\alpha_1}\cdots z_n^{\alpha_n},$$

where $\alpha_1+\cdots+\alpha_n = k$ are its elements. With the usual 2-norm this implies that $\|Z^k\| = \|Z\|^k$. Further, Z^{-k} denotes the column matrix whose elements are reciprocal to the elements of Z^k. The derivative d^kW/dZ^k is a matrix with the elements $(k!\alpha_1!^{-1}\cdots\alpha_n!^{-1})^{1/2}\partial^k w_l/\partial z_1^{\alpha_1}\cdots\partial z_n^{\alpha_n}$ arranged in m rows. With these notations every integral mapping $W: C^n \to C^m$ has a Taylor development

$$W(Z) = \sum \frac{1}{k!}\frac{d^kW(0)}{dZ^k} Z^k,$$

and the derivatives are expressed by Cauchy integrals as in the classical theory. In § 2 the author studies an entire mapping defined by a power series $W = \sum A_k Z^k$ and he introduces order and type as in the classical theory. Next, he develops the Wiman-Valiron theory of the maximal term of the power series in close analogy with the representation of the classical theory of H. Wittich [*Neuere Untersuchungen über eindeutige analytische Funktionen*, Springer, Berlin, 1955; MR **17**, 1067]. The maximal term $m(R)$ is defined as $\max_{0\leqq k<\infty}\max_{\|Z\|=R}\|A_k Z^k\|$, and the maximum modulus as $M(R) = \max_{\|Z\|=R}\|W(Z)\|$. The inequalities $m(R)\leqq M(R) < m(R)(\ln m(R))^{1/2+\varepsilon}$ hold for $\varepsilon>0$ except for a set of R-values with finite logarithmic measure. The first inequality sign would be wrong if $m(R)$ were defined by $\|A_k\|\,\|Z\|^k$ instead of $\|A_k Z^k\|$. Finally, a generalization of Hadamard's three-circle theorem is derived. In § 3 the author considers a system of partial differential equations

$$P_k\left(z_1,\cdots,z_n; w_1,\cdots,w_m; \frac{\partial w_1}{\partial z_1},\cdots,\frac{\partial^q w_j}{\partial z_1^{\alpha_1}\cdots\partial z_n^{\alpha_n}},\cdots\right) = Q_k(w_1,\cdots,w_m),\quad k=1,\cdots,m,$$

where $P_1, \cdots, P_m$ are polynomials and $Q_1, \cdots, Q_m$ are entire functions. The equations can be written as one vectorial equation with an entire function $Q(W) = \sum A_k W^k$ on the right side. The author proves that the system will have no entire solutions if a term $A_j W^j$, where j is larger than the degree with respect to W and the partial derivatives of the polynomial on the left side satisfy the condition $\|A_j W^j\| > 0$ for every W with $\|W\| = 1$. *H. Tornehave* (Copenhagen)

Aleksandrov, I. A. 273

Some estimates for holomorphic functions of several complex variables. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 3–13.

The author gives elementary proofs for a number of simple theorems concerning estimates of the coefficients in a power series in several variables and of the sum and its derivatives. The following are typical examples: If a power series $F(z_1, \cdots, z_n) = \sum c_{k_1, \cdots, k_n} z_1^{k_1} \cdots z_n^{k_n}$ satisfies the condition $|F| \leqq 1$ in a Reinhardt domain D, then

$$|c_{k_1, \cdots, k_n}| \leqq (1 - |c_{0, \cdots, 0}|^2)(\sup_D |z_1|^{k_1} \cdots |z_n|^{k_n})^{-1};$$

if F is holomorphic in the unit polycylinder and $|F| \leqq 1$, then

$$\left|\frac{\partial}{\partial z_p} F(z_1, \cdots, z_n)\right| \leqq (1 - |F(z_1, \cdots, z_n)|^2)(1 - |z_p|^2)^{-1}.$$

H. Tornehave (Copenhagen)

Alibekov, G. A. 274

Generalization of some results related to the Abel-Gončarov problem to the case of functions of several complex variables. (Russian. Azerbaijani summary)

Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Tehn. Mat. Nauk **1964**, *no.* 1, 13–24.

The author proves the following generalization of a theorem on convergence of Abel-Gončarov interpolation series [cf. M. A. Evgrafov, *The Abel-Gončarov interpolation problem* (Russian), GITTL, Moscow, 1954; MR **16**, 1104]: Let α_{pq}, $p = 1, \cdots, k$; $q = 0, 1, 2, \cdots$, be complex numbers, let $n_p(r)$ for $p = 1, \cdots, k$; $0 \leqq r < \infty$ denote the number of elements in the set $\{q \mid |\alpha_{p1} - \alpha_{p0}| + \cdots + |\alpha_{pq} - \alpha_{pq-1}| \leqq r\}$, let $f(z_1, \cdots, z_k)$ be an entire function. If for every r there exist a number θ satisfying $0 < \theta < \frac{1}{2}$ and a number $C(\theta) < \ln \theta^{-1}(1 - \theta)$ such that

$$|f(z_1, \cdots, z_k)| \leqq C(\theta) \min \{n_1(\theta r), \cdots, n_k(\theta r)\}$$

for every $(z_1, \cdots, z_k)$ in the r-polycylinder, then f possesses a development in an everywhere convergent interpolation series

$$f(z_1, \cdots, z_k) = \sum \frac{\partial^{q_1 + \cdots + q_k}}{\partial z_1^{q_1} \ldots \partial z_k^{q_k}} f(\alpha_{1q_1}, \cdots, \alpha_{kq_k}) P_{q_1}(z_1) \cdots P_{q_k}(z_k),$$

where

$$P_q(z) = \int_{\alpha_0}^{z} \int_{\alpha_1}^{z'} \cdots \int_{\alpha_{q-1}}^{z^{(q-1)}} dz^{(q)} dz^{(q-1)} \ldots dz'.$$

H. Tornehave (Copenhagen)

Kuhlmann, Norbert 275

Über die Auflösung der Singularitäten 3-dimensionaler komplexer Räume. I, II.

Math. Ann. **151** (1963), 304–331; *ibid.* **154** (1964), 387–405.

Let X be a complex analytic space [as defined for instance in Grauert and Remmert, Math. Ann. **136** (1958), 245–318; MR **21** #2063]. The problem of resolution of singularities is that of finding a complex analytic manifold X^* together with a proper holomorphic mapping $\pi: X^* \to X$ such that, for some analytic subvariety $S \subset X$ which contains the singular locus of X and whose complement $X - S$ is dense in X, the set $\pi^{-1}(S) = S^*$ is an analytic subvariety of X^* whose complement $X^* - S^*$ is dense in X^*, and the mapping $\pi: X^* - S^* \to X - S$ is an analytic isomorphism; the space X^* is called a proper analytic modification of X. This problem had already been solved for normal complex analytic spaces of dimensions 1 and 2; 1-dimensional normal analytic spaces are necessarily non-singular, and 2-dimensional normal analytic spaces were treated by Hirzebruch [Math. Ann. **126** (1953), 1–22; MR **16**, 26]. There is, of course, an extensive literature on the analogous problem in algebraic geometry. The present papers solve the problem of resolution of singularities for 3-dimensional normal analytic spaces which are analytic covers of 3-dimensional complex manifolds; an analytic cover is as defined in Grauert and Remmert [loc. cit.]. The solution is carried out in two steps. The first paper solves the problem when the branch locus of the covering is of a simple standard form, namely, when the singularities of the branch locus are ordinary double or triple points; this problem can be handled locally in all dimensions, but the global form appears to use the fact that X is 3-dimensional in a critical way. The second paper contains a proof of the fact that any analytic cover can be reduced to this standard form, by means of a proper analytic modification; this step consists in simplifying the singularities of an imbedded 2-dimensional analytic subvariety of a 3-dimensional manifold by proper modification of the ambient 3-dimensional manifold, and follows a method used by Zariski in the algebraic case [Ann. of Math. (2) **45** (1944), 472–542; MR **6**, 102].

This same problem has also been handled by Hironaka in a recent paper [ibid. (2) **79** (1964), 109–203; ibid. (2) **79** (1964), 205–326], and the reviewer is of course obliged to compare these two independent solutions. The results obtained by Hironaka are considerably more general, treating algebraic varieties over any field of characteristic zero as well as real and complex analytic spaces. Considering only the case of complex analytic spaces here, Hironaka solves the resolution of singularities problem for an arbitrary 3-dimensional complex analytic space; and he also solves the problem for complex Stein spaces of arbitrary dimension, and for arbitrary analytic subvarieties of a product $C^n \times P^m$ of the n-dimensional complex number space and the m-dimensional complex projective space, thus completely handling the local question. Hironaka's solution, being the more general, is of course the more difficult to read; at present, indeed, it is probably accessible only to the most devoted students of the subject. For this reason, it is still quite useful to have the author's solution of the special case available for reference, since it requires only rather standard background in several complex variables to read. On the other hand, the methods used by the author do not generalize easily,

so for the general case one must still read Hironaka's papers. *R. C. Gunning* (Princeton, N.J.)

Norguet, F. 276
Intégrales de formes différentielles extérieures non fermées. (Généralisation de la série de Taylor).
Rend. Mat. e Appl. (5) **20** (1961), 355–372.

The space of differentiable p-chains on a manifold can, as for any function space, be considered as an infinite-dimensional manifold. Integration of a p-differential form on a p-chain defines a complex-valued function on this manifold. The author describes how to define a "Taylor's series" expansion for this function restricted to certain subsets of the space of chains in case the underlying manifold is complex, using the residue formalism of Leray [Bull. Soc. Math. France **87** (1959), 81–180; MR **23** #A3281]. As special cases, he obtains classical formulas of Poincaré and Stieltjes. *R. Hermann* (Evanston, Ill.)

Rossi, Hugo 277
Vector fields on analytic spaces.
Ann. of Math. (2) **78** (1963), 455–467.

This paper contains a clear and concise account of some fundamental properties of analytic vector fields and differential forms on complex analytic spaces with singularities. Of particular interest are some of the suggestive unsolved problems involving analytic differential forms, which are discussed here. As applications of this circle of ideas, the author proves the following theorems. If A is a connected normal Stein space of dimension at least two, K is a compact subset of A, and h is a holomorphic function on $A-K$, then there exists a unique holomorphic function H on A which coincides with h on the unbounded component of $A-K$. If A is a normal analytic subvariety in an open ball B in C^n, and if there exists an open neighborhood W of the boundary of B in C^n together with a holomorphic map $\pi: W\cap B\to W\cap B\cap A$ which is the identity map on $W\cap B\cap A$, then A has no singularities; the converse assertion had been proved earlier by Docquier and Grauert [Math. Ann. **140** (1960), 94–123; MR **26** #6435]. *R. C. Gunning* (Princeton, N.J.)

SPECIAL FUNCTIONS

See also 40, 191, 194, 416, 431, 432, 435.

★Handbook of mathematical tables. 278
First edition. Supplement to "Handbook of Chemistry and Physics".
Chemical Rubber Publishing Co., Cleveland, Ohio, 1962. x+579 *pp.* $7.50.

This is the latest version of a very popular American collection: it has something for almost everybody, the numerical tables being usually to 5D. The printing and binding seem very satisfactory. For more detailed reviews see [MTAC **12** (1958), 146–147 and Math. Comp. **17** (1963), 303–304]. *John Todd* (Pasadena, Calif.)

Bhowmick, K. N. 279
Some integrals involving a generalized Struve's function.
J. Sci. Res. Banaras Hindu Univ. **13** (1962/63), 244–251.

The author determines various integrals involving the generalized Struve function $H_\nu{}^\lambda(z)$ defined by means of

$$H_\nu{}^\lambda(z)=\sum_{r=0}^{\infty}\frac{(-1)^r(\tfrac{1}{2})^{\nu+2r+1}}{\Gamma(r+\tfrac{3}{2})\Gamma(\nu+\lambda r+\tfrac{3}{2})}.$$

W. A. Al-Salam (Durham, N.C.)

Agarwal, R. P. 280
A note on evaluation of the integral $\int_0^\infty e^{-kt}I_0{}^n(t)\,dt$.
Proc. Edinburgh Math. Soc. (2) **14** (1964/65), 85–86.

The author proves that

$$\int_0^\infty e^{-3t}I_0{}^3(t)\,dt=\tfrac{1}{3}F_C(\tfrac{1}{2},1;1,1,1;\tfrac{1}{9},\tfrac{1}{9},\tfrac{1}{9})$$
$$=(18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6})\left(\frac{2K_2}{\pi}\right)^2,$$

where K_2 is the complete elliptic integral with modulus $(2-\sqrt{3})(\sqrt{3}-\sqrt{2})$ and F_C is one of Lauricella's hypergeometric functions in three variables.
W. A. Al-Salam (Durham, N.C.)

Krätzel, Ekkehard 281
Umformung mehrfach-unendlicher Reihen mit Anwendungen insbesondere auf die Thetareihen. (Romanian and Russian summaries)
An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I (N.S.) **8** (1962), 57–64.

Gegenstand dieser Abhandlung ist die Umformung mehrfachunendlicher Reihen

$$(1)\qquad \sum_{n_i=-\infty}^{+\infty} f(n_1,n_2,\cdots,n_r)\quad (r\geqq 2).$$

Hierzu denke man sich in einem r-dimensionalen Raum ein rechtwinkliches Koordinatensystem gegeben, auf dessen Achsen die ganzen Zahlen $n_1, n_2, \cdots, n_r$ abgetragen sind. Durch Drehung dieses Koordinatensystems wird bei gewissen Voraussetzungen eine Umformung von (1) erreicht. Die Menge der Gitterpunkte zerfällt in eine endlich Anzahl von Teilmengen, über die dann einzeln summiert wird, wodurch eine Zerlegung von (1) in endlich viele Teilsummen erreicht wird. Von van der Pol wurde der Spezialfall zweifach-unendlicher Reihen mit einer Drehung des Koordinatensystems um $\frac{1}{4}\pi$ betrachtet [B. van der Pol, Enseignement Math. (2) **1** (1956), 258–261; MR **17**, 1082; *Selected scientific papers*, Vol. 2, North-Holland, Amsterdam, 1960; MR **22** #10898]. Das allgemeine Ergebnis ist:

$$(2)\qquad \sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} f(m,n)=\sum_{\beta_i}\sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} f(ak-bl+\beta_1,\,bk+al+\beta_2).$$

Hier sind a und b ganze Zahlen, $(a,b)=1$. In der ersten Summe auf der rechten Seite von (2) ist über diejenigen β_1, β_2 zu summieren, die sich aus den Beziehungen $\beta_1=-(a\alpha_1-b\alpha_2)/(a^2+b^2)$, $\beta_2=-(b\alpha_1+a\alpha_2)/(a^2+b^2)$ ergeben, wenn α_1 und α_2 alle Lösungen der Kongruenz $a\alpha_1-b\alpha_2\equiv 0$ [mod (a^2+b^2)] durchlaufen. Spezialfall $a=b=1$:

$$(2')\qquad \sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} f(m,n)=\sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} f(k-l,k+l)+\sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} f(k-l,k+l-1).$$

Anwendungen: (I)

$$\sum_{n^2>m^2} q^{n^2-m^2} = 2\sum_1^\infty{}_k \sum_1^\infty{}_l \{q^{4kl}+q^{(2k-1)(2l-1)}\}$$

(Zahlentheoretisches Resultat). (II) Jacobische Thetafunktionen. Völlig elementare Herleitung der Differentialgleichung $\theta_1' = \pi\theta_0\theta_2\theta_3$ (abgesehen vom letzten Schritt, bei dem noch eine Integration notwendig wird). (III) Beweis der Beziehung $\pi^2\theta_3^4 = \theta_0''/\theta_0 - \theta_2''/\theta_2$ woraus bekanntlich eine Aussage über die Anzahl der Darstellungen von natürlichen Zahlen als Summe von vier Quadraten gefunden wird. Diese Betrachtungen werden vom Verfasser auf beliebigen Drehungen des r-dimensionalen Raumes ausgedehnt, und angewendet auf der Ableitung der Jacobischen Fundamentalformel der Thetafunktionen

$$\prod_{k=1}^4 \theta_3(v_k,\tau)+\prod_{k=1}^4 \theta_2(v_k,\tau) = \prod_{k=1}^4 \theta_3(w_k,\tau)+\prod_{k=1}^4 \theta_2(w_k,\tau),$$

$$2w_1 = v_1+v_2+v_3+v_4,$$
$$2w_2 = v_1+v_2-v_3-v_4,$$
$$2w_3 = v_1-v_2+v_3-v_4,$$
$$2w_4 = v_1-v_2-v_3+v_4.$$

S. C. van Veen (Delft)

Edelstein, L. A. 282a

On the one-centre expansion of Meijer's G-function.

Proc. Cambridge Philos. Soc. **60** (1964), 533–538.

Edelstein, L. A. 282b

On the one-region two-centre expansions of a generalized Coulomb field in spherical polar and parabolic coordinates.

Proc. Cambridge Philos. Soc. **60** (1964), 543–546.

The author presents a heuristic derivation of the addition formula

$$G^{m,n}_{p,q}\left[x+y\,\middle|\,\begin{matrix}a_1,\cdots,a_p\\ b_1,\cdots,b_q\end{matrix}\right] = G^{m,n}_{p,q}\left[x\,\middle|\,\begin{matrix}a_1,\cdots,a_p\\ b_1,\cdots,b_q\end{matrix}\right] + \sum_{t=1}^\infty \left(\frac{y}{x}\right)^t \frac{1}{t!}\, G^{m,n+1}_{p+1,q+1}\left[x\,\middle|\,\begin{matrix}0,\,a_1,\cdots,a_p\\ b_1,\cdots,b_q,\,t\end{matrix}\right]$$

for Meijer's G-functions "provided the G-functions exist, and the series converges". He illustrates this formula with a number of special cases of particular interest for molecular quantum mechanics, and presents some computer results tending to support the view that his methods may have some advantages over alternate expansion methods. Proofs would, of course, be desirable.

T. Erber (Chicago, Ill.)

Du Val, P. 283

On certain elliptic functions of order three.

Acta Arith. **9** (1964), 133–147.

The author discusses the parametrization of the general plane cubic curve in terms of homogeneous coordinates (x, y, z)

(1) $$x^3+y^3+z^3+6mxyz = 0$$

by means of certain elliptic functions of order 3. Precisely, the author introduces the following elliptic functions of order 3:

$$g_i(u) = \zeta_i(u)+\varepsilon\zeta_i(u-\tfrac{2}{3}\omega_i)+\varepsilon^2\zeta_i(u+\tfrac{2}{3}\omega_i),$$

$$i = 1, 2, 3, 4;\quad \varepsilon = \exp(\tfrac{1}{3}2\pi\sqrt{-1}),\quad \zeta_i(u) = \zeta(u)-\frac{\eta_i}{\omega_i}u,$$

where ζ is the Weierstrass zeta function with fundamental periods $2\omega_1$, $2\omega_2$ having quasi-periodicity

$$\zeta(u+2\omega_i) = \zeta(u)+2\eta_i,\qquad i = 1, 2,$$

and $\omega_3 = -\omega_1-\omega_2$, $\omega_4 = \omega_1-\omega_2$, $\eta_3 = -\eta_1-\eta_2$, $\eta_4 = \eta_1-\eta_2$. Various properties of the functions $g_i(u)$ are discussed and finally the curve (1) is parametrized as

$$x:y:z = g_i(u):g_i(-u):-b_i$$

with suitable choice of constants b_i, ω_i, etc. A theorem on a unirational transformation of the curve (1) is attached.

S. Hitotumatu (Tokyo)

Watanabe, Yoshikatsu 284

Zur einigen Integral-Abschätzung.

J. Gakugei Tokushima Univ. **13** (1962), 49–51.

The following integral formula is established:

$$\int_0^1 \frac{\tan^{-1}\sqrt{(\alpha(1-\zeta^2))}}{1+\beta\zeta^2}\,d\zeta = \frac{\pi}{2}-\frac{1}{\sqrt{\beta}}\left[\tan^{-1}\frac{\sqrt{(\beta(1+\alpha))}}{\alpha}-\tan^{-1}\frac{\sqrt{\beta}}{\alpha(1+\beta)}\right]$$

where, for $\sqrt{\tfrac{2}{3}}<t<2$, $\alpha = 3t^2/2-1$, $\beta = (3t^2-2)/5t^2$.

A. E. Danese (Buffalo, N.Y.)

Ragab, F. M. 285

The inverse Laplace transform of the product

$$\exp[-\tfrac{1}{2}a^{1/n}p^{1/n}]K_{n\nu}[\tfrac{1}{2}a^{1/n}p^{1/n}],\qquad n = 2, 3, 4, \cdots.$$

Boll. Un. Mat. Ital. (3) **19** (1964), 26–30.

The Laplace transform of

$$\sum_{i,-i}\frac{1}{i}\,\mathbf{E}\left[\Delta(n;n\nu),\Delta(n;-n\nu):\Delta(n;1/2):\frac{ae^{i\pi}}{n^nt}\right]$$

is shown to be

$$2^{(n+1)/2}\pi^{n/2}p^{-1}\exp[-a^{1/n}p^{1/n}/2]K_{n\nu}[a^{1/n}p^{1/n}/2],$$

where n is any positive integer >1, $\operatorname{Re}a>0$, $\operatorname{Re}p>0$, $\Delta(n;\alpha)$ represents the set of parameters α/n, $(\alpha+1)/n$, $\cdots$, $(\alpha+n-1)/n$, $\sum_{i,-i}$ means that in the expression following it i is to be replaced by $-i$ and the two expressions are then added, the function $\mathbf{E}$ is MacRobert's $\mathbf{E}$-function, and K is the modified Bessel function of the second kind.

A. E. Danese (Buffalo, N.Y.)

Kane, J. 286

A note on the quadrature of some exponential transforms of cylindrical functions.

J. Math. and Phys. **43** (1964), 140–143.

The author points out that the indefinite integral of $e^{\pm iz}z^mC_n$, where m and n are integers and C_n is any suitably normalized cylindrical function satisfying Bessel's equation, may be expressed as a finite combination of the two cylindrical functions C_0 and C_1. Such indefinite integrals

are tabulated for $n=0, 1$ $(m=1, \cdots, 4)$, and recurrence relations are given to extend these results to arbitrary m for $n=0, 1$.

Similar results for higher-order cylindrical functions are included and, in addition, there are brief remarks concerning related integrals and use of the formulas.

R. A. Valentin (Providence, R.I.)

Judina, A. S. 287

Cylindrical functions of two variables. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 222–231.

Von mehreren der in demselben Z. **1** (1961), 571–589 [MR **25** #1307] von Kuznecov und Verfasser [ibid. **1** (1961), 1099–1104; MR **25** #5199] aufgestellten asymptotischen Entwicklungen gewisser Zylinderfunktionen von zwei reellen Veränderlichen wird bewiesen, daß sie auch für komplexe Veränderliche richtig sind. Anschließend wird gezeigt, daß zweidimensionale Zylinderfunktionen beispielsweise bei der Lösung eines Systems von Telegraphengleichungen mit passenden Nebenbedingungen auftreten, die mit Hilfe der Laplace-Transformation durchgeführt wird.

L. Berg (Halle)

Deutsch, E. 288

Evaluation of certain integrals involving Bessel functions.

Proc. Edinburgh Math. Soc. (2) **13** (1962)/63), 285–290.

The integral

$$\left(\frac{\pi a^3}{2}\right)^{1/2}\int_0^\infty t^{m-1/2}e^{-tz}J_{3/2}(at)J_n(rt)\,dt,$$

in which a, r and z are real and positive, is evaluated in closed form in terms of elementary functions for various combinations of integer values of m and n in the ranges $-2\leq m\leq 2$, $0\leq n\leq 3$. Recurrence relations in the (m, n)-plane are also given.

F. W. J. Olver (Washington, D.C.)

Czarnecki, Adam 289

A note on the zeros of Bessel functions.

Amer. Math. Monthly **71** (1964), 403–404.

In Pierpont's *Functions of a complex variable* [Ginn, Boston, 1914; Dover, New York, 1959; MR **21** #4224] appears an invalid proof of the fact that the Bessel function $J_n(x)$, n real, has an infinite number of positive zeros. The "proof" is amended so as to be valid.

A. E. Danese (Buffalo, N.Y.)

Osipov, S. I. 290

Expansion of a polynomial of Bessel operators. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 149–155.

A Bessel operator is $B_k=(1/t)(tD)^k$, where $D=d/dt$. If P_n is a polynomial of degree n, the author gives explicit expressions for the coefficients of the powers of D in the polynomial $P_n(B_k)$. This is closely related to earlier work of the author [same Ž. **3** (1963), 191–194; MR **27** #6093].

A. Devinatz (St. Louis, Mo.)

Glasser, M. L. 291

A note on the integral $\int_0^1 s^\nu(1-s)^\nu|\sin xs|\,ds$ with an application to Schlomilch series.

J. Math. and Phys. **43** (1964), 158–162.

Closed expressions are obtained for $I_\nu(x)$—this denoting the integral in the title—for $0\leq x\leq\pi$ and $x=2\pi$, and the asymptotic behavior for large x is derived. In addition, numerical results are presented for various fractional values of ν when $x=n\pi$ $(n=1, \cdots, 10)$, and the function's general behavior is illustrated graphically. In obtaining an expression for $I_\nu(x)$ for arbitrary x, certain new formulas involving series of Bessel functions of fixed order are derived and special cases of these formulas are related to previously known series.

R. A. Valentin (Providence, R.I.)

Srivastava, K. N. 292

Inversion integrals involving Jacobi's polynomials.

Proc. Amer. Math. Soc. **15** (1964), 635–638.

Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial of order n with parameters α and β, let k and n be integers such that $0<k<n$, and let $0<\beta<1$. It is shown that the solution of

$$\int_x^1 F_n^{(k,\beta)}(t/x)g(t)\,dt = f(x),$$

where

$$F_n^{(k,\beta)}(x) = \frac{n!}{2^{k-1/2}\Gamma(k-\beta+n+1)}\,x(x^2-1)^{k-\beta}P_n^{(k-\beta,\beta)}(2x^2-1),$$

$f^{(m)}(1)=0$, $0\leq m\leq 2k$, $f^{(2k+1)}(x)$ piecewise continuous in $0<a\leq x\leq 1$, is

$$\int_t^1 G_n^{(k,\beta)}(t/y)y^{-2n+2k+2\beta-1}\left(-y^{-1}\frac{d}{dy}\right)^{2k+1}\times(y^{2n+2k-2\beta+1}f(y))\,dy,$$

where

$$G_n^{(k,\beta)}(x)=\frac{\Gamma(n-k+1)}{2^{k-1/2}\Gamma(n+\beta)}(1-x^2)^{k+\beta-1}P_{n-k}^{(k+\beta-1,-\beta)}(2x^2-1).$$

The solution for a similar integral equation is also indicated.

A. E. Danese (Buffalo, N.Y.)

Szász, Paul 293

On a sum concerning the zeros of the Jacobi polynomials with application to the theory of generalized quasi-step parabolas.

Monatsh. Math. **68** (1964), 167–174.

Put $\omega_n(x)=P_n^{(\alpha,\beta)}(x)$, the Jacobi polynomial of degree n. Assume $\alpha>-1$, $\beta>-1$. Let $x_{1n}, x_{2n}, \cdots, x_{nn}$ denote the roots of $\omega_n(x)$ and put

$$s_n=\sum_{k=1}^n\frac{1}{(1-x_{kn}^2)(\omega_n'(x_{kn}))^2}.$$

The author shows that

$$s_1=\frac{1}{(\alpha+1)(\beta+1)},\quad s_n=\frac{1}{(\alpha+1)(\beta+1)}\prod_{k=2}^n\left\{1-\frac{\alpha\beta}{(\alpha+k)(\beta+k)}\right\},$$

$$s_n\to\frac{1}{2^{\alpha+\beta+1}}\int_{-1}^1(1-x)^\alpha(1+x)^\beta\,dx\qquad(n\to\infty).$$

He then proves the following theorem. Let $0\leq\alpha<1$, $0\leq\beta<1$ and let $y_{1n}', y_{2n}', \cdots, y_{nn}'$ $(n=1, 2, 3, \cdots)$ be numbers such that $|y_{kn}'|\leq\Delta$, where Δ is independent of n and k. Let $f(x)$ be a function continuous in the interval

$-1 \leqq x \leqq 1$ and $S_n(x)$ a polynomial of degree $2n+1$ such that

$$(*)\qquad S_n(-1) = f(-1), \qquad S_n(1) = f(1), \qquad S_n(x_{kn}) = f(x_{kn}), \qquad S_n'(x_{kn}) = y_{kn}' \qquad (k = 1, 2, \cdots, n;\ n = 1, 2, 3, \cdots).$$

Then

$$S_n(x) \to f(x) \qquad (n \to \infty),$$

the convergence being uniform in each subinterval $-1+\delta \leqq x \leqq 1-\delta$ $(0<\delta<1)$. In particular, if $\alpha=\beta$ the convergence is uniform in the entire closed interval.

In an earlier paper [Acta Math. Acad. Sci. Hungar. **10** (1959), 413–419; MR **22** #3910], the author called a polynomial $S_n(x)$ of degree $2n+1$ satisfying (*) a generalized quasi-step polynomial. In this paper the theorem quoted was proved for the case of the Legendre polynomial and for the Chebyshev polynomial of the second kind.

L. Carlitz (Durham, N.C.)

Parodi, Maurice 294

Sur quelques propriétés des zéros des polynomes d'Hermite et de Legendre.

C. R. Acad. Sci. Paris **258** (1964), 5563–5565.

It is proved that if α and $\alpha+(\pi/\sqrt{2n})$ are in $[-1, 1]$, then there exists at least one zero of the Hermite polynomial $H_n(x)$ between α and $\alpha+(\pi/\sqrt{(2n)})$; if α and $\alpha+(\pi/\sqrt{(n(n+1))})$ are in $[-1, 1]$, then there exists at least one zero of the Legendre polynomial $P_n(x)$ between these values.

A. E. Danese (Buffalo, N.Y.)

Nörlund, N. E. 295

Sur les valeurs asymptotiques des nombres et des polynômes de Bernoulli.

Rend. Circ. Mat. Palermo (2) **10** (1961), 27–44.

Les polynômes de Bernoulli d'ordre ρ, soit $B_\nu^{(\rho)}(x)$, sont définis par la fonction génératrice

$$(1)\qquad \frac{t^\rho e^{xt}}{(e^t-1)^\rho} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu^{(\rho)}(x), \qquad |t| < 2\pi,$$

ρ étant un nombre quelconque. Pour $x=0$, ils se réduisent aux quantités $B_\nu^{(\rho)}$ définies par

$$(2)\qquad \left(\frac{t}{e^t-1}\right)^\rho = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu^{(\rho)}$$

(polynômes en ρ qui coïncident pour $\rho=1$ avec les nombres de Bernoulli classiques B_ν et qui comprennent comme cas particuliers les nombres de Stirling de première et de deuxième espèce. L'auteur se demande comment les polynômes de Bernoulli se comportent asymptotiquement. Il y a deux cas à distinguer.

(I) Comment se comportent les $B_\nu^{(\rho+\nu)}(x)$ quand ν tend vers l'infini, ρ et x ayant des valeurs finies quelconques? Les résultats principaux sont

$$(3)\qquad \frac{(-1)^\nu B_\nu^{(\rho+\nu+1)}(x)}{\nu!} = \nu^{-x}\left[\sum_{s=0}^{n-1}(-1)^s\binom{\rho}{s}(\log\nu)^{\rho-s}\frac{d^s}{dx^s}\frac{1}{\Gamma(1-x)} + O\{|(\log\nu)^{\rho-n}|\}\right].$$

On peut introduire les coefficients γ_s définis par [le] développement de la fonction entière $1/\Gamma(x)$ [=] $\sum_{s=0}^{\infty}\gamma_s x^{s+1}$ ($\gamma_0=1$; γ_1 est la constante d'Euler) donc

$$\gamma_s = \frac{(-1)^s}{s!}\left(\frac{d^s}{dx^s}\frac{1}{\Gamma(1-x)}\right)_{x=0}.$$

En posant $x=0$ dans (3) on obtient

$$(4)\qquad (-1)^\nu\frac{B_\nu^{(\rho+\nu+1)}}{\nu!} = \sum_{s=0}^{n-1}\rho(\rho-1)\cdots(\rho-s+1)\gamma_s(\log\nu)^{\rho-s}+O\{|(\log\nu)^{\rho-n}|$$

(II) L'auteur considère ensuite les polynômes $B_\nu^{(\rho)}(x)$ [en] admettant que x et l'indice supérieur aient des valeu[rs] finies quelconques pendant que ν augmente indéfinimen[t]. En tenant compte d'un théorème dû à O. Perr[on] [S.-B. Heidelberger Akad. Wiss. Math.-Natur. Kl. **191** Abt. A, Abh. 9, p. 7 en particulier] l'auteur obtient

$$\frac{B_\nu^{(\rho)}(x)}{\nu!} = \frac{2\cos\pi(2x+\rho-\frac{1}{2}\nu)}{\Gamma(\rho)(2\pi)^\nu}\{\nu^{\rho-1}+O(|\nu^{\rho-2}|)\}.$$

En particulier

$$B_{2\nu}^{(\rho)}(x) \sim \frac{(2\nu)!(-1)^\nu(2\nu)^{\rho-1}2\cos\pi(2x+\rho)}{\Gamma(\rho)(2\pi)^{2\nu}},$$

$$B_{2\nu+1}^{(\rho)}(x) \sim \frac{(2\nu+1)!(-1)^\nu(2\nu+1)^{\rho-1}2\sin\pi(2x+\rho)}{\Gamma(\rho)(2\pi)^{2\nu+1}}.$$

Dans le cas trivial n entier non négatif, $B_\nu^{(-n)}(x)$ [=] $(\nu!/(\nu+n)!)\Delta^n x^{\nu+n}$. À la fin du mémoire l'auteur don[ne] plusieurs exemples où interviennent les polynômes [de] Bernoulli, par exemple des expressions pour le rapp[ort] entre deux fonctions gamma, représenté asymptotiqu[e]ment dans l'angle $|\arg x|<\pi$. Ces séries sont des c[as] particuliers d'un développement plus général obtenu p[ar] Tricomi et Erdélyi [Pacific J. Math. **1** (1951), 133–14[...] MR **13**, 343]

$$\frac{\Gamma(x+\alpha)}{\Gamma(x+\beta)} \sim \sum_{\nu=0}^{\infty}(-1)^\nu B_\nu^{(\alpha-\beta+1)}(\alpha)\frac{(\beta-\alpha)_\nu}{\nu!}x^{\alpha-\beta-\nu}.$$

S. C. van Veen (Del[ft])

ORDINARY DIFFERENTIAL EQUATIONS

See also 272, 443, 445, 491, 493, 494, 495, 496, 500, 561, 736, 739, 766, 787, 1091, 1094, 1101, 1104.

Aliev, F. S. 2[96]

General solution of the Euler equation in generaliz[ed] functions. (Russian. Azerbaijani summary)

Akad. Nauk Azerbaĭdžan. SSR Dokl. **20** (1964), *no.* [...] 9–13.

The author considers the Euler equation

$$Ly \equiv x^n y^{(n)}(x)+a_1x^{n-1}y^{(n-1)}(x)+\cdots+a_ny(x) = 0$$

and obtains its fundamental system of solutions in ter[ms] of generalized functions.

V. Lakshmikantham (Aurangaba[d])

Bureau, Florent 297

Sur les équations différentielles dont l'intégrale générale est à points critiques fixes.

C. R. Acad. Sci. Paris **257** (1963), 1215–1217.

Let c_i, $i=1,\cdots,p$, be constants, $k>0$ an integer, $P(x;z)$ a polynomial in z of degree $\leqq k-2$, $H_i(x;z;u)$, $i=1,2,\cdots$, polynomials in $u,\dot{u},\cdots,u^{(p-1)}$. It is also assumed that $h(x;z)$, $H_i(x;z;u)$ are holomorphic functions of z at $z=0$ and that $P(x;z)$, $h(x;z)$, $H_i(x;z;u)$ are analytic in x on a certain domain D. The author states without proof the following theorem: In order that the general integral of the differential system

$$\dot{z} = 1+zP(x;z)+z^k u,$$

$$z^r\frac{d^r u}{dt^r}+c_1 z^{r-1}\frac{d^{r-1}u}{dt^{r-1}}+\cdots+c_{r-1}z\frac{du}{dt}+c_r u = h(x;z)+zH_1(x;z;u)$$

have all its critical points fixed, it is necessary that (i) the roots of the determinantal equation

$$\theta(\theta-1)\cdots(\theta-r+1)+c_1\theta(\theta-1)\cdots(\theta-r+2)+\cdots+c_{r-1}\theta+c_r = 0$$

be integers, no two of which are equal; (ii) the fact that one of these roots be zero imply that $h(x;0)\equiv 0$.

M. M. Peixoto (Providence, R.I.)

Roos, B. W. 298

On a class of singular second order differential equations with a non linear parameter.

Pacific J. Math. **14** (1964), 285–294.

The system of equations

$$x_1'(r)-(\lambda+1)E^{-1}(r)x_2(r) = 0,$$

$$x_2'(r)+[\lambda g(r)+k(r)]E(r)x_1(r) = 0,$$

where $E(r)=\exp[\int_0^r f(r)\,dr]$, can be shown by differentiation and substitution to be equivalent to the second-order equation

$$x''(r)+f(r)x'(r)+[\lambda^2 g(r)+\lambda h(r)+k(r)]x(r) = 0,$$

where $h(r)=g(r)+k(r)$. This paper obtains some asymptotic properties for this type of second-order equation with a non-linear parameter λ for a particular singular case over the interval $[0,\infty]$. The methods used are those followed by the author and reviewer in papers investigating spectral and asymptotic properties for the system of two first-order equations. The main result is a spectral theorem which gives a solution and a λ in the continuous spectrum.

W. Sangren (San Diego, Calif.)

Levitan, B. M.; Gasymov, M. G. 299

Determination of a differential equation by two spectra. (Russian)

Uspehi Mat. Nauk **19** (1964), *no.* 2 (116), 3–63.

A survey is given, with some new and simplified proofs, of results obtained by the authors and others on the following problem (the inverse Sturm-Liouville problem): What are necessary and sufficient conditions that two given sequences of real numbers $\{\lambda_n\}$ and $\{\mu_n\}$ be two spectra of a Sturm-Liouville operator $-d^2/dx^2+q(x)$ $(0\leqq x\leqq b\leqq\infty)$ for different boundary conditions, and how can one practically construct q and the boundary conditions? A complete solution of the problem is given in the regular case, and a solution for a particular class of potentials q is given in the singular case.

In the Introduction the history of the problem is briefly reviewed. Chapter 1 is concerned with the determination of a differential operator by its spectral function. Let $\rho(\lambda)$ be a monotone increasing function. It is shown that ρ is a spectral function of a certain problem

$$\text{(P)}\quad -y''+q(x)y = \lambda y,\quad y'(0)-hy(0) = 0,\quad 0\leqq x<\infty,$$

where q has m locally summable derivatives and where h is a real number, if and only if the following conditions are fulfilled: (a) if f is any finite function from $L_2(0,\infty)$ and if $\int_{-\infty}^{\infty}E^2(\lambda)\,d\rho(\lambda)=0$, where $E(\lambda)=\int_0^\infty f(x)\cos\lambda^{1/2}x\,dx$, then $f(x)=0$ (a.e.); (b) if $\sigma(\lambda)=\rho(\lambda)$ for $\lambda<0$ and $\sigma(\lambda)=\rho(\lambda)-(2/\pi)\lambda^{1/2}$ for $\lambda\geqq 0$, then the limit $\Phi(x)=\lim_{N\to\infty}\int_{-\infty}^{N}\cos\lambda^{1/2}x\,d\sigma(\lambda)$ exists boundedly in any finite region of the variable x, Φ has $m+1$ locally summable derivatives, and $\Phi(0)=-h$. The proof involves solving for $K(x,t)$ the integral equation $F(x,t)+K(x,t)+\int_0^x K(x,s)F(s,t)\,ds=0$, where

$$F(x,t) = \lim_{N\to\infty}\int_{-\infty}^{N}\cos\lambda^{1/2}x\cos\lambda^{1/2}t\,d\sigma(\lambda).$$

Then h and q are determined by the equations $q(x)=2(d/dx)K(x,x)$, $h=K(0,0)$. Also used is the fact that the solution $\varphi(x,\lambda,h)$ of $-y''+q(x)y=\lambda y$, $y(0)=1$, $y'(0)=h$, is given by $\varphi(x,\lambda,h)=\cos\lambda^{1/2}x+\int_0^x K(x,t)\cos\lambda^{1/2}t\,dt$. The last section of Chapter 1 is concerned with the regular Sturm-Liouville problem

$$\text{(Q)}\quad \begin{aligned} -y''+q(x)y &= \lambda y,\quad y'(0)-hy(0) = 0,\\ y'(\pi)+Hy(\pi) &= 0.\end{aligned}$$

It is known that the spectral function of (Q) is determined by the eigenvalues $\{\lambda_n\}$ and the numbers $\alpha_n=\int_0^\pi\varphi^2(x,\lambda_n,h)\,dx$. It is proved that two given sequences $\lambda_0,\lambda_1,\cdots$ and $\alpha_0,\alpha_1,\cdots$ correspond to a problem (Q), where $q^{(m)}\in L_1(0,\pi)$, if and only if $\lambda_n\neq\lambda_m$ for $n\neq m$, $\alpha_n>0$ for all n, $\lambda_n^{1/2}=n+a_0/n+o(1/n)$, $\alpha_n=\pi/2+o(1/n)$, and the function $F(x,t)=(1/\alpha_0)\cos\lambda_0^{1/2}x\cos\lambda_0^{1/2}t-(1/\pi)+\sum_{n=1}^{\infty}[(1/\alpha_n)\cos\lambda_n^{1/2}x\cos\lambda_n^{1/2}t-(2/\pi)\cos nx\cos nt]$ has $m+1$ summable derivatives in $0\leqq x,t\leqq\pi$. Several theorems whose statements do not involve the function $F(x,t)$ are proved under the assumption that the asymptotic formulas for $\lambda_n^{1/2}$ and α_n are known more precisely.

Chapter 2 is dedicated to the determination of the regular Sturm-Liouville problem (Q) by two spectra. Let (Q_i) $(i=1,2)$ be the problem (Q) with $h=h_i$. Let $\lambda_0,\lambda_1,\cdots$ be the eigenvalues of (Q_1) and $\mu_0,\mu_1,\cdots$ the eigenvalues of (Q_2). If $\alpha_n=\int_0^\pi\varphi^2(x,\lambda_n,h_1)\,dx$, it is shown that

$$(+)\quad \alpha_n = (h_2-h_1)(\mu_n-\lambda_n)^{-1}\prod_{k=0}^{\infty}{}'[(\lambda_k-\lambda_n)(\mu_k-\lambda_n)^{-1}],$$

where the prime indicates omission of the term $k=n$. Using this formula, an asymptotic expansion for α_n is obtained from the asymptotic expansions of λ_n and μ_n. On the basis of the results of the last section of Chapter 1, then, the following theorem is proved: If $\lambda_0,\lambda_1,\cdots$ and $\mu_0,\mu_1,\cdots$ are two sequences of numbers which alternate and for which there are fulfilled the asymptotic formulas $\lambda_n^{1/2}=n+a_0/n+a_1/n^3+O(1/n^4)$ and $\mu_n^{1/2}=n+a_0'/n+a_1'/n^3+O(1/n^4)$, where $a_0\neq a_0'$, then there exists an

absolutely continuous function q and numbers h_1, h_2, H such that $\{\lambda_n\}$ is the spectrum of (Q_1) and $\{\mu_n\}$ is the spectrum of (Q_2).

Chapter 3 is concerned with the determination of a singular Sturm-Liouville operator by two spectra. Assume that $q(x)$ is a real, locally summable function on $0 \leqq x < \infty$ such that problem (P) has a discrete spectrum for any real number h. (If necessary, a boundary condition at infinity is given also.) Let (P_i) ($i=1, 2$) be the problem (P) with $h=h_i$. Let $\{\lambda_n(h_i)\}$ be the eigenvalues of (P_i), enumerated in the order of growth, and suppose $\alpha_n(h_i)=\int_0^\pi \varphi^2(x, \lambda_n(h_i), h_i)\, dx$. The analogue of equation (+) is then proved, and thus, using the results of Chapter 1, a conditional solution of the inverse problem is obtained, based upon the assumption that $\{\lambda_n(h_1)\}$ and $\{\lambda_n(h_2)\}$ are known to be eigenvalues of some problems (P_1) and (P_2). In order to obtain a complete solution of the inverse problem, the potential q is restricted to the class of functions $\Omega_{1/4}$ consisting of all functions q for which there exists a constant h_0 such that for the boundary problem (P) with $h=h_0$ the following are true: (1) there exists a constant $\alpha>0$ such that $\lambda_n(h_0)-\lambda_{n-1}(h_0) \geqq \alpha$; (2) if $\rho_0(\lambda)$ is the spectral function of (P) with $h=h_0$, then $\rho_0(\lambda)=(2/\pi)\lambda^{1/2}-h_0+O(\lambda^{-(1/4)-\delta})$ for large $\lambda>0$, where $\delta>0$;

$$(3)\quad \lim_{n\to\infty} (\{\rho_0[\lambda_{n+1}(h_0)]-\rho_0[\lambda_n(h_0)]\} \times \{[\lambda_{n+1}(h_0)]^{1/2}-[\lambda_n(h_0)]^{1/2}\}^{-1}) = 2/\pi.$$

(It is shown that if properties (1), (2), (3) are fulfilled for one number h_0, then they are fulfilled for all numbers h.) In particular, potentials of the form $q(x)=\varepsilon x^2+p(x)$, where $\varepsilon>0$ and $p(x)$ is a finite summable function, belong to $\Omega_{1/4}$. Supposing, for simplicity, that the spectrum of (P) is bounded below and using the theorem proved in Chapter 1, it is shown that two sequences $\{\lambda_n(h_1)\}$ and $\{\lambda_n(h_2)\}$ of alternating numbers are eigenvalues of problems (P_1) and (P_2) with a locally summable potential $q \in \Omega_{1/4}$ if and only if the functions

$$\sigma_i(\lambda) = (h_2-h_1)^{-1} \sum_{\lambda_n(h_i)<\lambda} \{\lambda_n(h_2)-\lambda_n(h_1)\} \qquad (i = 1, 2)$$

are spectral functions of two singular problems of type (P) with locally summable potentials from the class $\Omega_{1/4}$. At the end of Chapter 3 an "if and only if" version of the theorem of Chapter 2 is stated.

There are two appendices. In Appendix 1 there is given a proof of the theorem of V. A. Ambarcumjan that if $q \in L_1(0, \pi)$ and if the eigenvalues $\lambda_0, \lambda_1, \cdots$ of the boundary problem $-y''+q(x)y=sy$, $y'(0)=y'(\pi)=0$ coincide with the numbers n^2 ($n=0, 1, \cdots$), then $q(x)=0$ a.e. In Appendix 2 there is given a new derivation of the asymptotic formulas for the numbers $\lambda_n^{1/2}$ and α_n for problem (Q). The article concludes with remarks on the literature.

R. C. Gilbert (Fullerton, Calif.)

Kestens, Jean 300

Le problème naturel aux valeurs propres.

Acad. Roy. Belg. Cl. Sci. Mem. Coll. in-8° **32**, *no.* 5, xiv + 122 *pp.* (1961).

This paper is an exposition of the theory of eigenvalues of self-adjoint problems connected with ordinary differential operators and some methods for bounding these eigenvalues. Some care is taken to allow jump conditions in the interval, and to allow the eigenvalue to occur in the boundary conditions. There is a slightly more thorough discussion of the Stiefel-Ziegler method of Schwarz quotients than is usually found in such an exposition. In fact, the author appears to be quite partial to this method. He states in the preface that it is more powerful than the Rayleigh-Ritz method, since the latter just involves the same Rayleigh principle at each step, and so does not make any progress. But this can be said of any iterative method, including the Schwarz constant method, which can, after all, be considered as a special case of the Rayleigh-Ritz method.

H. F. Weinberger (Minneapolis, Minn.)

Sahnovič, L. A. 301

Some properties of the discrete spectrum of the radial Schrödinger equation. (Russian)

Dokl. Akad. Nauk SSSR **153** (1963), 286–289.

In this note the author considers the radial equation of Schrödinger

$$(1)\quad \frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left[2E+\frac{2A}{r}-\frac{l(l+1)}{r^2}-V(r)\right]R = 0,$$

$$R(0) = 0.$$

If $\int_0^\delta r|V(r)|\,dr<\infty$ and $\int_N^\infty |V(r)|\,dr<\infty$ for arbitrary $\delta>0$, $N>0$, then, as is well known, $\lim_{r\to\infty}[R(r)e^{-\varepsilon r}r^{A/\varepsilon+1}]=C(\varepsilon, l)$, where $\varepsilon=\sqrt{(-2E)}$, $\operatorname{Re}(\varepsilon)>0$, and $R(r)$ is the solution of (1) regular at the origin and so normalised that $\lim_{r\to 0} R(r)/r^l=1$. The author indicates some properties regarding the analytic character of $C(\varepsilon, l)$ in the ε-plane and l-plane under a certain assumption regarding $V(r)$. Some analytic properties of $E_k(l)$ (eigenvalues) are also stated in the case of a suitable form of $V(r)$. Conditions for a unique solution of the inverse problem, i.e., determination of $V(r)$, are also indicated.

P. K. Ghosh (Calcutta)

Kac, I. S. 302

Behaviour of spectral functions of differential systems with boundary conditions at a singular endpoint. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 34–37.

The eigenfunction expansion of the differential operator defined formally by $f(x)\to df'(x)/dM$ is studied in $L_M{}^2(-\infty, b)$ under the hypothesis that $M(x)$ is non-decreasing and integrable. The expansion is written as $f(x)\to F(\lambda)=\int_{-\infty}^b f(x)u(x, \lambda)\,dM(x)$, where $u(x, \lambda)$ is a particular choice of eigenfunction for the eigenvalue λ. A spectral function $\tau(x)$ is a non-decreasing function of real x such that the expansion is an isometry of $L_M{}^2(-\infty, b)$ into $L_\tau{}^2(-\infty, +\infty)$. The main result is that the condition $\int_{-\infty}^c (\int_{-\infty}^x M(s)\,ds)^{\alpha-1}\,dx<\infty$ for $-\infty<c<b$ characterizes the singular differential operator whose spectral functions satisfy $\int_1^\infty \lambda^{-\alpha-1}\tau(\lambda)\,d\lambda<\infty$ ($\alpha \geqq 1$). A general method is also given for constructing spectral functions from entire functions of genus 1 having given zeros, but the result is already contained in Theorem V of the reviewer's paper on Hilbert spaces of entire functions [Trans. Amer. Math. Soc. **99** (1961), 118–152; MR **24** #A3289b] where the terminology is, however, different. See also the study of symmetry in such spaces [the reviewer, Duke Math. J. **29** (1962), 383–392; MR **26**

#585] and the construction of such spaces from entire functions having given zeros [the reviewer, Canad J. Math. **15** (1963), 563–583; MR **27** #3801]. {The reviewer's papers do not properly acknowledge previous work of M. G. Kreĭn and the author because he did not read Russian at the time they were written.}

L. de Branges (Lafayette, Ind.)

Freĭ, T. [Frey, T.]; Obadovič, Ĭ. D'. [Obádovics, J. Gy.] 303

Several principal problems concerning the eigenvalues with respect to systems of differential equations. (Russian. English summary)

Acta Math. Acad. Sci. Hungar. **15** (1964), 1–28.

Soient

$$M[y] = \sum_{i=0}^{m} (-1)^i [F_i(x) y^{(i)}]^{(i)},$$

$$N[y] = \sum_{i=0}^{n} (-1)^i [G_i(x) y^{(i)}]^{(i)}$$

deux opérateurs différentiels, où F_i et G_i sont des matrices et y est un vecteur. On considère le système (S) $M[y] = \lambda N[y]$, dans lequel λ est un paramètre, avec les conditions aux limites

$$\text{(L)} \qquad U_j[y] = \sum_{k=0}^{2n-1} [A_{kj} y^{(k)}(a) + B_{kj} y^{(k)}(b)] = 0 \qquad (j = 1, 2, \cdots, 2m).$$

La condition $m > n$ est admise. Le but de ce travail est l'étude, sous différents aspects, du problème des valeurs propres du système (S), (L). On construit la matrice de Green à l'aide de laquelle le problème peut être abordé dans la manière classique. Il y a pourtant certaines différences par rapport au cas scalaire. Le travail contient encore quelques résultats intéressants concernant la théorie extrémale des valeurs propres, un très utile théorème de comparaison et l'exposé d'une méthode numérique de calcul des valeurs propres. Il faut noter que plusieurs problèmes de la technique conduisent au problème mathématique étudié par les auteurs [voir, par exemple, L. Collatz, *Eigenwertaufgaben mit technischen Anwendungen*, Akademische Verlagsgesellschaft, Leipzig, 1949; MR **11**, 137].

C. Corduneanu (Iași)

Maksudov, F. G. 304

Expansion in eigenfunctions of non-selfadjoint, singular, second-order differential operators depending on a parameter.

Dokl. Akad. Nauk SSSR **153** (1963), 1001–1004.

M. A. Naĭmark [Trudy Moskov. Mat. Obšč. **3** (1954), 181–270; MR **15**, 959; Amer. Math. Soc. Transl. (2) **16** (1960), 103–193; MR **22** #8162] investigated the spectrum and expansion in eigenfunctions of a singular differential operator L_θ generated in $L^2[0, \infty)$ by the differential expression $l(y) = -y'' + p(x)y$ and the boundary condition $y'(0) - \theta y(0) = 0$, where $p(x)$ is a complex-valued function defined on $[0, \infty)$ and θ is a complex number. A. V. Štraus [Izv. Akad. Nauk SSSR Ser. Mat. **20** (1956), 783–792; MR **19**, 277] considered the case in which p is real-valued but $\theta = \theta(\lambda)$ is a complex-valued function of the complex variable λ. In the present article p is complex-valued and also θ is a complex-valued function of a complex variable. The nature of the spectrum of $L_{\theta(\lambda)}$ is investigated and the resolvent $R_\lambda = (L_{\theta(\lambda)} - \lambda I)^{-1}$ is shown to be an integral operator with kernel $K(x, t, \lambda)$ satisfying the inequalities $\int_0^\infty |K(x, t, \lambda)|^2\, dx < \infty$, $\int_0^\infty |K(x, t, \lambda)|^2\, dt < \infty$. By approximating the singular problem by boundary problems on finite intervals, an expression is obtained for $K(x, t, \lambda)$ in terms of the eigenfunctions of $L_{\theta(\lambda)}$ and solutions of the equation $l(y) = \lambda y$. From this expression an expansion theorem and Parseval equality are obtained.

R. C. Gilbert (Fullerton, Calif.)

Fountain, Leonard; Jackson, Lloyd 305

A generalized solution of the boundary value problem for $y'' = f(x, y, y')$.

Pacific J. Math. **12** (1962), 1251–1272.

Consider the second-order equation (1) $y'' = f(x, y, y')$, where $f(x, y, z)$ is defined on the domain R: $-\infty < a \leqq x \leqq b < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$. The paper deals with the problem of finding conditions on f that imply that given arbitrarily α and β, then there is at least one solution of (1) such that (2) $y(a) = \alpha$, $y(b) = \beta$. The function $\varphi(x)$ is said to be an under-function with respect to the boundary-value problem (2) if $\varphi(a) \leqq \alpha$, $\varphi(b) \leqq \beta$ and $\varphi(x) \leqq y(x)$ on any interval $[x_1, x_2] \subset [a, b]$ for all solutions $y(x)$ of (1) such that $\varphi(x_i) \leqq y(x_i)$, $i = 1, 2$. Similarly, a function $\psi(x)$ is said to be an over-function with respect to (2) if $\psi(a) \geqq \alpha$, $\psi(b) \geqq \beta$ and $\psi(x) \geqq y(x)$ on any interval $[x_1, x_2] \subset [a, b]$ for all solutions of (1) such that $\psi(x_i) \geqq y(x_i)$. Let $\{\varphi\}$ represent the collection of all under-functions with respect to (2) which are continuous on $[a, b]$ and call $H(x) = \sup[\varphi(x) : \varphi \in \{\varphi\}]$.

The main result of this paper is the theorem: Suppose the following. (i) For each fixed x and z, $f(x, y, z)$ is a nondecreasing function of y. (ii) $f(x, y, z)$ satisfies a Lipschitz condition with respect to y and z on each compact subset of R and there is a continuous function $g(x, y)$ which is nondecreasing in y and is such that $g(x, y) \leqq f(x, y, y')$ for all $(x, y, y') \in R$. Assume further that there exist functions φ and ψ and h such that (iii) $\psi(x)$ is continuous on $[a, b]$, is of class C' on (a, b) and is an over-function with respect to (2), (iv) $\varphi(x)$ is a continuous under-function with respect to (2), (v) $h(t)$ is positive and continuous for $t \geqq 0$, $|f(x, y, y')| \leqq h(|y'|)$ for $a \leqq x \leqq b$, $\varphi(x) \leqq y \leqq \psi(x)$, $|y'| < \infty$ and

$$\int_c^\infty \frac{t\, dt}{h(t)} > \max \psi(x) - \min \varphi(x),$$

where

$$c = \max \left| \frac{|\psi(b) - \varphi(a)|}{b - a}, \frac{|\psi(a) - \varphi(b)|}{b - a} \right|.$$

Then $H(x)$ is a solution of the boundary-value problem (2).

The method of proof is based on that presented by Perron [Math. Z. **18** (1923), 42–54] for the boundary-value problem for Laplace's equation.

M. M. Peixoto (Providence, R.I.)

Olver, F. W. J. 306

Error bounds for asymptotic expansions in turning-point problems.

J. Soc. Indust. Appl. Math. **12** (1964), 200–214.

In an earlier paper [same J. **11** (1963), 748–772; MR **28**

#4190] the author established error bounds for first approximations, in terms of Airy functions, to solutions of

$$\frac{d^2w}{dx^2} = [u^2p(x, u)+q(x, u)]w \qquad (u \to \infty, u > 0)$$

for x ranging over a real interval containing no singularities of p or q and just one zero of p. He now extends these results to higher approximations (truncations of complete asymptotic expansions of solutions) and to allowing complex values of x. The higher-order theory is illustrated by deriving error bounds for uniform asymptotic expansions of $J_n(x)$ and $Y_n(x)$ $(n\to\infty)$.

N. D. Kazarinoff (Ann Arbor, Mich.)

Mano, Kenji 307

On a stable solution of $(u'')^\lambda = P(x, u)(u')^\mu + Q(x, u)u^\nu$.

Sci. Rep. Fac. Lit. Sci. Hirosaki Univ. **9** (1962), 1–4.

For λ even, μ and ν odd, $Q \geqq P > 0$, the above equation has no solution $u(x)$ such that $u(x)\to 0$ as $x\to\infty$.

M. M. Peixoto (Providence, R.I.)

Mirak'jan, M. G. 308

The existence of a regular solution of a system of ordinary differential equations in the neighbourhood of a singular fixed point. (Russian)

Ukrain. Mat. Ž. **16** (1964), 246–250.

Consider the system

$$z^2\frac{dw_1}{dz} = za_{00}(z)+a_{10}(z)w_1+a_{01}(z)w_2 + \sum_{k_1+k_2=2}^{\infty} a_{k_1k_2}(z)w_1^{k_1}w_2^{k_2},$$

$$z^2\frac{dw_2}{dz} = zb_{00}(z)+b_{10}(z)w_1+b_{01}(z)w_2 + \sum_{k_1+k_2=2}^{\infty} b_{k_1k_2}(z)w_1^{k_1}w_2^{k_2},$$

where $a_{ij}(z)$, $b_{ij}(z)$ are analytic in a neighborhood of $z=0$. The author studies the asymptotic behavior of the solutions of this system in the neighborhood of $z=0$. The results are analogous to those obtained in the linear case.

J. C. Lillo (W. Lafayette, Ind.)

Kato, Junji 309

The asymptotic relations of two systems of ordinary differential equations.

Contributions to Differential Equations **3** (1964), 141–161.

The first part of the paper is devoted to a theorem on asymptotic equivalence of two systems of differential equations on a product space. The precise result is too complicated to be stated here. As an application, the author then deduces the following result on perturbations of linear systems. Suppose that all solutions of (1) $x'=Ax+p(t)$ are bounded on $0\leqq t<\infty$, where A is a constant matrix and p is a continuous vector function. Suppose that $|X(t, x)| \leqq \lambda(t)\phi(|x|)$, where λ and ϕ are continuous non-negative functions, ϕ is an increasing function, and $\int^\infty \lambda(t)\,dt<\infty$, $\int^\infty [dr/\phi(r)]=\infty$. Finally, suppose $|X(t,x)-X(t,x')| \leqq \lambda(t)L(r)|x-x'|$ for $|x|\leqq r$, $|x'|\leqq r$. Then the systems (1) and (2) $x'=Ax+p(t)+X(t,x)$ are asymptotically equivalent. This application generalizes a recent result of the reviewer [Proc. Amer. Math. Soc. **15** (1964), 758–765] and renders it obsolete.

F. Brauer (Madison, Wis.)

Smith, R. A. 310

Boundedness theorems for certain non-linear systems.

Rend. Circ. Mat. Palermo (2) **12** (1963), 239–256.

With the customary convention of summation, let

$$\text{(S)} \qquad a_{ij}x_j''+f_{ij}x_j'+\partial G/\partial x_i = p_i$$

be a system of m ordinary differential equations, where (a_{ij}) is a constant symmetric positive definite matrix and f_{ij}, G, p_i are known functions of $(t, x_1, \cdots, x_m)$, continuous throughout $T\times R^m$, T being the interval $0\leqq t<\infty$. The solutions of (S) are said to be ultimately bounded if there exists a function $\tau(r)$, finite for each $r>0$, and a constant N, such that every solution $x_i(t)$ of (S) with $\|x\|+\|x'\|\leqq r$ at $t=t_0$, exists for all $t>t_0$ and satisfies $\|x\|+\|x'\|\leqq N$ for $t>t_0+\tau(r)$. A typical result established in this paper is the following. The solutions of (S) are ultimately bounded if (i) there exist functions $F_i(t, x_1, \cdots, x_m)$ such that $f_{ij}=\partial F_i/\partial x_j$; $p_i=-\partial F_i/\partial t$; (ii) the F_i are bounded in $T\times Q$ for each bounded Q in R^m; (iii) $G(t, x_1, \cdots, x_m)\to+\infty$ uniformly on T as $|x|\to\infty$; (iv) there exist positive constants δ and ρ such that $\partial G/\partial t-\hat{a}_{ij}F_i\partial G/\partial x_j\leqq-\delta$ for all $(t, x_1, \cdots, x_m)$ with $t\geqq 0$, $\|x\|\geqq\rho$, where $(\hat{a}_{ij})$ is the inverse matrix of (a_{ij}). These results include those of Mizohata [Mem. College Sci. Univ. Kyoto Ser. A Math. **27** (1952), 115–121; MR **14**, 874] and Yamaguti [ibid. **28** (1954), 87–96; MR **16**, 477]. Other results are established, related to some known results of Reuter [J. London Math. Soc. **27** (1952), 48–58; MR **13**, 844], Amerio [Ann. Mat. Pura Appl. (4) **39** (1955), 97–119; MR **18**, 128], and Graffi [Ann. of Math. (2) **54** (1951), 262–271; MR **13**, 463]. The case of periodic solutions is also considered.

C. Corduneanu (Iași)

Harasahal, V. H. 311

On quasi-periodic solutions of differential equations. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 2 (39), 152–164.

Soit

$$\text{(S)} \qquad dx_i/dt = \sum_{j=1}^{n} P_{ij}(t)x_j$$

un système dont les coefficients $P_{ij}(t)$ sont des fonctions quasi-périodiques de base fréquentielle $\{\beta_1, \beta_2, \cdots, \beta_m\}$. Soient $F_{ij}(u_1, \cdots, u_m)$ des fonctions périodiques, de période $2\pi/\beta_k$ par rapport à la variable u_k, telles que $P_{ij}(t)=F_{ij}(t, \cdots, t)$. Au système (S) on associe le système d'équations aux dérivées partielles

$$\text{(S}'\text{)} \qquad \sum_{k=1}^{m} \partial x_i/\partial u_k = \sum_{j=1}^{n} x_jF_{ij}(u_1, \cdots, u_m).$$

Le problème des oscillations quasi-périodiques se réduit, de cette manière, au problème des oscillations périodiques. En utilisant cette méthode, l'auteur établit d'abord quelques résultats concernant le système (S), parmi lesquels mentionnons un théorème de réductibilité (à l'aide d'une transformation linéaire non singulière, à coefficients quasi-périodiques). La seconde partie du travail contient certains résultats pour les systèmes non linéaires à petit paramètre. On démontre l'existence d'une solution

quasi-périodique qui tend vers la solution quasi-périodique du système générateur, quand le paramètre tend vers zéro.
C. Corduneanu (Iași)

Krasnosel'skiĭ, M. A.; Strygin, V. V. 312
Some criteria for the existence of periodic solutions of ordinary differential equations. (Russian)
Dokl. Akad. Nauk SSSR **156** (1964), 1022–1024.
The authors consider a vector system of the form (1) $\dot{x}(t)=f(t,x)$, where f is continuous in both arguments and periodic in t of period ω. A guiding function $\Phi(x)$ for (1) is defined as a continuously differentiable function whose gradient is non-null on some ball $\|x\|\leq\rho_0$, satisfying $(\operatorname{grad}\Phi(x), f(t,x))>0$, $\|x\|\geq\rho_0$. The main theorem states that if there exists a guiding function for (1), with non-zero index on $\|x\|=\rho_0$, and if every solution of (1) is defined on $[0,\omega]$, then (1) has at least one solution of period ω. Later theorems give conditions which assure that at least for x in some region T, the second condition above is satisfied. These conditions also consist of relations involving the inner product of $f(t,x)$ and gradients of functions. The authors then extend these theorems to an equation with a periodic lag of the form (2) $\dot{x}(t)=f(t,x(t),x(t-h(t)))$. Again, sufficient conditions for the existence of a periodic solution of (2) are given, in terms of inequalities of the form (3) $(\operatorname{grad}\Phi(x), f(t,x,y))>0$, $\|x\|\geq\rho_0$, and all y. As (3) is assumed to be valid uniformly in y, the range of application of these theorems to systems of the form of (2) seem limited.
A. Stokes (Washington, D.C.)

Abdel Karim, Rahmi Ibrahim Ibrahim 313
★Über den Resonanzbegriff bei Systemen linearer und nichtlinearer gewöhnlicher Differentialgleichungen.
Von der Naturwissenschaftlich-Philosophischen Fakultät der Technischen Hochschule Carolo-Wilhelmina zu Braunschweig zur Erlangung des Grades eines Doktors der Naturwissenschaften, Braunschweig, 1963. 195 *pp.*
Consider the equation (1) $\dot{x}=A(t)x+f(t)$, where x is a vector, and $A(t), f(t)$ are periodic in t of period T. System (1) is said to be in the resonance case if there is a periodic solution $y(t)$ of period T of the adjoint equation (2) $\dot{y}=-A^T(t)y$ such that (3) $\int_0^T y^T(t)f(t)\,dt\neq 0$. System (1) is said to lie in the principal case when there are no periodic solutions of (2) of period T and in the exceptional case when there are no periodic solutions of (2) of period T satisfying (3). The author discusses in detail and in a standard way the behavior of the solutions of (1) in all three cases. Assuming that the nonlinear equation $\dot{u}=g(u,t)$ has a periodic solution u_0 of period T, the author discusses the system (4) $\dot{u}=g(u,t)+\beta f(t)$ for β small, provided the linear variational equation relative to u_0 is in either the resonance or principal case. The results for the principal case are well known. The author shows in the resonance case that the solutions of (4) with increasing values of t are such that $|u(t)-u_0(t)|$ is at least of the order of $\sqrt{\beta}$. *J. K. Hale* (Providence, R.I.)

Voĭlokov, V. I. 314
Periodic solutions of non-linear oscillatory systems with one degree of freedom. (Russian)
Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 1 (38), 19–27.
In an earlier work [same Izv. **1962**, no. 4 (29), 19–32; MR **25** #3231] the author introduced a procedure for studying the existence and location of periodic solutions for equations of the form $\ddot{x}+x+\alpha f(x,\dot{x})=\varphi(t)$. In this paper he uses these procedures to study the following three special cases:

$$\ddot{x}+x-\alpha[(1-x^2)\dot{x}-(\lambda+a)x] = \alpha^2x+\alpha\lambda_0\sin t,$$
$$\ddot{x}+n^2x-\alpha x^2+\alpha^2\lambda x = \alpha^3x+\alpha^2A\cos nt,$$
$$\ddot{x}+x-\alpha x^2+\alpha^2\lambda x = \alpha^3x+\alpha^2h\cos nt.$$

J. C. Lillo (W. Lafayette, Ind.)

Golokvosčjus, P. [Golokvosčus, P. B.] 315
Determination of the characteristic numbers of the solutions of a class of systems of differential equations with periodic coefficients. (Russian. Latvian and French summaries)
Litovsk. Mat. Sb. **3** (1963), *no.* 1, 77–101.
Résumé adapté de l'auteur: "Dans cet article on étudie les nombres caractéristiques de l'équation matricielle d'ordre deux, $\ddot{X}=XQ(t,\mu)$, où la matrice Q du second degré est développable en série $Q(t,\mu)=\sum Q_h(t)\mu^h$ avec les Q_h périodiques de période 1 et continues. La série est bornée par rapport à t, et convergente pour $|\mu|<R$. On suppose que les matrices $Q(t,\mu)$ et $Q_0(t)$ satisfont aux relations

$$Q(t,\mu)\cdot\int_0^t Q(\tau,\mu)\,d\tau \neq \int_0^t Q(\tau,\mu)\,d\tau\cdot Q(t,\mu),$$
$$Q_0(t)\cdot\int_0^t Q_0(\tau)\,d\tau = \int_0^t Q_0(\tau)\,d\tau\cdot Q_0(t),$$
$$Q_0(t) = \begin{pmatrix}\varphi_1(t)+b_1\varphi_2(t) & b_3\varphi_2(t)\\ b_2\varphi_2(t) & \varphi_1(t)\end{pmatrix}$$

de sorte que Q_0 a la forme canonique. Dans les cas discutés les b_h sont des constantes quelconques et les $\varphi_j(t)$ sont continues avec période 1 en t."
S. Lefschetz (Providence, R.I.)

Levin, J. J.; Shatz, S. S. 316
Nonlinear oscillations of fixed period.
J. Math. Anal. Appl. **7** (1963), 284–288.
In this paper, the authors improve the result of the reviewer's paper [1] [J. Math. Mech. **10** (1961), 569–578; MR **23** #A391]. However, the authors do not seem to have been aware of the reviewer's succeeding papers [2] [Arch. Rational Mech. Anal. **11** (1962), 27–33; MR **25** #5231], [3, 4] [J. Sci. Hiroshima Univ. Ser. A-I Math. **26** (1962), 93–109; MR **27** #400a; ibid. **26** (1962), 111–122; MR **27** #400b], before they read proofs.

In this paper, the authors prove the theorem: Given (*) $x''+g(x)=0$ ($'=d/dt$), where $g(x)\in C$ for x sufficiently small and $xg(x)>0$ for $x\neq 0$. Put $G(x)=\int_0^x g(u)\,du$, $X(x)=\eta\sqrt{(2G(x))}$ ($\eta=|x|/x$) and $h(X)=g[x(X)]$, where $x(X)$ is the inverse function of $X(x)$. Then every solution of (*) oscillates around $x=x'=0$ with the same period $\omega>0$ if and only if

$$h(X) = \frac{2\pi}{\omega}\cdot\frac{X}{1+S(X)} \qquad (X\neq 0)$$

for X sufficiently small, where $S(X)$ is an odd function,

defined and continuous for sufficiently small $X \neq 0$ and $S(X) \in L_1(0, a)$ for some $a > 0$.

This theorem was proved by the reviewer in [1] for analytic $g(x)$ and in [2] for $g(x)$ differentiable at $x = 0$. The authors point out that one needs no more smoothness on $g(x)$ than mere continuity if one pays attention to the integrability of the function $X/h(X)$.

For the proof of the lemma on which the proof of the theorem depends, the authors use properties of Fourier series, while the reviewer [2] uses a method of solving Abel's integral equation [E. T. Whittaker and G. N. Watson, *A course of modern analysis*, 4th ed., p. 229, Cambridge Univ. Press, Cambridge, England, 1927].

M. Urabe (Fukuoka)

Blaquière, Augustin 317

Étude de la stabilité d'un système non linéaire dans un espace de Riemann.

C. R. Acad. Sci. Paris **257** (1963), 57–59.

The investigation of the stability properties of the nonlinear autonomous system $\dot{x} = B(x)x$ is shown to be reducible to the search for a metric tensor $M(x)$ of a Riemannian phase space with the property that the set Σ:

$$2R\dot{R} = \langle x, M(x)B(x)x\rangle + \langle x, \dot{M}(x)x\rangle = 0$$

can be decomposed in two subsets Σ_1 and Σ_2, where Σ_1 is a double hyperplane going through the origin and Σ_2 does not contain the origin and has the property that on every trajectory which intersects it the sign of $R\dot{R}$ changes at the intersection point.

Examples are given. *George Szegö* (Baltimore, Md.)

Bogusz, Władysław 318

Asymptotic stability with no oscillation of autonomic systems with one degree of freedom. (Polish and Russian summaries)

Proc. Vibration Problems **4** (1963), 291–300.

A counterexample to the principal theorem can easily be constructed and this makes invalid the "method of limit curves" proposed in the paper.

J. P. LaSalle (Providence, R.I.)

Bhatia, Nam P. 319

Stability and Liapunov functions in dynamical systems.

Contributions to Differential Equations **3** (1964), 175–188.

The author discusses necessary and sufficient conditions for stability of closed sets (not necessarily compact) of dynamical systems defined on a locally compact separable metric space X, in terms of Liapunov functions. By a dynamical system the author means a continuous mapping $\pi: X \times R \to X$ (R is the real line) satisfying: (a) $\pi(x, 0) = x$, $x \in X$; (b) $\pi(\pi(x, t_1), t_2) = \pi(x, t_1 + t_2)$, $x \in X$, t_1, $t_2 \in R$. Several kinds of stability are defined. For example, a closed set $M \subset X$ is called semistable if for every $\varepsilon > 0$ and any compact set $K \subset X$, there exists a $\delta(\varepsilon, K) > 0$ such that $x \in K \cap S(M, \delta)$ implies $\gamma^+(x) \subset S(M, \varepsilon)$, where $S(M, \tau)$ is a τ-neighborhood of M and $\gamma^+(x)$ is the trajectory through x for $t \geqq 0$. In case δ is independent of K, M is called stable. Definitions of asymptotic stability and semi-asymptotic stability of M are also given. Relationships between these concepts are shown by examples. The author shows that necessary and sufficient conditions for stability, semistability, asymptotic stability and semi-asymptotic stability of M are the existence of Liapunov functions. A theorem is as follows. A necessary and sufficient condition for asymptotic stability of a closed set $M \subset X$ is the existence of a continuous function $v(x)$ defined on an invariant open set W ($\supset M$) satisfying the following conditions, where W has the property that there exists a $\delta > 0$ such that $W \supset S(M, \delta)$: (a) There exist two continuous increasing functions $\alpha(r)$ and $\beta(r)$ defined for $r \geqq 0$, $\alpha(0) = \beta(0) = 0$, such that $\alpha(d(x, M)) \leqq v(x) \leqq \beta(d(x, M))$ for $x \in W$; (b) $v(xt) \leqq f(t)v(x)$ for $t > 0$, $x \in W$, where xt means $\pi(x, t)$ and $f(t)$ is a continuous decreasing function defined on $[0, \infty)$, $f(t) \to 0$ as $t \to \infty$. Related papers are those of Auslander and Seibert [Internat. Sympos. Nonlinear Differential Equations and Nonlinear Mechanics, pp. 454–462, Academic Press, New York, 1963; MR **27** #416] and the reviewer [Funkcial. Ekvac. **5** (1963), 31–69; MR **28** #5223].

T. Yoshizawa (Ames, Iowa)

Živov, V. S. 320

On the stability of trajectories. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 2 (39), 79–81.

The trajectory stability of the title is essentially orbital stability, but in the non-autonomous case. A trajectory $q = f(t, t_0, q_0)$ of the system (1) $\dot{q} = Q(t, q)$ is said to be trajectory-stable if for a given ε every trajectory $q = f(t, t_0, q_0^*)$ for which $\|q_0 - q_0^*\|$ is sufficiently small has the property that for each t there is a $t^* \geqq t_0$ such that $\|f(t, t_0, q_0^*) - f(t^*, t_0, q_0)\| < \varepsilon$. If $x = f(t, t_0, q_0^*) - f(t^*, t_0, q_0)$, then (2) $\dot{x} = X(t, x)$, where $X(t, 0) \equiv 0$. Thus, the trajectory stability of a trajectory of (1) reduces to the Lyapunov stability of the trivial solution of (2). A simple example is given of a system with a trajectory stable in the above sense but unstable in the usual sense.

C. S. Coleman (Baltimore, Md.)

Ura, Taro 321

On the flow outside a closed invariant set; stability, relative stability and saddle sets.

Contributions to Differential Equations **3** (1964), 249–294.

The setting of the paper is the generalized dynamical system considered earlier by the author and I. Kimura [Comment. Math. Univ. St. Paul. **8** (1960), 23–39; MR **22** #11182]. Heretofore the author has considered the behavior of closed invariant sets isolated from closed invariant sets. In the present paper it is not required that the sets considered be isolated from other closed invariant sets.

Much of the paper is motivated by a proposition of V. I. Zubov [*The methods of A. M. Lyapunov and their application* (Russian), Izdat. Leningrad Univ. Moscow, 1957; MR **19**, 275] which, briefly put, says that a necessary and sufficient condition for positive stability of a closed invariant set is that the set does not intersect its negative limit set. The condition is necessary but the author shows, by examples in the plane, that the condition is not sufficient. Sufficient conditions for stability are then proved using the author's concept of prolongations [Funkcial. Ekvac. **2** (1959), 143–200; MR **22** #1727]. Necessary and sufficient conditions for relative stability are given in terms of relative prolongations.

A saddle set theory is developed to generalize the Bendixson theory in the euclidean plane from dynamical systems in the plane to these generalized dynamical systems in the plane. The Zubov proposition is shown to hold as both a necessary and sufficient condition on the line and in the plane provided, in the case of the plane, the closed invariant set is isolated from saddle singular points. *W. R. Utz* (Columbia, Mo.)

Dairov, G. 322

Stability of steady-state motion in the case of four zero roots of the characteristic equation. (Russian. Kazak summary)

Vestnik Akad. Nauk Kazah. SSR **1964**, *no.* 4 (229), 63–68.

The author treats a system in four-space which, after a series of transformations not changing stability properties, has the form $\dot{x}=y$, $\dot{y}=Y(x,y,z,u)$, $\dot{z}=u$, $\dot{u}=U(x,y,z,u)$, where the expansions at the origin of Y and U begin with terms of degree at least two. Assuming that $Y=a_0x^{\alpha_0}+\cdots$ and that $U=b_0z^{\beta_0}+\cdots$, where $a_0b_0\neq 0$, the stability properties of the trivial solution of the differential system are classified according to the nature of a_0, α_0, b_0, β_0 and certain additional coefficients in the expansions of Y and U. *C. S. Coleman* (Baltimore, Md.)

Imaz, Carlos; Vorel, Zdeněk 323

A certain type of stability. (Spanish)

Bol. Soc. Mat. Mexicana (2) 8 (1963), 46–52.

The system under consideration is the standard "control optimization" system

(1) $$\dot{x}=f(t,x,u(t)),$$

where x, f are n-vectors and $u(t)$ is a control p-vector belonging to an admissible collection Ω. Let S_R denote the solid sphere $\|x\|\leq R$, and $S_{\varepsilon R}$ the subset $\varepsilon\leq\|x\|\leq R$. Let $S_R\subset D$, where D is a region such that for $t\geq 0$, $u\in\Omega$, the usual existence and unicity conditions for the solutions of (1) prevail. One defines strong ε-stability relative to $u_0\in\Omega$ if given $x_0\in S_{\varepsilon R}$ and $t_0>0$, the solution $x(t)$ of (1), such that $x(t_0)=x_0$, reaches $\|x\|=\varepsilon$ at a time $t_1>t_0$ and remains in S_ε ever after. If there is a t_1 independent of x_0 and t_0, then the strong stability is uniform. Define $\Delta(\alpha,u)=\sup\alpha^{-1}x'f$ for $t\geq 0$, $\|x\|=\alpha$, $u\in\Omega$, and $\alpha\in[\varepsilon,R]$. Theorem 1: Let $u\in\Omega$ and let $f(t,x,u_1)=F(t,x)$ be lipschitzian in x, bounded and uniformly continuous for $t\geq 0$ and $x\in S_{\varepsilon R}$. Let also $\Delta(\alpha,u_1)=\Delta(\alpha)<0$, $\alpha\in[\varepsilon,R]$. Then (1) is uniformly strongly ε-stable in S_R relative to $u_1\in\Omega$. Theorem 2: Given $u_0\in\Omega$, and $\Omega_1\subset\Omega$ a neighborhood of u_0 in Ω. If $\Delta(\alpha,u_0)<0$, and there is strong ε-stability for all $u\in\Omega_1$, then there is a neighborhood $\Omega_2\subset\Omega$, of u_0 in Ω, such that (1) is strongly ε-stable relative to u in Ω_2.

A result analogous to Theorem 1 (but somewhat more complicated) is obtained for a semi-linear system

$$\dot{x}=A(t,u^1)x+g(t,x,u^2)$$

$(u^1,u^2\in\Omega_1\times\Omega_2)$. Furthermore, under certain specified conditions, the optimization problem is also solved. [References: J. B. Rosen, Internat. Sympos. Nonlinear Differential Equations and Nonlinear Mechanics, pp. 366–376, Academic Press, New York, 1963; MR **26** #6013; Ważewski, Studia Math. **10** (1948), 48–59; MR **10**, 40.] *S. Lefschetz* (Providence, R.I.)

Jataev, M.; Kurmašev, D. 324

A study of the critical cases of stability for steady-state motion. (Kazak. Russian summary)

Vestnik Akad. Nauk Kazah. SSR **1962**, *no.* 3 (204), 54–61.

The authors give a proof of a new method of investigating the critical cases of steady-state motion, in which they conclude that the stability or instability of the motion can be described by the behaviour of the first approximation to the right-hand side of the equations of motion in a sufficiently small neighbourhood of the origin or in a certain portion of a sufficiently small neighbourhood of the origin.

Derguzov, V. I. 325

On the stability of the solutions of the Hamilton equations with unbounded periodic operator equations. (Russian)

Mat. Sb. (N.S.) **63 (105)** (1964), 591–619.

Consider the equation (1) $J\,dx/dt=H(t)x$, where x is an element of a complex separable Hilbert space, $J=-J^*$ and J^{-1} are bounded operators, $H(t)$ is a symmetric unbounded operator, $H(t+T)=H(t)$, $T>0$ constant, $H(t)=H_0(t)+H_1(t)$, where $H_0(t)$ is positive definite, and $H_1(t)$ is a perturbation term. Both H_0 and H_1 are subjected to some additional technical assumptions. The present paper is a complete discussion of the concepts and results previously announced by the author [Dokl. Akad. Nauk SSSR **152** (1963), 1294–1296; MR **28** #314].

The general operator solution $X(t)$ of (1) can be written in the form $X(t)=H_0^{-1/2}(0)Z(t)H_0^{1/2}(0)$, where $Z(t)$, $Z^{-1}(t)$ are bounded operators, and $Z^*(T)FZ(T)=F$ with $F=-iH_0^{-1/2}(0)JH_0^{-1/2}(0)$ a bounded symmetric operator together with its inverse. An operator satisfying this latter property is called F-unitary. Equation (1) is called stable if there is a $C>0$ such that, for every generalized solution $x(t)$ of (1), $\||F|^{1/2}H_0^{1/2}(0)x(t)\|\leq C\||F|^{1/2}H_0^{1/2}(0)x(0)\|$, $-\infty<t<\infty$. If, in addition, this property is satisfied for all operators $\tilde{H}_1(t)$ close to $H_1(t)$ in an appropriate topology, then equation (1) is called strongly stable. The reason for introducing stability in this manner is explained in detail. The main result of the paper is the following: Equation (1) is strongly stable if the spectrum of the operator $Y(T)=|F|^{1/2}Z(T)|F|^{-1/2}$ is not of mixed type. The concept of spectrum of mixed type is introduced in a manner similar to that of Kreĭn (see Gel'fand and Lidskiĭ [Uspehi Mat. Nauk **10** (1955), no. 1 (63), 3–40; MR **17**, 482; Amer. Math. Soc. Transl. (2) **8** (1958), 143–181; MR **19**, 960]) for the finite-dimensional case. The proof involves a detailed discussion of F-unitary operators. *J. K. Hale* (Providence, R.I.)

Nemyckiĭ, V. V. 326

Oscillatory regimes of higher-dimensional dynamical systems. (Russian. English summary)

Qualitative methods in the theory of non-linear vibrations (*Proc. Internat. Sympos. Non-linear Vibrations, Vol. II*, 1961), *pp.* 308–314. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, 1963.

The author gives a definition of oscillatory regime for a system of n nonlinear differential equations $\dot{x}_i=f_i(x,t)$, $i=1,\cdots,n$, $x=(x_1,\cdots,x_n)$. A phase trajectory $x(t)$ is said to define an oscillatory regime if there is an axis L not crossing the trajectory and around which oscillations

take place. If L is an $(n-2)$-dimensional linear variety in the phase-space E_n, if (r, ϕ) are polar coordinates in a plane π orthogonal to L, if $(r(t), \phi(t))$ are the polar coordinates of the point $x^*(t)$ we obtain by projecting $x(t)$ on π, then the motion is said to be oscillatory around L if $\phi(t) \to \infty$ as $t \to +\infty$, is said to be damped (with respect to L) if $r(t) \to 0$, and so on. A Lyapunov type analysis (second method) can be established in many ways. To find oscillatory regimes, a Lyapunov rotating function might be constructed. Such a function V could be chosen so as to determine a family of surfaces $V(x) = \phi$ through L with $\sum_i V_{x_i} f_i \neq 0$, where $V_{x_i} = \partial V/\partial x_i$. To mark out the relaxing oscillations a second Lyapunov function W_1 might be constructed so that the surfaces $W_1(x) = C$ are cylinders with the axis L. Finally, to mark out auto-oscillations, a third Lyapunov function W_2 is constructed so that the surfaces $W_2(x_1, \cdots, x_n) = C$ intersect L and are homeomorphic to a plane. A criterion for the existence of periodic solutions is obtained.

L. Cesari (Ann Arbor, Mich.)

Ku, Y. H.; Puri, N. N. 327

On Liapunov functions of high order nonlinear systems.

J. Franklin Inst. **276** (1963), 349–364.

For autonomous nonlinear differential equations of the fourth order of the type $x^{(4)} + a_4 x^{(3)} + a_3 x^{(2)} + a_2 \dot{x} + a_1 x = 0$, where the a_i's are functions of x, $\dot{x}$, $x^{(2)}$, $x^{(3)}$, and similar equations of the third order, the authors present a procedure of writing Liapunov functions which cover some of the well-known cases considered by E. A. Barbashin [Automatic and Remote Control (Proc. First Internat. Congr. Internat. Federation of Automatic Control, Moscow, 1960), Vol. II, pp. 943–947] and by M. L. Cartwright [Quart. J. Mech. Appl. Math. **9** (1956), 185–194; MR **18**, 211]. The reviewer, however, did not see any relaxation of the already known conditions which guarantee asymptotic stability.

N. P. Bhatia (Cleveland, Ohio)

Arnol'd, V. I. 328

Proof of a theorem of A. N. Kolmogorov on the preservation of conditionally periodic motions under a small perturbation of the Hamiltonian. (Russian)

Uspehi Mat. Nauk **18** (1963), *no.* 5 (113), 13–40.

This paper, which is dedicated to Kolmogorov on his 60th birthday anniversary, contains a complete proof of a theorem formulated by Kolmogorov in 1954 [Dokl. Akad. Nauk SSSR **98** (1954), 527–530; MR **16**, 924; Proc. Internat. Congr. Mathematicians (Amsterdam, 1954), Vol. 1, pp. 315–333, Noordhoff, Groningen, 1957; MR **20** #4066]. This theorem, basic for the study of Hamiltonian differential equations, has many applications in celestial mechanics and other fields, and its proof has been long awaited.

The introduction contains an excellent discussion of the problem which puts the result into historical and logical perspective. Sections 2 to 4 contain the details of the proof of the main theorem, and the last section discusses an application to the motion of a non-symmetric heavy top.

We formulate the main result: Consider a Hamiltonian system $\dot{p} = H_q$, $\dot{q} = -H_p$, where p, q are both n vectors and $H = H(p, q, \varepsilon)$ is of period 2π in the angular variables $q_1, q_2, \cdots, q_n$. It is assumed that $H = H_0 + \varepsilon H_1 + \cdots$ is a real analytic function of all its arguments which for $\varepsilon = 0$ is independent of q. This means that for $\varepsilon = 0$ the system possesses n integrals $p_1, p_2, \cdots, p_n$, so that the given system is near an integrable one. Finally, it is assumed that the Hessian of $H_0(p)$ does not vanish. Then, for sufficiently small ε, the existence of almost periodic solutions of the form

$$p = \phi(Q_1, Q_2, \cdots, Q_n, \varepsilon), \qquad q = \psi(Q_1, Q_2, \cdots, Q_n, \varepsilon)$$

is asserted, where $Q_\nu = \omega_\nu t + \text{const}$ and ϕ, ψ are of period 2π in $Q_1, Q_2, \cdots, Q_n$. Moreover, in a preassigned domain $p \in F$ the set of solutions so obtained fills a set whose complement has arbitrarily small measure if ε is chosen sufficiently small. (The motion in the exceptional set has been studied in the paper reviewed below [#329].)

The difficult proof carried out in detail (sketchy indications had been given by Kolmogorov; see the remarks in MR **20** #4066, loc. cit.). The analytical difficulty involved stems from the occurrence of the so-called small divisors. The first definite results concerning such nonlinear small-divisor problems are due to C. L. Siegel [Ann. of Math. (2) **43** (1942), 607–612; MR **4**, 76; Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **1952**, 21–30; MR **15**, 222] in which, however, Hamiltonian systems have to be excluded. The present paper overcomes this difficulty for nearly integrable Hamiltonian systems. Undoubtedly these results will give rise to many new applications in celestial mechanics. We also refer to the paper by the reviewer [Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II **1962**, 1–20; MR **26** #5255] which contains an analogous result corresponding to systems of two degrees of freedom in the differentiable case.

J. Moser (New Rochelle, N.Y.)

Arnol'd, V. I. 329

Instability of dynamical systems with many degrees of freedom. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 9–12.

This note contains an example which is basic for the stability theory of Hamiltonian systems. In the previous work of Kolmogorov and the author [see Kolmogorov, Proc. Internat. Congr. Mathematicians (Amsterdam, 1954), Vol. 1, pp. 315–333, Noordhoff, Groningen, 1957; MR **20** #4066; the author, #328] for nearly integrable Hamiltonian systems of n degrees of freedom n-dimensional invariant tori had been constructed and were shown to form the majority of the solutions, the exceptional set having small measure. The present paper is concerned with the subtle problem of investigating the flow in the exceptional set.

The example is described by the Hamiltonian

$$H = \tfrac{1}{2}(I_1{}^2 + I_2{}^2) + \varepsilon(\cos\phi_1 - 1)(1 + \mu(\sin\phi_2 + \cos t)),$$

where I_ν, ϕ_ν are canonically conjugate variables and ε, μ small parameters. It is stated (Theorem 3): For given $0 < A < B$ and $\varepsilon > 0$ there exists a $\mu_0 > 0$ such that for $0 < \mu < \mu_0$ there exists a trajectory which connects the set $I_2 < A$ with $I_2 > B$. The time required to pass from one set to the other is extremely large, namely, of the order $\exp\{-\varepsilon^{-1/2}\}$.

For systems of two degrees of freedom (i.e., for a 4-dimensional phase space) such exceptional sets of solutions are also present but separated by invariant tori;

therefore in 4-dimensional phase space the above phenomenon cannot occur. The above example shows that already for a 5-dimensional phase space the exceptional set may be traversed by a trajectory. It is expected this example is typical.

To establish these statements the author constructs unstable invariant tori ("whiskered" tori) and investigates the intersection of the asymptotic manifold of solutions for different tori. They provide—similar to the intersection of separatrices in Poincaré's work—for a mechanism which passes solutions from the neighborhood of one torus to another. By forming a chain of such tori linked by intersecting asymptotic manifolds, a solution of the desired type can be constructed. The details of the proof must be formidable, although the idea of the proof is clearly outlined. *J. Moser* (New Rochelle, N.Y.)

Jaroševskiĭ, V. A. 330

Asymptotic solution of the equation of motion of certain conservative systems with slowly varying parameters. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 363–367.

Consider the system

$$(1)\qquad y''+\varepsilon f(\tau,y,y')+F(\tau,y,y') = 0,$$

where $\tau=\varepsilon t$, ε is small, and for $\varepsilon=0$, τ constant, the equation $y''+F(\tau,y,y')=0$ has a first integral. The aim of this paper is to obtain expansions of the solutions of (1) in power series in ε in such a way as to exhibit the adiabatic invariants of (1) in the sense of Landau and Lifšic [*Mechanics* (Russian), Fizmatgiz, Moscow, 1958; MR **21** #985; English transl., Pergamon, Oxford, 1960; MR **22** #11531]. Terms through order ε are discussed in detail. *J. K. Hale* (Providence, R.I.)

Onuchic, Nelson 331

On the uniform stability of a perturbed linear system.

J. Math. Anal. Appl. **6** (1963), 457–464.

Consider (1) $dx/dt=f(t,x)$, where x and f are n-dimensional vectors, $t\geqq 0$, x varies on an open set $\Omega\subset R^n$, and uniqueness of solutions is assumed; a solution $x(t)$ is said to be uniformly stable if given $\varepsilon>0$, $t_0>0$, there exists $\delta=\delta(\varepsilon)$ such that $\|x_0-x(t_0)\|<\delta$ implies that the solution $x(t;t_0,x_0)$ of (1) such that $x(t_0;t_0,x_0)=x_0$ exists for $t>t_0$ and $\|x(t;t_0,x_0)-x(t)\|\leqq\varepsilon$. Now particularize (1) to the two types (2) $dx/dt=A(t)x$ and (3) $dx/dt=A(t)x+g(t,x)$. Among other results, the author proves the following theorem: Every bounded solution of (3) is uniformly stable provided that (i) every solution of (2) is uniformly stable; (ii) for any $M>0$ there exists a function $h(t)\geqq 0$ such that $\|g(t,x)\|\leqq h(t)$, $\int_0^\infty h(t)\,dt<\infty$ whenever $t\geqq 0$, $\|x\|\leqq M$. *M. M. Peixoto* (Providence, R.I.)

Kovtun, I. I. 332

Some problems in the asymptotic solution of an operator differential equation. (Russian)

Approximate methods of solving differential equations, *pp.* 33–39. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, **1963.**

Consider the linear equation (1) $dX/dt=[A+\varepsilon B(t,\varepsilon)]X$, where X is an operator in a Hilbert space H, ε is a small parameter, A is a bounded operator, $\varepsilon B(t,\varepsilon)=\sum_{j=1}^{\infty}\varepsilon^j B_j(t)$, $B_j(t+2\pi)=B_j(t)$ and $\int_0^t\|B_j(\sigma)\|\,d\sigma\leqq\beta_j$, $0\leqq t\leqq 2\pi$, $\sum\varepsilon^j\beta_j<\infty$ for $|\varepsilon|<\varepsilon_0$. By standard arguments, the author shows that the absolutely continuous solution $X(t,\varepsilon)$, $X(0,\varepsilon)=I$, of (1) is analytic in ε uniformly in t. The main theorem is the following: If the spectrum of A lies in a horizontal strip of width 2π, then $X(t,\varepsilon)=P(t,\varepsilon)e^{K(\varepsilon)t}$, where $P(t+2\pi,\varepsilon)=P(t,\varepsilon)$, $P(t,0)=I$, $K(0)=A$, and P,K are analytic in ε uniformly in t. The author shows how to compute the coefficients in the power series expansion of $P(t,\varepsilon)$, $K(\varepsilon)$. *J. K. Hale* (Providence, R.I.)

Pul'kin, S. S.; Rozov, N. H. 333

The asymptotic theory of relaxation oscillations in systems with one degree of freedom. I. Calculation of the phase trajectories. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, *no.* 2, 70–82.

Authors' summary: "This paper is closely related to the known investigations of L. S. Pontrjagin and S. F. Miščenko on the general asymptotic theory of relaxation oscillations. The paper deals with the system of the second order $\varepsilon\dot x=f(x,y)$, $\dot y=g(x,y)$, where ε is a small positive parameter. A point S of the phase plane is called an irregular point of order m if $f(S)=f_x'(S)=\cdots=f_x^{(m-1)}(S)=0$; $f_x^{(m)}(S)\neq 0$; $f_y'(S)\neq 0$; $g(S)\neq 0$. A method is indicated for calculating (with any degree of accuracy) phase trajectories near an irregular point of order m. It is found, in particular, that in the vicinity of the point S the deviation of the trajectory from the curve $f(x,y)=0$ has the order $O(\varepsilon^{m/(2m-1)})$." *R. Conti* (Florence)

Hájek, Otomar 334

Cycle periods of plane dynamical systems.

J. Math. Anal. Appl. **9** (1964), 75–84.

For $f^n=(f_1^n,f_2^n)\to f=(f_1,f_2)$; f^n,f locally Lipschitzian (in effect he ends up with $f^n\neq f$ in a suitable bounded region), the author establishes a result concerning the periods of the periodic solutions of $dx/dt=f^n$ and $dx/dt=f$, $x=(x_1,x_2)$. If further $z=x_1+ix_2$ and $g=f_1+if_2$, the author gets a bound for the periodic solution by using the formula that the period T satisfies $T=\int(1/g(z))\,dz$ around the cycle. *S. P. Diliberto* (Berkeley, Calif.)

Valeev, K. G. 335

On the danger of combination resonances.

Prikl. Mat. Meh. **27** (1963), 1134–1142 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1745–1759.

Consider the system

$$(1)\qquad \ddot y+\mu N(\theta t)\dot y+(C+\mu P(\theta t))y = 0,$$

where y is an m-vector, $\mu\geqq 0$, $\theta\geqq 0$ are parameters, $C=\operatorname{diag}(\omega_1^2,\cdots,\omega_m^2)$, $\omega_j>0$, $N(\tau)$, $P(\tau)$ are periodic of period 2π. The author says system (1) belongs to class M (for mechanical systems) if the characteristic multipliers of (1) are symmetric with respect to the unit circle. This class is much larger than the Hamiltonian system and, in general, the solutions do not possess any special algebraic properties as in the case of Hamiltonian systems. Resonance points for system (1) are values of θ which lead to nonsimple characteristic multipliers of (1) for $\mu=0$. The author gives a procedure for calculating characteristic exponents near resonance and considers the

effect of the "friction" term $N(\tau)$ on the exponents. Following Schmidt and Weidenhammer [Math. Nachr. **23** (1961), 301–318; MR **25** #4666], the author shows that an introduction of an $N(\tau)$ with $N_0 = \int_0^{2\pi} N(\tau)\, d\tau$ positive definite may result in turning a stable system into an unstable system. (Related paper: the reviewer [*Contributions to the theory of nonlinear oscillations*, Vol. V, pp. 55–89, Princeton Univ. Press, Princeton, N.J., 1960; MR **25** #5224].)
J. K. Hale (Providence, R.I.)

Povzner, A. 336

A global existence theorem for a non-linear system and the defect index of a linear operator. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 377–386.

This paper contains the following interesting theorem: Consider a system of differential equations $\dot{x}_k = X_k(x)$ $(k = 1, \cdots, n)$ where the X_k are continuously differentiable for all $x_1, \cdots, x_n$. It is assumed that the divergence $\sum \partial X_k/\partial x_k = 0$. With this system the linear partial differential operator $\tilde{A} = \sum iX_k\, \partial/\partial x_k$ is associated and it is proven: Necessary and sufficient for the existence of almost all solutions for all $t > 0$ is that the symmetric operator $\tilde{A}$ has a defect index 0 in the upper half-plane.

A complete proof of this theorem and some generalizations of it are given. *J. Moser* (New Rochelle, N.Y.)

Aranson, S. H. 337

Rotation of a field in a theorem on the theory of structurally stable dynamical systems on a torus. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 995–998.

Those dynamical systems on a two-dimensional torus are considered which are structurally stable with respect to (1) the class of C^n $(n = 1, 2, \cdots)$ vector fields, (2) the class of systems with analytic right-hand sides, and (3) the class of systems whose right-hand sides are polynomials of given degree N.

Two theorems are proved in the paper for a dynamical system on a torus which is structurally stable in the class (1), (2) or (3). For such system: (I) There are only a finite number of structurally stable singular points and a finite number of closed orbits, and no separatrix can go from a saddle point to a saddle point; (II) There is no half-trajectory which is stable in the sense of Poisson and which is not a closed orbit [compare M. M. Peixoto, Topology **1** (1962), 101–120; MR **26** #426].

In the proof, the author argues on "a rotated field"; that is, the system $\dot{x} = P(x, y) - \mu Q(x, y)$, $\dot{y} = Q(x, y) + \mu P(x, y)$, where $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ is the original system.
C. Olech (Kraków)

Seibert, Peter 338

Zum Problem der Stabilität unter ständig wirkenden Störungen bei dynamischen Systemen.

Arch. Math. **15** (1964), 108–114.

The author calls a flow π^* on a locally compact metric space X a ϕ-approximation of a flow π on X if ϕ is a non-decreasing function in R_+, $\phi(0) = 0$ and $d(\pi(x, t), \pi^*(x, t)) < \phi(t)$ in $X \times R_+$; π^* is a ϕ-approximation of π in a domain D if this inequality holds for all $t \in R_+$ and those $x \in D$ for which $\pi(x, [0, t]) \cup \pi^*(x, [0, t]) \subset D$. Let M be a compact subset of X and write $S_r = \{x \in X : d(x, M) < r\}$. The author calls M strongly stable relative to π if for each $\varepsilon > 0$ and each ϕ there are positive constants δ, γ such that $\pi^*(S_\delta, R_+) \subset S_\varepsilon$ holds for every $(\gamma \cdot \phi)$-approximation π^* of π in S; if, in addition, for each $\varepsilon > 0$ and each ϕ there is a $\gamma > 0$ such that $\bigcap \overline{\pi^*(G, [t, \infty))} \subset S_\varepsilon$ holds for some $G \subset X$ and all $(\gamma \cdot \phi)$-approximations π^* of π in $S_\varepsilon(G)$, then M is strongly asymptotically stable with set of attraction G. The author proves the equivalence of (i) M is asymptotically stable and its set of attraction contains a compact neighborhood D of M, and (ii) M is strongly asymptotically stable with a set of attraction that contains D. For related results see the author's paper [Internat. Sympos. Nonlinear Differential Equations and Nonlinear Mech., pp. 463–473, Academic Press, New York, 1963; MR **26** #6527].
H. A. Antosiewicz (Los Angeles, Calif.)

Sell, George R. 339

A note on the fundamental theory of ordinary differential equations.

Bull. Amer. Math. Soc. **70** (1964), 529–535.

Given the differential equation (1) $x' = f(x)$, defined and continuous in some open subset W of real n-space R^n, the "integral funnel" through a given point p (also known as the "attainable set") is the set of all points (x, t) in R^{n+1} lying on the solutions of (1) through $x(0) = p$. Its sections for constant t are denoted by $F(p, t)$. Similarly, $F(A, t)$ is the t-section of the integral funnel constituted by all solutions starting at points of the set $A \subset W$. As equation (1) is not required to satisfy the conditions for uniqueness, $F(p, t)$ is, in general, a point-set in W, which might even be empty if all solutions through p leave W before the time t.

In the present paper, characteristic properties of integral funnels are stated, which are essentially modern versions of classical theorems going back to Kneser and Kamke. No proofs are included in this paper. A typical and comprehensive result is the following: The mappings $G: A \to F(A, t)$ and $H: p \to F(p, t)$ are bicontinuous injections of, respectively, the set K of all non-empty compact subsets of W, and the set W itself, into the set $\mathbf{F}$ of all continuous functions $g(t)$, defined in some interval I_g (which depends upon the function g), with values in the set of non-empty compact subsets of W. The continuity is understood in a certain topology defined there.
E. O. Roxin (Providence, R.I.)

Pugh, Charles C. 340

Cross-sections of solution funnels.

Bull. Amer. Math. Soc. **70** (1964), 580–583.

This paper deals with new and interesting questions concerning the "integral funnel" of differential equations. The definitions used are the following. Consider R^{n+1} as (t, y)-space, where t is real and $y = (y^1, \cdots, y^n)$, and let $\mathfrak{F}^n$ denote the set of all continuous maps $f: R^{n+1} \to R^n$ having compact support. Given $p = (t_0, y_0) \in R^{n+1}$ and $f \in \mathfrak{F}^n$, the integral funnel through p, denoted by $F(p)$, is the union of all curves $(t, y(t))$ which satisfy $dy(t)/dt = f(t, y(t))$, $y(t_0) = y_0$. The cross-section of the funnel at time t is denoted by $K_t(p)$. A set $A \subset R^m$ is a funnel-section if for some $n \geqq m$ there exist $f \in \mathfrak{F}^n$ and $p \in R^{n+1}$ such that $i(A) = K_s(p)$ for some real s, where $i: R^m \to R^n$ is the

usual injection of R^m onto the span of the first coordinate axes of R^n. Then the following is proved. Theorem 1: There exists a continuum which is not a funnel-section. Theorem 2: There exists a funnel-section which is not arcwise connected. Theorem 3: Any C^1 polyhedron (image of a finite abstract polyhedron, imbedded in Euclidean space by a map which is bi-C^1 on each simplex) is a funnel-section.

Then the definition is introduced: a, b being reals, $F(a \times A)$ is (a, b)-stable if $A = K_a(b \times K_b(a \times A))$. Theorem 4: Let f be in $\mathfrak{F}^n$ and let A be compact. Suppose that $F(a \times A)$ is (a, b)-stable for some $a, b \in R$. Then $R^n - A$ is diffeomorphic to $R^n - K_b(a \times A)$ by a diffeomorphism which is the identity on some neighborhood of infinity. Theorem 5: Suppose A is a continuum in R^n such that $R^n - A$ is diffeomorphic to $R^n - 0$ by a diffeomorphism which is the identity on a neighborhood of infinity. Then A is a stable funnel-section. If $n \neq 4$, the condition of the diffeomorphism to be the identity on a neighborhood of infinity, might be omitted. Theorem 6: S^1 is a "small funnel-section", i.e., there exist $f \in \mathfrak{F}^2$ and $p = (t_0, y_0) \in R^3$ such that $K_s(p)$ is C^1 equivalent to S^1 for all $s \neq t_0$.

A number of interesting questions are also raised. The proofs of the main results are not included in the paper.

E. O. Roxin (Providence, R.I.)

Pugh, Charles C. 341

The Closing Lemma and structural stability.

Bull. Amer. Math. Soc. **70** (1964), 584–587.

The author states two theorems he has proven which deal with the closing lemma. The closing lemma concerns itself with the following problem. Let M be a differentiable manifold and X a C^1 vector field on M. Let (M, X_t) be the flow whose trajectories are integral curves for X. Suppose now that there is a point p in M such that the orbit of p has a nontrivial recurrent trajectory, that is, p is in either its α or ω limiting set but the orbit of p is neither stationary nor periodic. The closing lemma is the assertion that it is possible to perturb the vector field X by an arbitrarily small amount in any small neighborhood of p such that the integral curve for the new vector field through p is periodic. Theorem 1 states that the closing lemma holds if M is any differentiable 2-manifold. Moreover, the perturbation of X is small in the C^1 sense. Theorem 2 concerns itself with the closing lemma on compact n-dimensional Riemannian manifolds. If M is a Riemannian manifold and f a diffeomorphism of M, then it makes sense to speak of the norm of the linear transformation df_p (the differential of f at p). Theorem 2 states that on a compact n-dimensional Riemannian manifold the closing lemma holds if $\|(dX_tp)^{-1}\|/t$ goes to zero as t goes to infinity. Again the perturbation is small in the C^1 sense. Proofs of these two theorems are not given.

The author concludes the paper by stating and proving the principal lemma used by him to prove Theorems 1 and 2. This is followed by an informal discussion of the relation of his results to structural stability. He indicates that he has proved various theorems. The paper ends with the following conjecture: If M is a differentiable 2-manifold and if the flow induced by the vector field X is crudely structurally stable, then the union of all the closed orbits of this flow is a closed set in M.

This paper is an announcement of results and gives essentially no proofs. *F. Hahn* (New Haven, Conn.)

Jones, G. Stephen 342

Periodic motions in Banach space and applications to functional-differential equations.

Contributions to Differential Equations **3** (1964), 75–106.

The author proves several fixed-point theorems. As an example we give the following theorem. Theorem 4: Let Y be a bounded open convex subset of a Banach space X and x_0 be a boundary point of Y. Let f be a completely continuous mapping of $\bar{Y}$ into itself. Suppose there exists a constant $k > 0$ such that $\|f(x) - x_0\| \geqq k\|x - x_0\|$ for $\|x - x_0\|$ sufficiently small. Suppose further that for any constant $M > 0$ there exists an integer n such that $\|f^n(x) - x_0\| \geqq M\|x - x_0\|$ for $\|x - x_0\|$ sufficiently small. Then f has a fixed point $x^* \neq x_0$ in $\bar{Y}$.

The author then discusses how these theorems may be used to establish the existence of nontrivial periodic solutions for functional-differential equations of the form $\dot{x}(t) = L(H_tx) + G(H_tx)$. In the last section of the paper the author considers several examples.

J. C. Lillo (W. Lafayette, Ind.)

Corduneanu, C. 343

Théorèmes d'existence globale pour les systèmes différentiels à argument retardé. (Russian summary)

Qualitative methods in the theory of non-linear vibrations (Proc. Internat. Sympos. Non-linear Vibrations, Vol. II, 1961), pp. 195–201. Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1963.

The system is

$$\dot{x}(t) = f(t, x(t), x(t-h)), \qquad h > 0,$$

subject to the condition $x(t) = \varphi(t)$, $t_0 - h \leqq t \leqq t_0$, x, f, φ being n-vectors, and f and φ being given. The author uses the Banach fixed-point theorem to establish three existence theorems. He indicates a fourth theorem which he does not state formally, although he outlines the proof. His condition on f is of a modified Lipschitz type that might be difficult to apply when f is nonlinear in its second and third arguments. The methods extend with only algebraic complications to the case of more than one lag h.

E. Pinney (Berkeley, Calif.)

Sędziwy, S. 344

On the trajectories and quasitrajectories of a two-dimensional orientor field.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 157–159.

Let x be an absolutely continuous real-valued function of a real variable t. Then the contingent derivative $Dx(t_0)$ of x at t_0 is the set of all limits of the difference quotient as t approaches t_0 through a sequence. If, moreover, $E(t, x)$ is a function defined on a domain G in the plane with values in the space of compact subsets of the line, we may study the "differential equation" $Dx(t) \subseteq E(t, x(t))$. Let $C(t, x)$ be the convex closure of $E(t, x)$. It is here shown that every solution of $Dx(t) \subseteq C(t, x(t))$ defined on an interval J is the limit of a sequence of solutions of $Dx(t) \subseteq E(t, x(t))$, uniformly on compact subsets of J. The proof is based upon comparison with solutions of $x' = \min E(t, x)$ and $x' = \max E(t, x)$.

B. L. Reinhart (College Park, Md.)

Sideriades, L. 345
Méthodes de topologie qualitative: Applications à l'étude des cheminées d'équilibre. (Russian summary)
Qualitative methods in the theory of non-linear vibrations (*Proc. Internat. Sympos. Non-linear Vibrations, Vol. II,* 1961), *pp.* 342–355. *Izdat. Akad. Nauk Ukrain. SSR, Kiev,* 1963.

The author announces the results of phase-plane and three-dimensional phase-space analyses of systems of differential equations representing hydraulic surge-tanks either running free, or with one of two types of governors.
D. Bushaw (Pullman, Wash.)

PARTIAL DIFFERENTIAL EQUATIONS

See also 245, 255, 256, 263, 743, 746, 794, 795, 831, 839, 897.

Bers, Lipman; John, Fritz; Schechter, Martin 346
★Partial differential equations.
With special lectures by Lars Gårding and A. N. Milgram. Proceedings of the Summer Seminar, Boulder, Colorado, 1957. Lectures in Applied Mathematics, Vol. III.
Interscience Publishers [*John Wiley & Sons, Inc.*], *New York-London-Sydney,* 1964. xiii + 343 *pp.* $10.70.

The lectures in this volume will be reviewed individually.

Petuhov, V. R. 347
A differential-functional equation in partial derivatives. (Russian)
Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 1 (38), 96–99.

The author considers the equation $u_t(t, x) = \lambda u(t, u_x(t, x))$ in the region $t \geqq 0$, $-\infty < x < \infty$, with the initial condition $u(0, x) = \phi(x)$. It is shown by the contraction principle that under suitable conditions there is one and only one solution.
W. A. Coppel (Canberra)

Nirenberg, L.; Treves, F. 348
Solvability of a first order linear partial differential equation.
Comm. Pure Appl. Math. **16** (1963), 331–351.

A linear partial differential operator L with complex C^∞ coefficients is called solvable on an open set $\Omega \subset R^n$ if for every $f \in C_0^\infty(\Omega)$ the equation $Lu = f$ has a solution $u \in D'(\Omega)$. It is solvable at a point ω if it is solvable on some neighborhood of ω.

In 1957 H. Lewy [Ann. of Math. (2) **66** (1957), 155–158; MR **19**, 551] discovered the fact that the operator $L = \partial/\partial x^1 + i\partial/\partial x^2 - 2i(x^1 + ix^2)\partial/\partial x^3$ is not solvable. In fact, for most C^∞ functions f the equation $Lu = f$ has no solution on any non-void open set.

In 1960 L. Hörmander [Math. Ann. **140** (1960), 124–146; MR **24** #A434; ibid. **140** (1960), 169–173; MR **26** #5279] gave a striking necessary condition for an operator L to be solvable at a point. Let L be of order m and let C_{2m-1} be the terms of order exactly $2m-1$ in the commutator $L\bar{L} - \bar{L}L$. (The coefficients in $\bar{L}$ are the conjugates of those in L.) If L is solvable at ω, then (for the characteristic polynomials) $C_{2m-1}(x; \xi) = 0$ if $L_0(x; \xi) = 0$ for x in a neighborhood of ω and $\xi \in R^n$. When L has order 1 Hörmander's condition becomes simply that C_1 is a linear combination of the leading parts L_0 and $\bar{L}_0$. If L_0 and $\bar{L}_0$ are linearly independent at ω, then this condition is also sufficient [L. Nirenberg, Seminars on Analytic Functions, Vol. I, pp. 172–189, Inst. for Advanced Study, Princeton, N.J., 1957].

In this paper the authors study the first-order case very carefully. They formulate a certain condition—(P) below—and show that it is sufficient for solvability, and another one, (P′), between (P) and the condition of Hörmander, and show that it is necessary for solvability. They conjecture that (P) is both necessary and sufficient.

Let C^k be the commutator $L_0C^{k-1} - C^{k-1}L_0$ ($C^1 = C_1$) and let $k(x)$ be the first value of k (if there is one) such that, at x, C^k is not a linear combination of L_0 and $\bar{L}_0$. The condition (P′) is as follows. (P′): In some neighborhood of ω, $k(x)$ if finite is even. Hörmander's condition is just that in some neighborhood of ω, $k(x)$ is never 1. In order to state condition (P) choose new (local) coordinates so that L_0 takes the form

$$L_0 = g\left(\frac{\partial}{\partial x^n} + i \sum_1^{n-1} b^k \frac{\partial}{\partial x^k}\right),$$

where g is a complex C^∞ function and the b^k are real C^∞ functions. The authors show that this is always possible and that the condition (P) does not depend on the way in which it is done. (It is assumed, of course, that L is strictly first-order at ω.)

(P): In some neighborhood of ω, the direction of the vector b is independent of x^n. In the new coordinates, L_0 and $\bar{L}_0$ are linearly independent at a point x if and only if $b(x) \neq 0$. In this case, (P), (P′), and the condition of Hörmander are all equivalent. At a point x where L_0 and $\bar{L}_0$ are linearly dependent, $k(x)$ is the least k such that $(\partial/\partial x^n)^k b \neq 0$, so that (P′) has a simple interpretation in terms of b.

In addition to the main theorem that (P) is sufficient for solvability and (P′) is necessary, there are interesting corollaries that give general circumstances under which (P) and (P′) are equivalent. One such is when b has no zero of infinite order as a function of x^n. In particular, (P) and (P′) are equivalent if the leading coefficients of L are analytic.

As an example, the theorems show that the operator $\partial/\partial t + it^k\partial/\partial x$ is solvable at points on the x-axis if and only if k is even (for in this case clearly $k(x) = k$).
K. T. Smith (Madison, Wis.)

Fal'kovič, S. V. 349
On the theory of quasilinear equations.
Prikl. Mat. Meh. **26** (1962), 571–572 (*Russian*); *translated as J. Appl. Math. Mech.* **26** (1962), 855–857.

The author briefly discusses the solution of the system

$$(*)\quad A_i\frac{\partial u}{\partial x} + B_i\frac{\partial u}{\partial y} + C_i\frac{\partial v}{\partial x} + D_i\frac{\partial v}{\partial y} + E_i = 0, \qquad i = 1, 2,$$

by the method of characteristics. He points out that a statement by V. V. Sokolovskiĭ [*Statics of granular media* (Russian), 3rd ed., rev. and suppl., Fizmatgiz, Moscow, 1960; MR **23** #B1774] implying a certain curve to be a characteristic of (*) is, in general, not correct unless $E_1 = E_2 = 0$.
P. Henrici (Zürich)

Heins, Albert E.; MacCamy, Richard C. 350

Integral representations of axially symmetric potential functions.

Arch. Rational Mech. Anal. **13** (1963), 371–385.

The authors consider the special case $k=1$ of the singular partial differential equation

$$(*) \qquad L_k[u]+q(r^2, z)u = 0,$$

where

$$L_k[u] = u_{rr}+kr^{-1}u_r+u_{zz},$$

assuming that the function q is analytic and entire in both arguments. Their general objective is to extend to this situation the results obtained by the reviewer [*Boundary problems in differential equations*, pp. 19–34, Univ. Wisconsin Press, Madison, Wis., 1960; MR **22** #2785] for arbitrary $k \geqq 0$, but with q depending on z only. Particular results include: (a) a uniqueness theorem, showing that a solution of (*) which is analytic on some segment of $r=0$ is uniquely determined by its values on that segment; (b) the following existence theorem: If $g=g(z)$ is real for z real and analytic in a simply connected region R symmetric with respect to the real axis, then there exists a solution $u(r, z)$ that is real-analytic for $z+ir \in R$ and satisfies $u(0, z)=g(z)$ for real $z \in R$; (c) a representation of the above solution in the form

$$u(r, z) = \int_{z-ir}^{z+ir} E(r, z, t)g(t)\,dt,$$

where the function E is constructed by a process of successive approximations; (d) the following regularity property: If u is a solution of (*) regular in a region R as above, then the function $g(\zeta)$ which is the extension to complex $\zeta=z+ir$ of $u(0, z)$ is analytic in R. There is a discussion of the special case where q depends on r only.

P. Henrici (Zürich)

Günther, Paul 351

Das iterierte Anfangswertproblem bei der Darbouxschen Differentialgleichung.

Math. Nachr. **25** (1963), 293–310.

The following problem is discussed: In a space of $n+2$ dimensions $(t_1, t_2, x^1, \cdots, x^n)$, $u(t_1, t_2, x)$ satisfies

$$\frac{\partial^2 u}{\partial t_1{}^2}+\frac{\lambda}{t_1}\frac{\partial u}{\partial t_1}-L[u] = 0,$$

where the function $u(0, t_2, x)$ is the solution of

$$\frac{\partial^2 u}{\partial t_2{}^2}+\frac{\lambda}{t_2}\frac{\partial u}{\partial t_2}-L[u] = 0,$$

$$u(0, 0, x) = \phi(x),$$

$$\frac{\partial u}{\partial t_2}(0, 0, x) = 0.$$

Here L is a linear elliptic second-order operator, and λ is a constant.

The solution is discussed as a function of λ by means of a recursion formula closely related to that used by A. Weinstein in his work on the Euler-Poisson-Darboux equation [Comm. Pure Appl. Math. 7 (1954), 105–116; MR **16**, 137]. No mention is made of this work.

H. F. Weinberger (Minneapolis, Minn.)

Bramble, J. H.; Payne, L. E. 352

Bounds for solutions of second-order elliptic partial differential equations.

Contributions to Differential Equations **1** (1963), 95–127.

The authors show how to find explicit bounds for the value at a point O of the solution of the Dirichlet problem $L(\psi)=(a^{ij}(x)\psi_{ij})_i=g$ in D, with ψ given on the boundary C of D. Their method is to approximate ψ by an arbitrary function φ, and derive integral estimates for the difference $u=-\varphi$.

For example, if P_1 is any point of C, the bound

$$(*) \quad |u(0)-u(P_1)| \leqq (I_1A^{1/2}d+I_2)\left\{g\left[a_1\bar{a}\oint_C l^{-1}r^{-(q+1)}[u(P)-u(P_1)]^2\,ds\right]^{1/2} + \left[\int_D r^{-(q-2)}(L(u))^2\,dv\right]^{1/2}\right\}$$

is derived. Here q is any number satisfying $N-2<q<N$, and if the boundary values of u are Hölder continuous, it can be chosen so that the boundary integral converges. The constants are all explicitly determined in terms of q, D, and the coefficients a_{ij} and their first derivatives.

This bound has the advantage over the bounds previously given by Payne and the reviewer [Pacific J. Math. **8** (1958), 551–573; MR **21** #2809] that it approaches zero, rather than infinity, as O approaches the boundary point P.

The bound (*) is derived by means of several inequalities involving weighted integrals. One example is

$$(**) \quad \int_D r^{-q}|u|^p\,dv \leqq \beta_3\oint_C r^{-(q-1)}|u|^p\,ds + \beta_4\int_D r^{-(q-2p)}|L(u)|^p\,dv,$$

which holds for any $p \geqq 2$ and $2 \leqq q < N$. The constants β_3 and β_4 are determined explicitly.

The technique for deriving this and other inequalities is to introduce an auxiliary potential function w which is the solution of

$$L(w) = r^{-q}|u|^{p/2} \quad \text{in } D$$
$$w = 0 \quad \text{on } C.$$

Then

$$\int_D r^{-q}|u|^p\,dv = \int_D |u|^{p/2}L(w)\,dv,$$

and an application of Green's theorem and Schwarz's inequality, together with some bounds previously derived by Payne and the reviewer [loc. cit.], leads to (**).

The last part of the paper improves the bound of Payne and the reviewer [loc. cit.] for the energy integral by rendering the constants involved independent of the derivatives of the coefficients.

H. F. Weinberger (Minneapolis, Minn.)

Bramble, J. H.; Payne, L. E. 353

Some integral inequalities for uniformly elliptic operators.

Contributions to Differential Equations **1** (1963), 129–135.

By techniques similar to those used in the paper reviewed above [#352], the authors show that if

$$\frac{\partial g}{\partial \nu}+\alpha g = 0 \text{ on } C,$$

where C is a closed surface with interior D, α is a given function on C, and $\partial g/\partial \nu$ is the conormal derivative with respect to the uniformly elliptic operator

$$L(u) \equiv (a^{ij}g_{,i})_{,j},$$

then

$$\oint_C \left|\frac{\partial g}{\partial \nu}\right| ds \leqq \int_D |L(g)|\, dv.$$

The same inequality in the case $g=0$ on D was found by G. Fichera [Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **19** (1955), 411–418; MR **18**, 131].

It is also shown that if $L(h)=0$ in D,

$$\oint_C \alpha|h|\, ds \leqq \oint_C \alpha \left|\frac{1}{\alpha}\frac{\partial h}{\partial \nu}+h\right| ds.$$

The authors also show that

$$\int_D u^{4p}\, dv \leqq 2^{4p-1}\left\{\frac{2p}{K(p)}\oint_C \alpha\left(\frac{1}{\alpha}\frac{\partial u}{\partial \nu}+u\right)^{4p} ds + \left[\frac{2p}{K(p)}\right]^{4p}\int_D [L(u)]^{4p}\, dv\right\},$$

where $K(p)$ is an explicitly known constant depending only on p, D and the a^{ij} but not on the derivatives of the a^{ij}. The constant p may be $\frac{1}{2}$ or a positive integer. This inequality can be used in conjunction with those of the paper reviewed above to approximate the solution of the boundary-value problem $L(\psi)=g$, $\partial\psi/\partial\nu+\alpha\psi$ given on C.

H. F. Weinberger (Minneapolis, Minn.)

Bramble, J. H.; Payne, L. E. 354

Upper and lower bounds in equations of forced vibration type.

Arch. Rational Mech. Anal. **14** (1963), 153–170.

The techniques developed in the two papers reviewed above [#352; #353] are extended to the case of the equation

$$L(\psi)+p(x)u = f,$$

where p need not be non-negative.

A bound of the form

$$\int_R w^2\, dV \leqq \left[\left[1+\frac{1}{\rho}\right]\left\{K\oint_\Gamma t^{-1}w^2\, dS\right\}^{1/2} + \frac{1}{\rho}\left\{\int_R [L(w)+pw]^2\, dV\right\}^{1/2}\right]^2$$

is derived, where K is an explicit constant depending on the domain and the equation, and ρ is the magnitude of the smallest eigenvalue of the operator $L+p$. (The domain is now R, its boundary Γ.) This is a generalization of the corresponding inequality with $w=0$ on Γ given by the reviewer [Proc. Sympos. Appl. Math., Vol. VIII, pp. 89–91, McGraw-Hill, New York, 1958; MR **24** #B1313].

By using an artificial dimension to make p non-positive, the authors obtain pointwise bounds from this inequality. Bounds are also given for the eigenfunction of the problem $Lu+p(x)u+\lambda u=0$, $u=0$ on Γ_1, corresponding to an eigenvalue that is only known approximately.

H. F. Weinberger (Minneapolis, Minn.)

Nakamori, Kanzi 355

Fundamental theorems for an elliptic system of partial differential equations of first order in two independent variables. (Esperanto)

Yokohama Math. J. **9** (1961), 1–27.

The author studies existence, uniqueness and comparison theorems for elliptic systems of the form

$$\frac{\partial u_1}{\partial x}-\frac{\partial u_2}{\partial y} = f_1(x, y, u_1, u_2),$$

$$\frac{\partial u_2}{\partial x}+\frac{\partial u_1}{\partial y} = f_2(x, y, u_1, u_2),$$

under several conditions, such as $u_1=0$, $|u_2|<A$ on the boundary. *G.-C. Rota* (Cambridge, Mass.)

Maz'ja, V. G. 356

On the boundary regularity of solutions of elliptic equations and of a conformal mapping. (Russian)

Dokl. Akad. Nauk SSSR **152** (1963), 1297–1300.

The author considers the Dirichlet problem for the uniformly elliptic equation

$$(*) \qquad a_{ij}u_{x_ix_j}+b_iu_{x_i}+(c_iu)_{x_i}+du = f_0+(f_i)_{x_i}$$

in a bounded region $D\subset R^n$ $(n\geqq 2)$, with boundary $\mathscr{F}D$. It is assumed that $a_{ij}\in c^{(0,\lambda)}(\bar{D})\cap W_{2q}{}^{(1)}(D)$, $b_i, c_i, f_i\in L_{2q}(D)$, $d, f_0\in L_q(D)$, $2q>n$. A weak solution with zero Dirichlet data is a $u\in \mathring{W}_2{}^{(1)}(D)$ satisfying the equation obtained by multiplying (*) by a test function and integrating by parts once. Let Σ_ρ be the ball of radius ρ centered at the boundary point 0. It is shown that if

$$\liminf_{N\to\infty}\frac{1}{N}\sum_1^N 2^{(n-2)\nu}\operatorname{cap}(\mathscr{C}D\cap\Sigma_{2^{-\nu}}) > 0,$$

then u is Hölder continuous at 0 (neglecting null sets). If $\sum 2^{(n-2)\nu}\operatorname{cap}(\mathscr{C}D\cap\Sigma_{2^{-\nu}})=\infty$, then u is continuous at 0. ($\mathscr{C}$ denotes the complement and "cap" the Wiener capacity.) Results on boundary behavior of quasiconformal mappings are also obtained.

W. Littman (Minneapolis, Minn.)

Nečas, Jindřich 357

Sur une méthode pour résoudre les équations aux dérivées partielles du type elliptique, voisine de la variationnelle.

Ann. Scuola Norm. Sup. Pisa (3) **16** (1962), 305–326.

Soient Ω un ouvert de R^n et $W^{(k)}_{p,\alpha}(\Omega)$ l'espace des fonctions u telles que $D^iu\in L_p$ sur Ω pour la mesure $\rho^\alpha(x)\,dx$, où $\rho(x)$ est la distance de x à la frontière de Ω. Etude des traces des éléments de $W^{(k)}_{p,\alpha}(\Omega)$ sur des variétés de dimension $n-1$, sans caractérisation de ces traces; théorèmes de plongement. L'auteur montre que si Ω est de type $\mathscr{N}^{0,1}$ (frontière Lipschitzienne avec une petite condition supplémentaire) il existe une fonction σ continue dans $\bar{\Omega}$, indéfiniment différentiable dans Ω et "équivalente" à ρ et telle que $|D^i\sigma(x)|\leqq c_i/\sigma(x)^{|i|-1}$, pour chaque i, où c_i est une constante dépendant de i.

Soit ensuite (dans les notations de l'auteur) une forme sesquilinéaire

$$B(v, u) = \sum_{|i|,|j|\leqq k}\int_\Omega a_{ij}(x)D^iv\overline{D^ju}\, dx, \qquad a_{ij}\in L_\infty(\Omega).$$

On suppose que $|B(\varphi, \varphi)| \geq c|\varphi|^2_{W_2^{(k)}}$, $\varphi \in \mathscr{D}(\Omega)$, $W_2^{(k)} = \mathring{W}^{(k)}_{2,0}$. Soient $\mathring{W}^{(k)}_{2,\alpha}(\Omega)$ l'adhérence de $\mathscr{D}(\Omega)$ dans $W^{(k)}_{2,\alpha}(\Omega)$ et $W_{2,\alpha}^{-(k)}(\Omega)$ le dual de $\mathring{W}^{(k)}_{2,-\alpha}(\Omega)$. L'auteur montre alors, à l'aide des résultats précédents, des inégalités valables pour α assez petit, disons $\alpha \in J$, qui impliquent que $D = \sum_{i,j} (-1)^{|i|} D^i(a_{ij} D^j)$ est, pour $\alpha \in J$, un isomorphisme de $\mathring{W}^{(k)}_{2,\alpha}(\Omega)$ sur $W^{-k}_{2,\alpha}(\Omega)$. *J. L. Lions* (Paris)

Chen, Liang-jing [Chen, Liang-ching] 358
A boundary-value problem for a degenerate elliptic equation.
Acta Math. Sinica **13** (1963), 332–342 (*Chinese*); *translated as Chinese Math.* **4** (1964), 360–371.

The author proves that the solution of the boundary-value problem

$$\mathscr{L}(u) \equiv y^m u_{yy} + u_{xx} + au_x + bu_y + cu = 0,$$

$$u|_\Gamma = \varphi,$$

$$\lim_{\substack{P\in D\\ P\to P_i}} \frac{u}{W} = \lim_{\substack{P\in\Gamma\\ P\to P_i}} \frac{\varphi}{W},$$

$$\lim_{P\to Q\in\gamma} \frac{u_y}{H} = \psi$$

is unique, where $a(x, y)$, $b(x, y)(>0)$, and $c(x, y)$ are analytic in D, a simple connected domain in the upper half-plane whose boundary is $\Gamma + \gamma + \{P_i\}$, where Γ consists of a finite number of open Jordan arcs consisting of a finite number of open intervals of the X-axis with $\{P_i\}$ the set of all corners; $1 < m < 2$, φ and ψ are given functions on Γ and γ, respectively, H and W are functions satisfying the following conditions:

$$\lim_{y\to 0} H(x, y) = \infty,$$

$$\frac{y^m H_{yy} + my^{m-1}H_y + H_{xx} + a_0 H_x + b_0 H_y}{H} = O(1),$$

$$\mathscr{L}(W) < 0, \quad W > 0 \text{ in } D.$$

If φ and ψ are continuous on Γ and γ, respectively, and the relationship

$$\lim_{\substack{P\in\Gamma\\ P\to P_i}} \frac{\varphi}{W} = -\lim_{\substack{P\in\gamma\\ P\to P_i}} \psi$$

is satisfied, then there exists a solution for the boundary-value problem. *M. Coroi-Nedelcu* (Bucharest)

Kuks, L. M. 359
Some properties of the solutions of nonlinear elliptic equations. (Russian)
Izv. Vysš. Učebn. Zaved. Matematika **1958**, *no.* 4 (5), 131–139.

L'auteur donne quelques résultats concernant les équations aux dérivées partielles du second ordre, résultats qui rappellent des faits bien connus de la théorie des équations ordinaires (théorèmes de comparaison, identité de Picone, théorèmes de séparation, etc.). Mentionnons le résultat suivant. Soit

$$\text{(E)} \quad \sum_{i,k=1}^{2} \frac{\partial}{\partial x_i}\left(A_{ik}\frac{\partial u}{\partial x_k}\right) + f_1\left(x_1, x_2, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right) = 0,$$

$$\text{(E')} \quad \sum_{i,k=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial z}{\partial x_k}\right) + f_2\left(x_1, x_2, z, \frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}\right) = 0$$

deux équations telles que $A_{ik} = A_{ki}$ et $a_{ik} = a_{ki}$ soient définis dans un domaine D du plan (x_1, x_2). Supposons que $A_{11} \geq a_{11}$, $(A_{11} - a_{11})(A_{22} - a_{22}) - (A_{12} - a_{12})^2 \geq 0$ dans D et qu'il existe une solution $u \not\equiv 0$ de (E), s'annullant sur la frontière de D. Alors, toute solution z de (E'), linéairement indépendante de u dans D, admet des zéros dans D dès que

$$\iint_D u^2 \left[\frac{f_2\left(x_1, x_2, z, \frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}\right)}{z} - \frac{f_1\left(x_1, x_2, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right)}{u}\right] dx_1 dx_2 \geq 0.$$

La convergence de l'intégrale résulte de l'identité semblable à l'identité de Picone. *C. Corduneanu* (Iași)

Ovsepjan, S. G. 360
The first boundary-value problem for certain differential equations with constant coefficients. (Russian)
Dokl. Akad. Nauk SSSR **155** (1964), 524–527.

This note takes up the relative differential boundary problem $Mu - \lambda Lu = h$, under the circumstances that (1) L and M are partial differential operators with constant coefficients and common order $2s$, (2) M is elliptic, and (3) the solution u and its normal derivatives up to order $s-1$ vanish on the surface of an ellipsoidal region in n-space. The analysis depends upon the author's earlier study of polynomial eigenfunctions of the homogeneous equation [Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk **16** (1963), no. 3, 23–38; MR **27** #5019]. *Robert McKelvey* (Madison, Wis.)

Wallin, N. O. 361
A problem for elliptic partial differential operators defined on an arbitrary open set.
Math. Scand. **9** (1961), 185–196.

Let a be a linear elliptic operator of arbitrary order with smooth coefficients, and $V(x)$ a real function defined on a domain Ω in R^n. The domain of the operator $a + V$ is defined as the set of square integrable functions f such that $(a + V)f$ is again square integrable in Ω. The author shows that if V increases sufficiently rapidly near the boundary of Ω, the adjoint of $a + V$ is given by $(a + V)^* = \bar{a} + V$, where $\bar{a}$ is the formal adjoint of a. (The required speed of increase depends on a and Ω.) This generalizes a result of F. Browder [C. R. Acad. Sci. Paris **246** (1958), 526–528; MR **20** #5348] for $\Omega = R^n$. *H. F. Weinberger* (Minneapolis, Minn.)

Pini, Bruno 362
Osservazioni sulla ipoellitticità.
Boll. Un. Mat. Ital. (3) **18** (1963), 420–432.

It is well known that if $P(D_1, D_2) = P(-i\partial/\partial x, -i\partial/\partial y) = \sum a_{rs} D_1^r D_2^s$ $(r \leq p, s \leq q)$ is a hypo-elliptic operator with constant coefficients, then $a_{p0}, a_{0q} \neq 0$, and $a_{rs} = 0$ when $r/p + s/q > 1$. The author considers (for $p \geq q$) the associated operators $P_k(D_1, D_2) = \sum a_{rs} D_1^r D_2^s$ $(1 - k/p < r/p + s/q \leq 1$, $k = 0, 1, \cdots)$. He shows that if m is the maximal multiplicity of the real zeros of $P_0(\lambda_1, \lambda_2) = \lambda_2^p P_0(\lambda_1/\lambda_2, 1) = 0$, then P_m is the principal part of P, in the sense that the hypo-ellipticity of P is not influenced by the terms in the remainder $P - P_m$. Some non-trivial examples are also discussed. *J. Friberg* (Göteborg)

Vaghi, Carla 363
Sulla regolarizzazione delle soluzioni dell'equazione non omogenea delle onde.
Ist. Lombardo Accad. Sci. Lett. Rend. A **96** (1962), 883–902.

The author generalizes a result of O. A. Ladyženskaya [Dokl. Akad. Nauk SSSR **74** (1950), 417–420; MR **14**, 280; ibid. **85** (1952), 481–484; MR **14**, 474] concerning the regularity of weak solutions of the wave equation

$$(1)\qquad \frac{\partial^2 u}{\partial t^2} = Au + f(x,t), \quad x\in\Omega,\ t\in[0,T],$$

where Ω is a bounded domain in R^n and A is the second-order elliptic operator on Ω given by

$$(2)\qquad Av = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial v}{\partial x_j}\right) - a(x)v,$$

where $a_{ij}(x) = a_{ji}(x)$ and $a(x) \geqq 0$ are bounded real measurable functions on Ω such that

$$\inf_{x\in\Omega} \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqq \nu \sum_{i=1}^{n} \xi_i^2 (\nu > 0).$$

The problem treated here is that of sufficient conditions on the coefficients of A, the domain Ω, and the given function f so that a weak solution of (1) satisfying generalized homogeneous initial conditions and zero Dirichlet condition on $\partial\Omega$ has certain prescribed regularity properties. Specifically, conditions are stated in order that the formal eigen-expansion solution

$$(3)\qquad \sum_{k=1}^{\infty} \lambda_k^{-1/2} \int_0^t f_k(\tau) \sin \lambda_k^{1/2}(t-\tau)\, dt\, v_k(x),$$

with $f_k(t) = (f(0,t), v_k)_{L_2(\Omega)}$ and $Av_k = \lambda_k v_k$, be a weak solution of the given problem and that the formally differentiated series to order $\leqq m$ converge in $L_2(\Omega)$ uniformly in $t\in[0,T]$.

Assuming Ladyženskaya's conditions on A and Ω, the author weakens the Ladyženskaya conditions on f by introducing certain Banach spaces $Z^{(m)}(Q)$, $Q = \Omega\times(0,T)$, analogous to those of L. Amerio [Ann. Mat. Pura Appl. (4) **49** (1960), 393–417; MR **22** #8210] which topologically include the usual Sobolev L_2-spaces $H_m(Q)$ in terms of which the Ladyženskaya conditions are expressed. Let $Z(Q)$ denote the L_1 space of $L_2(\Omega)$-valued functions on $[0,T]$. For $m\geqq 1$, $Z^{(m)}(Q)$ is the linear space of all f such that the distribution derivatives of f (with respect to both x and t) for all orders $\leqq m$ belong to $Z(Q)$. Suitably normed as a product space, $Z^{(m)}(Q)$ is a Banach space such that $Z^{(m)}(Q) \supset H_m(Q)$. Finally, let $Z_0^{(1)}(Q)$ be that subspace of $Z^{(1)}(Q)$ for which $f\in \mathring{H}_1(\Omega)$ for almost all t. The author's conditions on f are then expressible as follows: $f\in Z^{(m-1)}(Q)$ and $A^j f \in Z_0^{(1)}(Q)$, $j = 0, 1, \cdots, [\frac{1}{2}m]-1$. *C. R. DePrima* (Pasadena, Calif.)

Matsumura, Mutsuhide 364
Existence locale de solutions pour quelques systèmes d'équations aux dérivées partielles.
Japan. J. Math. **32** (1962), 13–49.

The author considers systems of first-order equations, and obtains results analogous to those of Hörmander in the case of a single equation of higher order [Math. Ann. **140** (1960), 124–146; MR **24** #A434; ibid. **140** (1960), 169–173; MR **26** #5279]. The bulk of the paper is devoted to sharpened versions and extensions of results of Calderón and Zygmund on singular integral operators [Amer. J. Math. **79** (1957), 901–921; MR **20** #7196]. If H is a singular integral operator, then the symbol of H is denoted by $\sigma(H) = h(x,\xi)$. Λ denotes $\sqrt{-\Delta}$, and an operator K is said to be of order $\geqq 1$ if ΛK and $K\Lambda$ are bounded from L_2 to L_2. $H\circ K$ is defined by $\sigma(H\circ K) = \sigma(H)\cdot\sigma(K)$. Then it is shown that $\Lambda H - H\Lambda = H_0 + H_1$ where

$$\sigma(H_0) = -i\sum \frac{\partial h}{\partial x_j}\frac{\partial}{\partial \xi_j}(|\xi|),$$

and H_1 is of order $\geqq 1$. Similarly, $(HK - H\circ K)\Lambda = J_0 + J_1$ where

$$\sigma(J_0) = -i\sum |\xi| \frac{\partial h}{\partial \xi_j}\frac{\partial k}{\partial x_j},$$

and J_1 is of order $\geqq 1$.

The operator δ_H is defined by $\delta_H = \partial/\partial t - iH\Lambda$. A train of reasoning based on the preceding results shows that if

$$\operatorname{Im}\left[\frac{\partial h}{\partial t} - \sum \frac{\partial}{\partial \xi_j}(\bar{h}|\xi|)\frac{\partial h}{\partial x_j}\right] = f(t,x,\xi)\operatorname{Im}(h),$$

where f is a smooth function of degree zero in ξ, then using L_2 norms for functions of x and t, there exists a number C depending on the support of φ, such that

$$\|\varphi\| \leqq C\|\delta_H\varphi\| \text{ and } \|\varphi\| \leqq C\|\delta_H{}^*\varphi\|$$

for all $\varphi\in C_0^\infty$. This result implies an a priori estimate, and hence an existence theorem for certain first-order systems satisfying a commutator condition. Finally, using a construction similar to that of Hörmander, a sufficient condition for non-existence of solutions of a system of first-order partial differential equations is obtained.
D. Ludwig (Berkeley, Calif.)

Freĭdlin, M. I. 365
The Dirichlet problem for an equation with periodic coefficients depending on a small parameter. (Russian. English summary)
Teor. Verojatnost. i Primenen. **9** (1964), 133–139.

Author's summary: "This paper studies the limiting behaviour of the solution $u^\varepsilon(x)$ of Dirichlet's problem for the equation

$$L^\varepsilon u^\varepsilon = \frac{1}{2}\sum a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial^2 u^\varepsilon}{\partial x^i\,\partial x^j} + \sum b_i\left(\frac{x}{\varepsilon}\right)\frac{\partial u^\varepsilon}{\partial x^i} - c\left(\frac{x}{\varepsilon}\right)u^\varepsilon = 0$$

when $\varepsilon\to 0$. The coefficients of the operator L^1 are assumed to be periodical. It is proved that $\lim_{\varepsilon\to 0} u^\varepsilon(x) = u(x)$ exists. The function $u(x)$ is a solution of Dirichlet's problem for the equation $\bar{L}u = 0$, where the coefficients of the operator $\bar{L}$ are obtained by averaging the coefficients of the operator L^ε." *W. H. Fleming* (Providence, R.I.)

Carroll, Robert 366
On some singular quasi-linear Cauchy problems.
Math. Z. **81** (1963), 135–154.

Consider the equation

$$(1)\qquad (L + Q(t)\Lambda)w + f(t,w) = 0,$$

where

$$L = \frac{d^2}{dt^2} + (\alpha+\beta)\frac{d}{dt} + \gamma$$

with α, β, γ, Q numerical functions of t and α singular at $t=0$, Λ is a self-adjoint linear operator, semi-bounded below, with domain $D(\Lambda)$ dense in the separable Hilbert space H, and $f(t, w)$ is non-linear. Conditions are given under which there exists a solution w with $t \to w(t) \in E^2(H)$ ($E^m(G)$ is the space of m-times continuously differentiable functions of t with values in G), $w(\tau) = T \in D(\Lambda)$ $(0 \leqq \tau \leqq t \leqq b < \infty)$, $t \to w(t) \in E^0(D(\Lambda))$, where $D(\Lambda)$ has the graph topology, and $\partial w(\tau)/\partial t = 0$. The last condition arises from the singular nature of α at $t=0$. Assume $\alpha, \beta, \gamma, Q \in C^0(0, b]$ and $t \to f(t, w): R \to H$ is continuous if, for example, $t \to w(t) \in E^0(D(\Lambda))$. If $\tau > 0$, the problem is regular and may be treated as the first-order system

$$(2) \qquad \frac{\partial}{\partial t}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ Q\Lambda + \gamma & \alpha + \beta \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix} = 0,$$

where $w_1 = w$ and $w_2 = w_t$. For $\tau > 0$ conditions are given for the construction of a Green's operator for (2) in the form

$$(3) \qquad G(t, \tau) = \begin{pmatrix} Z(t, \tau) & Y(t, \tau) \\ Z_t(t, \tau) & Y_t(t, \tau) \end{pmatrix},$$

where $Z(\tau, \tau) = I$, $Z_t(\tau, \tau) = 0$, $Y(\tau, \tau) = 0$, and $Y_t(\tau, \tau) = I$ (because of the singularity, $Y_t(t, \tau)$ is not continuous in both variables as $\tau \to 0$). Using $G(t, \tau)$, the non-linear equation

$$(4) \qquad w = Z(t, \tau)T - \int_\tau^t Y(t, \xi) f(\xi, w)\, d\xi$$

is derived and studied. From the study of (4) follows the existence of $w(t)$ satisfying (1) and the previously stated conditions. *J. R. Cannon* (Upton, N.Y.)

Carroll, Robert 367
Some degenerate Cauchy problems with operator coefficients.
Pacific J. Math. **13** (1963), 471–485.
Using an energy method based on the work of Lions [Rend. Sem. Mat. Fis. Milano **28** (1959), 149–188; ibid. **29** (1959), 235–239; MR **23** #A1947], the author proves the existence of weak solutions of some degenerate Cauchy problems. For a summary of the results, see the author [C. R. Acad. Sci. Paris **253** (1961), 1193–1195; MR **24** #A3443]. *J. R. Cannon* (Upton, N.Y.)

Chu, Sherwood C.; Diaz, J. B. 368
Remarks on a mixed boundary-value problem for linear hyperbolic partial differential equations in two independent variables.
Arch. Rational Mech. Anal. **16** (1964), 187–195.
The differential equation is

$$u_{xy} + au_x + bu_y + cu = d,$$

to be satisfied inside a rectangle, with subsidiary conditions

$$\alpha_0 u + \alpha_1 u_x + \alpha_2 u_y = \sigma \text{ on } y = f_1(x),$$
$$\beta_0 u + \beta_1 u_x + \beta_2 u_y = \tau \text{ on } x = f_2(y),$$
$$u(x^*, y^*) = \gamma.$$

All coefficients are functions of x and y. The point (x^*, y^*) generally does not lie either on the rectangle or on the curves $y = f_1(x)$ or $x = f_2(y)$. A number of different cases, involving assumptions on the coefficients, are discussed, e.g., $\alpha_2 \equiv 0$, $\beta_1 \equiv 0$, α_1 and β_2 never vanish. One result is an existence and uniqueness theorem, provided that the rectangle is sufficiently small. A counterexample is exhibited for the case of a large rectangle. The proofs are based on Riemann's representation formula and Picard iterations. *D. Ludwig* (Berkeley, Calif.)

Mihaĭlov, V. P. 369
The first boundary-value problem for a class of hypoelliptic equations. (Russian)
Mat. Sb. (N.S.) **63 (105)** (1964), 238–264.
The following linear equation is treated:

$$(*) \qquad (-1)^{[s/2]+1}\frac{\partial^s u}{\partial t^s} + (-1)^{m+1}\sum_{|i|=|j|=m} D^i A^{ij}(x, t) D^j u + B(u) = f.$$

Here m and s are arbitrary integers $\geqq 1$, the second term (sum) is an elliptic operator with respect to the space variables, and B involves lower-order terms. A solution (a function satisfying (*) a.e. whose derivatives appearing in (*) belong to L_2) is sought in a cylindrical region $Q = \Omega \times [0, T]$ with prescribed normal derivatives up to order $m-1$ on the lateral boundary, and prescribed t-derivatives up to order $[\frac{1}{2}(s-1)]$ at $t=0$ and $[\frac{1}{2}(s-2)]$ at $t=T$. The Fredholm theorems are proved for this problem. First the existence of a weak solution for the case when the spatial part of the operator is Δ^m and $B \equiv 0$ is established by a common functional-analytic method. Then its regularity is proved. An a priori estimate is established for the general case by a method similar to that given by Agmon, Douglis and Nirenberg [Comm. Pure Appl. Math. **12** (1959), 623–727; MR **23** #A2610]. This enables the Schauder-Leray continuity argument to be applied, yielding the final result. *P. C. Fife* (Minneapolis, Minn.)

Duff, G. F. D. 370
On wave fronts, and boundary waves.
Comm. Pure Appl. Math. **17** (1964), 189–225.
The author considers hyperbolic equations of the form

$$P_m\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right) U(t, x) = 0,$$

where P_m is a homogeneous polynomial of degree m. An appropriate number of homogeneous boundary conditions are imposed on the hyperplane $x_1 = 0$. If a point source is located away from the boundary, then the singularities of the resulting solution may be described in terms of waves issuing from the source point, waves induced on the boundary, secondary boundary waves, and waves reflected from the boundary. Using Fourier analysis, distribution theory and asymptotic expansion by means of stationary phase, the author produces the leading or most singular term corresponding to the various kinds of wave fronts. The results provide a firm and important link between qualitative descriptions of propagation phenomena (familiar to seismologists) and the often uninformative expressions which are produced by Fourier analysis. *D. Ludwig* (Berkeley, Calif.)

Brands, J. J. A. M. 371

Bounds for the ratios of the first three membrane eigenvalues.

Arch. Rational Mech. Anal. **16** (1964), 265–268.

Let D be a bounded domain in the (x, y)-plane with sufficiently smooth boundary C. Consider the membrane eigenvalue problem

$$\Delta u+\lambda u = 0 \quad \text{in } D,$$
$$u = 0 \quad \text{on } C,$$

where Δ denotes the Laplace operator. The eigenvalues λ_i are ordered, $\lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \lambda_4 \leqq \cdots$. The reviewer, Pólya and Weinberger [J. Math. and Phys. **35** (1956), 289–298; MR **18**, 905] have shown that $\lambda_{n+1}/\lambda_n < 3$, $(\lambda_2+\lambda_3)/\lambda_1 \leqq 6$. The author obtains the improved inequalities

$$\lambda_2/\lambda_1 \leqq \tfrac{1}{4}[5+\sqrt{33}],$$
$$(\lambda_2+\lambda_3)/\lambda_1 \leqq (3+\sqrt{7})\lambda_1,$$
$$\lambda_3/\lambda_1 \leqq \tfrac{1}{3}[7+2\sqrt{7}].$$

L. E. Payne (College Park, Md.)

Gu, Chao-hao [Ku, Chao-hao] 372

A boundary value problem for hyperbolic systems and its applications.

Acta Math. Sinica **13** (1963), 32–48 (*Chinese*); *translated as Chinese Math.* **4** (1963), 35–53.

The existence, in the small, of solutions of a class of systems of quasi-linear hyperbolic equations in three dependent variables and two independent variables, is demonstrated. This class of equations is typified by the equations governing the flow of a perfect gas between a piston and the shock wave it generates when a position co-ordinate and a variable, constant on particle paths, are used as independent variables. The characteristic form of the equations are integrated formally to generate a set of integral equations and an iterative procedure is applied to the solution of these integral equations. The bulk of the paper is devoted to obtaining the estimates of the approximations required to establish the existence of the solution in the small. The author indicates generalizations of the system considered to which the results obtained may be extended. *J. J. Mahony* (Nedlands)

Pini, Bruno 373

Problema ridotto di Cauchy per certe equazioni pseudo-paraboliche. I.

Atti Accad. Sci. Ist. Bologna Cl. Sci. Fis. Rend. (11) **9** (1961/62), *no.* 1, 60–83.

$P_j(\xi)$ is a homogeneous polynomial of degree j in $\xi=(\xi_1, \cdots, \xi_r)$; D is the elementary differential operator $(\partial/\partial x_1, \cdots, \partial/\partial x_r)$; m, n, q are positive integers, m is even, $n<m$ is odd. The differential operator

$$(1) \qquad L = \sum_{k=0}^{q} P_{m(q-k)}(D)\frac{\partial^{nk}}{\partial t^{nk}},$$

which acts on functions defined on R^{r+1} is hypo-elliptic if the roots $\lambda_{p(s)}$, $p=1, 2, \cdots, nq$, of

$$(2) \qquad \sum_{k=0}^{q} P_{m(q-k)}(-is)\lambda^{nk} = 0$$

have the property that, for each p and all real s, Re $\lambda_p(s) \neq 0$. The author calls operators (1) with this property pseudoparabolic. Among these the parabolic operators are those for which Re $\lambda_p(s)$ are all of the same sign (necessarily $n=1$ in this case).

For the properly pseudoparabolic operators (non-parabolic) of the form (1) the author derives, by explicit constructions, a priori bounds for certain derivatives of sufficiently smooth solutions (including specified behavior as $t\to\infty$) of $Lu=0$, $t\geqq 0$, in terms of certain Hölder norms of derivatives of $f_j(x)=\partial^j u(x, 0)/\partial t^j$, $j=0, 1, \cdots, M-1$. Here M is precisely the number of roots of (2) whose real parts are negative. These results are related to those of Dikopolov and Šilov on certain well-posed problems [Izv. Akad. Nauk SSSR Ser. Mat. **24** (1960), 369–380; MR **22** #11219; errata, MR **22**, p. 2547].

C. R. DePrima (Pasadena, Calif.)

Morozov, V. A. 374

Differential-difference schemes of second-order precision for quasilinear problems of parabolic type with discontinuous elements. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, *no.* 2, 12–22.

In the rectangle Π $(0\leqq x\leqq l, 0\leqq t\leqq T)$, the parabolic equation

$$(1) \qquad Lu \equiv r(x, t, u)\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(s(x, t)\frac{\partial u}{\partial x}\right) -f\left(x, t, u, 2s(x, t)\frac{\partial u}{\partial x}\right) = 0,$$

where $r(x, t, u)\geqq c_r>0$, $s(x, t)\geqq c_s>0$, $f_u(x, t, u, p)\leqq c_f\leqq 0$ for $(x, t)\in\Pi$, is investigated with the initial condition

$$(2) \qquad u(x, 0) = \varphi(x)$$

and with one of the boundary conditions

$$(3a) \qquad u|_{x=0} = w_1(t), \quad u|_{x=l} = w_2(t),$$
$$(3b) \qquad u|_{x=0} = w_3(t), \quad (s\partial u/\partial x+\sigma_0 u)|_{x=l} = w_4(t),$$
$$(3c) \qquad (s\partial u/\partial x-\sigma_1 u)|_{x=0} = w_5(t),$$
$$(s\partial u/\partial x+\sigma_2 u)|_{x=l} = w_6(t),$$

where $\sigma_i=\sigma_i(t)\geqq 0$, $i=0, 1, 2$; $\sigma_1(t)+\sigma_2(t)\geqq\sigma>0$, $0\leqq t\leqq T$. Certain smoothness conditions are imposed on the functions r, s, f. These functions may have discontinuities of the first order with respect to x on a finite number of straight lines $x=\xi_\nu$ $(\nu=1, 2, \cdots, n)$.

Under the assumption that the boundary-value problem (1), (2), (3a), respectively, (1), (2), (3b), respectively, (1), (2), (3c) is solvable and its solution is unique, it is shown that a differential-difference scheme may be constructed for the approximate solution with second-order accuracy. Estimates of the error are presented.

K. Rektorys (Prague)

Čeremnyh, Ju. N. 375

Behaviour of solutions of the first boundary-value problem for certain quasi-linear parabolic equations. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 1037–1040.

Two important estimates are presented for solutions of

quasilinear parabolic differential equations of the form $L_\varepsilon u(x, t, \varepsilon) - \partial u(x, t, \varepsilon)/\partial t = 0$ which degenerate for $\varepsilon \to 0$.

In the first part of the paper, the first boundary-value problem with zero boundary conditions is considered in the region $G_T\{(x, t)|\varphi_1(t) < x < \varphi_2(t), 0 < t < T\}$ for the equation

$$(1)\quad \frac{\partial u(x, t, \varepsilon)}{\partial t} = L_\varepsilon u(x, t, \varepsilon) \equiv$$

$$\frac{\partial}{\partial x}\left(\left[a\left(x, t, u, \frac{\partial u}{\partial x}\right) + \varepsilon\left(\left(\frac{\partial u}{\partial x}\right)^2 + 1\right)^{(\nu-2)/2}\right]\frac{\partial u}{\partial x}\right),$$

$$\frac{\partial a(x, t, u, v)}{\partial v} v \geqq 0,$$

$$a(x, t, u, v) \geqq a_0(u)|v|^\beta,$$

$$\nu \geqq 2, \quad \beta \geqq 0, \quad a_0(\lambda) > 0, \quad \lambda \neq 0, \quad a_0(0) = 0.$$

Denote $m(\varepsilon) = m(T, \varepsilon) = \max_{(x,T)\in G_T} |u(x, T, \varepsilon)|$ and $M = \max_{(x,0)\in G_T} |u(x, 0, \varepsilon)|$. Suppose that in G_T a solution $u(x, t, \varepsilon) \in C^{(2)}$ (with respect to x and to t) exists, and that $|u(x, t, \varepsilon)| \leqq M$. The following estimate holds:

$$(2)\qquad a_0(\xi_0)(m(\varepsilon))^{\beta+1} \leqq \frac{(\mu_2 G_T)^{\beta+1}}{T^{\beta+2}} CM,$$

where $C = 2^{\beta+1}(\varphi_2(0) - \varphi_1(0))$, $m(\varepsilon)/2 < \xi_0 < m(\varepsilon)$. Namely, if $(\mu_2 G_T)^{\beta+1}/T^{\beta+2} \to 0$ for $T \to \infty$, then $m(T, \varepsilon) \to 0$ for $T \to \infty$.

The estimate (2) holds for generalised solutions of the equation $L_0 u(x, t) - \partial u(x, t)/\partial t$, which is the limiting case of (1) for $\varepsilon \to 0$. The special case of (2) is then investigated for $a(x, t, u, \partial u/\partial x) = u^\alpha$, $\alpha > 0$ and $\varepsilon = 0$.

In the second part of the work, an estimate similar to (2) is presented for the equation

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x, t, u)\frac{\partial u}{\partial x_j}\right) + \varepsilon\Delta u - \frac{\partial u}{\partial t} = 0.$$

K. Rektorys (Prague)

Slobodeckiĭ, L. N. 376

Continuous dependence on a parameter of the fundamental solution of a parabolic system. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 742–744.

Soit $B(\sigma)$ une famille de systèmes paraboliques au sens de Petrowsky, dépendant du paramètre σ variant dans un voisinage de 0, d'ordre $2p$ indépendant de σ. On suppose que $B(\sigma)$ est uniformément parabolique au sens de Petrowsky, i.e., les constantes intervenant dans la condition de Petrowsky sont indépendantes de σ. On suppose aussi que lorsque $\sigma \to 0$ les coefficients de $B(\sigma)$ convergent uniformément en $x \in R^n$ et $t \geqq 0$, vers ceux de $B(0)$. Il est connu (Èĭdel'man) qu'il existe une solution fondamentale $U(t, x; \tau, y; \sigma)$ de $B(\sigma)$ telle que pour $0 \leqq l \leqq 2p$, on ait

$$|D_x^l U(t, x; \tau, y; \sigma)| \leqq$$

$$\frac{c}{(t-\tau)^{(n+l)/2n}} \exp\left\{a(t-\tau) - \eta\left[\frac{|x-y|}{(t-\tau)^{1/2p}}\right]^{2p/(2p-1)}\right\} = G_l^\sigma,$$

où les constantes dépendent des constantes intervenant dans la condition de Petrowsky et de σ; alors, pour tout $\varepsilon > 0$, il existe un voisinage de 0 tel que, pour σ dans ce voisinage, on ait:

$$(1)\quad |D_x^l U(t, x; \tau, y; \sigma) - D_x^l U(t, x; \tau, y; 0)| \leqq \varepsilon G_l^0$$

pour $0 \leqq l \leqq 2p-1$. Moyennant des conditions supplémentaires sur les coefficients de $B(\sigma)$, (1) est encore valable si $l = 2p$.

Applications au problème de Cauchy, pour lequel l'auteur retrouve, en les améliorant, des résultats qu'il avait déja établis antérieurement [Mat. Sb. (N.S.) **46 (88)** (1958), 229–258; MR **22** #830]. *J. L. Lions* (Paris)

Wang, Guang-yin [Wuang, Kuang-ying]; Chen, Dun-dong [Chen, Tun-tung]; Mei, Duo-wen [Mei, To-wen] 377

On a Monge-Ampère equation of parabolic type.

Acta Math. Sinica **14** (1964), 78–92 (*Chinese*); *translated as Chinese Math.* **5** (1964), 85–100.

The authors study the Monge-Ampère equation of parabolic type that can be reduced to the form

$$(*)\quad F(x, y, u, p, q, r, s, t) = Hr + Lt + HL + rt - s^2 = 0,$$

where H and L are functions of x and y only, and there exists a constant k such that $kL_y = H_x$. By first considering this problem as the limit when $a \to 0$ in the hyperbolic equation $F + a^2 = 0$, the authors are led to a transformation of the characteristic initial-value problem of (*) into an equivalent initial-value problem for a quasi-linear parabolic equation. This latter equation has been studied by O. A. Oleĭnik and T. P. Ventcel' [Mat. Sb. (N.S.) **41 (83)** (1957), 105–128; MR **19**, 149]. In this way the existence and uniqueness of the solution for the original equation is established.

Similarly, to study the so-called first boundary-value problem for (*), the authors apply a contact transformation which takes (*) into an equivalent first boundary problem for a quasi-linear parabolic equation studied by M. Gevrey [J. Math. Pures Appl. (6) **9** (1913), 305–471; ibid. (6) **9** (1913), 473–475].

R. Barrar (Santa Monica, Calif.)

Edmunds, D. E. 378

Two uniqueness theorems associated with parabolic differential operators.

J. London Math. Soc. **39** (1964), 345–354.

The author proves a "backward" uniqueness theorem for parabolic differential inequalities of the form

$$\left(\frac{\partial v}{\partial t} - Av\right)^2 \leqq c_1 v^2 + c_2 \sum_{i=1}^{n}\left(\frac{\partial v}{\partial x_i}\right)^2,$$

where

$$A = \sum_{i,j=1}^{n} a_{ij}(x, t)\frac{\partial^2}{\partial x_i \partial x_j}$$

is uniformly elliptic in $D_0 = B \times \{t| -\tau \leqq t \leqq 0\}$ and B is the complement of some bounded, open subset of R^n. The proof is based on the methods of the reviewer and M. H. Protter [Duke Math. J. **28** (1961), 369–382; MR **25** #4254] and D. Graffi [Ann. Mat. Pura Appl. (4) **50** (1960), 379–387; MR **22** #12924]. *M. Lees* (Pasadena, Calif.)

Koval'čik, I. M. 379

Representation of the solution of the Cauchy problem for a parabolic system in the form of a Wiener integral. (Russian)

Dokl. Akad. Nauk SSSR **138** (1961), 1284–1286.

The author solves the problem: $\partial u/\partial t = \Delta u + p(x, t)u$, $u(x, 0+) = \varphi(x)$, where u and φ are n-vectors and p is an $n \times n$ matrix. Under suitable assumptions on p and φ the solution is given by a vector-valued Wiener integral. It is an extension to the vector case of the solution formulae for the scalar case of R. H. Cameron [Ann. of Math. (2) **59** (1954), 434–462; MR **15**, 799] (one space dimension) and A. J. Tingley [Proc. Amer. Math. Soc. **7** (1956), 846–851; MR **19**, 149] (many space dimensions). For a discussion of the integral and some of its properties the author refers to a previous paper [Ukrain. Mat. Ž. **12** (1960), 25–31; MR **23** #A4183]. See also the paper of Ju. Daleckiĭ [Dokl. Akad. Nauk SSSR **134** (1960), 1013–1016] for a presentation in a more general setting.

R. K. Juberg (Minneapolis, Minn.)

Gu, Ljan'-kun' [Ku, Lien-kun] 380

The behavior of the solution of Stefan's problem as time goes to infinity. (Russian)

Dokl. Akad. Nauk SSSR **138** (1961), 263–266.

Let the functions $s(t)$, $u_1(x, t)$, $u_2(x, t)$ be a solution of a problem of the following type: (1) u_i bounded and continuous in $\bar{D}_i$, where $D_1 = \{(x, t) : x < s(t), t > 0\}$ and $D_2 = \{(x, t) : x > s(t), t > 0\}$, except possibly at (0, 0); (2) $\partial u_i/\partial x$, $\partial^2 u_i/\partial x^2$, $\partial u_i/\partial t$ exist in D_i and $L_i(u_i) \equiv \partial^2 u_i/\partial x^2 + B_i(x, t)\partial u_i/\partial x + C_i(x, t)u_i - A_i(x, t)\partial u_i/\partial t = 0$ in D_i, where A_i, B_i, and C_i are sufficiently smooth and bounded, $A_i(x, t) \geqq a > 0$ and $C_i(x, t) \leqq 0$; (3) $u_1(x, 0) = \varphi_1(x)$ when $x < 0$, $u_2(x, 0) = \varphi_2(x)$ when $x > 0$, $u_i(s(t), t) = 0$,

$$(\partial u_1/\partial x - \partial u_2/\partial x)(s(t), t) = ds(t)/dt,$$

where the φ_i are sufficiently smooth and bounded and s is differentiable with $s(0) = 0$. The author presents a collection of theorems, where, under various combinations of hypotheses on the coefficients A_i, B_i, C_i and the data φ_i, the limiting values, as t approaches infinity, of appropriate members of the set of functions $s(t)$, $u_1(x, t)$, $u_2(x, t)$ are described. A comparison method obtained from the maximum principle is employed in deducing these results. The known explicit solution of the problem with $B_i = C_i = 0$, $A_i(x, t) = a_i^2$ (constant), $\varphi_1(x) = u_-$ (constant) < 0, and $\varphi_2(x) = u_+$ (constant) > 0 is used in the construction of comparison functions. Some similar results have been presented in papers by A. Friedman [J. Math. Mech. **9** (1960), 19–66; MR **26** #1627; ibid. **9** (1960), 327–345; MR **26** #1628; ibid. **9** (1960), 885–903; MR **26** #1629]. The author states at the end that some of the results can be extended to hold also for generalized solutions.

R. K. Juberg (Minneapolis, Minn.)

Bitsadze, A. V. [Bicadze, A. V.] 381

★Equations of the mixed type.

Translated by P. Zador; translation edited by I. N. Sneddon. A Pergamon Press Book.

The Macmillan Co., New York, 1964. xiii + 160 *pp.* \$8.50.

This is a translation of the original Russian [Izdat. Akad. Nauk SSSR, Moscow, 1959]. It is intended as an introduction to partial differential equations of mixed type and is largely self-contained. The subject of Chapter 1 is the reduction to canonical form of the equation

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = E(x, y, u, u_x, u_y),$$

where A, B and C are real-analytic functions on some domain D and parabolic degeneracy occurs on a simple curve in D. Chapter 2 is concerned with the singular Cauchy problem for the equation

$$y^m u_{xx} - u_{yy} + au_x + bu_y + cu = 0,$$

with m a positive constant. The existence of a unique solution is established under the condition

$$\lim_{y \to 0+} y^{(1-m)/2} a(x, y) = 0.$$

In Chapter 3, the Dirichlet problem for the equation

$$y^m u_{xx} + u_{yy} + au_x + bu_y + cu = 0$$

is solved on a domain D bounded by a simple curve σ in the upper half-plane with endpoints at $A(0, 0)$ and $B(1, 0)$ and by the segment AB. The solution is also shown to exist for an equation with principal part $u_{xx} + y^m u_{yy}$ on the same domain for the case where all solutions of the equations are bounded. If all solutions are not bounded, the problem E of Keldyš [Dokl. Akad. Nauk SSSR **77** (1951), 181–183; MR **13**, 41], in which values are prescribed on σ and u is required to be bounded on D, is shown to be correct. Chapter 4 is devoted to the Tricomi problem and some of its generalizations. The treatment of the equation

$$u_{xx} + (\operatorname{sgn} y)u_{yy} = 0 \qquad (1)$$

appears to be essentially the same as that in the author's earlier monograph [Trudy Mat. Inst. Steklov. **41** (1953); MR **16**, 43], but a discussion of Tricomi's equation $yu_{xx} + u_{yy} = 0$ is added in the present work. Chapter 5 treats the problem M (see the author's earlier work) and Frankl's problem for equation (1). A brief discussion of Morawetz's proof [Comm. Pure Appl. Math. **11** (1958), 315–331; MR **20** #3375] of the existence of a weak solution to Frankl's problem is included. The author's solution [Dokl. Akad. Nauk SSSR **109** (1956), 1091–1094; MR **18**, 743] of a problem proposed by Frankl is also presented. References to a number of related results are given. {The book is marred by a large number of misprints.}

H. Ogawa (Riverside, Calif.)

Bakievič, N. I. 382

Singular Tricomi problems for the equation $\eta^\alpha u_{\xi\xi} + u_{\eta\eta} - \mu^2\eta^\alpha u = 0$. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 2 (39), 7–13.

L'équation mentionnée dans le titre est considérée pour $\alpha > 0$, $(-1)^\alpha = -1$, μ étant réel ou purement imaginaire. Soit D le domaine borné du plan (ξ, η) dont la frontière est formée par un intervalle AB de l'axe $O\xi$ et les arcs AC et BC des caractéristiques passant par A et B. On considère les problèmes suivants: (I) Trouver une solution (dans D) de l'équation considérée telle que $u = \tau(\xi)$ sur AB et $u = \varphi(\xi)$ sur AC; (II) $\partial u/\partial\eta = \nu(\xi)$ sur AB et $u = \varphi(\xi)$ sur AC; (III) (Problème singulier de Cauchy) $u = \tau(\xi)$ pour $\eta = 0$, $\partial u/\partial\eta = \nu(\xi)$ sur AB. On obtient des formules intégrales pour les solutions (assez compliquées pour être reproduites ici) à l'aide desquelles on établit certaines propriétés de régularité.

C. Corduneanu (Iași)

Besala, P. 383
On solutions of first order partial differential equations defined in an unbounded zone.
Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 95–99.

The Cauchy problem for a system of nonlinear partial differential equations of the form

$$(1)\qquad z_x^i = f^i(x, Y, z^1, \cdots, z^m, z_{y_1}^i, \cdots, z_{y_n}^i) \quad (i = 1, \cdots, m),\ Y = (y_1, \cdots, y_n),$$

in $D\{|x| < a(a \leq +\infty), Y \text{ arbitrary}\}$ is investigated. Some theorems are presented concerning properties of solutions of this problem, namely, relations between solutions corresponding to different right-hand sides of the system (1). The solutions are supposed to be continuous and to have first partial derivatives in D. The theorems presented, which are based on a fundamental lemma proved in the paper, are generalizations of the corresponding theorems by Ważewski, Szarski and Besala, proved under rather stronger assumptions. *K. Rektorys* (Prague)

Prodi, Giovanni 384
Résultats récents et problèmes anciens dans la théorie des équations de Navier-Stokes.
Les Équations aux Dérivées Partielles (Paris, 1962), pp. 181–196. *Éditions du Centre National de la Recherche Scientifique, Paris,* 1963.

This is an expository paper, and deals with the mixed problem (in Hadamard's sense), as well as with other problems (periodic solutions, stability) of the Navier-Stokes equations. Special references are made to some papers written by E. Hopf [Math. Nachr. **4** (1951), 213–231; MR **14**, 327] and by A. A. Kiselev and O. A. Ladyženskaja [Izv. Akad. Nauk SSSR Ser. Mat. **21** (1957), 655–680; MR **20** #6881]. At the end of the paper, some personal contributions are also given, which are taken from the author's earlier paper [Ann. Mat. Pura Appl. (4) **48** (1959), 173–182; MR **23** #A3384].
V. N. Constantinescu (Bucharest)

Itô, Seizô 385
The existence and the uniqueness of regular solution of non-stationary Navier-Stokes equation.
J. Fac. Sci. Univ. Tokyo Sect. I **9**, 103–140 (1961).

On étudie les équations de Navier-Stokes dans un ouvert D de R^3, borné, de frontière C^4. L'auteur démontre l'existence et l'unicité, dans un intervalle de temps assez petit, d'une solution $u = \{u^1, u^2, u^3\}$ vérifiant (entre autres) les conditions de régularité suivantes: u^i, $\partial u^i/\partial t$, $\partial^2 u^i/\partial x^i \partial x^j$ sont continues dans $D \times]0, T[$ (T étant la limite suffisamment petite de l'intervalle d'existence). u^i est continue dans $\bar{D} \times [0, T]$, $u = a$ pour $t = 0$ et $u = b$ pour $t > 0$, $x \in S$, frontière de D (on suppose que a est continue dans $\bar{D}$ et C^2 dans D, que Δa est d'intégrale de Dirichlet finie dans D et C^1 dans D, avec div $a = 0$; on suppose que b est C^2 sur $S \times \{t \geqq 0\}$ avec $\int_S b \cdot n\, dS = 0$ pour tout $t \geqq 0$, et on suppose enfin que b satisfait à une condition de type Lipschitzien.

Ce théorème, où seule l'existence est délicate, est démontré par la méthode des approximations successives, possible à l'aide d'estimations très précises sur le problème linéarisé (§§ 2 à 5) étudié à l'aide des potentiels d'Odqvist.

Depuis la parution de cet article, des compléments, dans le sens de ce travail, ont été apportés notamment par Kato, Fujita et Serrin. *J. L. Lions* (Paris)

Ladyženskaja, O. A. 386
Solvability in the large of boundary-value problems for linear and quasi-linear parabolic equations and the Navier-Stokes equations. (Russian)
Proc. 4th All-Union Math. Congr. (Leningrad, 1961), *Vol. I, pp.* 134–157. *Izdat. Akad. Nauk SSSR, Leningrad,* 1963.

An outline of the current state of knowledge on non-linear parabolic and elliptic equations of second order, including a valuable exposition of the principal ideas and methods used in obtaining the important a priori estimates in the author's papers, and in her papers with Uraltseva. Applications of the ideas to mechanics—notably diffraction theory and hydrodynamics—are discussed briefly. One of the open problems indicated by the author, that of determining the asymptotic behavior at infinity of stationary solutions of the Navier-Stokes equations in an exterior domain, has since been partially settled by the reviewer [Nonlinear Problems (Proc. Sympos., Madison, Wis., 1962), pp. 99–115, Univ. Wisconsin Press, Madison, Wis., 1963; MR **27** #2739]. *R. Finn* (Stanford, Calif.)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 201, 481, 1098.

Mauldon, J. G. 387
Continuous functions satisfying linear recurrence relations.
Quart. J. Math. Oxford Ser. (2) **15** (1964), 23–31.

The author shows that for any continuous real function f, the following are equivalent:

$$(1)\qquad f(x) = \sum_{k=1}^{s} A_k(x)e^{b_k x} \quad (-\infty < x < \infty),$$

where, for $k = 1, \cdots, s$, the b_k are distinct constants and the A_k are polynomials of degree n_k;

$$(2)\qquad \Delta_n(x, h) = \det[f(x+(i+j)h)] = 0,$$

where $n = \sum_{k=1}^{s}(n_k+1)$ and $i, j = 1, \cdots, n$. (3) Every sequence $\{f_m = f(x+mh): m = \cdots, -2, -1, 0, 1, 2, \cdots\}$ satisfies a linear recurrence relation with constant coefficients, i.e., there exist two-place functions λ_j, $j = 1, \cdots, n$, independent of m, such that

$$f(x+mh) = \sum_{j=1}^{n} \lambda_j(x, h) f(x+(m-j)h)$$

$(m = \cdots, -2, -1, 0, 1, 2, \cdots)$, for all x and h. The special cases $n = 1, 2$ have previously been considered by D. V. Ionescu [Mathematica (Cluj) **1 (24)** (1959), 11–26; MR **26** #1643]. *B. Schweizer* (Tucson, Ariz.)

Meynieux, Robert 388
Sur l'analyticité de solutions continues d'une équation fonctionnelle.
C. R. Acad. Sci. Paris **254** (1962), 4413–4414.

In continuation of his work [same C. R. **254** (1962), 1726–1728; MR **25** #225; ibid. **254** (1962), 3301–3303; MR **26** #6630] the author examines what effects the dropping of a condition of dimensions and one of certain generalized regularity have upon the behavior of the solutions of the functional equation $F[f(u), g(v)] = h(u+v)$ in multidimensional spaces. The main result is that h remains analytic even then. *J. Aczél* (Stanford, Calif.)

Kuczma, M.; Zajtz, A. 389

Über die multiplikative Cauchysche Funktionalgleichung für Matrizen dritter Ordnung.

Arch. Math. **15** (1964), 136–143.

The authors obtain the most general solution of the functional equation $f(xy) = f(x)f(y)$ when the domain of f is the set of real numbers and the range of f the set of real 3×3 matrices. In this general solution, the elements of $f(x)$ are determined as solutions, or combinations of solutions, of various well-known ordinary functional equations. *B. Schweizer* (Tucson, Ariz.)

Vincze, E. 390

Beitrag zur Theorie der Cauchyschen Funktionalgleichungen.

Arch. Math. **15** (1964), 132–135.

Let $(Q, *)$ be a semigroup and $(K, +, \cdot)$ a commutative field which is "biquadratically closed" in the sense that every polynomial over K of degree at most four has at least one root in K. Let g be a mapping from Q to K. Then g is a solution of $[g(u*v)]^2 = [g(u)+g(v)]^2$ if and only if g is a solution of $g(u*v) = g(u)+g(v)$, i.e., in the set of all mappings from Q to K, the Cauchy equation and its square are equivalent. The author's argument is elementary and elegant. The result is purely algebraic and it should be noted that the commutativity of $(Q, *)$ is a consequence, not an assumption.

B. Schweizer (Tucson, Ariz.)

Đoković, Dragomir 391

On certain classes of cyclic functional equations. (Serbo-Croatian. French summary)

Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 114 (1963), 48 *pp.*

Soient S un ensemble et M un groupe abélien. L'auteur a considéré les équations fonctionnelles suivantes

$$(1)\quad F(x_1, x_2, \cdots, x_p)+F(x_2, x_3, \cdots, x_{p+1})+\cdots +F(x_n, x_1, \cdots, x_{p-1}) = 0 \quad (n \geqq 2p-1),$$

$$(2)\quad F_1(x_1, x_2, \cdots, x_p)+F_2(x_2, x_3, \cdots, x_{p+1})+\cdots +F_n(x_n, x_1, \cdots, x_{p-1}) = 0 \quad (n \geqq 2p-1),$$

$$(3)\quad F(x_1, x_2, \cdots, x_n, x_{n+1})-F(x_2, x_3, \cdots, x_{n+1}, x_1) = f(x_1, x_2, \cdots, x_{n-1}, x_n\cdot x_{n+1}),$$

$$(4)\quad f(x_1, x_2, \cdots, x_{n-1}, x_n\cdot x_{n+1}) +f(x_2, x_3, \cdots, x_n, x_{n+1}\cdot x_1)+\cdots +f(x_{n+1}, x_1, \cdots, x_{n-2}, x_{n-1}\cdot x_n) = 0,$$

$$(5)\quad F(x_1, x_2, x_3)-F(x_2, x_3, x_1) = f(x_1, x_2+x_3),$$

$$(6)\quad f(x_1, x_2+x_3)+f(x_2, x_3+x_1)+f(x_3, x_1+x_2) = 0,$$

$$(7)\quad \sum_{i=1}^{m+n+p} C^{i-1}F(x_1+x_2+\cdots+x_m, x_{m+1}+x_{m+2}+\cdots +x_{m+n}, x_{m+n+1}+x_{m+n+2}+\cdots+x_{m+n+p}) = 0,$$

$$(8)\quad \sum_{i=1}^{m+n+p} C^{i-1}F(x_1+x_2+\cdots+x_m, x_{m+1}+x_{m+2}+\cdots +x_{m+n}) = 0,$$

$$(9)\quad \sum_{i=1}^{m+n+p} C^{i-1}F_i(x_1+x_2+\cdots+x_m, x_{m+1}+x_{m+2}+\cdots +x_{m+n}) = 0,$$

où $x_i \in S$, f et F_i sont les fonctions inconnues dont les valeurs appartiennent au groupe M, et C désigne un opérateur cyclique, de période $m+n+p$ défini par

$$Cf(x_1, x_2, \cdots, x_{m+n+p}) = f(x_2, x_3, \cdots, x_{m+n+p}, x_1),$$

où f est une fonction arbitraire. Dans le cas des équations (4) on suppose que $(S, \cdot)$ est un groupoïde associatif à élément unité. Les équations (5)–(9) sont considérées dans le domaine réel R ($S = M = R$). J. Aczél, M. Ghermanescu et M. Hosszú [Magyar Tud. Akad. Mat. Kutató Int. Közl. **5** (1960), 215–221; MR **23** #A2660] ont trouvé la solution générale de l'équation (1) sous l'hypothèse que l'équation $mX = A$ ($A, X \in M$) possède pour tout nombre naturel m ($\leqq n$) une solution unique en X. L'auteur obtient la solution générale des équations (1) et (2) dans le cas où M est un groupe abélien arbitraire.

Les résultats de l'auteur relatifs à l'équation (3) sont la généralisation de ses résultats antérieurs, obtenus avec D. S. Mitrinović [C. R. Acad. Sci. Paris **252** (1961), 1090–1092; MR **24** #A2160]. Dans les cas $n=5$ et $n=7$ il a déterminé la solution générale de ces équations sous l'hypothèse que l'équation $(n+1)X = A$ $(X, A \in M)$ possède une solution unique en X. L'auteur a obtenu les solutions générales des équations (5), (6) et (8) en utilisant la solution générale de l'équation de Cauchy $f(x+y) = f(x)+f(y)$. Il a trouvé aussi la solution générale de (7) sous l'hypothèse de la continuité de la fonction F, de même que celle de (9) pourvu que les fonctions F soient continues et qu'il soit $m=2$, $n=p=1$.

S. B. Prešić (Belgrade)

SEQUENCES, SERIES, SUMMABILITY

See also 415.

Gál, S.; Gál, L. 392

The discrepancy of the sequence $\{(2^n x)\}$.

Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 129–143.

Given any sequence $\mathscr{S} = (s_1, s_2, \cdots)$ of real numbers, let $\mathscr{N}(N, a_1, a_2, \mathscr{S})$ denote the number of indices i not exceeding N such that $a_1 \leqq (s_i) < a_2$. Put

$$R(N, a_1, a_2, \mathscr{S}) = \mathscr{N}(N, a_1, a_2, \mathscr{S}) - (a_2-a_1)N$$

and define the discrepancy $D(N)$ of the sequence $\mathscr{S}$ through

$$ND(N) = \text{lub } R(N, a_1, a_2, \mathscr{S}),$$

where the lub is taken over every pair of real numbers a_1, a_2 satisfying $0 \leqq a_1 \leqq a_2 \leqq 1$. This paper is devoted to proving that if $n_1 < n_2 < \cdots$ are non-negative integers

and if $D(N, x)$ is the discrepancy of the sequence $(2^{n_1}x, 2^{n_2}x, \cdots)$, then there is a constant c such that for almost all real x,

$$\limsup_{N\to\infty} ND(N, x)/\sqrt{(N \log\log N)} \leq c.$$

This is achieved by majorising $|\int_0^1 R^p\,dx|$ using combinatorial arguments concerning products of Walsh functions, and by estimating the measure of sets of the form $\{x: R(x) \geqq y\}$. It is suggested that $c = 1$.
E. J. Akutowicz (Bologna)

Maddox, Ivor J. 393
On absolute Riesz summability factors. II.
Tôhoku Math. J. (2) **16** (1964), 60–71.

The author had given in an earlier paper [same J. **15** (1963), 116–120; MR **27** #3968] sufficient conditions for a series $\sum a_n\epsilon_n$ to be summable by the absolute Riesz method $|R, \lambda, \kappa|$ whenever $\sum a_n$ is bounded (R, λ, κ), where κ is a non-negative integer. This paper extends the result to non-integral values of $\kappa \geqq 0$ and for $\lambda = \{\lambda_n\}$ "reasonably regular", i.e., satisfying the conditions (a) $0 < a \leqq \Delta\lambda_n/\Delta\lambda_{n-1} \leqq A$ (a, A constants), and (b) λ_{n+1}/λ_n decreases to 1. References to several other related results are also given. The author's main theorem generalizes results on absolute Cesàro summability factors due to Chow [J. London Math. Soc. **29** (1954), 459–476; MR **16**, 464] and Ahmad [Math. Z. **76** (1961), 295–310; MR **24** #A2780]. *M. R. Parameswaran* (E. Lansing, Mich.)

Gorst, Ju. G. 394
Extension of a theorem of Mazur and Orlicz to semi-continuous and integral summability methods. (Russian. English summary)
Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 745–749.

Brudno [Mat. Sb. (N.S.) **16 (58)** (1945), 191–247; MR **7**, 12] established the well-known theorem: If A and B are regular matrix methods such that every bounded sequence summed by A is also summed by B, then it is summed by B to the same value. A very short proof of this theorem was then given by Petersen [J. London Math. Soc. **31** (1956), 324–326; MR **18**, 31] and then, by means of functional analysis, by Mazur and Orlicz [Studia Math. **14** (1954), 129–160; MR **16**, 814]. Altman [ibid. **13** (1953), 233–243; MR **15**, 697] generalized this theorem to semi-continuous methods (of the form $\sum_{n=0}^{\infty} a_n(x)s_n$, where the limit is taken as $x\to\infty$) under the additional hypothesis of the existence of an integer-valued, non-decreasing function $\omega(x)$ such that $\lim_{x\to\infty} \sum_{n=\omega(x)}^{\infty} a_n(x) = 0$.

The author establishes a similar theorem for integral methods (of the form $\int_0^\infty K(x, t)s(t)\,dt$); the corresponding condition is now the existence of a non-decreasing function $\omega(x)$ such that $\lim_{x\to\infty} \int_{\omega(x)}^{\infty} |K(x, t)|\,dt = 0$. By counter-examples he shows also that Brudno's theorem cannot be established for semi-continuous methods without some additional condition. (Altman's condition is not necessary.) *V. Vučković* (Notre Dame, Ind.)

Cross, R. W. 395
On the conditions for a T-matrix to evaluate no bounded divergent sequence.
Bull. Soc. Math. Belg. **15** (1963), 243–252.

Der Verfasser beginnt mit dem Beweis des folgenden Satzes: Ist A nulltreu und besitzt $\sum_k a_{nk}x_k = 0$, $x_k = O(1)$ $(n = 0, 1, \cdots)$ nur die Lösung $x_k = 0$, so wird genau dann eine beschränkte divergente Folge limitiert, wenn die Bilder der Nullfolgen im Raum der Nullfolgen (mit der Norm $\sup |x_n|$) einen abgeschlossenen Teilraum bilden. Beim hinreichenden Teil des Beweises darf sogar die Eindeutigkeit weggelassen werden. (Dieses Ergebnis überschneidet sich mit Resultaten von J. Copping [Studia Math. **21** (1961/62), 177–194; MR **25** #3296].) Daraus wird u. a. gefolgert: Ist A permanent, perfekt und stärker als Konvergenz, so wird von A eine beschränkte divergente Folge limitiert. Weitere Ergebnisse beziehen sich auf die Existenz und Permanenz einer inversen Matrix zu A.
A. Peyerimhoff (Marburg)

Meir, Amram 396
Analytic continuation by summation-methods.
Israel J. Math. **1** (1963), 224–228.

Given a complex sequence $\{d_n\}$ $(d_n \neq -1)$ and a sequence of polynomials $\{P_n\}$ $(P_n(1) \neq 0)$, the $[F, d_n]$ and $[F^*, P_n]$ transforms of a sequence $\{s_n\}$ are given by

$$t_n = \left\{\prod_{k=1}^{n} (d_k+1)^{-1}(d_k+E)\right\}s_0,$$

$$t_n^* = \left\{\prod_{k=1}^{n} (P_k(1))^{-1}P_k(E)\right\}s_0,$$

respectively, where E is the shift operator $E^p s_k = s_{k+p}$. The author investigates $[F, d_n]$ and $[F^*, P_n]$ transforms of the partial sums of $\sum z^k$, and considers the following question: Given a subset M of the complex plane, does there exist an $[F, d_n]$ ($[F^*, P_n]$) transform which sums $\sum z^k$ to $(1-z)^{-1}$ if z is in M, and does not sum it if z lies in the complement of the closure of M? Affirmative answers are found in several cases, of which the simplest is the following. Let the polynomial $P(z)$ satisfy $\operatorname{Re}\{P(1)\} = 0$. Then there exists a fixed sequence $\{d_n\}$ such that the $[F, d_n]$ transform sums the geometric series to the value $(1-z)^{-1}$ for every z for which $\operatorname{Re}\{P(z)\} > 0$, and does not sum it if $\operatorname{Re}\{P(z)\} < 0$. *J. D. Buckholtz* (Lexington, Ky.)

Meir, Amram 397
On two problems concerning the generalized Lototsky transforms.
Canad. J. Math. **16** (1964), 339–342.

Let $\{d_n\}$ $(d_n \neq -1)$ be a fixed complex sequence and let

$$\prod_{i=1}^{n} \frac{d_i+z}{d_i+1} = \sum_{k=0}^{n} c_{nk}z^k.$$

Then the sequence t_n given by the formula $t_n = \sum_{k=0}^{n} c_{nk}s_k$ is called the $[F, d_n]$ transform of the sequence $\{s_n\}$. The sequence $\{s_n\}$ is said to be $[F, d_n]$ summable to s if $\lim t_n = s$. Let $d_n = \rho_n e^{i\theta_n}$, $n = 1, 2, \cdots$, where $-\pi < \theta_n \leqq \pi$ and $\rho_n > 0$. Miracle and the reviewer [same J. **14** (1962), 418–435; MR **25** #4281] left unanswered the following questions concerning the $[F, d_n]$ transform. Do the conditions $\sum \rho_n^{-1} = +\infty$ and $\lim \theta_n = 0$ imply regularity of the $[F, d_n]$ transform? Does there exist a sequence $\{d_n\}$ such that θ_n does not tend to zero and the $[F, d_n]$ transform is regular? The author provides negative and affirmative answers, respectively, to these questions. The

author also clears up a number of inaccuracies in the above-cited paper.

V. F. Cowling (New Brunswick, N.J.)

Baron, S. 398

Generalizations of a theorem of E. Landau. (Russian. Estonian and German summaries)

Tartu Riikl. Ül. Toimetised No. 129 (1962), 357–364.

A classical theorem of E. Landau asserts that the factorial series $\sum n!/(x+1)\cdots(x+n)$ $(x \neq -1, -2, \cdots)$ converges [absolutely] if and only if the Dirichlet series $\sum a_n(n+1)^{-x}$ converges [absolutely]. Here it is proved that if either series is C^α [$|C^\alpha|$] summable so is the other. This result is then extended to more general methods of summability. Similar results are obtained for double series, as well as for (single and double) Newton series and their corresponding Dirichlet series.

A. Dvoretzky (New York)

Slepenčuk, K. M. 399

Non-linear transformations of some classes of sequences (products). (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 2 (39), 144–151.

An infinite product $\prod_{k=1}^\infty (1+u_k)$ is (A)-summable if $(A)P_n = \prod_{k=1}^\infty (1+a_{nk}u_k)$ converges to a finite limit different from 0, where $A = \|a_{nk}\|$ is a given matrix [M. D. Kalašnikov, Ukrain. Mat. Ž. **3** (1951), 477–488; MR **14**, 1079]. For certain classes of infinite products and certain matrices, the author gives some necessary and sufficient conditions for (A)-summability.

C. V. Stanojevic (Detroit, Mich.)

Pospeev, V. E. 400

On the evaluation of double sums and integrals. (Russian. Uzbek summary)

Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk **1963**, *no.* 4, 39–45.

G. S. Salehov [*The computation of series* (Russian), GITTL, Moscow, 1955; MR **17**, 839] established, with the help of the Euler summation formula, integral estimates for the sums of certain series. Here similar estimates are deduced from the Laplace-Gauss summation formula. Both types of estimates are extended to double series.

A. Dvoretzky (Jerusalem)

Miesner, Walter 401

Sätze zur absoluten Summierung von Laplace-Stieltjes-Integralen.

Mitt. Math. Sem. Giessen. Heft 61 (1964), ii + 36 *pp.*

This paper contains three principal theorems. Theorem 1: Let $b(t)$ be of bounded variation in any interval $(1, R)$ $(R>0)$. If $f(s) = \int_1^\infty e^{-st}\,db(t)$ converges for $\Re s > 0$ and is regular at $s=0$, and if the integral $\int_1^\infty t^{-k}|db(t)| < \infty$ for some k, then $\int_1^\infty db(t)$ is $|C, k|$ summable, that is, $x^{-k}B_k(x) = (1/\Gamma(k+1))\int_0^x (1-t/x)^k\,db(t)$ converges to a limit as $x \to \infty$ and is of bounded variation in the interval $(1, \infty)$. This is the absolute-summability analogue of the M. Riesz theorem for the Laplace integral and of the Fatou theorem for power series. If we write $z = e^{-s}$ and $b(t) = \sum_{n \leq t} a_n$, then the Laplace-Stieltjes integral reduces to a power series and the theorem reduces to a power-series theorem which is due to W. B. Jurkat and A. Peyerimhoff [Math. Z. **60** (1954), 255–270; MR **16**, 351].

Theorem 2: Suppose that the power series $f(z) = \sum_{n=1}^\infty a_n z^n$ is convergent for $|z|<1$ and is regular at $z=1$, and that the series $\sum_1^\infty n^{-k}(w(n))^{-k}|a_n| < \infty$ for some k, where $w(t)$ is positive, monotone increasing,

$$\int_1^x t^{-1}(w(t))^{-1}\,dt \to \infty \qquad \text{as } x \to \infty,$$

$w(t+1) = O(w(t))$ and $(w(t))^{m-k}$ $(m = 0, 1, 2, \cdots, [k])$ are completely monotone. Let us write $\mu(x) = \exp(\int_1^x (tw(t))^{-1}\,dt)$ and suppose that $(\mu(t))^{-1}$ and $\mu'(t)$ are completely monotone. Then the series $\sum_1^\infty a_n$ is $|R, \mu, k|$ summable, that is, $\int_1^\infty db^\mu(t)$ is $|C, k|$ summable, where $b^\mu(t) = \sum_{\mu(n) \leq t} a_n$. This is the absolute-summability analogue of the Zygmund theorem [*Trigonometric series*, 2nd ed., Vols. I, II, Cambridge Univ. Press, New York, 1959; MR **21** #6498]. As a corollary, the author proves that if $f(z) = \sum a_n z^n$ is convergent for $|z|<1$ and is regular at $z=1$ and further if (*) $a_n = n^k(\log n)^k b_n$, where (b_n) is of bounded variation, then $\sum a_n$ is $|R, \log n, k|$ summable. In this theorem the coefficient condition (*) is best possible.

Theorem 3: Two Fourier series $f(x) \sim \sum \Re(a_n e^{inx})$ and $g(x) \sim \sum \Re(b_n e^{inx})$ are $|R, \log n, k|$ equisummable for some k in the interval $(-\varepsilon, \varepsilon)$ $(0<\varepsilon \leqq \pi)$ when $a_n - b_n = (n \log n)^k c_n$, (c_n) being of bounded variation and there is a periodic function $h(x)$ which coincides with $f(x)-g(x)$ almost everywhere in the interval $(-\varepsilon, \varepsilon)$ and whose Fourier series is $|R, \log n, k|$ summable there. This contains many known theorems as special cases [S. N. Bhatt, Tôhoku Math. J. (2) **11** (1959), 13–19; MR **21** #6504; S. Izumi, ibid. (2) **1** (1950), 136–143; MR **12**, 254; R. N. Mohanty and S. Izumi, ibid. (2) **8** (1956), 201–204; MR **20** #4142].

S. Izumi (Tokyo)

APPROXIMATIONS AND EXPANSIONS

See also 211, 274, 413, 417, 497, 744.

Jager, H. 402

A multidimensional generalization of the Padé table. I, II, III.

Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 193–198, 199–211, 212–225.

This series of papers is based on some unpublished work of K. Mahler, and concerns the development of an idea of Hermite ["Sur la fonction exponentielle", *Œuvres de Charles Hermite*, Tome III, pp. 150–181, Gauthier-Villars, Paris, 1912]. The author's starting point is the observation that for any m positive integers ρ_i $(i = 1, 2, \cdots, m)$ there exists a non-trivial set of polynomials $a_i(x)$ (where $a_i(x)$ is of degree $\rho_i - 1$ at most) such that $r(x) = \sum_{i=1}^m a_i(x)f_i(x)$ is $O(x^{\sigma-1})$ as $x \to 0$, where $\sigma = \sum_{i=1}^m \rho_i$. Also there exists a set of non-trivial polynomials $\bar{a}_i(x)$ (where $\bar{a}_i(x)$ is of degree $\sigma - \rho_i$ at most) such that $\bar{r}_{kl}(x) = \bar{a}_l(x)f_k(x) - \bar{a}_k(x)f_l(x)$ for all $k, l = 1, 2, \cdots, m$ is $O(x^{\sigma+1})$ as $x \to 0$. Hermite considered the case in which $f_i(x) = \exp(w_i x)$; the author here systematically generalizes the theory and extends the investigations to cover the cases in which $f_i(x)$ is a formal power series. The study of the Padé table is included in the case $m=2$ with $f_1(x) = 1$.

C. W. Clenshaw (Teddington)

Curtiss, J. H. 403

Polynomial interpolation in points equidistributed on the unit circle.

Pacific J. Math. **12** (1962), 863–877.

Let distinct points $S_n = \{z_{n1}, z_{n2}, \cdots, z_{nn}\}$ be given on the unit circle $|z| = 1$ in the complex z-plane, let a function f also be given on $|z| = 1$ and let $L_n = L_n(f; z)$ denote the polynomial of degree at most $n-1$ found by interpolation to f at the points S_n. Walsh [*Interpolation and approximation by rational functions in the complex domain*, 2nd ed., Amer. Math. Soc., Providence, R.I., 1956] proved that if the points S_n are equally spaced for each n, and if f is Riemann integrable, then

$$\lim_{n\to\infty} L_n(f; z) = \frac{1}{2\pi i}\int_{|t|=1} \frac{f(t)}{t-z}\,dt$$

uniformly on any closed point set on the region $|z| < 1$. If f is analytic on $|z| \leq 1$, the conditions on S_n are much less restrictive.

Let the first $n-1$ points of S_n be the points of S_{n-1}, and let $\theta_1, \theta_2, \cdots$ be an infinite sequence of mutually independent random variables each with the uniform marginal probability distribution on $[0, 2\pi]$. Let $z_k = e^{i\theta_k}$ $(k = 1, 2, \cdots)$. Let the function f be given everywhere on $|z| = 1$ and let

$$L_n(f; z) = \sum_{k=1}^{n} f(z_k)\frac{\omega_n(z)}{(z-z_k)\omega_n'(z_k)},$$

where $\omega_n(z) = (z-z_1)(z-z_2)\cdots(z-z_n)$. If g is a mapping from a vector space of k independent vectors to the complex plane, $E_k g$ denotes the mean value

$$E_k g = \left(\frac{1}{2\pi}\right)^k \int_0^{2\pi}\cdots\int_0^{2\pi} g(\alpha_1, \alpha_2, \cdots, \alpha_k)\,d\alpha_1\cdots d\alpha_k.$$

The following theorems are proved. Theorem 1: If f is continuous on $|z| = 1$ and possesses there an $(n-2)$nd-order derivative satisfying a Lipschitz condition with exponent one, then for all z

$$E_n L_n(f; z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}, \qquad n \geq 2,$$

where $\sum_0^n a_k z^k$ is the Taylor expansion of the analytic function

$$F(z) = \frac{1}{2\pi i}\int_{|t|=1}\frac{f(t)}{t-z}\,dt, \qquad |z| < 1.$$

Thus if f is infinitely differentiable on $|z| = 1$, $\lim_{n\to\infty} E_n L_n(f; z)$ exists and equals $F(z)$ for $|z| \leq 1$. If f is analytic on $|z| \leq 1$, then this limit exists uniformly and equals $f(z)$ on each closed disk with center at the origin which does not contain a singularity of f.

Theorem 3: Let f be analytic for $|z| < R$ $(R > 1)$ but have a singularity on $|z| = R$. Let $P_n(z)$ be the polynomial of degree at most $n-1$ found by interpolation to f in the nth roots of unity. Then

$$\lim_{n\to\infty} [P_n(z) - E_n L_n(f; z)] = 0$$

for $|z| < R^2$, uniformly for $|z| \leq R' < R^2$.

These results are then discussed and extended to the case where the points of S_{n-1} are not necessarily points of S_n. Various other connected questions are raised and explained. An example is given to show that if the set S_n consists of the $(n-2)$nd roots of unity and two additional points not equally spaced, then Walsh's result is upset; the result remains true if $f(z)$ is analytic for $|z| \leq 1$.

G. M. Petersen (San Francisco, Calif.)

Brudnyĭ, Ju. A.; Gopengauz, I. E. 404

Approximation by piecewise polynomial functions. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **27** (1963), 723–746.

Etant donné un entier $r > 0$ et une suite $-1 \leq t_1^{(r)} \leq t_2^{(r)} \leq \cdots \leq t_r^{(r)} \leq 1$, on construit à partir d'une fonction bornée $f(x)$ définie sur $[0, 1]$ une fonction $P_{r,n}(f; x)$ qui coïncide sur chaque intervalle $[\nu/n, (\nu+1)/n]$, $\nu = 0, 1, \cdots, n-1$, avec le polynôme interpolant $f(x)$ aux points $(1/2n)[2\nu + 1 + t_\mu^{(r)}]$, $\mu = 1, 2, \cdots, r$. On étudie alors la quantité

$$\tau_{r,n}(f) = \sup_{k\geq n}\ \sup_{0\leq x\leq 1} |f(x) - P_{r,k}(f; x)|$$

en fonction des propriétés différentielles de $f(x)$. Posons

$$\omega_s(f; t) = \sup\left|\sum_{\nu=0}^{s}(-1)^\nu\binom{s}{\nu}f(x+\nu h)\right|,$$

le supremum étant contraint par $|h| \leq t$ et $x, x+sh \in [0, 1]$. Parmi les résultats citons: (1) Pour chaque entier $r > 0$ on a $\omega_r(f; 1/n) \sim \tau_{r,n}(f)$, $n \to \infty$; (2) Si la série

$$\sum_{k=1}^{\infty} k^{m-1}\tau_{r,k}(f)$$

converge pour $m < r$. alors la $m^{\text{ième}}$ dérivée $f^{(m)}$ existe et

$$\omega_{r-m}\left(f^{(m)}; \frac{1}{n}\right) = O\left(\sum_{k=n+1}^{\infty} k^{m-1}\tau_{r,k}(f)\right);$$

(3) Pour que la fonction $f(x)$ soit r fois dérivable et $\sup_x |f^{(r)}(x+h) - f^{(r)}(x)| \leq Mh$, il faut et il suffit que

$$\tau_{r+1,n}(f) \leq \frac{M}{(r+1)!2^{2r+1}n^{r+1}};$$

(4) Soit k un entier positif fixe, alors

$$\sup_{0\leq x\leq 1-1/n} E_{k-1}(f; x, x-1/n) \sim \tau_{k,n}(f), \qquad n \to \infty;$$

(5) Pour qu'une fonction $f(x)$ indéfiniment dérivable appartienne à la classe de Gevrey d'indice $\alpha > 0$ sur $[0, 1]$ il faut et il suffit qu'il existe une constante $A > 0$ telle que pour chaque intervalle $[c, d] \subset [0, 1]$ on a

$$E_{n-1}(f; c, d) = O(\exp\{-[Ar/(d-c)]^{1/\alpha}\}), \qquad r \to \infty.$$

E. J. Akutowicz (Bologna)

Hitotumatu, Sin 405

Some considerations on the best-fit polynomial approximations. I.

Comment. Math. Univ. St. Paul. **12** (1964), 59–73.

This paper presents basic theorems for the theory of best-fit (in the L_∞ norm) approximation. Though the results are known, the presentation is original, concise, modern and well done.

J. R. Rice (Warren, Mich.)

van der Corput, J. G. 406

Asymptotic developments. III. Again on the fundamental theorem of asymptotic series.

J. Analyse Math. **9** (1961/62), 195–204.

Part II appeared in same J. **5** (1956/57), 315–320 [MR **21** #1479]. The fundamental theorem on asymptotic series can be formulated as follows. Theorem I: A series $a_0 + a_1 + \cdots$ converges asymptotically if and only if a_k tends asymptotically to zero as $k \to \infty$. Here the terms

$a_k(\omega)$ are defined for each point ω of an unbounded set Ω lying in the complex plane or on a Riemann surface. That the series $a_0+a_1+\cdots$ converges asymptotically means that it is possible to define on Ω a function $s(\omega)$ such that $s-(a_0+a_1+\cdots+a_{h-1})$ tends asymptotically to zero as $h\to\infty$. s is called the asymptotic sum and $s\sim\sum'^{\infty}_{h=0} a_h$. If each term $a_k(\omega)$ is assumed to be infinitely differentiable on Ω and if for each fixed integer $h\geqq 0$ the hth derivate $a_k^{(h)}(\omega)$ tends asymptotically to zero as $k\to\infty$, it is possible, according to the fundamental theorem, to define on Ω functions $s_h(\omega)$ such that, for each fixed integer $h\geqq 0$,

$$s_h(\omega) \sim \sum_{k=0}^{\infty}{}' a_k^{(h)}(\omega).$$

The deeper question arises whether it is possible to define on Ω an infinitely differentiable function $s(\omega)$ such that, for each fixed integer $h\geqq 0$,

$$(1)\qquad s^{(h)}(\omega) \sim \sum_{k=0}^{\infty}{}' a_k^{(h)}(\omega).$$

In his Theorem 2 the author states that the answer is affirmative. In the proof he uses a fixed, monotonically nonincreasing, infinitely differentiable function $\sigma(t)$ equal to 1 for $t\leqq 0$ and equal to 0 for $t\geqq 1$. Such a function he calls a smoothing function. A special case of Theorem 2 is Theorem 3: If $a_0, a_1, \cdots$ are arbitrary fixed complex numbers, then there exists a function $f(x)$ infinitely differentiable for $x\geqq 0$ with $f^{(h)}(0)=a_h$ $(h=0, 1, \cdots)$. Further considerations lead to the question: If we know that a function $s(\omega)$ is infinitely differentiable for $\omega\geqq 0$, and if the asymptotic behaviour of each derivative $s^{(h)}(\omega)$ is known, what do we know about the behaviour of $s(\omega)$ in the interval $0\leqq\omega\leqq 10^{10}$? The answer is "nothing", apart from the fact that the function is infinitely differentiable in that interval. The following obvious theorem (Theorem 4) is stated. r is a fixed integer $\geqq 0$. The terms $a_h(\omega)$ defined on Ω are r *times* differentiable with respect to ω, and for $k=0, 1, \cdots, r$ the function $a_h^{(k)}(\omega)$ tends asymptotically to zero as $h\to\infty$. Then it is possible to construct on Ω an r-times differentiable function $s(\omega)$ which satisfies (1) for $k=0, 1, \cdots, r$. This theorem remains true if the two expressions "differentiable" are replaced by "continuously differentiable". Finally, these theorems are generalized to functions of two or more real independent variables. The notions of asymptotic equality and of asymptotic limits are defined in a natural way.

S. C. van Veen (Delft)

Buadze, A. I. 407

Approximation of a function by Bernstein-Rogosinski sums. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **33** (1964), 285–292.

This paper contains direct and inverse theorems concerning approximation of functions in L^p $(1\leqq p\leqq\infty)$ on the two-dimensional torus through various linear operators formed in the usual way with triangular matrices and Fourier series. Results of a similar nature are also given concerning approximation of functions in L^p on the real line. There appear generalisations of Bernstein's inequality concerning the derivative of a trigonometric polynomial. No proofs and not all definitions are given.

E. J. Akutowicz (Bologna)

FOURIER ANALYSIS

See also 76, 182, 195, 223, 401, 462, 463, 499, 726.

Littlewood, J. E. 408

On the real roots of real trigonometrical polynomials.

Studies in mathematical analysis and related topics, pp. 219–226. *Stanford Univ. Press, Stanford, Calif.*, 1962.

Suppose f is a real trigonometric polynomial of degree $n>0$. Define

$$M_\lambda(f) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^\lambda\, d\theta\right)^{1/\lambda},$$

and denote by $N(f)$ the number of real roots of f in $[-\pi, \pi)$. The author announces the following upper bound for $N(f)$. There exists a positive $c_0\geqq 2^{3/2}/\pi$ such that if $c_0<c<1$ and $M_1(f)\geqq cM_2(f)$, then $N(f)\leqq(1-A(c))n$, where $A(c)$ depends only on c, and $A(c)\to 1$ as $c\to 1$. Going the other way around, the following lower bound is obtained. If f is a trigonometric polynomial of degree n, we can write

$$f(x) = \sum_{i=0}^{n} a_i \cos(ix+\alpha_i).$$

Now let c and k be positive constants $(k\geqq 1)$, and let F be the class of trigonometric polynomials f satisfying

$$\sum_{i=0}^{m} a_i^2 \leqq k\frac{m}{n}(M_2(f))^2$$

for $n\varepsilon_n\leqq m\leqq n$, where n is the degree of f. Suppose now a sequence $\{f_n\}$ of functions in F satisfies $M_1(f_n)\geqq cM_2(f_n)$, and degree $f_n=n$ for $n=1, 2, \cdots$. Then for some $A>0$, all but a finite number of the functions in the sequence satisfy $N(f_n)>Ak^{-1}c^8n$.

B. Randol (New Haven, Conn.)

Hsiang, Fu Cheng 409

On a convergence test of Hardy-Littlewood's type for Fourier series.

J. Austral. Math. Soc. **4** (1964), 49–55.

Let $\phi(t)$ be an even L-integrable function, periodic with period 2π and let its Fourier series be $\frac{1}{2}a_0+\sum_{n=1}^{\infty} a_n\cos nt$. The author [Proc. Amer. Math. Soc. **9** (1958), 37–44; MR **20** #2567] has proved that if, for some $\Delta>0$, (i) $\int_0^t\phi(u)\,du=o\{t(\log(1/t))^{-\Delta}\}$ as $t\to+0$ and (ii) $a_n>-kn^{-1}(\log n)^{\Delta}$, then $\sum a_n$ converges to the sum $s=0$. In this note, by showing that for each $\Delta>0$ and each $\eta>0$ there exists an even function $\phi(t)$ satisfying (i), with its Fourier series diverging at $t=0$, and such that $a_n=O\{(\log n)^{\Delta+\eta}/n\}$, he proves that it is essential to use the same Δ in conditions (i) and (ii) of the above theorem.

B. N. Prasad (Allahabad)

Teljakovskiĭ, S. A. 410

Some bounds for trigonometric series with quasi-convex coefficients. (Russian)

Mat. Sb. (*N.S.*) **63** (**105**) (1964), 426–444.

Als Erweiterung eines bekannten Ergebnisses für den Fall $a_k\searrow 0$ (vgl. Zygmund, *Trigonometric series*, 2nd ed., Vols. I, II, Cambridge Univ. Press, New York, 1959; MR **21** #6498], Theorem (1.14), S. 185) zeigt Verfasser,

dass $\sum a_k \sin kx$ mit $\sum_{k=1}^{\infty} k|\Delta^2 a_{k-1}| < \infty$, $a_k \to 0$ genau dann eine Fourierreihe ist, wenn gilt $\sum (|a_k|/k) < \infty$. Es wird ferner eine hinreichende Bedingung dafür angegeben, dass eine trigonometrische Reihe in mehreren Veränderlichen eine Fourierreihe ist. Aus dieser Bedingung erhält der Verfasser eine Verallgemeinerung eines Ergebnisses von Babenko [Dokl. Akad. Nauk SSSR **132** (1960), 982–985; MR **22** #12342]. Es ist im Rahmen eines Referates nicht möglich, die umfangreichen Einzelheiten zu besprechen.
A. Peyerimhoff (Marburg)

Talaljan, A. A. 411
Rearranged trigonometrical systems which are systems of convergence in the weak sense. (Russian)
Mat. Sb. (N.S.) **63 (105)** (1964), 620–638.

Le présent travail est la continuation des études importantes de Kolmogorov, Zahorski et Ul'janov [voir Ul'janov, Uspehi Mat. Nauk **16** (1961), no. 3 (99), 61–142; MR **23** #A2701] relatives aux permutations des termes des séries de Fourier d'une fonction à carré sommable et de Men'šov [N. K. Bari, *Trigonometric series* (Russian), Ch. VI, Fizmatgiz, Moscow, 1961; MR **23** #A3411] concernant la rectification de $f(x) \in L[-\pi, \pi]$ sur un ensemble de mesure $< \varepsilon$. Le principal résultat de cet article est le théorème suivant.

Il existe une série

$$(*) \qquad a_0 + \sum_{k=1}^{\infty} (a_k \cos \mu_k x + b_k \sin \mu_k x),$$

où μ_k sont des nombres naturels différents entre eux, telle que (a) Pour une suite des nombres entiers $k_1 < k_2 < \cdots < k_n < \cdots$ la série $\sum_{i=1}^{\infty} (a_{k_i} \cos \mu_{k_i} x + b_{k_i} \sin \mu_{k_i} x)$, formée par des termes correspondants de (*) est presque partout divergente et $\sum (a_{k_i}^2 + b_{k_i}^2) < \infty$; (b) Quel que soit $f(x) \in L_2[-\pi, \pi]$ et $\varepsilon > 0$, il existe une fonction $\psi(x) \in L_2[-\pi, \pi]$ et une suite des entiers $n_1 < n_2 < \cdots$ telles que mes $E[\psi(x) \neq f(x)] < \varepsilon$ et la série $\sum_{j=1}^{\infty} (a_{n_j} \cos \mu_{n_j} x + b_{n_j} \sin \mu_{n_j} x)$, $\sum (a_{n_j}^2 + b_{n_j}^2) < \infty$, converge presque partout vers $\psi(x)$ sur $[-\pi, \pi]$.
M. Tomić (Belgrade)

Szüsz, P. 412
Über die absolute Konvergenz lakunaerer trigonometrischer Reihen. (Russian summary)
Acta Math. Acad. Sci. Hungar. **12** (1961), 215–220.

Suppose K is an arbitrary given positive number. The author shows that there exists a lacunary sequence of integers $\{n_k\}$, with $n_{k+1}/n_k > K$, with the property that if $\{a_k\}$ is any non-increasing sequence of positive numbers, then $\sum |a_k \sin \pi n_k x| < \infty$ for some $x \not\equiv 0 \pmod 1$ implies $\sum a_k < \infty$. If, however, one insists on "large gaps", the analogous conclusion is false. More particularly, the author shows that if $\{n_k\}$ is any sequence of integers with the property that $n_{k+1}/n_k \to \infty$, then there exists a non-increasing sequence $\{a_k\}$ so that $\sum |a_k \sin \pi n_k x| < \infty$ is a set of x of the power of the continuum, while $\sum a_k = \infty$.
E. M. Stein (Princeton, N.J.)

Petrov, I. M. 413
On the order of approximation of functions of class Z_1 by positive linear polynomial operators. (Russian)
Uspehi Mat. Nauk **19** (1964), no. 2 (116), 151–154.

Let Z_1 be the class of 2π-periodic functions for which

$$|f(x+h) - 2f(x) + f(x-h)| \leqq 2|h|,$$

and let

$$L_n(f, x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) u_n(t-x)\, dt,$$

where $u_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \rho_k^{(n)} \cos kt \geqq 0$. Put

$$g_n(L) = \sup_{f(x) \in Z_1} n\left[\max_x |f(x) - L_n(f, x)|\right].$$

The author proves

$$1.81 < \liminf_{n=\infty} \inf_{L_n} g_n(L) \leqq \limsup_{n=\infty} \inf_{L_n} g_n(L) < 2.38.$$

The proof depends on L. Fejér's [J. Reine Angew. Math. **146** (1916), 53–64; ibid. **146** (1916), 65–82] representation for $u_n(t)$ with the author's special choice of coefficients, a previous result of the author,

$$g_n(L) = \frac{n}{\pi} \int_{-\pi}^{\pi} |t| u_n(t)\, dt$$

[Izv. Vysš. Učebn. Zaved. Matematika **1960**, no. 1 (14), 188–193; MR **24** #A967], and Szász's inequality

$$\rho_k^{(n)} \leqq \cos \frac{\pi}{[n/k] + 2}.$$

One could get a sharper estimation

$$\liminf_{n=\infty} \inf_{L_n} g_n(L) > 2,$$

using P. P. Korovkin's inequality $\rho_k^{(n)} + \rho_{n-k}^{(n)} \leqq 1$ [Uspehi Mat. Nauk **13** (1958), no. 6 (84), 99–103; MR **21** #253].
C. V. Stanojevic (Detroit, Mich.)

Levinson, N.; McKean, H. P., Jr. 414
Weighted trigonometrical approximation on R^1 with application to the germ field of a stationary Gaussian noise.
Acta Math. **112** (1964), 99–143.

Let $\Delta = \Delta(a)$ be a given, non-trivial, even, non-negative weight function with $\int_{-\infty}^{\infty} \Delta\, da < \infty$. The paper is concerned with the trigonometric approximation of functions in Z with respect to the weight Δ, where Z is the (real) Hilbert space $L^2(R^1, \Delta da)$ of Lebesgue measurable functions f with $f^*(-a) = f(a)$, $\|f\| = \|f\|_\Delta = (\int_{-\infty}^{\infty} |f|^2 \Delta\, da)^{1/2} < \infty$ (*: conjugate complex). We write Z^{cd} for the space (in Z) of e^{iat} $(c \leqq t \leqq d)$ and consider the following seven subspaces of Z: $Z^- = Z^{-\infty 0}$, $Z^+ = Z^{0\infty}$, $Z^{+/-}$ = the projection of Z^+ onto Z^-, $Z^\bullet$ = the class of entire functions $f(a+ib)$ with

$$\limsup_{R \uparrow \infty} R^{-1} \max_{0 \leqq \theta \leqq 2\pi} \log |f(Re^{i\theta})| \leqq 0,$$

which belong to Z on the line $b = 0$, $Z^{0+} = \bigcap_{\delta > 0} Z^{0\delta}$, $Z_\bullet$ = the span of $(ia)^d$, $d = 0, 1, 2, \cdots$, $\int_{-\infty}^{\infty} a^{2d} \Delta\, da < \infty$ and $Z^{-\infty} = \bigcap_{t<0} Z^{-\infty t}$. In this paper the authors give extensive and complete discussions about the inclusion relations among these subspaces. These depend, as is expected, on whether or not the weight function Δ is a Hardy function. An even Hardy function Δ is such that $\Delta = |h|^2$, $h^*(-a) = h(a)$, where $h = h(\gamma)$, $\gamma = a + ib$, belongs to the Hardy class H^2 in the upper half-plane. The main results are the following: $Z \supset Z^- \supset Z^{+/-} \supset Z^- \cap Z^+$ is always true and obvious, but if Δ is a Hardy function,

$Z \neq Z^- \neq Z^- \cap Z^+$ and in the non-Hardy case, $Z = Z^- \cap Z^+ = Z^{-\infty}$. In the Hardy case, conditions are given for $Z^- \neq Z^{+/-}$ or $Z^{+/-} = Z^- \cap Z^+$ and also for $Z^{+/-} = Z^{0+}$. $Z^- \cap Z^+ \supset Z^{0+}$ is true whether or not Δ is Hardy. As to the relations concerning $Z_\bullet$, the following results are obtained: $Z^\bullet$ is dense in Z or alternately $Z^\bullet$ is a closed subset of Z; conditions are given under which the alternative holds. $Z^\bullet \subset Z^{0+}$ is always true. If Δ is Hardy, then $Z^\bullet = Z^{0+}$, and $Z^\bullet = Z^- \cap Z^+$ with the condition that $\int_{-d}^{d} \Delta^{-1}\, da < \infty$ $(d < \infty)$. Also conditions are given for which $Z^\bullet = Z^{+/-}$. In connection with $Z_\bullet$, conditions are given for $Z_\bullet = Z$ and for $Z_\bullet \neq Z$. This is nothing more than S. Bernstein's problem on trigonometric approximation discussed by many authors [e.g., S. N. Mergeljan, Uspehi Mat. Nauk **11** (1956), no. 5 (71), 107–152; MR **18**, 734; Amer. Math. Soc. Transl. (2) **10** (1958); MR **20** #1146]. In the case $Z_\bullet \neq Z$, a necessary and sufficient condition is given for $Z_\bullet = Z^\bullet$. The special case with a particular form of Δ and some other problems are discussed. The applications of these results to the centered stationary Gaussian process $x(t)$ $(t \in R^1)$ with covariance function $Ex(s)x(t) = \int_{-\infty}^{\infty} e^{ia(t-s)}\Delta(a)\, da$ are straightforward if the (real) Hilbert space Q which is the closed span of $x(t)$ with norm $(E|x|^2)^{1/2}$ is mapped 1-1 onto Z with map $x(t) \to e^{iat}$ $(\in Z)$, inner products being preserved and, using the notation Q^{cd} = the span of $x(t)$ $(c \leq t \leq d)$ and B^{cd} = the smallest Borel field of the probability field B measuring $x(t)$ $(c \leq t \leq d)$, a perfect correspondence is obtained between Z^- or $Q^-(=Q^{-\infty 0})$ and $B^- = B^{-\infty 0}$ = the past; Z^+ or $Q^+(=Q^{0\infty})$ and $B^* = B^{0\infty}$ = the future; $Z^{+/-}$ or $Q^{+/-}$ and $B^{+/-}$ = the smallest splitting field of past and future; Z^{0+} or Q^{0+} and $B^{0+} = \bigcap_{\delta > 0} B^{0\delta}$ = germ; $Z_\bullet$ or $Q_\bullet$ and the associated field B; and $Z^{-\infty}$ or $Q^{-\infty}$ and B^- = the distant past. *T. Kawata* (Washington, D.C.)

Wang, Si-lei 415

On the functions represented by Rademacher series.

Acta Math. Sinica **13** (1963), 647–652 (*Chinese*); *translated as Chinese Math.* **4** (1964), 703–708.

It was proved by S. B. Stečkin and P. L. Ul'janov [Izv. Akad. Nauk SSSR Ser. Mat. **26** (1962), 211–222; MR **25** #385] that if $F(x)$ is analytic in $(0, 1)$ and the Rademacher series (*) $\sum_1^\infty a_m r_m(x)$ converges to $F(x)$ for all x in the set $E \subset [0, 1]$ whose measure $mE > \frac{1}{2}$, then the series (*) converges almost everywhere in $[0, 1]$ to $F(x) = c(1-2x)$, c being a certain constant. The present author asks whether a continuous function in $[0, 1]$ can generally be represented almost everywhere by the series (*) and shows that if $F(x)$ is continuous in $[0, 1]$ and the series (*) converges almost everywhere in $[0, 1]$ to $F(x)$, then $F(x) = F(0)\cdot(1-2x)$, $a_m = F(0)/2^m$ $(m = 1, 2, \cdots)$, and that the condition "almost everywhere convergent" cannot be replaced by "convergent in $E \subset [0, 1]$, $mE < 1$". *R. S. L. Srivastava* (Kanpur)

Taberski, R. 416

Some properties of Fourier-Bessel series. I.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 151–156.

Der Verfasser betrachtet Fourier-Bessel Reihen

$$\sum_m d_m^{(\nu)} \sqrt{(j_m x)} J_\nu(j_m x) \sim f(x)$$

$$(f(x) \in L(0, 1),\ \nu > -1,\ J_\nu(j_m) = 0).$$

Ist $f \in \mathrm{Lip}\,\alpha$ $(0 < \alpha < 1)$, so ist $d_m^{(\nu)} = O(m^{-\alpha})$ und aus $\sum_{m=n}^\infty |d_m^{(\nu)}| = O(n^{-\alpha})$ folgt $f \in \mathrm{Lip}\,\alpha$ für jedes Intervall $[\varepsilon, 1]$, $\varepsilon > 0$. Ist $f(t)t^{-\nu-1/2} \in \mathrm{Lip}\,\alpha$ $(\nu \geq -\frac{1}{2})$ und $f(t) = O(1-t)$ für $t \to 1-0$, so gilt $\mathrm{Max}_{0 \leq x \leq 1} |F_n^{(\nu)}(x, f) - f(x)| = O(n^{-\alpha})$ $(0 < \alpha < 1)$ bzw. $= O(\log n/n)$ $(\alpha = 1)$, wo $F_n^{(\nu)}(x, f)$ die C_1-Mittel der Teilsummen der Fourier-Bessel Reihe seien. Ist noch $\alpha > \frac{1}{2}$, so ist analog zum Satz von Bernstein $\sum_m |d_m^{(\nu)}| < \infty$. *A. Peyerimhoff* (Marburg)

Ul'janov, P. L. 417

On the approximation of functions. (**Russian**)

Sibirsk. Mat. Ž. **5** (1964), 418–437.

The author gives statements on the approximation of 2π-periodic functions f by the Cesàro means $\sigma_n^\alpha(x, f)$ of order $\alpha > 0$ of their Fourier series

$$\tfrac{1}{2}a_0 + \sum_{k=1}^\infty (a_k \cos kx + b_k \sin kx).$$

Examples: (i) If $\omega_p(\delta, f) \equiv \sup_{|h| \leq \delta} \| f(x+h) - f(x) \|_p$ and $f \in L^p(0, 2\pi)$, then for $n = 1, 2, \cdots$,

$$(1) \qquad \|\sigma_n^\alpha(x, f) - f(x)\|_p \leq B(\alpha)\, \omega_p\left(\frac{\log(n+2)}{n+1}, f\right)$$

for $p = 1$, $p = \infty$ and every $\alpha > 0$;

$$(2) \qquad \|\sigma_n^\alpha(x, f) - f(x)\|_p \leq B(\alpha, p)\omega_p\left(\frac{1}{n+1}, f\right)$$

for $1 < p < \infty$ and $\alpha \geq 1$. Here the B's are constants depending only on α, respectively, α and p. (ii) If $f \in \mathrm{Lip}_M\, \beta = \{f : |f(x) - f(y)| \leq M|x-y|^\beta\ \forall\, x, y\}$, then for $\alpha > 0$ and $n = 1, 2, \cdots$,

$$(1) \quad \|\sigma_n^\alpha(x, f) - f(x)\|_c \leq B(\alpha) \frac{M}{(n+1)^\beta} \quad \text{for} \quad 0 < \beta < 1,$$

$$(2) \quad \|\sigma_n^\alpha(x, f) - f(x)\|_c \leq B(\alpha)\, M \frac{\log(n+2)}{n+1} \quad \text{for} \quad \beta = 1.$$

At the end of the paper there are given some statements about general methods of summation.

G. Goes (Lawrence, Kans.)

Ruban, P. I.; Krasil'nikov, K. V. 418

A method of approximating even functions satisfying a Lipschitz condition by trigonometrical polynomials. (**Russian**)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 2 (39), 136–138.

Let f be a 2π-periodic even function which fulfills a Lipschitz condition of order α $(0 < \alpha \leq 1)$ with constant K. The authors give an answer to the following question: Which trigonometric polynomial of degree n has the smallest deviation from f in the metric of L_p? This polynomial can be written in the form $T_n^N(f, x) = \frac{1}{2}a_0^N + \sum_{\nu=1}^n a_\nu^N \cos \nu x$, where $a_0^N = (2/N) \sum_{k=1}^N f(x_k)$, $a_\nu^N = (2/N) \sum_{k=1}^N f(x_k) \cos \nu x_k$ with $x_k = x_k^N = 2k\pi/N$ $(k = 0, 1, \cdots, N)$ and N any natural number $\geq 2n+1$. Let $F_n^N(f, x) = (1/n) \sum_{k=1}^n T_k^N(f, x)$, i.e., $F_n^N(f, x)$ is the nth Fejér partial sum formed with the T_k^N $(k = 1, 2, \cdots, n)$. The authors give estimates for $\sup |f(x) - F_n^N(f, x)|$, where the supremum is taken with respect to all 2π-periodic even functions belonging to certain Lipschitz classes of order α $(0 < \alpha \leq 1)$ and for $\lim_{n\to\infty} n/N \neq 0$ or $\lim_{n\to\infty} n/N = 0$.

G. Goes (Lawrence, Kans.)

Žižiašvili, L. V. 419

Some properties of the (C, α)-means of Fourier series and the conjugate trigonometric series. (Russian)

Mat. Sb. (N.S.) **63 (105)** (1964), 489–504.

Soient $f(x) \in L[-\pi, \pi]$, et 2π-périodique, $S[f]$ sa série de Fourier, $\tilde{f}$ et $\tilde{S}[f]$ la fonction conjuguée de f, respectivement la série conjuguée de $S[f]$, et notons $\varphi(x, t) = f(x+t) + f(x-t) - 2s$. Si f satisfait au point x aux conditions

$$\text{(a)} \qquad \int_0^h \varphi(x, t)\, dt = o(h),$$

$$\text{(b)} \qquad \int_h^\pi \frac{|\varphi(x, t) - \varphi(x, t+h)|}{t^{1-\alpha}}\, dt = o(h^\alpha), \qquad -1 \leqq \alpha < 1,$$

alors $S[f]$ est $(C, -\alpha)$-sommable vers s. Ce résultat généralise un théorème de Hardy et Littlewood [Math. Z. **28** (1928), 612–634; Theorem 7], où l'on suppose que $f \in \mathrm{Lip}(\beta, p)$, $0 < \beta \leqq 1$, $\beta p > 1$, $\alpha < \beta$. Dans ce cas, de $\beta p > 1$ résulte que f est équivalente à une fonction continue et (b) s'ensuit immédiatement, tandis que la condition (a) d'après Hardy et Littlewood [ibid., Théorème 1] est nécessaire pour la sommabilité $(C, -\alpha)$.

L'auteur donne aussi l'estimation de $\|S_n^\alpha(x) - f(x)\|_C$ et de $\|t_n^\alpha(x) - \tilde{f}(x)\|_C$ en fonction du module de continuité de f. S_n^α, respectivement t_n^α, sont les sommes de Cesàro d'ordre α de $S[f]$, respectivement de $\tilde{S}[f]$.

M. Tomić (Belgrade)

Tihonov, A. N. 420

On stable summability methods for Fourier series. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 268–271.

Es sei $\{u_n(x)\}$, $a \leqq x \leqq b$, ein Orthonormalsystem mit $u_n'' - q^2 u_n = -\lambda_n u_n$, $u_n(a) = u_n(b) = 0$, $0 \leqq q^2 \leqq M$. Nach Vorgabe von $f(x) \in L_2(a, b)$, $\tilde{f}_n = \int_a^b \tilde{f}(x) u_n(x)\, dx$ und Zahlen $\tilde{f}_n$, $\sum \tilde{f}_n^2 < \infty$ gilt: Ist x_0 ein Stetigkeitspunkt von $\tilde{f}(x)$, so gibt es zu jedem $\varepsilon > 0$ ein δ_0 mit $|f^\alpha(x_0) - f(x_0)| < \varepsilon$ $(f^\alpha(x_0) = \sum_1^\infty (\tilde{f}_n^\alpha/(1+\alpha\lambda_n)) u_n(x)$, $\alpha = r\delta^2$, $r = \text{const})$ für $\sum_{n=1}^\infty (\tilde{f}_n - f_n)^2 < \delta \leqq \delta_0$.

A. Peyerimhoff (Marburg)

Pogodičeva, N. A. 421

Study of a class of summability methods of interpolation processes. (Russian. English summary)

Ukrain. Mat. Ž. **16** (1964), 164–184.

Es sei $f(x)$ stetig und mit 2π periodisch, ferner sei

$$U_n(f; x; \lambda) = \tfrac{1}{2} a_0 + \sum_{k=1}^n \lambda_k^{(n)} (a_k \cos kx + b_k \sin kx),$$

$$\tilde{U}_n(f; x; \lambda) = \tfrac{1}{2} a_0^{(n)} + \sum_{k=1}^n \lambda_k^{(n)} (a_k^{(n)} \cos kx + b_k^{(n)} \sin kx)$$

$(a_k^{(n)} = (2/(2n+1)) \sum_{\nu=0}^{2n} \cos kx_\nu^{(n)} f(x_\nu^{(n)})$, $k = 0, 1, \cdots, n$, $x_\nu^{(n)} = 2k\pi/(2n+1)$, $\nu = 0, 1, \cdots, 2n$ und entsprechend für $b_k^{(n)})$, wo $\lambda_k^{(n)} = l_k^{(n)} \cos k\alpha_n$ ist mit $l_0^{(n)} = 1$, $l_{n+1}^{(n)} = 0$, $l_k^{(n)}$ für jedes feste n konvex oder konkav in k, $0 \leqq \alpha_n \leqq \frac{1}{2}\pi$, $\alpha_n = O(1/n)$. Ist $L_n(\alpha_n) = \sup_{|f(x)| \leq 1} |U_n(f; x; \lambda)|$, $\tilde{L}_n(x, \alpha_n) = \sup_{|f(x)| \leq 1} |\tilde{U}_n(f; x; \lambda)|$, so gilt

$$\tilde{L}_n(x, \alpha_n) = \tfrac{1}{2}\pi |\sin \tfrac{1}{2}(2n+1)x| L_n(\alpha_n) + O(1)$$

gleichmässig in n, x und α_n. *A. Peyerimhoff* (Marburg)

Cigler, Johann 422

Einige Bemerkungen zur Theorie der fastperiodischen Funktionen.

Arch. Math. **15** (1964), 155–160.

Author's summary: "In der vorliegenden Note werden die Beziehungen zwischen einer topologischen Gruppe $\mathfrak{G}$ und ihrer fastperiodischen Kompaktifizierung $\tilde{\mathfrak{G}}$ vom Standpunkt der Theorie der Gleichverteilung aus untersucht. Es zeigt sich, dass im Falle der Gruppe $\mathfrak{G}$ der ganzen (reellen) Zahlen $\mathfrak{G}$ gleichverteilt in $\tilde{\mathfrak{G}}$ ist. Daraus wird auf einfache Weise die Existenz der Verteilungsfunktion einer reelwertigen fastperiodischen Funktion auf $\mathfrak{G}$ gefolgert. Im Falle einer beliebigen topologischen Gruppe wird die Existenz des Mittelwertes einer fastperiodischen Funktion als Gleichverteilungsaussage gedeutet. Daraus wird wieder die Existenz der Verteilungsfunktion fur eine reelwertige fastperiodische Funktion hergeleitet."

E. Følner (Copenhagen)

Kung, Sun 423

Fourier analysis on unitary groups. I. Partial sums and summability of Fourier series.

Sci. Record (N.S.) **4** (1960), 317–322.

The author determines explicitly the Dirichlet kernel and related kernels for summability of Fourier series on the n-dimensional unitary group. Suppose $U \to A_f(U)$ is the irreducible representation of signature $f = (f_1, f_2, \cdots, f_n)$, $f_1 \geqq f_2 \geqq \cdots \geqq f_n$, and $\sum_f \operatorname{tr}(C_f \Phi_f'(U))$ is the Fourier expansion of the function $u(U)$ with $C_f = (N(f)/c)^{1/2} \int u(U) A_f(\bar{U}) \dot{U}$, where $N(f)$ is the degree of the representation; $\dot{U}$ is the invariant volume element whose total volume is c; and $\Phi_f(U) = (N(f)/c)^{1/2} A_f(U)$. He shows that if

$$S_N(U) = \sum_{N \geqq l_1 > l_2 > l_n > -N} \operatorname{tr}(C_f \Phi_f'(U)), \quad l_j = f_j + n - j,$$

then

$$S_N(U) = \frac{1}{c} \int u(VU) \mathscr{D}_N(V) \dot{V},$$

with

$$\mathscr{D}_N(V) = \frac{c'}{D(e^{i\theta_1}, \cdots, e^{i\theta_n})} \det[d_N^{(i-1)}(\theta_j)].$$

Here $e^{i\theta_1}, \cdots, e^{i\theta_n}$ are the characteristic roots of V, $D(x_1, \cdots, x_n) = \prod_{i<j} (x_i - x_j)$; $d_N(\theta) = \sin(N + \frac{1}{2})\theta / \sin \theta/2$, $d_N^{(i-1)}(\theta)$ its $(i-1)$st derivative; c' is an appropriate constant. A similar result dealing with more general summability is also given. *E. M. Stein* (Princeton, N.J.)

Kung, Sun 424

Fourier analysis on unitary groups. II. Abel summability.

Sci. Sinica **10** (1961), 26–30.

For the notation see the preceding review [#423]. Hua [Sci. Record (N.S.) **2** (1958), 280–284; MR **22** #1792] considered an analogue of Abel summability for Fourier expansions on the unitary group given by

$$\sum_f \rho^f(r) \operatorname{tr}(C_f \Phi_f'(U)), \qquad 0 \leqq r < 1,$$

with

$$\rho^f(r) = \frac{1}{N(f)c} \int \frac{(1-r^2)^{n^2}}{|\det(1-rU)|^{2n}} \chi_f(U) \dot{U},$$

where χ_f is the character of the representation A_f. He showed that $\rho^f(r) \to 1$ as $r \to 1$; $\rho^f(r) = r^{f_1+f_2+\cdots+f_n}$ if $f_n \geqq 0$, similarly $\rho^f(r) = r^{-f_1-f_2-\cdots-f_n}$ if $f_1 \leqq 0$.

In this note the author determines an explicit (although complicated) expression for $\rho^f(r)$, holding in general, and using it he announces a Tauberian condition for the convergence of the Fourier series.

E. M. Stein (Princeton, N.J.)

Kung, Sun 425

Fourier analysis on unitary groups. I. Abel summability and the Dirichlet kernel of a Fourier series. (Chinese. English summary)

Acta Math. Sinica **10** (1960), 239–261 (*Chinese*); *translated as Chinese Math.* **1** (1962), 249–272.

From the author's summary: "This paper contains part of the results obtained in studying the Fourier series of an integrable function on the unitary group, chiefly through further investigation of Abel summability as defined by Hua and with the establishment of an expression for Abel summability. The Dirichlet kernel for a Fourier series is also given. To sum up, the final expression for Abel summability, given in § 1.3–§ 1.7, has been obtained through complicated calculation. In § 1.8, we obtain the Dirichlet kernel of a Fourier series by an analytical method, with the aid of the Poisson kernel in Abel summability. In § 1.9, we give another proof (an algebraic proof) of the foregoing result, which is due to Hua. By using the same method we have defined, in § 1.10, a method of summability and the corresponding kernels."

M. Fukamiya (Sendai)

Gong, Sheng [Kung, Sun] 426

Fourier analysis on unitary groups. II. Cesàro summability of Fourier series.

Acta Math. Sinica **12** (1962), 17–31 (*Chinese*); *translated as Chinese Math.* **3** (1963), 19–33.

From the author's summary: "If $u(U)$ is an integrable function on a unitary group U_n of order n, its Fourier series is

$$(2.1.1) \qquad \sum_{f_1 \geqq f_2 \geqq \cdots \geqq f_n} \operatorname{tr}(C_{f_1 f_2 \cdots f_n} \Phi'_{f_1 \cdots f_n}(U)),$$

where

$$C_{f_1\cdots f_n} = \int_{U_n} u(V)\overline{\Phi_{f_1\cdots f_n}(V)}\dot{V}.$$

In this paper, we will investigate the Cesàro summability of (2.1.1) to supplement the Abel summability. § 2.2 is devoted to the definition of the Cesàro summability and its corresponding kernel. The following theorem of Riesz type is proved in § 2.3–§ 2.4: If $u(U)$ is continuous on U_n, the Fourier series (2.1.1) of $u(U)$ is summable (c, α) for $\alpha > (n-1)/n$. This is one of the main subjects in this paper.

"In order to have some insight of the Cesàro summability in the general case, we will study in more detail the typical and most important example, the Fejér summability. From this example, some other coefficients and constants in the Cesàro summability of general orders can similarly be calculated. This is the second subject of this paper, and will be the contents of § 2.5–§ 2.7. The same techniques of matrix integration as employed in the preceding paper [#425 above] will be utilized further in the present paper."

M. Fukamiya (Sendai)

Hewitt, Edwin 427

A new proof of Plancherel's theorem for locally compact Abelian groups.

Acta Sci. Math. (*Szeged*) **24** (1963), 219–227.

Plancherel's theorem for arbitrary locally compact abelian (LCA) groups was established by M. G. Kreĭn [Dokl. Akad. Nauk SSSR **30** (1941), 482–486; MR **2**, **316**] and A. Weil [*L'intégration dans les groupes topologiques et ses applications*, Actualités Sci. Indust., no. 869, Hermann, Paris, 1940; MR **3**, 198]. For the case of vector groups $\boldsymbol{R}^a$, we have a beautiful proof due to F. Riesz [Acta Sci. Math. (Szeged) **3** (1927), 235–241] in which the sequence of functions $\exp(-(v_1^2+v_2^2+\cdots+v_a^2)/2n^2)$ plays an important rôle.

The present author constructs quite explicitly, for an arbitrary LCA group, a sequence of functions which plays an analogous rôle, and extends Riesz's idea to get a new proof of the Plancherel-Kreĭn-Weil Theorem.

Such functions are described essentially in the following Theorem (2.2): Let H be an LCA group containing a compact open subgroup J, Y the dual group of H, and A the annihilator of J in Y. The Haar measures on H and Y are taken such that both J and A have measure 1. Then, for any σ-compact open subgroup Δ of Y, there is a sequence of functions $w_n(\chi)$ on Y with the properties: (i) each $w_n(\chi)$ is continuous with compact support and $w_n(Y) \subset [0, 1]$; (ii) the sequence $w_n(\chi)$ is non-decreasing and $\lim w_n(\chi) \geqq 1$ for $\chi \in \Delta$; (iii) if $\varphi_n(x) = \int_Y w_n(\chi)\chi(x)\,dx$, then $\varphi_n(x) \geqq 0$ on H and $\int_H \varphi_n(x)\,dx = 1$; (iv) for $\chi \in \Delta$, $\lim \int_H \varphi_n(x^{-1}y)\chi(y)\,dy = \chi(x)$, $x \in H$, and for $\chi \in Y \cap \Delta'$, $\int_H \varphi_n(x^{-1}y)\chi(y)\,dy = 0$, for $n = 1, 2, 3, \cdots$.

M. Fukamiya (Sendai)

Herz, Carl S. 428

Fonctions opérant sur les fonctions définies-positives.

C. R. Acad. Sci. Paris **255** (1962), 2560–2561.

Let G be a locally compact abelian group, $P(G)$ the continuous positive definite functions on G. Π denotes the closed unit complex disc and $\Gamma = \{z : |z| = 1\}$ the circle group considered as discrete. The following sets of complex-valued functions defined on Π are considered. $M_c : f$ is representable by a series convergent for $|z| \leqq 1$, $f(z) = \sum_{m,n=0} a_{m,n} z^m \bar{z}^{-n}$, where $a_{m,n} \geqq 0$; $S : f(z) = 0$ for $|z| < 1$ and f restricted to Γ belongs to $P(\Gamma)$; $M : f = g + h$, where $g \in M_c$ and $h \in S$; $F(G) : f(\varphi) \in P(G)$ for $\varphi \in P$ and $|\varphi| \leqq 1$; $F_c(G) : f \in F(G)$ and is continuous on Π. F is the intersection of the sets $F(G)$, where G runs through all discrete groups. Theorem 1: $F = M$. Theorem 2: If G has an element of infinite order, then $F(G) = M$ if G is discrete and $F(G) = M_c$ otherwise.

R. P. Hunter (University Park, Pa.)

INTEGRAL TRANSFORMS, OPERATIONAL CALCULUS

See also 285, 286, 290, 292, 470, 478, 725.

Duffin, R. J.; Duris, C. S. 429

A convolution product for discrete function theory.

Duke Math. J. **31** (1964), 199–220.

A function f is called discrete analytic on the lattice of points of the complex plane with integer coordinates if

on each unit square the difference quotient across one diagonal equals that on the other diagonal. A theory of such functions is given by Duffin [Duke Math. J. **23** (1956), 335–363; MR **17**, 1193]. This theory has many points in common with the classical function theory. However, the product of two discrete analytic functions belongs to the same class only in very special cases.

Using a discrete analogue of a line integral, the authors succeed in defining a convolution product $f * g$ which, for discrete analytic factors, is likewise discrete analytic. Using this definition, a theory similar to that of Volterra integral equations is developed. The convolution product is also used for defining a discrete analogue of the Laplace transform. It turns out that the classical complex inversion formula has also its discrete counterpart.

H. A. Lauwerier (Amsterdam)

Gadžiev, A. D. 430

On the convergence of integral operators. (Russian. Azerbaijani summary)

Akad. Nauk Azerbaĭdžan. SSR Dokl. **19** (1963), *no.* 12, 3–7.

On étudie des opérations de la forme

$$f(x) \to W_\lambda(f; x) = \int_a^b f(t)\Phi_\lambda(t-x)\,dt.$$

Moyennant des hypothèses convenables concernant le noyau $\Phi_\lambda(t)$ (remplies par le noyau de Poisson) on démontre que

$$\lim_{\substack{(x,\lambda)\to(x_0,0)\\ (x,\lambda)\in L}} \frac{\partial W_\lambda(f; x)}{\partial x} = \left[\frac{f_+'(x_0)+f_-'(x_0)}{2}\right] + \frac{\theta}{\pi}\left[f_+'(x_0)-f_-'(x_0)\right],$$

θ étant l'angle entre la droite L et la perpendiculaire à l'axe OX en x_0, $f_+'(x_0)$, respectivement $f_-'(x_0)$, étant la dérivée à gauche, respectivement à droite, de la fonction f en x_0.

E. J. Akutowicz (Bologna)

Masood, S. 431

Generalised Laplace transforms and self-reciprocal functions.

Gaṇita **13** (1962), 51–73.

Let $W_{l,n}(x)$ denote Whittaker's function and let

$$(1)\qquad \phi(s) = s\int_0^\infty (st)^{n-1/2}e^{-st/2}W_{l,n}(st)f(t)\,dt.$$

When $l+n=\frac{1}{2}$, equation (1) becomes the Laplace transform. If

$$(2)\qquad f(t) = \int_0^\infty K(ty)f(y)\,dy,$$

we say that $f(t)$ is s.r. (i.e., self-reciprocal) for the kernel $k(x)$. The author confines himself to the case where $k(x)$ is the sum of two hypergeometric functions of the type ${}_2F_3(\alpha, \beta; \gamma, \delta, \epsilon; -x)$.

Assuming suitable convergence conditions the author substitutes the expression for $f(t)$ from (2) into (1) and by interchanging the order of integration establishes the following result: If (i) $\phi(s)$ and $f(s)$ are related as in (1), and (ii) $f(s)$ is s.r. for $k(x)$, then $\phi(s)$ and $f(t)$ are also related by a formula of the type

$$(3)\qquad \phi(s) = \int_0^\infty h(s, y)f(y)\,dy,$$

where the kernel $h(s, y)$ is an infinite integral involving $k(ty)$ and $W_{l,n}(st)$.

The author then discusses special cases and evaluates a large number of infinite integrals, many of which involve Bessel and related functions.

C. Fox (Montreal, Que.)

Naylor, D. 432

On a Lebedev expansion theorem.

J. Math. Mech. **13** (1964), 353–363.

The author continues his work on the application of integral transforms to differential equations [same J. **12** (1963), 375–383; MR **26** #4127]. He considers the transform

$$\Psi(u) = \int_a^\infty K_u(kr)\Psi(r)\frac{dr}{r}, \qquad a > 0,$$

where $K_u(kr)$ is a modified Bessel function. The eigenfunctions and Green's function bounded in the interval (a, ∞) of Bessel's differential equation subject to the condition of vanishing on $r=a$ are constructed. These are given in terms of K_u and I_u, where I_u is also a modified Bessel function. Expansions of functions are given in terms of these solutions. Applications are made to the solution of wave equations.

A. Devinatz (St. Louis, Mo.)

Bhowmick, K. N. 433

An integral associated with H_ν-transform.

Gaṇita **13** (1962), 41–49.

Let

$$H_\nu(z) = \sum_{r=0}^\infty \frac{(-1)^r(\frac{1}{2}z)^{\nu+2r+1}}{\Gamma(r+\frac{3}{2})\Gamma(\nu+r+\frac{3}{2})}.$$

The author evaluates the integral

$$\int_0^\lambda xH_\nu(\alpha x)H_\nu(\beta x)\,dx.$$

He then shows that

$$\int_0^\lambda (xu)^{1/2}H_\nu(xu)\,du\int_0^{x-\delta}(uy)^{1/2}H_\nu(uy)g(y)\,dy = O(1/x)$$

as $\lambda \to \infty$,

$$\int_0^\lambda (xu)^{1/2}H_\nu(xu)\,du\int_{x+\delta}^\infty (uy)^{1/2}H_\nu(uy)g(y)\,dy = O(1/x)$$

as $\lambda \to \infty$,

where $\frac{1}{2} < \operatorname{Re}\nu < 1$ and all the integrals in question are absolutely convergent.

W. A. Al-Salam (Durham, N.C.)

Singh, S. P. 434

Certain kernel functions.

J. Sci. Res. Banaras Hindu Univ. **13** (1962/63), 252–266.

In this paper the author proves three theorems connecting

different classes of functions which are self-reciprocal under the Fourier kernel

$$\psi_{\nu,k+1/2,m}(x) = \sum_{m,-m} \frac{\Gamma(2m)\Gamma(\nu+1-m-3k)}{\Gamma(m-k)\Gamma(\nu+1-m-2k\pm m)} x^{-k-m} \times {}_2F_3\left[\begin{matrix}1+k-m, \nu+1-m-3k \\ 1-2m, \nu+1-m-2k\pm m\end{matrix}; -x\right].$$

In Theorem 2, the author gives only a formal procedure and not a rigorous proof. {There are many printing errors.}

V. P. Mainra (Pilani)

Iwasaki, Koziro 435

On Bochner transforms. II. A generalization attached to $M(n, R)$ and "an" n-dimensional Bessel function.

Proc. Japan Acad. **39** (1963), 711–714.

In this paper Bochner transforms [same Proc. **39** (1963), 257–262; MR **28** #1452] attached to a function space defined on matrices are studied together with n-dimensional Bessel functions and their differential equations.

P. L. Butzer (Aachen)

Pandey, R. N. 436

On intero-exponential transform of two variables.

J. Sci. Res. Banaras Hindu Univ. **13** (1962/63), 332–343.

The "intero-exponential function" is defined as

$$E_v(x) = e^{-x} \int_0^\infty (1+y)^{-v} e^{-xy}\, dy.$$

The author considers a double integral transform with the kernel $(px)^\sigma (qx)^\rho E_v(px) E_w(qx)$. {Reviewer's remarks: E_v is a multiple of the incomplete gamma function and hence of a special confluent hypergeometric function; the transform considered here is a trivial variant of the so-called Whittaker transform; the work set out in the paper consists of the usual kind of formal manipulations; the formulae are badly written and the text contains many misprints.}

A. Erdélyi (Edinburgh)

Bellert, S. 437

On the continuation of the idea of Heaviside in the operational calculus.

J. Franklin Inst. **276** (1963), 411–440.

Die Arbeit ist im wesentlichen eine Übersetzung einer in Rozprawy Elektrotechn. **6** (1960), 169–212 stehenden polnischen Fassung. An Słowikowski [Bull. Acad. Polon. Sci. Cl. III **4** (1956), 643–647; MR **19**, 145] anknüpfend, wird die Übertragung der Mikusińskischen Operatorenrechnung auf lokal konvexe Räume wie auch bei Bittner [Studia Math. **20** (1961), 1–18; MR **25** #4316] weiterentwickelt. In diesen werden Gleichungen der Form $W(T)x = a$ unter passenden Anfangsbedingungen nach x aufgelöst, wobei $W(T)$ ein Polynom in bezug auf einen linearen Operator T ist, dessen Koeffizienten Konstanten oder auch mit T vertauschbare Endomorphismen des Raumes sind. Besitzt der zu $W(T)$ inverse Operator eine Partialbruchzerlegung, so kann die Lösung x durch unendliche Reihen dargestellt werden, deren Konvergenz mit Hilfe einer Folge von Halbnormen untersucht wird. Als Spezialfälle werden gewöhnliche und partielle Differentialgleichungen mit konstanten Koeffizienten, Eulersche Differentialgleichungen, Differenzengleichungen mit konstanten und gewissen variablen Koeffizienten, Differential-Differenzengleichungen sowie nichtlineare Differentialgleichungen, die sich durch eine Substitution auf lineare zurückführen lassen, betrachtet.

L. Berg (Halle)

INTEGRAL EQUATIONS

See also 253, 254, 480, 503, 632, 728.

Litvinčuk, G. S.; Hasabov, È. G. 438

On a type of singular integral equations. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 608–625.

The integral equation

$$(1) \qquad a(t)\overline{\varphi(t)} + \frac{b(t)}{\pi i}\int_L \frac{\varphi(\tau)}{\tau-\alpha(t)}\, d\tau = c(t)$$

is considered, where $a(t)$, $b(t)$, $c(t)$ satisfy Hölder's condition on the closed Ljapunov contour L. It is assumed that $a(t)$ and $b(t)$ do not vanish on L. The function $\alpha(t)$ is a homeomorphic (direction-preserving) mapping of L upon itself, $\alpha'(t) \neq 0$ on L and satisfies Hölder's condition on L. Under the additional hypotheses that $\alpha[\alpha(t)] = t$ on L, and $a(t)\overline{a[\alpha(t)]} = b(t)\overline{b[\alpha(t)]}$ on L, the author gives a qualitative analysis of the given integral equation. The number of linearly independent solutions of the homogeneous equation corresponding to (1) is found, and an algorithm is derived for finding these solutions. Conditions for the solvability of equation (1) are determined. On the basis of these results the normal solvability and the vanishing of the index of the given integral equation are established. Cases are given when the given equation can be solved in closed form.

H. P. Thielman (Alexandria, Va.)

Avantaggiati, Antonio 439

Su di una classe di trasformazioni integrali singolari.

Matematiche (Catania) **17** (1962), 56–84.

Let γ be a simple closed plane curve of class $C^{(2)}$ and $\{A_l\}_{l=1}^N$ $(N \geq 3)$ a covering of γ by open arcs such that $A_l \cap A_k$ $(l \neq k)$ is non-vacuous if and only if $k = l-1$ or $k = l+1$ (where $A_0 = A_N$, $A_1 = A_{N+1}$, etc.). We consider a set $\{\Gamma\}_{A,n}$ of N square matrices $\Gamma^{(l,l+1)} = (\gamma_{rs}^{(l,l+1)})$ of order n, where the $\gamma_{rs}^{(l,l+1)}$ are functions of class $C^{(0,\alpha)}$ and $\det \Gamma^{(l,l+1)} \neq 0$ on $\overline{A_l \cap A_{l+1}}$, and a similar set $\{\Theta\}_{A,n}$. Let $C^{(0,\alpha)}(\{\Gamma\}_{A,n})$ $[L^{(2)}(\{\Gamma\}_{A,n})]$ be the space of all N-tuples $((\varphi^{(1)}, \cdots, \varphi^{(N)}))$ of vectors $\varphi^{(l)} = (\varphi_1^{(l)}, \cdots, \varphi_n^{(l)})$ such that $\varphi_r^{(l)}$ is a function of class $C^{(0,\alpha)}$ $[L^{(2)}]$ on every closed arc contained in A_l and verifying $\varphi^{(l)} = \Gamma^{(l,l+1)} \cdot \varphi^{(l+1)}$, i.e., $\varphi_r^{(l)} = \sum_s \gamma_{rs}^{(l,l+1)} \varphi_s^{(l+1)}$ $(r = 1, \cdots, n;\ l = 1, \cdots, N)$. We denote by $\mathbf{C}^{(0,\alpha)}(\{\Gamma\}_{A,n})$ the set of all N-tuples $\{m^{(1)}, \cdots, m^{(N)}\}$ of square matrices of order n such that, taking from each $m^{(l)}$ the kth column vector $m_{.k}^{(l)}$, we have $((m_{.k}^{(1)}, \cdots, m_{.k}^{(N)})) \in C^{(0,\alpha)}(\{\Gamma\}_{A,n})$ and such that $m^{(l)} = \Gamma^{(l,l+1)} \cdot m^{(l+1)}$ on $A_l \cap A_{l+1}$. The set $\mathbf{C}^{*(0,\alpha)}(\{\Gamma^*\}_{A,n})$ is similarly defined by replacing the matrices Γ by their transposes and considering the row vectors of the $m^{(l)}$. Finally, we denote by $\mathbf{C}^{(0,\alpha)}(\{\Theta\}_{A,n}; \{\Gamma^*\}_{A,n})$ the set of all systems $\{\{a^{(h;k)}\}\}$ of N^2 square matrices of order n such that each coefficient $a_{rs}^{(h;k)}$ is a function of class $C^{(0,\alpha)}$ on $A_h \times A_k$ and such that (1) for each fixed h and $\xi \in A_h$, $\{a^{(h;1)}(\xi, \cdot), \cdots, a^{(h;N)}(\xi, \cdot)\} \in \mathbf{C}^{*(0,\alpha)}(\{\Gamma^*\}_{A,n})$, and (2) for

each fixed k and $\eta \in A_k$, $\{a^{(1;k)}(\cdot, \eta), \cdots, a^{(N;k)}(\cdot, \eta)\} \in \mathbf{C}^{(0,\alpha)}(\{\Theta\}_{A,n})$.

Let $\{\{a^{(h;k)}\}\}$ and $\{\{b^{(h;k)}\}\}$ belong to $\mathbf{C}^{(0,\alpha)}(\{\Theta\}_{A,n}; \{\Gamma^*\}_{A,n})$ and let (D_k) be a partition of γ into closed arcs such that $D_k \subset A_k$. The author calls $\mathscr{S}$ the continuous linear map from $L^{(2)}(\{\Gamma\}_{A,n})$ into $L^{(2)}(\{\Theta\}_{A,n})$ defined by

$$(\mathscr{S}\varphi)^{(h)}(\xi) = a^{(h;h)}(\xi, \xi)\cdot\varphi^{(h)}(\xi) + \frac{1}{\pi i}\sum_{k=1}^{N}\int_{D_k}\frac{b^{(h;k)}(\xi, \eta)\cdot\varphi^{(k)}(\eta)}{\eta-\xi}\,d\eta.$$

Setting $s^{(h;k)} = a^{(h,k)} + b^{(h;k)}$ and $t^{(h;k)} = a^{(h;k)} - b^{(h;k)}$ he proves that if $(\det s^{(h;h)}(\xi, \xi))\cdot(\det t^{(h,h)}(\xi, \xi)) \neq 0$ for all $\xi \in A_h$ and $h = 1, \cdots, N$, then there exists a linear map $\mathscr{S}'$: $L^{(2)}(\{\Theta\}_{A,n}) \to L^{(2)}(\{\Gamma\}_{A,n})$ such that $\mathscr{S}'\mathscr{S} = \mathscr{J} + \mathscr{T}$, where $\mathscr{J}$ is the identity map and $\mathscr{T}$ is a completely continuous endomorphism of $L^{(2)}(\{\Gamma\}_{A,n})$. He also calculates the index of $\mathscr{S}$ and applies his result to singular integral equations with discontinuous coefficients.

J. Horváth (Nancy)

Janikowski, Józef 440

Équation intégrale non linéaire d'Abel.

Bull. Soc. Sci. Lettres Łódź **13** (1962), *no.* 11, 8 *pp.*

The author considers the integral equation

$$(1) \qquad \int_0^x \frac{F[x, y, \phi(y)]}{(x-y)^\alpha}\,dy = f(x),$$

with $0<\alpha<1$, and proves the following theorem. Let $F(x, y, u)$ be of class C_1 in the domain $0 \leq x \leq a$, $0 \leq y \leq x$, $0 \leq u \leq b$, satisfy $F(0, 0, 0) = 0$, $F_u'(0, 0, 0) \neq 0$, and have continuous second partial derivatives F_{yx}'', F_{ux}'' in this domain. Let $f(x)$ be of class C_1 on $0 \leq x \leq a_1$ and let $f(0) = 0$. Then for some positive number $a \leq a_1$, there is a unique solution $\phi(x)$ of (1) which is of class C_1 for $0 \leq x \leq a$ and satisfies $\phi(0) = 0$. The proof involves the substitution $x = at$ and the application of an implicit function theorem for operations in Banach spaces.

D. H. Hyers (Los Angeles, Calif.)

Tihonov, A. N. 441

On the solution of non-linear integral equations of the first kind. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 1296–1299.

A stable method for solving the integral equation

$$\int_a^b K(x, s, z(s))\,ds = u(x), \qquad c \leq x \leq d,$$

is considered. The author states that this problem is not well posed because it does not have a solution for every given $u(x)$, and because it may happen that, even though a solution $\bar{z}(s)$ exists for some $\bar{u}(x)$, a small change of the norm (in L_2) of $\bar{u}$ may cause a large change in the norm (in C) of $\bar{z}$. The object of this article is to develop methods for the regularization of such non-linear integral equations of the first kind. These methods permit one to find uniform approximations for $\bar{z}(s)$ on the basis of the perturbed values $\tilde{u}(x)$ of $u(x)$. This work is an extension of the author's earlier results [same Dokl. **151** (1963), 501–504; MR **28** #5576; ibid. **153** (1963), 49–52; MR **28** #5577] for linear integral equations of non-linear ones.

H. P. Thielman (Alexandria, Va.)

Thielman, H. P. 442

Applications of the fixed point theorem by Russian mathematicians.

Nonlinear Integral Equations (*Proc. Advanced Seminar Conducted by Math. Research Center, U.S. Army, Univ. Wisconsin, Madison, Wis.*, 1963), *pp.* 35–48. *Univ. Wisconsin Press, Madison, Wis.*, 1964.

An expository article containing a brief survey of some of the material contained in the early chapters of the book by M. A. Krasnosel'skiĭ [*Topological methods in the theory of nonlinear integral equations* (Russian), GITTL, Moscow, 1956; MR **20** #3464; English transl., Macmillan, New York, 1964; MR **28** #2414], together with references to some more recent work of Russian authors.

D. H. Hyers (Los Angeles, Calif.)

Błaż, J. 443

Sur l'existence d'une solution d'une équation différentielle à argument avancé.

Ann. Polon. Math. **15** (1964), 1–8.

Using the principle of contraction mapping, two existence and uniqueness results are established for the equation:

$$(1) \qquad \phi'(t) = \int_0^\infty f[t, \phi(t), \phi(t+s)]\,dr(t, s), \qquad s \geqq 0.$$

The list of conditions, which includes a sort of Lipschitz condition, is long, but the results are an improvement over those of S. H. Doss and S. K. Nasr [Amer. J. Math. **75** (1953), 713–716; MR **15**, 324].

N. P. Bhatia (Cleveland, Ohio)

Nohel, John A. 444

Problems in qualitative behavior of solutions of nonlinear Volterra equations.

Nonlinear Integral Equations (*Proc. Advanced Seminar Conducted by Math. Research Center, U.S. Army, Univ. Wisconsin, Madison, Wis.*, 1963), *pp.* 191–214. *Univ. Wisconsin Press, Madison, Wis.*, 1964.

An exposition of some results concerning the global existence, boundedness, and asymptotic behavior of solutions of certain nonlinear integral equations of Volterra type, together with a bibliography of fifty-six items.

D. H. Hyers (Los Angeles, Calif.)

Levin, J. J.; Nohel, J. A. 445

On a nonlinear delay equation.

J. Math. Anal. Appl. **8** (1964), 31–44.

Given

$$(1) \qquad x'(t) = -\frac{1}{L}\int_{t-L}^{t}(L-(t-\tau))g(x(\tau))\,d\tau \qquad \left(' = \frac{d}{dt},\ L > 0,\ 0 \leqq t < \infty\right),$$

where $g(x)$ is locally Lipschitzian and $xg(x) > 0$ $(x \neq 0)$, $G(x) = \int_0^x g(\xi)\,d\xi \to \infty$ $(|x| \to \infty)$. The principal result of this paper is the following theorem. Let $\psi(t) \in C[-L, 0]$ be given. (i) Then (1) has a unique solution $x = u(t)$ on $[-L, \infty)$ such that $u(t) \equiv \psi(t)$ on $[-L, 0]$ and $u(t)$ is continuous at $t = 0$. Moreover, $|u^{(j)}(t)| \leqq K_1 < \infty$ $(j = 0, 1, 2)$ on $[0, \infty)$ for some constant K_1. (ii) The limit set Ω of the curve $x = u(t)$, $y = u'(t)$ in the xy-plane is an orbit of the

equation (2) $x''+g(x)=0$. If this orbit Γ is not an origin, then L is an integral multiple of the period ω (>0) of Γ. Further, if we write the equation of Γ as $x=\varphi(t)$, $y=\varphi'(t)$, then

$$\lim_{n\to\infty}\left[\max_{0\leqq t\leqq K_2}\left|u^{(j)}(nL+t)-\varphi^{(j)}(t+t_n)\right|\right] = 0 \qquad (j=0,1,2)$$

for any positive constant K_2 and some sequence $\{t_n\}$, where $0\leqq t_n\leqq\omega$ if $\Gamma\neq(0,0)$ and t_n is arbitrary if $\Gamma=(0,0)$.

In the proof, the non-negative function

$$E[x(t)] = G[x(t)]+\frac{1}{2L}\int_{t-L}^{t}\left[\int_{\tau}^{t} g(x(s))\,ds\right]^2 d\tau$$

plays an important role. For the solution $x=u(t)$ of (1), $E'[u(t)]\leqq 0$. This property is used for proving the boundedness of $x=u(t)$. The boundedness of $E''[u(t)]$ implies $\int_{t-L}^{t} g[u(\tau)]\,d\tau\to 0$ $(t\to\infty)$, which implies $u''(t)+g[u(t)]=(1/L)\int_{t-L}^{t} g[u(\tau)]\,d\tau\to 0$ $(t\to\infty)$. From this, it is proved that the limit set Ω contains an orbit of (2). Further, for an orbit $x=\varphi(t)$ of (2) contained in Ω, it is proved that $\int_{t-L}^{t} g[\varphi(\tau)]\,d\tau=0$ and $E[\varphi(t)]=\lim_{s\to\infty}E[u(s)]$. Using these two equalities, (ii) of the theorem is proved.

The authors also discuss the equation (3) $x'(t)=-\int_{t-L}^{t} a(t-\tau)g(x(\tau))\,d\tau$ in a similar way. Here $g(x)\in C$, $xg(x)>0$ $(x\neq 0)$, $G(x)=\int_0^x g(\xi)\,d\xi\to\infty$ $(|x|\to\infty)$ and $a(t)\in C[0,L]$, $a(L)=0$, $(-1)^k a^{(k)}(t)\geqq 0$ $(0<t\leqq L$, $k=0,1,2,3)$, $a''(t)\not\equiv 0$. The authors' result is as follows. For any given $\psi(t)\in C[-L,0]$, (3) has a unique solution $x=u(t)$ on $[-L,\infty)$ such that $u(t)\equiv\psi(t)$ on $[-L,0]$ and $u(t)$ is continuous at $t=0$. Moreover, for such solution $x=u(t)$, $\lim_{t\to\infty}u^{(j)}(t)=0$ $(j=0,1,2)$. If $a(t)=(L-t)/L$, then (3) reduces to (1); however, this is precisely the case excluded in the above result. *M. Urabe* (Fukuoka)

FUNCTIONAL ANALYSIS

See also 182, 209, 210, 213, 215, 253, 256, 299, 300, 302, 304, 325, 361, 363, 364, 428, 584, 636, 649.

Cudia, Dennis F. 446

The geometry of Banach spaces. Smoothness.

Trans. Amer. Math. Soc. **110** (1964), 284–314.

This is a detailed analysis of certain aspects of the geometry of Banach spaces. These center about the generalizations of uniform convexity and the differentiability of the norm function. The paper is divided into five sections and contains over 40 theorems, propositions, and corollaries. The principal contributions of the author are in Sections 3 and 4 which contain, respectively, the dual differentiability of the norm and the analysis of the differentiability of norm functionals using generalized Gaussian spherical image maps. The material at points is somewhat complex—a necessity of the subject—but which has the effect of slowing progress for the non-expert. Thus one definition of fifteen lines introduces five types of weakly uniform rotundity for the unit sphere S at the point f in the direction g, starting with the above and ending with "S is weakly uniformly rotund uniformly at M uniformly in the directions N if and only if . . .". We give two theorems as examples of the material covered: (1) Let M and N be non-vacuous subsets of X^* (the adjoint of X). Then $\|\cdot\|^*$ is differentiable [uniformly] in M with respect to N if and only if S is weakly uniformly rotund [uniformly] at M in the directions N. (2) X and X^* with their norm topologies are homeomorphic under the extended spherical image map T if and only if X is a Banach space, $\|\cdot\|$ is Fréchet differentiable, and U is weakly uniformly convex. Here, U is the unit ball. The spherical image of a vector $x\in X$ will not be defined here. However, the notion of spherical image is a part of the classical theory of convexity in finite-dimensional spaces. The author deploys it here to its full generality. The second theorem gives an answer to a question of Klee [same Trans. **74** (1953), 10–43, esp. p. 35; MR **14**, 989]. *E. R. Lorch* (New York)

Yamamuro, Sadayuki 447

On fixed point theorems. (Japanese)

Sûgaku **15** (1963), 105–107.

Let G be an open set in a locally convex real linear topological space X and F a continuous mapping of $\bar G$ (the closure of G) into X such that $F(\bar G)$ lies in a compact set. Assume also $0\in G$. The author proves the following two theorems. Theorem 1: If $x\in\partial G$ and $F(x)=\alpha x$ imply $\alpha\leqq 1$, then there exists a fixed point of F in $\bar G$. Theorem 2: Let G also be symmetric $(G=-G)$ and assume that $x\in\partial G$ and $f(x)=\alpha f(-x)$, with $f(x)=x-F(x)$, imply $\alpha\leqq 1$. Then either F has a fixed point in $\bar G$, or F_1, defined by $F_1(x)=\frac{1}{2}\{F(x)-F(-x)\}$, has a fixed point in ∂G.

For the proof of these theorems the author applies theorems on the degree of a mapping in locally convex linear topological spaces. *M. Nagumo* (Osaka)

Belobrov, P. K. 448

On the Chebyshev center of a set. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 1 (38), 3–9.

Let M be a bounded subset of a normed space X; the Čebyšev center of M is a point y_0 such that $\sup_{x\in M}|x-y_0|=\inf_{y\in X}\sup_{x\in M}|x-y|$. The set of Čebyšev centers is the same for M, the convex extension $C(M)$ of M, and for the closure of $C(M)$. The Čebyšev center exists for any bounded subset M of X if X is reflexive. If H is a real Hilbert space, a set $M\subset H$ is said to satisfy condition (A) if, for each $z_0\in M$, there exists an $x_0\in M$ and a positive η such that $(x-z_0, x_0-z_0)\geqq\eta$ for all $x\in M$. For a closed M, condition (A) is satisfied if and only if M is convex. The main result: If $M\subset H$ satisfies (A), then the Čebyšev center of M is contained in M. Together with the preceding lemmas it follows that, for a bounded $M\subset H$, the Čebyšev center of M belongs to the closure of $C(M)$. The main result may be generalized for uniformly convex spaces with an additional condition. *V. Pták* (Prague)

Belobrov, P. K. 449

On the Chebyshev center of a set in Banach space. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 2 (39), 25–30.

Let M denote a bounded set in a Banach space X. A point $y^*\in X$ is called the Chebyshev centre of M if $\sup_{x\in M}\|x-y^*\|=\inf_{y\in X}\sup_{x\in M}\|x-y\|$. The Banach space X is of $(\mathfrak{N})$-type if the Chebyshev centre of any bounded, closed and convex set $M\subset X$ belongs to M. A point

$x_0 \in Z \subset X$ is called, according to D. P. Mil'man, an **attainable point** of Z if there exists a functional $f \in X^*$ **such that** $f(x) < f(x_0)$ for any $x \in Z$, $x \neq x_0$. The author **proves** the following two theorems.

If a Banach space is uniformly convex with strongly differentiable norm at every point different from the zero-point, then the Chebyshev centre of any compact, closed and convex set is identical with the Chebyshev centre of the set of its attainable points.

If M is a closed, convex and compact set in X of ($\mathfrak{N}$)-type and if $\{C_n\}$ is a sequence of continuous operators from X into X which is uniformly convergent on M to the identity operator, then $\lim_{n\to\infty} y_n^* = y^*$, where y_n^* and y^* denote the Chebyshev centres of $C_n(M)$ (the image of M by means of C_n) and M, respectively.

J. Albrycht (Poznań)

Husain, Taqdir 450

$B(\mathscr{T})$-spaces and the closed graph theorem.

Math. Ann. **153** (1964), 293–298.

The general form of the open mapping theorem [see the reviewer, Bull. Soc. Math. France **86** (1958), 41–74; MR **21** #4345] is the following. Let f be an almost open mapping from a B-complete space E onto a space F, the graph of which is closed in $E \times F$. Then f is open. The author replaces B-complete by $B(\mathscr{T})$-complete and takes F to be a t-space. Then f will be automatically almost open. Now if f were continuous, the conclusion would follow from the definition. Here the closed graph only is assumed, so the author adds the assumption that f be almost continuous. An analogous statement is given for the closed graph theorem. Both of them are based on a general result concerning $B(\mathscr{C})$-spaces. Some more remarks on $B(\mathscr{T})$-spaces and t-spaces are added, including an improvement of an earlier result of the author [Math. Ann. **146** (1962), 413–422; MR **26** #2861].

V. Pták (Prague)

Henkin, G. M. 451

Stability of unconditional bases in a uniformly convex space. **(Russian)**

Uspehi Mat. Nauk **18** (1963), *no.* 6 (114), 219–223.

The main result is the following extension of a theorem of N. K. Bari [Moskov. Gos. Univ. Učen. Zap. **148** Mat. **4** (1951), 69–107; MR **14**, 289]. Let E be a uniformly convex Banach space with a differentiable modulus of convexity whose derivative $\delta(\varepsilon)$ is convex from below. Put $\delta^{-1}(t) = \sup\{\varepsilon : \delta(\varepsilon) \leq t\}$ and let $\delta^*(\varepsilon) = \int_0^\varepsilon \delta^{-1}(t)\,dt$. Let $\varphi_k \in E$ ($k = 1, 2, \cdots$) be such that $\sum_{k=1}^\infty c_k\varphi_k = 0$ implies $c_k \equiv 0$. Then if (f_k) is an unconditional basis of E and

$$\sum_{k=1}^{\infty} \delta^*(\|f_k - \varphi_k\|) < \infty,$$

(φ_k) is also an unconditional basis of E.

A. Dvoretzky (Jerusalem)

Pełczyński, A.; Szlenk, W. 452

Sur l'injection naturelle de l'espace (l) dans l'espace (l_p).

Colloq. Math. **10** (1963), 313–323.

The main result of this paper is that the identity mapping of (l) into (l_p) satisfies

$$\sum_{i=1}^{n} \|x_i\|_p \leq C \sup_{|\eta_i| \leq 1} \left\| \sum_{i=1}^{n} \eta_i x_i \right\|_p$$

for some C and any finite set of $x_i \in l$ if and only if $p \geq 2$. The "if" part is essentially due to S. Mazur, and his (unpublished) proof is reproduced. The "only if" part utilizes a theorem of the reviewer and Rogers [Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 192–197; MR **11**, 525] and an inequality of Landau.

A corollary of this result is the following: If the matrix a_{ij} $(i, j = 1, 2, \cdots)$ satisfies $\sum_{i=1}^\infty |\sum_{j=1}^\infty a_{ij}\xi_j| < \infty$ for every bounded sequence (ξ_j), then $\sum_{i=1}^\infty |a_{ii}| < \infty$. (This answers a problem of Mazur that started the present investigation.) The following generalization is also established. Let X be a Banach space with a commutative basis (e_k), and let (f_k) be the linear functionals biorthogonal to (e_k). Then if $\sum_{i=1}^\infty x_i$ is any commutatively convergent series in X, the series $\sum_{i=1}^\infty f_{k_i}(x_i)e_{k_i}$ are also commutatively convergent for arbitrary choices of the sequence (k_i).

A. Dvoretzky (Jerusalem)

Hanani, H.; Netanyahu, E.; Reichaw-Reichbach, M. 453

The sphere in the image.

Israel J. Math. **1** (1963), 188–195.

The authors give theorems for Hilbert and Banach spaces similar in spirit to the well-known theorem of Bloch in one complex variable. Theorem: Let $f: S \to X$ be a mapping of the sphere $S = \{x \mid \|x\| \leq r\}$ of the (real) Hilbert space X into X. If for some $\bar{y} \in X$, (*) $(x, f(x)) \leq (x, \bar{y})$ for $\|x\| = r$ and if $x + f(x)$ is completely continuous, then there exists an $\bar{x} \in S$ with $f(\bar{x}) = \bar{y}$.

The theorem is established by applying a previous fixed-point theorem to the mapping $x + f(x) - \bar{y} = g(x)$. Clearly if $g(\bar{x}) = \bar{x}$, then $f(\bar{x}) = \bar{y}$. Applications of the theorem are then given to specific mappings for which equation (*) is verifiable for a set of $\bar{y}$'s containing a ball. Thus the "sphere in the image" of the title.

Ideas along this line are also applied to mappings $f(x)$, where $f(x) + x$ is assumed to be a contraction. Theorem: Let $f(x)$ be a mapping of the sphere $S(\bar{x}, \bar{r}) = \{x \mid \|x - \bar{x}\| \leq \bar{r}\}$ of the Banach space X into X. If $f(x)$ has a derivative at each point of a sphere $S(x_0, r_0) \subset S(x, r)$, and if for some $\beta \neq 0$ and γ, $0 < \gamma < 1$, the inequality $\|\beta f'(\xi) - I\| \leq \gamma$, $\xi \in S(x_0, r_0)$, holds (operator norm), then $f(s)$ contains a sphere of radius $r \geq ((1-\gamma)/|\beta|)r_0$.

W. A. Veech (Princeton, N.J.)

Schmets, Jean 454

Semi-ordre linéaire dans les espaces linéaires semi-normes.

Bull. Soc. Roy. Sci. Liège **33** (1964), 17–44.

On a linear lattice, every order bounded linear functional can be decomposed into positive linear functionals. The author considers this fact on complex coefficient linear spaces with positive cones. The reviewer cannot find anything new in this paper. *H. Nakano* (Detroit, Mich.)

Goodner, Dwight B. 455

The closed convex hull of certain extreme points.

Proc. Amer. Math. Soc. **15** (1964), 256–258.

Let $C(H)$ denote the space of bounded continuous real-valued functions on the Hausdorff space H. Theorem: The unit ball in $C(H)$ is the norm closed convex hull of its

extreme points if and only if any two disjoint zero-sets in H are contained in complementary open-and-closed sets.

L. Gillman (Rochester, N.Y.)

Grisvard, Pierre 456

Commutativité des procédés d'interpolation "réel" et "complexe"; applications.

C. R. Acad. Sci. Paris **258** (1964), 4900–4902.

In this note some most interesting new results on interpolation spaces are announced. The "real" interpolation methods referred to in the title are those which lead to the interpolation spaces $S(p_0, \xi_0, A_0; p_1, \xi_1, A_1)$ [cf. Lions and the reviewer, Inst. Hautes Études Sci. Publ. Math. No. **19** (1964), 5–68; MR **29** #2627]. According to a result by the reviewer [C. R. Acad. Sci. Paris **256** (1963), 1424–1426; Ricerche Mat. **12** (1963), 248–261] they depend, effectively, on only two parameters θ, p defined by $0=\xi_0(1-\theta)+\xi_1\theta$, $1/p=(1-\theta)/p_0+\theta/p_1$, and we may thus set $S(p_0, \xi_0, A_0; p_1, \xi_1, A_1)=(A_0, A_1)_{\theta,p}$. The "complex" methods are those which lead to the spaces $[A_0, A_1]_\theta$ [cf. Calderón, Studia Math. **24** (1964), 113–190]. Here A_0, A_1 are any two Banach spaces both continuously imbedded in a given Hausdorff topological vector space $\mathscr{A}$. Let there now be given two more such spaces B_0, B_1. Then we may form $C_i=(A_i, B_i)_{\theta_i,p_i}$, $i=0, 1$. Under a certain auxiliary assumption (H) on C_0, C_1, the formulation of which we omit, the following inclusions now hold:

(1) $$[C_0, C_1]_\eta \subset ([A_0, A_1]_\eta, [B_0, B_1]_\eta)_{\theta,p},$$

(2) $$(C_0, C_1)_{\eta,q} \subset ((A_0, A_1)_{\eta,q}, (B_0, B_1)_{\eta,q})_{\theta,p},$$

where $\theta=(1-\eta)\theta_0+\eta\theta_1$, $1/p=(1-\eta)/p_0+\eta/p_1$, $q\leqq p$. The straightforward proof is outlined in the case of (1). The assumption (H) is fulfilled in the so-called "semi-group case", i.e., A_i is the domain of the n_ith power $\Lambda_i^{n_i}$ of the infinitesimal generator Λ_i of a continuous semi-group of operators $G_i(t)$ in B_i, $i=0, 1$, provided $G_0(t)=G_1(t)$ in $B_0 \cap B_1$, $n_0=n_1$, and, indeed, in this case, assuming also the reflexivity of B_i and using the duality theorems of Calderón and Lions, one even gets equality, instead of only inclusion, with $q=p$ in the case of (2). In particular, one can apply the result to the translation operation in R^m. Thus taking (Sobolev space) $A_i=W_{P_i}{}^k$, $B_i=W_{P_i}{}^{-k}$, $1<P_i<\infty$, and writing (Besov space) $C_i=(W_{P_i}{}^k, W_{P_i}{}^{-k})_{\theta_i,p_i} = B^{s_i}_{P_i,p_i}$, $B_{P_i}{}^{s_i}=B^{s_i}_{P_i,P_i}$, $s_i=k(1-2\theta_i)$, $1<p_i<\infty$, $i=0, 1$, one obtains

(1′) $$[B^{s_0}_{P_0,p_0}, B^{s_1}_{P_1,p_1}]_\eta = B^s_{P,p},$$

(2′) $$(B_{p_0}{}^{s_0}, B_{p_1}{}^{s_1})_{\eta,p} = B_p{}^s,$$

where $s=(1-\eta)s_0+\eta s_1$, $1/P=(1-\eta)/P_0+\eta/P_1$, $1/p=(1-\eta)/p_0+\eta/p_1$. This last result is an important contribution to the problem of interpolating between spaces $B^s_{P,p}$ with P and s varying simultaneously. Hitherto, essentially only the special case when P or s is fixed was known.

J. Peetre (Lund)

Cantwell, John 457

Partially ordered groups of continuous functions.

Math. Ann. **151** (1963), 252–261.

Continuing work of Fan [Ann. of Math. (2) **51** (1950), 409–427; MR **11**, 525], Fleischer [ibid. (2) **64** (1956), 260–263; MR **18**, 136], and Kadison [Mem. Amer. Math. Soc. No. 7 (1951); MR **13**, 360], the author obtains characterizations of a compact space X in terms of the partially ordered additive group (p.o.a.g.) structure of $C(X)$, the continuous functions from X to the reals. The main results are as follows. Theorem 3: An infinitely divisible, completely integrally closed, archimedean p.o.a.g. A is isomorphic to $C(X)$ for some compact space X if and only if (i) every closed maximal subsemigroup of A is singular, and (ii) for every Dedekind cut (A_1, A_2) of A there exists $f \in A$ such that $a_1 \leqq f \leqq a_2$ for all $a_1 \in A_1$ and $a_2 \in A_2$. (All the special terms here have simple definitions in terms of the order structure.) Theorem 5: A compact space X is homeomorphic to S^n if and only if there exists a set of $n+1$ functions that generate $C(X)$ as a closed linear lattice, but no such set contains a function that is $\geqq 0$.

L. Gillman (Rochester, N.Y.)

Dang, Song-shi [Tang, Sung-shih] 458

On the imbedding theorems of a class of function spaces.

Acta Math. Sinica **13** (1963), 433–440 (*Chinese*); *translated as Chinese Math.* **4** (1964), 470–477.

Let $D_{\mathbf{p}}{}^{\mathbf{l}}$ denote a space containing functions $\phi \in L_p(E_n)$, with $\alpha_i{}^{l_i}\phi \in L_{p_i}(E_n)$, $i=1, 2, \cdots, n$, the point $\alpha=(\alpha_1, \cdots, \alpha_n) \in E_n$ (n-dimensional Euclidean space), $l_i>0$, p and p_i are positive numbers not less than 1, $\mathbf{l}=(l_1, \cdots, l_n)$, $\mathbf{p}=(p_1, \cdots, p_n; p)$. The space $D_{\mathbf{p}}{}^{\mathbf{l},\mathbf{m}}$ differs from $D_{\mathbf{p}}{}^{\mathbf{l}}$ in that the condition $\alpha_i{}^{l_i}\phi \in L_{p_i}$ becomes $(\alpha_1, \cdots, \alpha_n)^{m_i}\alpha_i{}^{l_i}\phi \in L_{p_i}$, where $m_i \geqq 0$, $\mathbf{m}=(m_1, \cdots, m_n)$. Finally, the space $\mathscr{D}_{\mathbf{p}}{}^{\mathbf{l}}$ differs from $D_{\mathbf{p}}{}^{\mathbf{l}}$ in that the condition $\phi \in L_p$ becomes $\phi \in L_{p_i}$, $\mathbf{p}=(p_1, \cdots, p_n)$. The author proves imbedding theorems for these spaces and gives some applications to differentiable function spaces. All these results are too complicated to be quoted here, but we cite the first theorem. Let $\phi \in D_{\mathbf{p}}{}^{\mathbf{l}}$. Then for $1 \leqq q<p$, $q<p_i$, $h=1-\sum_{i=}^{n}\ (1/l_i)(1/q-1/p_i)>0$, we have $\phi \in L_q$; moreover,

$$\|\phi\|_{L_q} \leqq K\|\phi\|_{L_p}{}^{1-\Sigma_i \eta/l_i} \prod_{j=1}^{n} \|\alpha_j{}^{l_j}\phi\|_{L_{p_j}}^{\eta/l_j}.$$

Here

$$K = \left(\beta+\frac{1}{l_1q}+\cdots+\frac{1}{l_nq}\right)^{1/q}$$
$$\times\{\beta^{-\beta}(l_1q)^{1/l_1q}\cdots(l_nq)^{1/l_nq}\}^{(q^\beta+1/l_1+\cdots+1/l_n)-1}$$
$$\times\{A^\beta A_1{}^{1/l_1q}\cdots A_n{}^{1/l_nq}\}^{(q^\beta+1/l_1+\cdots+1/l_n)-1},$$

$$\beta = ph(p-q)^{-1}, \quad \eta = \left(\frac{1}{q}-\frac{1}{p}\right)\left[1-\sum\frac{1}{l_i}\left(\frac{1}{p}-\frac{1}{p_i}\right)\right]^{-1}.$$

The constants A and A_i depend only on q, p, p_i, l_i and not on ϕ. Explicit formulas for these constants are given.

J. Albrycht (Poznań)

Fiorenza, Renato 459

Ulteriori risultati sulla interpolazione tra spazi di funzioni con derivate hölderiane.

Ricerche Mat. **11** (1962), 224–253.

Various results are given for "Banach space interpolation" of spaces defined by Hölder norms; the results are related to the work of E. Gagliardo.

G.-C. Rota (Cambridge, Mass.)

Bear, H. S. 460

An abstract potential theory with continuous kernel.

Pacific J. Math. **14** (1964), 407–420.

The author considers a generalization of potential theory in the unit ball of Euclidean space. Let Γ be a compact Hausdorff space, C the normed linear space of all real continuous functions on Γ, C^* the dual space of C. The author postulates a probability measure μ on Γ and a family Δ of positive continuous functions z on Γ satisfying $\int z(\theta)\,d\mu(\theta)=1$, such that the constant 1 is in Δ. He now embeds Γ and Δ in C^* by sending each θ in Γ into the functional of evaluation at θ and each z in Δ into the element $z\mu$ of C^*. He assumes: (I) Denoting the images of Γ and Δ by Γ^*, Δ^*, respectively, $\Delta^*\cup\Gamma^*$ is compact in C^* (in the weak-* topology); (II) Γ^* is the boundary of Δ^* in $\Gamma^*\cup\Delta^*$; and (III) the map $z\to z\mu$ is a homeomorphism of Δ (in the topology of uniform convergence) on Δ^*. The author shows that the embedding of $\Gamma\cup\Delta$ is one-one, so one may drop the stars on Γ^* and Δ^*. For u in C, extend u to a function $\bar u$ on $\Gamma\cup\Delta$ by $\bar u(z)=\int u(\theta)z(\theta)\,d\mu(\theta)$ for z in Δ. Let H denote the space of all these functions $\bar u$. Define $\mathscr{H}$ as the set of all functions on Δ which are uniform limits on compact sets of Δ of functions in H. The author calls the functions of $\mathscr{H}$ "harmonic". He then proves for the classes H [respectively, $\mathscr{H}$] analogues of the strict maximum principle, Harnack's inequality, Harnack's convergence theorem, Herglotz's theorem, and related results.

J. Wermer (Providence, R.I.)

Mamedov, R. G. 461

Local saturation of a family of positive linear operators. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 499–502.

Let $(L_\lambda)_{\lambda\in R}$ be a family of positive operators defined on the class H of functions $f:[a,b]\to R$ having continuous derivative $f^{(2m-1)}$. For every $\lambda\in R$ define on H the operator M_λ by

$$M_\lambda(f) = \left[L_\lambda(f)-\sum_{k=0}^{2m-1} f^{(k)}\tau_\lambda^{[k]}/k!\right]\Big/\tau_\lambda^{[2m]}$$

for $f\in H$, where $\tau_\lambda^{[k]}$ is the transform of the function $(t-x)^k$ by L_λ and $m\geqq 1$ is a given integer. Under certain additional conditions on M_λ, it is proved that (1) if $M_\lambda(f)=o(1)$ for $\lambda\to\infty$, uniformly on $[a,b]$, then f is a polynomial of degree $\leqq 2m-1$; (2) $M_\lambda(f)=O(1)$ for $\lambda\to\infty$, uniformly on $[a,b]$ if and only if $f^{(2m-1)}\in$ Lip 1. In the second part of the paper there are similar results for positive operators on a space of periodic functions.

N. Dinculeanu (Bucharest)

Taibleson, Mitchell H. 462

On the theory of Lipschitz spaces of distributions on Euclidean n-space. I. Principal properties.

J. Math. Mech. **13** (1964), 407–479.

This is a detailed exposition of most of the results contained in an earlier research announcement [Bull. Amer. Math. Soc. **69** (1963), 487–493; MR **27** #577]. If $g(x,y)$ is a measurable function for $x\in E_n$, and $0\leqq y<\infty$, $1\leqq p\leqq\infty$, set

$$\|g(x,y)\|_{pq} = \left[\int_0^\infty (\|g(\cdot,y)\|_p)^q\,dy/y\right]^{1/q}$$

if $1\leqq q<\infty$ and $\|g(x,y)\|_{p\infty}=\text{ess sup}_{y>0}\|g(\cdot,y)\|_p$. For any real α denote by $\bar\alpha$ the smallest non-negative integer $\geqq\alpha$. First let $\alpha>0$. For any measurable $f(x)$, $x\in E_n$, let $f(x,y)=f(x)*P(x,y)$ be its Poisson integral, where

$$P(x,y) = \pi^{-(n+1)/2}\Gamma\left(\frac{n+1}{2}\right)y(|x|^2+y^2)^{-(n+1)/2}.$$

$\Lambda(\alpha;p,q)$ is the space of all $f(x)\in L_p(E_n)$ for which the norm

$$\|f(x)\|_{\alpha;p,q} = \|y^{\bar\alpha-\alpha}f^{(\bar\alpha)}(x,y)\|_{pq}+\|f(x)\|_p$$

is finite, where $f^{(k)}(x,y)=\partial^k f(x,y)/\partial y^k$. This norm is equivalent to $\|y^{k-\alpha}f^{(k)}(x,y)\|_{pq}+\|f(x)\|_p$, where k is any integer $\geqq\alpha$. The author exhibits for certain ranges of α a great number of other norms on $\Lambda(\alpha;p,q)$ equivalent to $\|f(x)\|_{\alpha;p,q}$ which show in particular that $\Lambda(\alpha;p,\infty)=$ Lip (α,p) for $0<\alpha<1$, $\Lambda(1;p,\infty)$ is the Zygmund space Lip $(*,p)$ of smooth functions and $\Lambda(\alpha;p,p)$ is Besov's space B_p [Trudy Mat. Inst. Steklov. **60** (1961), 42–81; MR **24** #A3501] for $0<\alpha\leqq 1$.

For $\alpha>0$ the function $G^\alpha(x)$ is defined by its Fourier transform $\widehat{G^\alpha}(x)=(1+4\pi^2|x|^2)^{-\alpha/2}$. If $f(x)\in L_p(E_n)$, then its Bessel potential in the sense of Aronszajn and Smith [Ann. Inst. Fourier (Grenoble) **11** (1961), 385–475; MR **26** #1485] is defined by $J^\alpha f(x)=G^\alpha(x)*f(x)$; this definition is extended to negative α and $J^{-\alpha}J^\alpha f=f$. The Gauss-Weierstrass kernel is the function

$$W(x,y) = 2^{-n}\pi^{-n/2}y^{-n/2}\exp(-|x|^2/4y),$$

and $f(x,y)=f(x)*W(x,y)$ is the Gauss-Weierstrass integral of $f(x)$. Set

$$\|g(x,y)\|_{pq}^* = \left[\int_0^1 \|g(\cdot,y)\|_p{}^q\,dy/y\right]^{1/q}$$

if $1\leqq q<\infty$ and $\|g(x,y)\|_{p\infty}^*=\text{ess sup}_{0<y\leqq 1}\|g(\cdot,y)\|_p$.

For α real we define $L_{p,\alpha}(E_n)$ as the image of $L_p(E_n)$ under the map $f\to J^\alpha f$, equipped with the norm $\|f\|_{p,\alpha}=\|J^{-\alpha}f\|_p$. For $\alpha\leqq 0$ let $\Lambda(\alpha:p,q)$ be the space of all $f\in L_{p,\alpha-1/2}$ for which the norm

$$\|f\|_{\alpha;p,q} = \|y^{\overline{\alpha/2}-\alpha/2}f^{(\overline{\alpha/2})}(x,y)\|_{pq}^*+\|f\|_{p,\alpha-1/2}$$

is finite, where this time $f(x,y)$ is the Gauss-Weierstrass integral of $f(x)$. For any integer $k\geqq\alpha/2$ and real $\beta<\alpha$ the norm $\|f\|_{\alpha;p,q}$ is equivalent to $\|y^{k-\alpha/2}f^{(k)}(x,y)\|_{pq}^*+\|f\|_{p,\beta}$. For α, β real, $1\leqq p$, $q\leqq\infty$, J^β maps $\Lambda(\alpha;p,q)$ isomorphically onto $\Lambda(\alpha+\beta;p,q)$.

Numerous results concerning the spaces $\Lambda(\alpha;p,q)$ are proved, among them a convexity theorem, a necessary and sufficient condition that $\Lambda(\alpha;p,q)\subset\Lambda(\beta;r,s)$ {whose statement the reviewer could not understand}, necessary and sufficient conditions that $L_{p,\alpha}\subset\Lambda(\alpha;p,q)$ and $\Lambda(\alpha;p,q)\subset L_{p,\alpha}$, the completeness of $\Lambda(\alpha;p,q)$, the separability of $\Lambda(\alpha;p,q)$ for $1\leqq p,q<\infty$, and a substitute result for $p=\infty$. The author investigates the map which associates with $f\in\Lambda(\alpha;p,q)$ its restriction to a k-dimensional Euclidean subspace of E_n. He also proves that if the continuous linear operator $T:\Lambda(\alpha;p,q)\to\Lambda(\beta;r,s)$ is translation-invariant and $r<p$, then $T=0$ on the closure of the Schwartz space $\mathscr{D}$ in $\Lambda(\alpha;p,q)$.

Analogous definitions and theorems are given also for the case when the n-dimensional torus T_n replaces E_n.

J. Horváth (Nancy)

Chen, Yung-ming 463
On two-functional spaces.
Studia Math. **24** (1964), 61–88.

Let $1 \leqq p_1 \leqq p_2 < \infty$ and φ be a positive function defined on $[0, \infty)$ such that $\varphi(x)x^{-p_1}$ is increasing, $\varphi(x)x^{-p_2}$ is decreasing and $\varphi_1(x) = \varphi(x)x^{-1} \to \infty$ as $x \to \infty$, and let ψ_1 be the inverse function of φ_1. Let $\Phi_1(x) = \int_0^x \varphi_1(t)\,dt$ and $\Psi_1(x) = \int_0^x \psi_1(t)\,dt$. Consider the Orlicz class $L_{\Psi_1}(a, b)$ of measurable functions $y:[a, b] \to R$ such that $\rho_y = \int_a^b \Psi_1(|y(t)|)\,dt < \infty$. The author defines the spaces $L_\varphi^{**}(a, b)$ of functions $x:[a, b] \to R$ such that the function xy is integrable for every $y \in L_{\Psi_1}(a, b)$ with norm $\|x\|_{\langle\varphi\rangle} = \sup\{\int_a^b x(t)y(t)\,dt; y \in L_{\Psi_1}(a, b), \rho_y \leqq 1\}$. We have $L_\varphi^* \subset L_\varphi^{**}$, where L_φ^* is the usual Orlicz space [see W. Orlicz, Bull. Internat. Acad. Polon. Sci. Lett. Cl. Sci. Math. Natur. Sér. A **1932**, 207–220]. In some cases we have $L_\varphi = L_\varphi^* = L_\varphi^{**}$. The main properties of the spaces L_φ^* are extended to the space L_φ^{**}.

In the second part of the paper the author considers some operations defined on classes of Fourier series of functions satisfying certain generalized Lipschitz conditions, which are generalizations of results of G. G. Lorentz [Math. Z. **51** (1948), 135–149; MR **10**, 33].

N. Dinculeanu (Bucharest)

Cristescu, Romulus 464
Opérateurs engendrés par des familles de distributions.
C. R. Acad. Sci. Paris **258** (1964), 3151–3153.

Author's summary: "Nous introduisons une opération de composition entre une famille de distributions et une distribution et nous donnons ses propriétés. La convolution de deux distributions s'obtient comme un cas particulier."

D. Laugwitz (Darmstadt)

Stummel, F. 465
★Einführung in die Distributionstheorie.
Vorlesung im Wintersemester 1962/63 an der Technischen Universität Berlin. HMI-B 30.
Hahn-Meitner-Institut für Kernforschung Berlin, Berlin, 1963. 83 *pp.* (*not consecutively numbered*).

Expository.

Pilkey, Walter D. 466
Some properties and applications of singularity functions based on the theory of distributions.
J. Franklin Inst. **277** (1964), 464–497.

Unter Singularitätenfunktionen werden die Heavisidesche Sprungfunktion, ihre (im Distributionssinne genommenen) Ableitungen und Linearkombinationen solcher verallgemeinerten Funktionen verstanden. Es wird im Anschluß an die Schwartzsche Distributionstheorie eine leicht lesbare Einführung in die Behandlung solcher spezieller Distributionen gegeben, wobei auch stückweise differenzierbare Funktionen besonders behandelt werden. Anwendungen auf einfache Fragen der Balkenbiegung und der Plattenbiegung werden skizziert. Im Literaturverzeichnis, das 46 Titel unfaßt, fehlen Hinweise auf die Operatorenrechnung von Mikusiński, in der entsprechende Probleme sehr einfach behandelt werden [J. Mikusiński, *Operational calculus*, Pergamon, New York, 1959; MR **21** #4333].

D. Laugwitz (Darmstadt)

Gol'dman, M. A.; Kračkovskiĭ, S. N. 467
Invariance of certain spaces associated with the operator $A - \lambda I$. (Russian)
Dokl. Akad. Nauk SSSR **154** (1964), 500–502.

Let A be a linear operator in a complex Banach space $\mathfrak{B}$. Notation: $\mathfrak{Z}(A) = A^{-1}(0)$, $\mathfrak{R}(A) = A\mathfrak{B}$, $\mathfrak{N}(A) = \bigcup_{n=1}^{\infty} \mathfrak{Z}(A^n)$, $\mathfrak{M}(A) = \bigcap_{n=1}^{\infty} \mathfrak{R}(A^n)$, $\mathfrak{Z}_1(A) = \mathfrak{Z}$ (A restricted to $\mathfrak{M}(A)$). For any complex λ, put $A_\lambda = A - \lambda I$. Let Φ denote the set of λ such that $\mathfrak{R}(A_\lambda)$ is closed and either $\mathfrak{Z}(A_\lambda)$ or $\mathfrak{B}/\mathfrak{R}(A_\lambda)$ is finite-dimensional; let Γ denote the set of $\lambda \in \Phi$ such that $\mathfrak{Z}(A_\lambda) \neq \mathfrak{Z}_1(A_\lambda)$.

For a closed A the following theorems are proved (for some related results see I. C. Gohberg and M. G. Kreĭn [Uspehi Mat. Nauk **12** (1957), no. 2 (74), 43–118; MR **20** #3459] and also the authors [Dokl. Akad. Nauk SSSR **86** (1952), 15–17; MR **14**, 478]): (1) Γ is discrete; (2) if λ, μ do not belong to Γ and are contained in the same component of Φ, then the closures of $\mathfrak{N}(A_\lambda)$ and $\mathfrak{N}(A_\mu)$ are equal and $\mathfrak{M}(A_\lambda) = \mathfrak{M}(A_\mu)$.

M. Katětov (Prague)

Tkačenko, V. A. 468
On the spectral analysis of the one-dimensional Schrödinger operator with a periodic complex-valued potential. (Russian)
Dokl. Akad. Nauk SSSR **155** (1964), 289–291.

The author states certain theorems regarding the operator in question. The circumstances under which its resolvent is an integral operator are also stated. It is also stated that the operator is the closure of a certain operator and is unitarily equivalent to another operator; all these latter operators are defined in the note.

P. K. Ghosh (Calcutta)

Gribanov, Ju. I. 469
Coordinate spaces and infinite systems of linear equations. IV. (Russian)
Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 2 (39), 53–64.

Part III appeared in same Izv. **1963**, no. 3 (34), 27–39 [MR **27** #1798]. The author considers approximations of a solution of a linear equation in a Banach coordinate space by solutions of "approximate" equations. Let K be a Banach space and A a bounded linear operator in K. An operator equation $X = AX + H$ with $X, H \in K$ is called uniquely solvable in K if it has a unique solution $X \in K$ for any $H \in K$, i.e., $E - A$, where E is the identity operator, has a bounded inverse. A simple operator-theoretical lemma is used to prove the following result. Suppose $\|A - A_n\| \to 0$. Then $X = AX + H$ is uniquely solvable in K if and only if there exists an $N > 0$ such that $X = A_nX + H$ is uniquely solvable in K for $n \geqq N$ and

$$\sup[\|(E - A_n)^{-1}\| : n \geqq N] < +\infty.$$

If this is the case, then the solutions of approximate equations $X_n = A_nX_n + H_n$ converge in norm to the solution of $X = AX + H$. Let l be a Banach coordinate space as defined in Part I [ibid. **1962**, no. 4 (29), 38–48; MR **26** #4171]. Then the equation $X = AX + H$ has the form $x_m = \sum_{n=1}^{\infty} a_{mn}x_n + h_m$, $m = 1, 2, \cdots$, where $X = (x_n)$, $H = (h_n)$ and $A = (a_{mn})$. The author applies the above observations to linear equations of this type in which A is ω- (or α-, β-, γ-) completely continuous [for definitions, see Part III cited above]. These are discussed in detail, especially for $l = l_p$ ($1 \leqq p \leqq \infty$).

M. Hasumi (Berkeley, Calif.)

Golovkin, K. K.; Solonnikov, V. A. 470
Bounds for integral operators in translation-invariant norms. (Russian)
Trudy Mat. Inst. Steklov. **70** (1964), 47–58.

The authors prove estimations in the L_p-norm and (in cases (2) and (3)) in an arbitrary translation-invariant norm for the following integral transforms:

$$(1)\qquad v(x) = \int_{E_{n-1}} K(x, y)u(y)\,dy,$$

$$(2)\qquad v(x) = \int_{E_n} K(x-y)u(y)\,dy,$$

$$(3)\qquad v(x, \lambda) = \int_{E_n} K(\lambda, y)u(x+y)\,dy;$$

E_n is the real Euclidean n-space, $x \in E_n$ and $\lambda \in E_1$. The kernels satisfy certain stated conditions. Applications to partial differential equations are given.

Z. Zieleźny (Wroclaw)

Gindler, Herbert A. 471
Extensions of linear transformations.
Amer. Math. Monthly **71** (1964), 525–529.

Let (X, Y) denote the class of continuous linear transformations on X into Y, where X and Y are normed linear spaces with completions $\hat{X}$ and $\hat{Y}$; and for each $A \in (X, Y)$ let $\hat{A}$ be the unique continuous extension of A to all of $\hat{X}$. Each such transformation is assigned one of 9 states according to whether its range is all of the image space, dense in it or non-dense, and whether its inverse is continuous, discontinuous or non-existent.

Of the 81 possible pairs of states for $(A, \hat{A})$, $A \in (X, Y)$, 65 cannot occur (Theorem 1). Examples are given which show that the remaining 16 pairs can occur. The results are displayed via the state diagram device introduced by A. E. Taylor [*Introduction to functional analysis*, Wiley, New York, 1958; MR **20** #5411].

T. R. Jenkins (Los Altos, Calif.)

Goulaouic, Charles 472
Interpolation pour les opérateurs compacts et pour les opérateurs continus.
C. R. Acad. Sci. Paris **258** (1964), 2248–2250.

Let A_0 and A_1 be Banach spaces with continuous injections j_0 and j_1 into a locally convex Hausdorff topological vector space $\mathscr{A}$. By $A_0 \cap A_1$ is meant the kernel of the map $(x_0, x_1) \to j_0(x_0) - j_1(x_1)$ from $A_0 \times A_1$ to $\mathscr{A}$ (the suffixes on the j's are misprinted in the definition as given in the paper). The maps $(x_0, x_1) \to x_0$, $(x_0, x_1) \to x_1$ are injections of $A_0 \cap A_1$ into A_0, A_1, respectively; it is thus possible to regard $A_0 \cap A_1$ as a subset of both A_0 and A_1. It is assumed that the images of $A_0 \cap A_1$ are dense in A_0 and A_1.

A Banach space A is said to have the interpolation property for compact [continuous] operators relative to the pair (A_0, A_1) if (1) $A_0 \cap A_1 \subseteq A$; (2) every operator on $A_0 \cap A_1$ that is compact [continuous] in the topologies induced by A_0 and A_1 is also compact [continuous] in the topology induced by A. The author states a theorem giving conditions under which a Banach space having the interpolation property for compact operators also has the interpolation property for continuous operators. Three lemmas on which the theorem depends are stated; there are no proofs.

J. H. Williamson (Cambridge, England)

Foguel, Shaul R. 473
Powers of a contraction in Hilbert space.
Pacific J. Math. **13** (1963), 551–562.

Since the work of Nagy, Foiaş, Schreiber and others on dilations of a contractive operator in Hilbert space, it has been known that the part of the spectrum of such an operator that lies on the unit circle behaves roughly like the spectrum of a unitary operator. This paper derives results of this type without use of dilation theory, by an ingenious and very short trick. Various applications are made to Markov processes and the spectra of positive operators.

G.-C. Rota (Cambridge, Mass.)

Ryff, John V. 474
On the representation of doubly stochastic operators.
Pacific J. Math. **13** (1963), 1379–1386.

It is well known [see, e.g., Hardy, Littlewood and Pólya, *Inequalities*, Cambridge Univ. Press, London, 1934] that if y and x are n-dimensional real vectors, one can obtain an intrinsic criterion for whether $y = Tx$ for some doubly stochastic matrix T. This paper studies possible generalizations of this criterion to spaces of type L. The main result is a representation theorem for a class of doubly stochastic operators which is shown to coincide with the one introduced by the reviewer [Bull. Amer. Math. Soc. **68** (1962), 95–102; MR **24** #A3671].

G.-C. Rota (Cambridge, Mass.)

Schaefer, Helmut H. 475
On the point spectrum of positive operators.
Proc. Amer. Math. Soc. **15** (1964), 56–60.

Some generalizations of the theorem due to the reviewer [Bull. Amer. Math. Soc. **67** (1961), 556–558; addendum, ibid. **68** (1962), 49; MR **24** #A1621] are given, leading to results which appear to be best possible. Samples: Theorem 1: Let E be a space of type $L_p(S, \Sigma, \mu)$ $(1 \leq p < \infty)$ or $C(X)$, and let T be a positive linear operator on E, with spectral radius 1, such that $T'\psi \leq \psi$ for some strictly positive linear form ψ on E, where T' is the adjoint operator. Then if $\alpha f = Tf$ and $|\alpha| = 1$, one has $\alpha^n |f| g^n = T(|f| g^n)$ for every integer n, where $f = |f| g$.

G.-C. Rota (Cambridge, Mass.)

Widom, Harold 476
On the spectrum of a Toeplitz operator.
Pacific J. Math. **14** (1964), 365–375.

Let L^2 and H^2 be the usual Lebesgue and Hardy spaces on the circle group, respectively. If $\phi \in L^\infty$, P is the projection of L^2 onto H^2, and $f \in H^2$, define $T_\phi f = P(\phi f)$. T_ϕ is called a Toeplitz or Wiener-Hopf operator, since on the dual of the circle group T_ϕ defines a convolution operator. If ϕ is real, then the spectrum of T_ϕ is the closed interval [ess inf ϕ, ess sup ϕ]. This latter fact is a result of Hartman and Wintner [Wintner, Amer. J. Math. **76** (1954), 867–882; MR **17** #499]. By means of an elegant and ingenious argument the present author establishes the fact that for any bounded ϕ the spectrum of T_ϕ is always connected.

A. Devinatz (St. Louis, Mo.)

Devinatz, Allen 477

Toeplitz operators on H^2 spaces.

Trans. Amer. Math. Soc. **112** (1964), 304–317.

Let A be a Dirichlet algebra of continuous functions on the compact Hausdorff space X, $d\mu$ a positive measure on X arising from a maximal ideal of A, $L_2 = L_2(d\mu)$, H_2 the closure of A in L_2, and P the projection from L_2 to H_2. The Toeplitz operator T_ϕ associated with a bounded measurable function ϕ on X is the operator in H_2 defined by $T_\phi(g) = P(\phi g)$. In the classical case X is the unit circle, A is the uniform closure of all trigonometric polynomials of the form $\sum_{n=0}^N a_n e^{in\theta}$, $d\mu = (2\pi)^{-1} d\theta$, and the matrix of T_ϕ relative to the basis $\{e^{in\theta} : n \geqq 0\}$ is the Toeplitz matrix (c_{j-k}), where $c_k = \int \phi(\theta)\, d\mu(\theta)$. This paper is concerned with the question of invertibility of T_ϕ, and, among other things, the spectrum of T_ϕ with ϕ real is found (extending a theorem due to Hartman and Wintner [Amer. J. Math. **76** (1954), 867–882; MR **17**, 499] in the classical case); a representation theorem is established for those ϕ for which T_ϕ is invertible (the proof here depending on an extension of a theorem of Helson and Szegö [Ann. Mat. Pura Appl. (4) **51** (1960), 107–138; MR **22** #12343] concerning the conjugation operator); and a theorem of the reviewer [Illinois J. Math. **4** (1960), 88–99; MR **24** #A432], which gives in the classical case a criterion for invertibility if ϕ has an argument with only jump discontinuities, is here proved more simply and in a sharpened form.

H. Widom (Ithaca, N.Y.)

Putnam, C. R. 478

Toeplitz matrices and invertibility of Hankel matrices.

Pacific J. Math. **14** (1964), 651–658.

Let (c_n) be a real sequence in l^2, $c_0 = 0$, $c_{-n} = c_n$, $f_k(\theta) = \sum c_n n^{-k} \cos n\theta$. The matrices (c_{i-j}), (c_{i+j-1}), (c_{i+j}) are written T, H, K, respectively, and $A = (a_{m,n})$, where $a_{m,n} = 1$ if $|m-n| = 1$, 0 otherwise. Let E_0 designate the spectral resolution of A in the sense that

$$A = \int_0^\pi 2\cos\theta\, dE_0(\theta).$$

An operator is called absolutely continuous if $\|E(\lambda)x\|^2$ is absolutely continuous for each x when $E(\lambda)$ is the spectral resolution of the operator. Let $F_k = \int_0^\pi f_k(\theta)\, dE_0(\theta)$. Then if 0 is in the point spectrum of H, F_0 has no point spectrum, and each of $F_2, F_3, \cdots$ is absolutely continuous. If, in addition, $\sum |c_n| < \infty$, then F_1 is absolutely continuous; if $\sum n|c_n| < \infty$, F_0 is absolutely continuous.

J. L. B. Cooper (Madison, Wis.)

Čumakin, M. E. 479

Generalized resolvents of an isometric operator. (Russian)

Dokl. Akad. Nauk SSSR **154** (1964), 791–794.

Let U be a closed isometric operator in Hilbert space H. A system R_ζ of bounded operators in H is said to be a generalized resolvent for U if $R_\zeta h = P(E - \zeta\tilde{U})^{-1}h$ for all $h \in H$ and some unitary extension $\tilde{U}$ of U to some superspace $\tilde{H}$. Here P is the orthogonal projection of $\tilde{H}$ onto H. An intrinsic characterization of a generalized resolvent for some U and for a given U is given by means of conditions similar to those characterizing a resolvent. A direct formula: Every generalized resolvent of the closed isometric operator U may be written in the form

$$R_\zeta = [E - \zeta(U \oplus \Phi(\zeta))]^{-1}, \quad |\zeta| < 1,$$

where $\Phi(\zeta)$ depends analytically on ζ in $|\zeta| < 1$ and is an operator from $H \ominus D(U)$ into $H \ominus R(U)$ of norm $\leqq 1$. Conversely, for each Φ the above formula yields a generalized resolvent of U. This contains the result of Nagy on the powers of a contraction. A further result is given on isometries which do not have 1 as proper value. Using this, a similar representation formula is given for generalized resolvents of closed symmetric operators. No proofs.

V. Pták (Prague)

Schwartz, J. 480

Some results on the spectra and spectral resolutions of a class of singular integral operators.

Comm. Pure Appl. Math. **15** (1962), 75–90.

The author considers operators T of the form

$$(*)\quad (Tf)(s) = m(s)f(s) + \lim_{\varepsilon\to 0^+} \frac{1}{2\pi i}\int_0^1 \frac{K(s,\sigma)f(\sigma)\,d\sigma}{s-\sigma}, \quad f \in L_2(0,1),$$

where $K \in C^1$. Such an operator can be written in the form

$$(**)\quad (Tf)(s) = [m(s)f(s) + k(s)(I^+f)(s)] + \int_0^1 k(s,\sigma)f(\sigma)\,d\sigma,$$

$$2\pi i k(s) = K(s,s),$$

$$2\pi i(s-\sigma)k(s,\sigma) = K(s,\sigma) - K(s,s),$$

where $k(s)$ and $k(s,t)$ are continuous and $(I^+f)(s)$ denotes the integral in (*) with $K(s,\sigma) \equiv 1$. Let ρ denote the rectangle in the $z = x+iy$ plane consisting of those z for which $0 \leqq x \leqq 1$, $-2\pi \leqq y \leqq 0$. It is first shown that the essential spectrum of T is the image of the boundary of ρ under the mapping $x+iy \to m(x) + ik(x)y$. In case $m(s)$ and $K(s,s) = 2\pi i k(s)$ are real, the essential spectrum is real. If, also, $K(0,\sigma) = K(1,\sigma) = 0$ and $m(s) - s$ and $K(s,\sigma)$ are sufficiently small, together with their first four derivatives, then T is similar to the operation M defined by $M(f)(s) = m(s)f(s)$. Finally, it is shown that the limits

$$\lim_{t\to+\infty} \exp\{itT\}\cdot\exp\{-itm(S)\},$$

$$\lim_{t\to-\infty} \exp\{itT\}\cdot\exp\{-itm(S) - K(S,S)\}$$

exist in the strong topology of operators, S being the operator defined by $(Sf)(s) = sf(s)$.

C. B. Morrey, Jr. (Berkeley, Calif.)

Schwartz, J. 481

Compact analytic mappings of B-spaces and a theorem of Jane Cronin.

Comm. Pure Appl. Math. **16** (1963), 253–260.

The principal result of this paper is the following. Let T be a compact analytic mapping of an open set U in a Banach space B into B, let V be a bounded subset of U whose closure is contained in U, and let $x_1 \in B$ be such that $x_1 \notin (I+T)(\text{bdry } V)$; further, let $\deg(I+T;\, V, x_1)$ be the degree of the mapping $I+T$ on V with respect to the point x_1 [see Leray and Schauder, Ann. Sci. École Norm.

Sup. (3) **51** (1934), 45–78]. Then $\deg(I+T;\ V, x_1)\geqq 0$; $\deg(I+T;\ V, x_1)>0$ if and only if $x_1\in(I+T)(V)$; and $\deg(I+T;\ V, x_1)>1$ if and only if either there is more than one solution in V of the equation $x_1=x+T(x)$, or the Fréchet derivative of T at the unique solution x of $x_1=x+T(x)$ in V has -1 in its spectrum. This provides a foundation for treating questions of existence and uniqueness of solutions of analytic functional equations of the form $y=x+T(x)$ for nonlinear transformations T of the above type, as is illustrated by a set of interesting corollaries derived almost immediately from the theorem. A particular case of this theorem, in which only the finite-dimensional topological degree appeared, was derived by J. Cronin [Ann. of Math. (2) **58** (1953), 175–181; MR **15**, 234] to handle similar problems for a special class of functional equations. *R. C. Gunning* (Princeton, N.J.)

Schwartz, J. 482

Non-isomorphism of a pair of factors of type III.

Comm. Pure Appl. Math. **16** (1963), 111–120.

The author shows that there are at least three non-isomorphic factors of type III, thus solving one of the outstanding problems of the theory of von Neumann algebras. The proof relies on some previous results of Pukánszky [Publ. Math. Debrecen **4** (1956), 135–156; MR **18**, 323]. *G.-C. Rota* (Cambridge, Mass.)

Harris, R. T., Jr. 483

The perturbation of group representations.

Duke Math. J. **28** (1961), 325–333.

Let G be a locally compact abelian group, and $\{R_t(\cdot)\}$ a family of unitary representations on the Hilbert space H, parametrized by t and depending continuously on this parameter. The main problem dealt with is to obtain conditions that the various representations are unitarily equivalent. The results obtained concern, typically, the case when one can distinguish a certain linear dense subspace D of H, and an integrable function $h(g)$ on G, so that $|\langle R_t(g)a, a\rangle|\leqq h(g)$ if $a\in D$, $|a|\leq 1$.

{In a letter to the reviewer the author points out that Proposition 4 on p. 329 is false. He adds that for the validity of Theorems 3 and 5 one needs the additional restriction that

$$|P(K_\lambda{}^t)-P(K_\lambda{}^{t'})|\leqq b|K_\lambda{}^t-K_\lambda{}^{t'}|,$$

where $p(A)$ is the projection on the closure of the range of A, and K^t is the vector-valued Fourier transform of the representation $R_t(\cdot)$, considered as an operator from D to D′.} *E. M. Stein* (Princeton, N.J.)

Herz, Carl S. 484

Fonctions opérant sur certains semi-groupes.

C. R. Acad. Sci. Paris **255** (1962), 2046–2048.

Let R be the real field and E a non-empty set. R^E is a topological algebra over R with the usual operations. Let $Q\subseteq R^E$ be stable under addition and contain all the non-negative constants but no others. $Q(B)$ is the set of $\varphi\in Q$ such that $\varphi(E)\subseteq B$. The set of functions F defined on B such that $\varphi\in Q(B)$ implies $F(\varphi)\in Q$ is denoted by $F(Q,B)$, and the set of continuous functions in $F(Q,B)$ is denoted by $F_c(Q,B)$. The case in which B is a closed interval $[-a,b]$, $0\leqq a\leqq b$, is considered. $M_c(B)$ denotes the functions F defined by a series convergent for $|u|\leqq b$: $F(u)=\sum_{k=0}^\infty c_ku^k$, where $c_k\geqq 0$. $M(B)$ is the set of functions F of the form $F=F_c+S$, where $F_c\in M_c(B)$ and S is defined on $[-a,b]$ and satisfies $S(u)=0$ for $|u|<b$; $S(b)\geqq 0$; $|S(-a)|\leqq S(b)$. According to Proposition 1, if Q is stable under multiplication and is closed in R^E, then $M(B)\subset F(Q;B)$. The author then seeks propositions which conclude $F(Q;B)\subset M(B)$ and formulates axioms in this direction.

Now let G be a topological group; $P(G)$ denotes the real continuous positive definite functions defined on G. Z_2 is the group of order 2. Theorem 1 states that

$$\bigcap_n F(P(Z_2{}^n):B)=M(B).$$

Theorem 4 states that if G is an abelian locally compact group having an element of infinite order, then $F(P(G);B)=M(B)$ if G is discrete, and $F(P(G);B)=M_c(B)$ otherwise. This last result for G the additive group of integers and $B=(-1,1)$ is due to W. Rudin [Duke Math. J. **26** (1959), 617–622; MR **22** #91]. *R. P. Hunter* (University Park, Pa.)

Dixmier, Jacques; Douady, Adrien 485

Champs continus d'espaces hilbertiens et de C^*-algèbres.

Bull. Soc. Math. France **91** (1963), 227–284.

The authors study fiber spaces whose fibers are (in general infinite-dimensional) Banach spaces. The ultimate purpose is the classification of C^*-algebras. Indeed, a C^*-algebra A may be regarded as an algebra of cross-sections of a fiber space whose base is the space P of primitive ideals of A, the fiber at a point I of P being the quotient algebra A/I.

The paper is divided into three chapters. Chapter I consists mostly of basic definitions and elementary consequences. Let B be a topological space. By a continuous field $\mathscr{E}$ of Banach spaces over B we mean a family $\{E(z)\}$ $(z\in B)$ of Banach spaces indexed by B, together with a family Γ of vector fields on B (that is, elements of $\prod_{z\in B}E(z)$) satisfying the following four conditions: (I) Γ is closed under multiplication by continuous complex functions on B; (II) $\{x(z)\,|\,x\in\Gamma\}=E(z)$ for each z in B; (III) $z\to\|x(z)\|$ is continuous for each x in Γ; (IV) if x is a vector field, and for each z in B and each $\varepsilon>0$ there is an x' in Γ such that $\|x-x'\|<\varepsilon$ on some neighborhood of z, then $x\in\Gamma$. If $E(z)$ is the same space D for all z, and Γ consists of all continuous functions on B to D, $\mathscr{E}$ is called a constant field. If $\mathscr{E}$ is (isometrically) isomorphic with a constant field, it is called trivial; if each z in B has a neighborhood Y such that $\mathscr{E}$ restricted to Y is trivial, $\mathscr{E}$ is locally trivial. If $\mathscr{E}'=(\{E'(z)\},\Gamma')$ is a continuous field over B, $E'(z)$ is a closed subspace of $E(z)$ for each z, and $\Gamma'\subset\Gamma$, then $\mathscr{E}'$ is called a subfield of $\mathscr{E}$. We say that $\mathscr{E}$ is separable if Γ has a countable subset Λ such that $\{x(z)\,|\,x\in\Lambda\}$ is dense in $E(z)$ for each z in B.

Now assume that the $E(z)$ are all Hilbert spaces. If $E(z)$ has the same Hilbert dimension a for all z in B, we shall say that $\mathscr{E}$ is of rank a. Chapter II is devoted to theorems asserting that under certain conditions a continuous field of Hilbert spaces is necessarily trivial. Theorem 1: If B is paracompact, a locally trivial continuous field of Hilbert spaces over B of infinite rank is trivial. This illustrates the striking difference (due to the contractibility of the unitary group of an infinite-dimensional Hilbert space) between fields of finite and of infinite rank. Theorem 4 asserts that if B is paracompact, every separable continuous field of Hilbert spaces over B is a subfield of a

trivial field. The most surprising result is Theorem 5: If B is paracompact and of finite dimension, every separable continuous field of Hilbert spaces over B of rank $\aleph_0$ is trivial. Chapter II concludes with some examples of non-trivial fields of infinite rank.

Chapter III deals with continuous fields of C^*-algebras. A C^*-algebra is elementary of rank a if it is isomorphic with the algebra of all compact operators on some Hilbert space of dimension a. A continuous field $\mathscr{A}=(\{A(z)\},\Theta)$ of elementary C^*-algebras over some topological space B is of rank a if $A(z)$ is of rank a for each z; $\mathscr{A}$ is said to satisfy Fell's condition (authors' terminology) if for each z_0 in B, there is a neighborhood Y of z_0, and a vector field p on Y continuous with respect to $\mathscr{A}$, such that, for each z in Y, $p(z)$ is a projection of rank one. Of course, if $\mathscr{A}$ is locally trivial, it satisfies Fell's condition. Now if $\mathscr{E}=(\{E(z)\},\Gamma)$ is a continuous field of Hilbert spaces over B, there is a natural way to construct from $\mathscr{E}$ a continuous field $\mathscr{A}=(\{A(z)\},\Theta)$ of elementary C^*-algebras on B satisfying Fell's condition such that $A(z)$ is just the algebra of all compact operators on $E(z)$; we write $\mathscr{A}=\mathscr{A}(\mathscr{E})$. The question arises: Given a continuous field $\mathscr{A}$ of elementary C^*-algebras over B, is $\mathscr{A}$ of the form $\mathscr{A}(\mathscr{E})$ for some $\mathscr{E}$? And if it is, is the $\mathscr{E}$ unique? We begin with local results. Theorem 7: $\mathscr{A}$ satisfies Fell's condition if and only if it is everywhere locally of the form $\mathscr{A}(\mathscr{E})$. Theorem 8: If $\mathscr{E}$ and $\mathscr{E}'$ are continuous fields of Hilbert spaces, $\mathscr{A}(\mathscr{E})$ and $\mathscr{A}(\mathscr{E}')$ are locally isomorphic if and only if $\mathscr{E}$ and $\mathscr{E}'$ are locally isomorphic. Next come global results. Corollary of Theorem 9: If B is paracompact and $H^2(B, Z)$ (the second cohomology group of B with coefficients in the group of integers) is 0, and if $\mathscr{E}$ and $\mathscr{E}'$ are continuous fields of Hilbert spaces over B, then $\mathscr{E}$ and $\mathscr{E}'$ are isomorphic if and only if $\mathscr{A}(\mathscr{E})$ and $\mathscr{A}(\mathscr{E}')$ are isomorphic. From now on we assume B paracompact. For each continuous field of C^*-algebras $\mathscr{A}$ on B satisfying Fell's condition the authors construct in a natural way an element $\delta(\mathscr{A})$ of $H^3(B, Z)$. Theorem 10: $\mathscr{A}$ is of the form $\mathscr{A}(\mathscr{E})$ if and only if $\delta(\mathscr{A})=0$. Corollary: If $H^3(B, Z)=0$, every such $\mathscr{A}$ is of the form $\mathscr{A}(\mathscr{E})$ for some $\mathscr{E}$. Theorem 11: If a is an infinite cardinal, the correspondence $\mathscr{A}\rightarrow\delta(\mathscr{A})$ sets up a bijective correspondence between $H^3(B, Z)$ and the classes of locally trivial fields of elementary C^*-algebras of rank a. From Theorem 5 follows Theorem 12: If B is paracompact and of finite dimension, every continuous separable field of elementary C^*-algebras which is of rank $\aleph_0$ and satisfies Fell's condition is locally trivial.

In the final paragraph, the authors rephrase the preceding results of Chapter III in terms of C^*-algebras with continuous trace [see the reviewer, Acta Math. **106** (1961), 233–280; MR **29** #1547]. *J. M. G. Fell* (Seattle, Wash.)

Segal, I. E. 486

Transforms for operators and symplectic automorphisms over a locally compact abelian group.

Math. Scand. **13** (1963), 31–43.

The mathematical content of this paper deals with the situation in which G is a Hausdorff locally compact Abelian group with dual group G^*. To each element $z=(a, a^*)$ of the direct sum A of G and G^* is assigned the unitary operator $W(z)$ on $L_2(G)$ defined by $f(x)\rightarrow a^*(2x+a)f(x+a)$. Then $W(z)W(z')=\chi(z, z')W(z+z')$, where $z=(a, a^*)$, $z'=(b, b^*)$, and the bicharacter χ is defined by $\chi(z, z')=a^*(b)(b^*(a))^{-1}$. The first main theorem asserts that, if the doubling map $x\rightarrow 2x$ of G multiplies the Haar measure on G by a constant c^2, then the map $f\rightarrow c\int W(z)f(z)\,dz$, defined initially for $f\in L_1(A)\cap L_2(A)$, extends into an isometry (the Weyl transform) of $L_2(A)$ into the Hilbert space of all Hilbert-Schmidt operators on $L_2(G)$; if the doubling map is an automorphism of G, the Weyl transform is onto, and any operator T on $L_2(G)$ with absolutely convergent trace is the Weyl transform of the function $f(z)$ defined by $f(z)=c\cdot\mathrm{tr}(W(-z)T)$.

A bicontinuous automorphism of A is termed symplectic [anti-symplectic] if it leaves χ invariant [changes χ into its inverse]. The second main theorem says that, if the doubling map is an automorphism of G, and if T is symplectic [anti-symplectic], then there exists a unitary [anti-unitary] operator $\Gamma(T)$ on $L_2(G)$, unique within a scalar multiplier, such that $W(Tz)=\Gamma(T)W(z)\Gamma(T)^{-1}$ $(z\in A)$; and $T\rightarrow\Gamma(T)$ is a continuous projective representation of the symplectic group (called here the harmonic representation). (Symplectic transformations are usually and more restrictedly defined as linear transformations on the direct sum of a finite-dimensional linear space $M=\{x\}$ and its dual $M^*=\{f\}$ which leave invariant the skew bilinear form $f(x')-f'(x)$.)

An introductory section of the paper draws analogies with the spin representations of orthogonal groups and discusses the quantum-mechanical significance of the harmonic representation. *R. E. Edwards* (Canberra)

Gohberg, I. C. 487

The factorization problem in normed rings, functions of isometric and symmetric operators, and singular integral equations. (Russian)

Uspehi Mat. Nauk **19** (1964), *no.* 1 (115), 71–124.

The essential content of this paper is a detailed exposition of the results announced earlier in the author's papers [Dokl. Akad. Nauk SSSR **145** (1962), 971–974; MR **27** #6147; ibid. **146** (1962), 284–287; MR **25** #4376]. *P. Saworotnow* (Washington, D.C.)

Sakai, Shôichirô 488

Weakly compact operators on operator algebras.

Pacific J. Math. **14** (1964), 659–664.

In the present paper the author uses results of A. Grothendieck [Canad. J. Math. **5** (1963), 129–173; MR **15**, 438] (concerning weakly compact operators on spaces $C(K)$ with K compact) to establish various properties of B^*-algebras. Except for the above theorems, the main tools are the author's results that a W^*-algebra M is the dual of a Banach space, the associated space of M, and weak sequential completeness of such spaces [cf. Pacific J. Math. **6** (1956), 763–773; MR **18**, 811; Proc. Japan Acad. **33** (1957), 439–444; MR **20** #5437].

The main theorems are: (a) A bounded operator from a B^*-algebra to an abstract L-space is weakly compact; (b) A weakly sequentially complete B^*-algebra is finite-dimensional. An application of (a) yields a neat proof for the triviality of spectrally continuous operators on $L^1(G)$ which commute with right translations, in the general setup of an arbitrary locally compact non-compact group G. As a corollary of (b), and by further arguments, the author obtains that infinite-dimensional B^*-algebras and B^*-algebras with infinite-dimensional irreducible representations are topologically distinct from the classical

Banach spaces and from the spaces associated with W^*-algebras. In this way a whole new class of Banach spaces is offered as testing ground for the various conjectures and unsolved problems. *P. C. Deliyannis* (Chicago, Ill.)

Nel, L. D. 489

Topological algebras of unbounded functions and unbounded operators.

J. London Math. Soc. **39** (1964), 220–230.

Let A be a commutative W^*-algebra of operators on a Hilbert space H, and M the structure space of A (in which the closure of each open set is open). Let $\bar{C}(M)$ be the algebra of all continuous functions f on M to the complex sphere such that $f^{-1}(\infty)$ is meager (that is, a countable union of nowhere dense sets). The reviewer and Kelley [Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 592–598; MR **14**, 480] showed how to extend the well-known correspondence between $C(M)$ (the algebra of continuous complex functions on M) and A to a correspondence between $\bar{C}(M)$ and an algebra $\bar{A}$ of unbounded normal operators on H. In this article the author defines a topology for $\bar{C}(M)$ as follows. By a quasi-decomposition of M we mean a countable disjoint family $\mathscr{D}$ of open-closed subsets of M whose union has meager complement in M. Let $C(\mathscr{D})$ be the subalgebra of $\bar{C}(M)$ consisting of those f in $\bar{C}(M)$ for which $f^{-1}(\infty)$ is disjoint with $\bigcup \mathscr{D}$. Note that the quasi-decompositions are directed by the relation of refinement, and that $\bar{C}(M) = \bigcup_{\mathscr{D}} C(\mathscr{D})$. The author topologizes each $C(\mathscr{D})$ with the Cartesian product of the uniform topologies of the $C(D)$ $(D \in \mathscr{D})$, and then gives to $\bar{C}(M)$ the inductive limit of the topologies of the $C(\mathscr{D})$. He also notes how this topology for $\bar{C}(M)$ can be described in terms of the isomorphic algebra $\bar{A}$. *J. M. G. Fell* (Seattle, Wash.)

Hirschman, I. I., Jr. 490

Finite section Wiener-Hopf equations on a compact group with ordered dual.

Bull. Amer. Math. Soc. **70** (1964), 508–510.

Let Θ be a compact Abelian topological group with dual Ξ on which there is given an order relation "$<$" compatible with the group structure. For $\xi \in \Xi$ and $\theta \in \Theta$, the value of the character ξ at θ is denoted by (ξ, θ). For rather general Banach algebras A of complex functions $f(\theta) = \sum_\xi f(\xi)(\xi, \theta)$, the author studies families of operators

$$E^+(\eta)f\cdot\theta = \sum_{\xi \geq \eta} f(\xi)(\xi, \theta); \quad E^-(\eta)f\cdot\theta = \sum_{\xi \leq \eta} f(\xi)(\xi, \theta).$$

In particular, for $c \in A$ the Wiener-Hopf operators

$$W_c^+ f = E^+(0)cf, \qquad f \in E^+(0)A,$$
$$W_c^- f = E^-(0)cf, \qquad f \in E^-(0)A$$

are discussed together with their finite sections

$$W_c^+(\eta)f = E^-(\eta)E^+(0)cf, \qquad f \in E^-(\eta)E^+(0)A,$$
$$W_c^-(\eta)f = E^+(-\eta)E^-(0)cf, \quad f \in E^+(-\eta)E^-(0)A.$$

The principal result states that $W_c^+(\eta)$ and $W_c^-(\eta)$ have bounded inverses for η large enough if W_c^+ and W_c^- have bounded inverses. With his result, the author can develop a detailed theory of Szegö polynomials on θ. The proof is based on a "sharp" form of a norm-inequality proved by the reviewer [Illinois J. Math. **7** (1963), 97–103; MR **26** #2818]. *G. Baxter* (Minneapolis, Minn.)

Hille, Einar 491

Remarks on differential equations in Banach algebras.

Studies in mathematical analysis and related topics, pp. 140–145. *Stanford Univ. Press, Stanford, Calif.*, 1962.

Some enlightening examples are given illustrating the behavior of solutions of analytic ordinary differential equations with isolated singularities in a non-commutative Banach algebra. Various results are given concerning the behaviour of solutions of differential equations whose coefficients are quasi-nilpotent elements of the algebra. *G.-C. Rota* (Cambridge, Mass.)

Dombrovskaja, I. N. 492

Linear operator equations of the first kind. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 2 (39), 74–78.

Let A be a continuous linear operator which transforms the elements x of a Hilbert space X into elements y of a Hilbert space Y and for which the inverse exists as an unbounded operator. The equation $Ax = y$ has for given y on a compact subset M of X a quasi-solution x_M defined by the property that $\|Ax - y\|$ is minimal for $x = x_M$. The following theorem is proved. Let M be a convex weakly compact subset of X. Then the quasi-solution x_M exists for all $y \in Y$ and is uniquely and continuously dependent on y.

This theorem is applied to the case of coinciding real separable X and Y, where A is a normal operator and M a sphere. The theory is illustrated by the example

$$\int_{-\infty}^{\infty} \exp(-|x-y|)\varphi(y)\,dy = f(x).$$

H. A. Lauwerier (Amsterdam)

Persidskaja-Bulaševa, L. K. 493

The solution of certain functional equations. (Russian. Kazak summary)

Izv. Akad. Nauk Kazah. SSR Ser. Fiz.-Mat. Nauk **1963**, *no.* 3, 39–44.

Differential equations in a Banach space involving the directional derivative

$$D_f V = \lim_{\Delta t \to 0} \frac{V(t+\Delta t, X+\Delta t\cdot f) - V(t, X)}{\Delta t},$$

are considered, where t is a scalar, and X and $f = f(t, X)$ belong to a Banach space, V taking on values in a Banach space, also. Differential equations of order n are transformed into systems of n first-order equations in a standard way, which is shown to preserve solutions of a given initial-value problem. For linear equations, a condition for boundedness of the solution in t is given. *L. B. Rall* (Madison, Wis.)

Dubinsky, E. 494

Differential equations and differential calculus in Montel spaces.

Trans. Amer. Math. Soc. **110** (1964), 1–21.

Extension of standard material of analysis of vector-valued functions to a class of topological vector spaces. Differential calculus, partial derivatives, products and reciprocals of functions, perfect differentials, existence and uniqueness theorems for ordinary differential equations. Thorough and readable exposition. *G.-C. Rota* (Cambridge, Mass.)

Coddington, Earl A.; Gilbert, Richard C. 495
Generalized resolvents of ordinary differential operators.
Trans. Amer. Math. Soc. **93** (1959), 216–241.
Application of the Naimark theory of generalized resolutions of the identity to the spectral theory of ordinary differential operators. Integral operator representations are given for generalized resolvents, and various deeper results are derived relating to spectral resolutions of various self-adjoint extensions of the same differential operators.
G.-C. Rota (Cambridge, Mass.)

Coffman, Charles V. 496
Non-linear differential equations on cones in Banach spaces.
Pacific J. Math. **14** (1964), 9–15.
Let Y be an ordered Banach space with closed positive cone K. In an earlier paper [same J. **12** (1962), 69–75; MR **25** #4210] the author had proved, assuming K to have a weakly compact cross-section H and $t \to A(t)$ to be a strongly continuous map of $I=[0, \infty)$ into the cone of bounded positive operators on Y, that the differential equation $y'(t)=-A(t)y$ has a non-trivial solution $t \to y(t) \in K$ on I with $y(0) \in H$. The main result of the present paper generalizes this to $y'=f(t, y)$, $f(t, 0)\equiv 0$, supposing that K has a compact cross-section H and $-f$ is continuous on $I \times K$ into K, both with respect to the weak topology defined by a total subset Γ of Y^*. He proves the existence of a "weak Γ-solution" $t \to y(t)$ having $y(0) \in H$. {Reviewer's remark: The assumption (made for several results) that the unit ball of Y be compact for the weak Γ-topology implies that Y can be identified as the strong dual of a suitable Banach space.} *H. H. Schaefer* (Tübingen)

Labsker, L. G. 497
Asymptotic equalities for an approximation of functions by a certain family of linear integral operators in the metric spaces C and L_p. (Russian. Azerbaijani summary)
Akad. Nauk Azerbaĭdžan. SSR Dokl. **20** (1964), *no.* 2, 3–10.
The author proves some equalities of the form

$$\lim_{\lambda\to\lambda_0} \|L_\lambda(f)-f\|/|R_\lambda(n)|\Delta_\lambda(n) = (n!)^{-1}[1+(-1)^n]\|f^{(n)}\|,$$

where the norm is either the sup norm or the L_p-norm, for a certain family (L_λ) of integral operators on a class of functions defined on $[a, b]$. For details the reader is referred to the paper. *N. Dinculeanu* (Bucharest)

Mil'man, D. P. 498
Separators of non-linear functionals and their linear extensions. (Russian)
Izv. Akad. Nauk SSSR Ser. Mat. **27** (1963), 1189–1210.
In this paper a generalization of a theorem of Mazur and Orlicz on moments [Studia Math. **13** (1953), 137–179, p. 147; MR **16**, 932] is proved, and some applications to linear inequalities are given. Relationships to other similar results are discussed. Unfortunately, bibliographical references are very incomplete. The main result is as follows. Let E be a linear space over R (the field of real numbers) and let functionals $\alpha_i : X_i \to R$ be given, where X_i are subsets of E ($i=0, 1, 2$). α_0 is said to be a separator for α_1 and α_2 if the following conditions are satisfied: (1) if $0 \in X_1$ [$0 \in X_2$], then $\alpha_1(0) \leqq 0$ [$0 \leqq \alpha_2(0)$]; (2) $\alpha_1(x) \leqq \alpha_0(x)$ for $x \in X_1 \cap X_0$; (3) $\alpha_1(x) \leqq \alpha_2(x)$ for $x \in X_1 \cap X_2$; (4) $\alpha_0(x) \leqq \alpha_2(x)$ for $x \in X_0 \cap X_2$. Let

$$X_1' = X_1 \setminus (X_0 \cup \{x \in X_1 \cap -X_2 : \alpha_2(-x)+\alpha_1(x) \leqq 0\})$$

and

$$X_2' = X_2 \setminus (X_0 \cup \{x \in X_2 \cap -X_1 : \alpha_2(x)+\alpha_1(-x) > 0\}).$$

Theorem: Let $\{y \in E : ty \in -X_1 \cup X_2 \cup X_0$ for some non-negative $t\}$ be a linear subspace of E and let $X_0=-X_0$. Then the following conditions are equivalent: (i) There is a linear functional $f: E \to R$ such that $f(x)=\alpha_0(x)$ for $x \in X_0$, $\alpha_1(x) \leqq f(x)$ for $x \in X_1$, $f(x) \leqq \alpha_2(x)$ for $x \in X_2$; (ii) for arbitrary non-negative numbers $\varepsilon_j^0, \varepsilon_j', \varepsilon_j''$ and arbitrary elements $x_j^0 \in X_0$, $x_j' \in X_1'$ and $x_j'' \in X_2'$ ($j=1, 2, \cdots, N$; $N=1, 2, \cdots$) the equality

$$\sum_{j=1}^N (\varepsilon_j^0 x_j + \varepsilon_j' x_j' - \varepsilon_j'' x_j'') = 0$$

implies the inequality

$$\sum_{j=1}^N [\varepsilon_j^0 \alpha_0(x_j^0) + \varepsilon_j' \alpha_1(x_j') - \varepsilon_j'' \alpha_2(x_j'')] \leqq 0.$$

The idea of the proof is similar to arguments of Pták [ibid. **15** (1956), 365–366; MR **18**, 320] and Sikorski [ibid. **13** (1953), 180–182; MR **16**, 932].
A. Pełczyński (Warsaw)

Khoan, Vo-Khac 499
Les $\tilde{Q}$-fonctions.
C. R. Acad. Sci. Paris **258** (1964), 3161–3164.
In an earlier note [same C. R. **257** (1963), 3800–3803; MR **28** #1480] the author has defined a Q^b-function as a function from $(-\infty, +\infty)$ to a Hilbert space H which has continuous autocorrelation function, and a Q-function as a function f from $(-\infty, +\infty)$ to a Banach space B such that $\Phi(f(t))$ is a (numerical-valued) Q^b-function for any $\Phi \in B^*$ (the conjugate space). Spectral (i.e., "Fourier") analysis was discussed in (roughly speaking) the spirit of almost-periodic functions, and a Q-function was found to be the sum of a "pseudo-random" function (possessing a continuous Fourier-Stieltjes transform of its autocorrelation-function) and an almost-periodic function. In the present paper, a polynomial (with arguments in a Banach space) is a sum of multilinear forms, and f is a Q^q-function if the substitution of arbitrary translates of f into any polynomial of degree $\leqq q$ produces a Q-function. Theorem: The concept Q^q is independent of q if B is the complex numbers.
A Q-function (with values in a Hilbert space H) is called regular if there is a base ψ_i for H such that

$$\lim_{n\to\infty} \sum_{i=1}^n f_i(t)\psi_i = f(t)$$

implies the validity of the same limit equation in the Besicovitch norm (here f_i are the ψ_i-components of f). A regular Q^2-function is called a $\tilde{Q}$-function. The author develops a theory of such functions, giving sufficient conditions for pure almost-periodicity and pure pseudo-randomness. For a numerical-valued $\tilde{Q}$-function, $|f(t)|^\alpha$ is stated to be a $\tilde{Q}$-function for any real positive α. A necessary and sufficient condition is given for $f(t)$ to be a Q^q-function with values in complex n-space.

The material is asserted to be background for work in non-linear problems [see #500 below].
G. J. Minty (Ann Arbor, Mich.)

Khoan, Vo-Khac 500

Q-solutions d'un système différentiel.

C. R. Acad. Sci. Paris **258** (1964), 3430–3433.

The terminology is that of #499 above. Here Z is the space of n-tuples of complex numbers. The author studies the linear differential equation (1) $z'(t)+Az(t)=f(t)$ and the nonlinear equation (2) $z'(t)+Az(t)=F(t, z(t))$, where the derivative is taken in a (very) weak sense and A is a matrix over the interval $[0, \infty)$. The theorem is stated: If no eigenvalue of A is pure imaginary, then (1) has a unique Q-solution when $f(t)$ is a Q-function. This theorem is used to show that (2) is equivalent to an "equation of Hammerstein type", (3) $z+\mathfrak{K}\mathfrak{F}z=0$, where $\mathfrak{K}$ is defined by the above theorem rather than by an explicit integral operator. If A is "strongly monotonic": Re $\langle Az, z\rangle \geq \alpha|z|^2$, with $\alpha>0$, then $\mathfrak{K}$ is "strictly monotonic": Re $\langle \mathfrak{K}z, z\rangle \geq 0$, and various sufficient conditions are given for the solvability of (3) by various methods: Picard (contractive), Leray-Schauder (topological), and a method attributed to the reviewer (monotonicity), under various hypotheses about the form of F, e.g., Lipschitzian, monotone, polynomial, etc. *G. J. Minty* (Ann Arbor, Mich.)

Kato, Tosio 501

Demicontinuity, hemicontinuity and monotonicity.

Bull. Amer. Math. Soc. **70** (1964), 548–550.

X is a (real or complex) Banach space, $D\subset X$. A function $G: D\to X^*$ is called "monotone" if Re $\langle Gu-Gv, u-v\rangle \geq 0$ for $u, v\in D$. In recent papers, F. E. Browder has solved functional equations involving such operators, first under the assumption of demicontinuity (from strong topology in X to weak* in X^*) and then the apparently weaker hemicontinuity (continuity of the restriction of G to straight-line segments in X, to weak* topology in X^*).

The author gives "sequential" versions of these definitions and shows that (under weak additional hypotheses) hemicontinuity implies demicontinuity for monotone functions. Here D is assumed "quasi-dense" (for $u\in D$, there exists M_u dense in X such that $v\in M_u$ implies $u+tv\in D$ for $t>0$ sufficiently small). In infinite dimensions, the author assumes G also to be locally bounded.
G. J. Minty (Ann Arbor, Mich.)

Browder, Felix E. 502

Continuity properties of monotone nonlinear operators in Banach spaces.

Bull. Amer. Math. Soc. **70** (1964), 551–553.

Definitions are as in the review above [#501]. Here G is called "vaguely continuous" if (roughly speaking) for any $u\in D$ and any closed straight-line segment S with end point u: If $S\cap D$ contains an open (relative to S) neighborhood of u, then there is a subset S' of S with u as a limit point such that G is continuous on S'. The author shows that for monotone functions (on a quasi-dense domain) even this very weak continuity hypothesis implies demicontinuity. For the infinite-dimensional case, one must also assume G locally bounded.

It might be mentioned that this paper and the one reviewed above seem to be the first investigations of "purely structural" questions in the theory of monotone functions. {Misprints: In one place on the first page, ($\rightarrow$) should read ($\rightharpoonup$).} *G. J. Minty* (Ann Arbor, Mich.)

Browder, Felix E. 503

Remarks on nonlinear functional equations.

Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 985–989.

Let X be a complex reflexive Banach space, $F: X\to X^*$, continuous from the strong topology of X to the weak topology of X^*. Suppose also (i) there is a continuous real-valued $c(r)$ with $c(r)\to\infty$ as $r\to\infty$ such that $|\langle Fu, u\rangle| \geq c(\|u\|)\|u\|$, and (ii) for each $N>0$, there is a continuous increasing function $k_N(r)$ with $k_N(0)=0$ such that

$$|\langle Fu-Fv, u-v\rangle| \geq k_N(\|u-v\|)\|u-v\|$$

for $\|u\|\leq N$, $\|v\|\leq N$. The author shows that F is a homeomorphism of F onto X^*.

This result generalizes to Banach spaces and eliminates a boundedness hypothesis from an earlier result of Zarantonello [Bull. Amer. Math. Soc. **70** (1964), 781–787] who gave sufficient conditions for $(\lambda I-F)$ to be a homeomorphism onto in the Hilbert-space case; this result in turn had generalized a famous theorem on linear operators. The main device of Browder's generalization is the use of a net over the directed set of finite-dimensional linear subspaces, in the place of an "ordinary" sequence.
G. J. Minty (Ann Arbor, Mich.)

CALCULUS OF VARIATIONS

See 1075, 1076, 1100.

GEOMETRY

See also 107, 171, 523, 564.

Jeger, M. 504

Über die gruppenalgebraische Struktur der Elementargeometrie.

Elem. Math. **19** (1964), 1–8; *ibid.* **19** (1964), 29–35.

Ausgehend von der Darstellung der Bewegungen einer euklidischen Ebene als Produkte von Spiegelungen diskutiert Verfasser in seiner Antrittsvorlesung die Formulierung geometrischer Relationen durch Gleichungen zwischen Spiegelungsprodukten, das Beweisen geometrischer Sätze durch gruppentheoretisches Rechnen mit Spiegelungen, das gruppentheoretische Axiomensystem der ebenen absoluten Geometrie aus des Referenten *Aufbau der Geometrie aus dem Spiegelungsbegriff* [Springer, Berlin, 1959; MR **21** #6557] euklidische und nichteuklidische Modelle dieses Axiomensystems, insbesondere das Minimalmodell, die Bewegungsgruppe der euklidischen Ebene über dem Primkörper von Charakteristik 3. {Einige Verbote bleiben unerwähnt, in den Definitionen S. 29 unten: $\Sigma_A\neq\Sigma_g$ bzw. $\Sigma_f\neq\Sigma_g$, in Axiom V: Σ_h weder zu Σ_f noch zu Σ_g senkrecht.} *F. Bachmann* (Kiel)

Freudenthal, Hans 505

Die Orientierung des Raumes.

Math.-Phys. Semesterber. **10** (1963/64), 161–174.

Freudenthal, Hans 506

Beziehungen der $\mathfrak{E}_7$ und $\mathfrak{E}_8$ zur Oktavenebene. X, XI.

Nederl. Akad. Wetensch. Proc. Ser. A **66** = *Indag. Math.* **25** (1963), 457–471; 472–487.

For previous papers in this series, see same Proc. **57** (1954), 218–230 [MR **16**, 108]; ibid. **57** (1954), 363–368 [MR **16**, 900]; ibid. **58** (1955), 151–157 [MR **16**, 900]; ibid. **58** (1955), 277–285 [MR **16**, 900]; ibid. **62** (1959), 165–201; ibid. **62** (1959), 447–474.

In VIII and IX of this series the author constructed the metasymplectic geometry over the reals, which gives a representation of the exceptional Lie group $\mathfrak{F}_4$. As was indicated in VIII, a similar geometry must exist over the complex numbers, quaternions and octaves, to which groups of type $\mathfrak{E}_6$, $\mathfrak{E}_7$ and $\mathfrak{E}_8$, respectively, correspond. In the present papers a uniform construction is given for the four metasymplectic geometries, departing from a real form of the Lie algebra $\mathfrak{F}_4$, $\mathfrak{E}_6$, $\mathfrak{E}_7$ or $\mathfrak{E}_8$, which is called $\mathfrak{R}_4$. For $\phi \in \mathfrak{R}_4$ the linear mapping $\langle\phi, \phi\rangle$ of $\mathfrak{R}_4$ into itself is defined by

$$\langle\phi, \phi\rangle\phi^* = (\mathrm{ad}\,\phi)^2\phi^* + \varepsilon_2{}^{-1}\,\mathrm{tr}(\mathrm{ad}\,\phi\;\mathrm{ad}\,\phi^*)\phi - \varepsilon_3{}^{-1}\,\mathrm{tr}((\mathrm{ad}\,\phi)^2)\phi^*,$$

where the constants ε_2 and ε_3 are so chosen in each of the four cases that $\mathrm{tr}\langle\phi, \phi\rangle = 0$ and $\langle\phi_\mu, \phi_\mu\rangle = 0$ if μ is the highest root of $\mathfrak{R}_4$. $\langle\phi_1, \phi_2\rangle$ is defined by linearization of $\langle\phi, \phi\rangle$. The Lie algebra generated by the $\langle\phi_1, \phi_2\rangle$ is called $\mathfrak{R}_1$. A symplecton is the class of scalar multiples of an element $\phi \in \mathfrak{R}_4$ such that $\langle\phi, \phi\rangle = 0$. A point is an $A \in \mathfrak{R}_1$ which can be written as $A = \langle\phi_1, \phi_2\rangle$ for $\phi_1, \phi_2 \in \mathfrak{R}_4$ with $[\phi_1, \phi_2] = 0$, again up to a scalar factor. Several geometric relations are defined between such elements: two points or two symplecta can be joined, tied or hinged, a point and a symplecton can be incident or semi-incident; each of these relations implies the following one. A maximal set of joined points is a plane in a symplecton. The points that are incident with a symplecton form a symplectic geometry (see V).

Concluding the paper the author proves a great number of properties of these geometries, e.g., the intersection of two symplecta consists of at least one point if and only if they are tied, of more than one point if they are joined; two points A and B are joined with at least one common point C if they are hinged; C is unique when A and B are not tied; a symplecton is a maximal set of mutually tied points. *F. D. Veldkamp* (New Haven, Conn.)

Coxeter, H. S. M. 507

Regular compound tessellations of the hyperbolic plane.

Proc. Roy. Soc. Ser. A **278** (1964), 147–167.

The author discusses relations between different regular tessellations of the hyperbolic plane. Consider two tessellations $\{p, q\}$ and $\{m, n\}$ such that the set of vertices of $\{p, q\}$ is a subset of the set for $\{m, n\}$. For the dual conception, consider the centers of faces instead of the vertices.

For metrical reasons, there are not too many tessellations which are in one of the above relations: in the first instance, the minimal distance of vertices in $\{p, q\}$ must occur as a distance of vertices in $\{m, n\}$, and in general, the cosines of distances belong to different algebraic number fields. The author compiles five one-parameter families and seventeen isolated cases. All these cases are discussed in detail, and each of them reveals some beautiful features. By comparing the areas of the two tessellations, one obtains a necessary condition for existence. This condition is comparatively weak, and does not allow a full enumeration.

The author conjectures that the list given in the paper is complete. *J. Mennicke* (Wolfenbüttel)

Benz, Walter 508

Das von Staudtsche Theorem in der Laguerregeometrie.

J. Reine Angew. Math. **214/215** (1964), 53–60.

In Verallgemeinerung der zweidimensionalen Laguerregeometrie wird jedem kommutativen Körper K eine Laguerregeometrie zugeordnet, deren Speere die homogenen geordneten Paare aus dem Ring der dualen Zahlen über K sind; rein duale Paare sind dabei auszuschließen. Durch diese Homogenisierung wird die sonst übliche Verwendung unendlich großer dualer Zahlen vermieden. Jene eineindeutigen Abbildungen der Menge aller Speere auf sich, die genau konzyklische Speermengen wieder in solche überführen, sind die Zykelverwandtschaften. Die Laguerregruppe aller Zykeltransformationen, also die Automorphiengruppe der Laguerregeometrie, wird hergeleitet. Ihre Darstellung ist die sinngemäße Verallgemeinerung der Formeln für die klassische Laguerregruppe über den Körper der reellen Zahlen, wobei natürlich die Automorphismen des Grundkörpers K eingehen. *H. Brauner* (Stuttgart)

Freese, Raymond W. 509

A three-point property in straight line spaces.

Amer. Math. Monthly **71** (1964), 529–530.

Marr and Stamey proved [same Monthly **69** (1962), 22–25; MR **25** #489] the theorem that "If M is a closed, connected subset of a Euclidean or hyperbolic space and has the double isosceles property, then M is convex". The author establishes the validity of the theorem in a slightly wider class of spaces, viz., straight-line spaces. *S. R. Mandan* (Kharagpur)

Godeaux, Lucien 510

Variétés algébriques privées de variété canonique.

Acad. Roy. Belg. Bull. Cl. Sci. (5) **49** (1963), 1045–1051.

Author's summary: "Construction, dans un espace linéaire à $n-1$ dimensions, d'une hypersurface algébrique privée de variété canonique mais ayant un système bicanonique infini."

Burniat, Pol 511

Surfaces algébriques de genres $p_a = p_g = 1$ et bigenre $P_2 = 2, 3, 4, 5$.

Acad. Roy. Belg. Bull. Cl. Sci. (5) **49** (1963), 1052–1058.

Author's summary: "On ne connaît pas les valeurs du bigenre compatibles avec une valeur donnée $pg \geqq 0$ du genre géométrique d'une surface algébrique. Une indication substantielle a été fournie grâce à la construction de

modèles projectifs multiples. On donne ici des modèles simples particulièrement élémentaires pour les invariants

$$p_a = p_g = 1, \quad p^{(1)} = 1, 2, 3, 4, \quad P_2 = 2, 3, 4, 5."$$

Abhyankar, Shreeram 512
Remark on Hessians and flexes.
Nieuw Arch. Wisk. (3) **11** (1963), 110–117.

Let $X_1, X_2, \cdots, X_m$ be indeterminates over a field k and let $F(X_1, X_2, \cdots, X_m)$ be a polynomial. If $F_i = \partial F/\partial X_i$ and $F_{ij} = \partial F_i/\partial X_j$, $G_{00} = F$, $G_{0i} = G_{i0} = F_i$ and $G_{ij} = F_{ij}$ for $i, j = 1, 2, \cdots, m$, then $H = \det(G_{ij})$, $0 \leqq i, j \leqq m$, is defined to be the affine Hessian of F. It is shown that as in the case of the ordinary Hessian, H is a covariant of F of weight two, and if the characteristic p of k is not 2, the classical Hessian criterion for flexes remains valid if the affine Hessian is used. That is, if $m = 3$ and if P is a simple point of the curve $F = 0$, then $H(P) = 0$ is a necessary and sufficient condition for P to be a flex. (It can happen when $p \neq 0$ that all simple points are flexes.)

H. T. Muhly (Iowa City, Iowa)

Hirschfeld, James W. P. 513
The double-six of lines over $PG(3, 4)$.
J. Austral. Math. Soc. **4** (1964), 83–89.

Over any field a skew hexagon $a_1b_3a_2b_1a_3b_2$ with vertices $A_{ij} = a_i \cap b_j$ determines a unique linear complex with respect to which it is self-polar, although over $GF(2^n)$ there is no unique quadric for self-polarity. Taking b_6 as a transversal of five lines a_1, a_2, a_3, a_4 and a_5, a double-six $\{a_i, b_j\}$ can be constructed in which a_i and b_j intersect when $i \neq j$, but the other 36 pairs are skew. Given a skew hexagon $a_1b_3a_2b_1a_3b_2$ with sets of alternate vertices collinear, a unique linear complex C exists so that a_i is polar to b_i for $i = 1, 2, 3$. Over $GF(2^n)$, this skew hexagon can be extended to a non-degenerate double-six in which the polarity is valid also for $i = 4, 5, 6$. The author proves conversely that over $GF(4)$, if a skew hexagon gives rise to a non-degenerate double-six, the sets of alternate vertices are collinear. For the field $GF(4)$ and for no other field, the double-six is uniquely determined by a skew hexagon. Each double-six is self-polar with respect to a unique linear complex.

In $PG(3, 4)$ a double-six $\{a_i, b_j\}$ determines an additional 15 lines c_{ij} in the linear complex C of 85 lines such that c_{ij} meets a_i, b_j and c_{kl} ($k, l \neq i, j$). The 27 lines contain 45 points lying on a unique cubic surface determined by the double-six; this surface contains 36 double-sixes. The author concludes by describing a representation in [5] in which the lines of [3] become tangent primes to a quadric Ω and the non-special linear complexes become points not on Ω.

J. S. Frame (E. Lansing, Mich.)

Groemer, Helmut 514
Über Würfel- und Raumzerlegungen.
Elem. Math. **19** (1964), 25–27.

Die von H. Hadwiger gestellte Frage, ob alle Zerlegungspolyeder des n-dimensionalen Euklidischen Raumes R^n mit einem n-dimensionalen Würfel zerlegungsgleich seien, beantwortete der Verfasser durch folgenden Satz: "Es sei K ein beschränktes Polyeder mit inneren Punkten und $\{K_i\}$ eine Zerlegung des R^n in zu K kongruente Polyeder K_i. Ist $\{K_i\}$ eine reguläre Lagerung oder zumindest die Vereinigung von endlich vielen (zur selben Gruppe gehörigen) regulären Lagerungen, so ist K mit einem Würfel zerlegungsgleich." Hierbei heißt eine Menge $\{G_i\}$ von einander nicht überlappenden Polyedern G_i des R^n eine reguläre Lagerung, wenn $\{G_i\}$ eine transitive Gruppe von Deckbewegungen besitzt, die ein beliebiges Polyeder $G_j \in \{G_i\}$ immer wieder in ein Polyeder von $\{G_i\}$ überführen und stets eine solche Bewegung enthalten, die irgend zwei Polyeder G_j, $G_k \in \{G_i\}$ zur Deckung bringt. Zum Beweis benützt Verfasser Sätze über Bewegungsgruppen des R^n und über die Zerlegungsgleichheit von Polyedern.

H. R. Müller (Braunschweig)

CONVEX SETS AND GEOMETRIC INEQUALITIES

See also 897.

Straus, E. G. 515
Two comments on Dvoretzky's sphericity theorem.
Israel J. Math. **1** (1963), 221–223.

Author's summary: "For any two positive integers k, l and any $\varepsilon > 0$ there exists an $N(k, l, \varepsilon)$ so that given any l convex bodies $C_1, \cdots, C_l$, symmetric about the origin in E^n with $n \geqq N$, there exists a subspace E^k so that each C_i intersects E^k, or has a projection into E^k, in a set which is nearly spherical (asphericity $> \varepsilon$). The measure of the totality of E^k which intersect a given body in E^n in a nearly ellipsoidal set is considered and an affine invariant measure is introduced for that purpose."

B. R. Gelbaum (Minneapolis, Minn.)

Firey, Wm. J. 516
The mixed area of a convex body and its polar reciprocal.
Israel J. Math. **1** (1963), 201–202.

In Euclidean n-space let K be a convex body and $\hat{K}$ its polar reciprocal with respect to a unit sphere centered at an interior point of K. The author proves that $(K + \hat{K})/2 \supseteq E$ and uses this to show that in 2-space, $A(K, \hat{K}) \geqq \pi$ (where $A(K, \hat{K})$ is the mixed area) with equality if and only if K is a circle concentric with E, and in 3-space $4\pi S(K, \hat{K}) \geqq \sqrt{[M(K)M(\hat{K})]} \geqq 16\pi^2$ (where $S(K, \hat{K})$ is the mixed surface area and $M(K)$ is the total mean curvature) with equality if and only if $K = rE$ for some $r > 0$. (For relevant definitions see Bonnesen and Fenchel [*Theorie der konvexen Körper*, Springer, Berlin, 1934].)

W. Moser (Winnipeg, Man.)

Chakerian, G. D.; Stein, S. K. 517
On the symmetry of convex bodies.
Bull. Amer. Math. Soc. **70** (1964), 594–595.

The authors announce a number of results on certain measures of symmetry and on some related combinatorial problems. A convex body in Euclidean n-space E^n is called k-symmetric provided it coincides with its reflection through some k-plane. For a convex body K in E^n, $c(K; k)$ is the maximum of the ratio of volumes of K^* and of K, where K^* is a k-symmetric convex body contained in K. Theorem 1 states that

$$c(K; k) \geqq (2^{n-k}n!)^{-1} \max\{k!, (n-k)!\}.$$

Other results deal with measures of symmetry of mass

distributions and with coverings of K by translates of $-K$. As a by-product the authors disprove a conjecture of the reviewer [Portugal. Math. **18** (1959), 155–164; MR **23** #A2792]. *B. Grünbaum* (Jerusalem)

Soltan, P. S. 518

On the illumination of the boundary of a convex body from within. (Russian)

Mat. Sb. (*N.S.*) **57** (**99**) (1962), 443–448.

Let K be a convex body; a subset A of the boundary of K is "illuminating K from within" provided there exists for each point of K a point of A such that the segment determined by them contains interior points of K. Let $p(K)$ denote the least p such that there exists a k-pointed set illuminating K from within. The author proves: (1) If K is an n-dimensional convex body, then $2 \leq p(K) \leq n+1$; (2) For each p with $2 \leq p \leq n+1$ there exists an n-dimensional convex body K satisfying $p(K)=p$. {A different proof of (1), and some related results, are given in the reviewer's paper [Acta Math. Acad. Sci. Hungar. 15 (1964), 161–163; MR **28** #3367].} *B. Grünbaum* (Jerusalem)

Supnick, Fred; Quintas, Louis V. 519

Extreme Hamiltonian circuits. Resolution of the convex-odd case.

Proc. Amer. Math. Soc. **15** (1964), 454–456.

The authors find the longest Hamiltonian circuit through n points when these lie on the boundary of their convex hull and n is odd. They further state that the corresponding case with n even has since been solved by them [Notices Amer. Math. Soc. **11** (1964), 335]. *H. T. Croft* (Cambridge, England)

Shephard, G. C. 520

Convex bodies and convexity on Grassmann cones. VI. The projection functions of a simplex.

J. London Math. Soc. **39** (1964), 307–319.

Part V (by the author, H. Busemann, and G. Ewald) appeared in Arch. Math. **13** (1962), 512–526 [MR **28** #522b]. Author's summary: "The first two sections of this paper are devoted to proving the following: (1) If $1 < r = n-2$, the projection function $\hat{P}(T^n, R)$ of a (non-degenerate) simplex T^n is a convex, but not totally convex, function on G^n_{n-2}. This, together with results from previous papers in the series, leads to the comprehensive statement: (2) The projection function $\hat{P}(T^n, R)$ of a (non-degenerate) n-dimensional simplex T^n $(r<n, n>3)$ is totally convex for $r=1$, $n-1$, convex but not totally convex for $r=n-2$, and is not convex for $1<r<n-2$.

"The final section of the paper uses these results, along with a general lemma (16) on convex functions to obtain additional information about maximal polyhedra in projections. The following extension of a previous result in the series is obtained: (2′) Let $f_s(K, R)$ be the function induced on $G_r{}^n$ by a maximal volume of a polyhedron with at most s vertices contained in the projection $P(K, \mathscr{R})$ of a convex body K in E^n onto an r-flat $\mathscr{R}$. Let $v(r, n)$ and $v_t(r, n)$ be, respectively, the suprema of the numbers s for which $f_s(K, R)$ is convex, and is totally convex, for all K. Then $v(n-1, n) = v_t(n-1, n) = \infty$, and $v(r, n) = r+3$, $v_t(r, n) = r+2$ for $2 \leq r \leq n-2$." *A. M. Macbeath* (Birmingham)

Cooke, G. 521

A problem in convexity.

J. London Math. Soc. **39** (1964), 370–378.

The author completely solves a problem of Klee, proving that in any plane domain of constant width λ one can inscribe a semicircle of diameter λ. He proves the result first for a general Reuleaux polygon P by considering the locus L of centres of diameters of P, and those cusp points of L where L touches its convex hull. Such points are shown to be suitable points for the centres of the semicircles. A standard limiting argument then extends the result to curves of constant width in general, and with a little more ingenuity, we find that there are always at least 3 distinct semicircles (3 is best possible). *H. T. Croft* (Cambridge, England)

Ehrhart, Eugène 522

Une généralisation probable du théorème fondamental de Minkowski.

C. R. Acad. Sci. Paris **258** (1964), 4885–4887.

The following conjecture is proposed by the author, and established for $n=2$: If the center of gravity G of an n-dimensional convex body K is a point of the n-dimensional lattice of points with integral coordinates, and if the n-content of K is greater than $(n+1)^n/n!$, then K contains in its interior a lattice point different from G. It is shown by examples that, for all n, the conjecture is not valid for bodies with n-content equal to $(n+1)^n/n!$. *B. Grünbaum* (Jerusalem)

Mąkowski, Andrzej 523

Inequalities for radii of inscribed, circumscribed and escribed circles.

Nieuw Arch. Wisk. (3) **12** (1964), 5–7.

Let r be the inradius, r_i the exradii, R the circumradius of a triangle. The author proves

$$\frac{1}{\sqrt{2}} r\sqrt{R} \leqq \frac{1}{3}\sqrt{r_1 r_2 r_3} \leqq \frac{1}{2} R\sqrt{r}.$$

The right-hand side follows from the cubic arithmetic mean-geometric mean inequality for the sides of a triangle. The left-hand side follows from an inequality of Leuenberger [Elem. Math. **16** (1961), 127–129]. The author also shows how the inequality $\sqrt{\frac{1}{3}(r_1{}^2+r_2{}^2+r_3{}^2)} \geqq \frac{3}{2}R$ [Fejes Tóth, Mat. Lapok **1** (1950), 313] can be obtained from inequalities due to Carlitz, Leuenberger, and Gerretsen. *H. W. Guggenheimer* (Minneapolis, Minn.)

Fejes Tóth, L. 524

What the bees know and what they do not know.

Bull. Amer. Math. Soc. **70** (1964), 468–481.

It was observed by Kepler that the bottom of the bee's cell consists of three equal rhombi like those at a trivalent vertex of the rhombic dodecahedron. Pappus and many others have suggested that, in building such cells, the bees aim at economy. The author proves that, if the three equal rhombi were replaced by two rhombi and two hexagons, the bees could save 0.35% of the amount of wax that is needed to cover the opening of the cell. In defence of the bees, he points out that this amount of saving is negligible in comparison with the inevitable errors caused by variations in the thickness of the cell

walls and by irregularities in the shape of the hexagonal opening.

He goes on to summarize a number of his earlier papers. In particular, he mentions the problem of decomposing Euclidean space into not necessarily convex cells of unit volume having the least average surface area. He recalls Lord Kelvin's discovery that, as a possible cell, the rhombic dodecahedron is superseded by the truncated octahedron, and that the truncated octahedron can advantageously be distorted (so that its dihedral angles become 120°) by making the sides of the square faces bulge outward while the flat hexagons are wrinkled into monkey-saddles. He raises the unanswered question to which S. Melmore has devoted many years of his life: Has Kelvin's solid the least surface area among all space-fillers of given volume?

Finally, he mentions Heppes's enumeration of all spherical tessellations with convex faces whose angles are all equal. These are the regular tessellations along with five irregular ones in which the angles are 120°. This enumeration enabled Heppes to prove that any lather which contains an interior bubble contains a non-convex bubble. *H. S. M. Coxeter* (Toronto, Ont.)

Groemer, Helmut 525

Existenzsätze für Lagerungen im Euklidischen Raum.

Math. Z. **81** (1963), 260–278.

In recent decades there has accumulated a large body of theorems having to do with densities of packings and coverings in R^n, Euclidean n-space. The classical one is that of the densest packing and thinnest covering of R^n by congruent spheres. These have been generalized in many ways, e.g., to spheres of different radii, convex and non-convex bodies, multiple packings and coverings, and so on. Most of these results are inequalities on the magnitudes of densities, and do not prove the existence of configurations having maximum (packing) or minimum (covering) densities. Indeed, the only existence theorems of this sort known to date are due to R. A. Rankin [Ann. of Math. (2) **48** (1947), 1062–1081; MR **9**, 226] for the case of packing and coverings of R^n by congruent spheres, and E. Hlawka [Monatsh. Math. **53** (1949), 81–131; MR **11**, 12] for the more general case of packings and coverings by convex bodies. The author considers the problem in a general way and obtains such general existence theorems that this paper is an important addition to this field; indeed, it is certain to be considered a classic. A few definitions and theorems will have to suffice for a review.

A finite or infinite sequence $\Sigma=(S_i)$ of point sets S_i in R^n is a configuration provided: (a) any bounded region of R^n contains only a finite number of S_i; (b) the set of all diameters of the S_i is bounded; (c) to each S_i there corresponds a positive number γ_i. If Σ is a finite sequence, $\gamma(\Sigma)$ denotes the sum of γ_i taken over all S_i in Σ.

For any point p the ρ-section of p relative to Σ is the set of all $S_i\in\Sigma$ whose distance from p is $\leqq\rho$.

A set $\mathscr{S}=(\Sigma_\nu)$ of configurations Σ_ν is said to be proper if the following conditions (I), (II), and (III) or (III') are satisfied. (I) If $\Sigma_\nu\in\mathscr{S}$, and Σ_ν' is obtained from Σ_ν by a translation, then $\Sigma_\nu'\in\mathscr{S}$. (II) There exists $\rho\geqq 0$ such that if Φ is a configuration whose ρ-sections coincide with the ρ-sections of some $\Sigma_\nu\in\mathscr{S}$, then $\Phi\in\mathscr{S}$. (III) If $\Sigma_\kappa\in\mathscr{S}$ and $S\in\Sigma_\kappa$, then $\Sigma_\kappa-(S)\in\mathscr{S}$. (III') If $\Sigma_\kappa\in\mathscr{S}$ and $S\in\Sigma_\lambda\in\mathscr{S}$, then $\Sigma_\kappa\cup(S)\in\mathscr{S}$.

K will denote a bounded point set with positive volu (Jordan content) $\mu(K)$. For any positive number σ, t points σp with $p\in K$ will be denoted by σK. $\bar K$ is t closure of the convex cover of K, and $\mathring{K}$ the interior of K is starlike if for all $0\leqq\sigma<1$, $\sigma\bar K\subset K$. Σ^*K and Σ_* denote the set of all $S\in\Sigma$ with $K\cap S\neq\varnothing$ and $S\subset$ respectively. Let

$$\delta^*(\Sigma,K)=\frac{\gamma(\Sigma^*K)}{\mu(K)},\qquad \delta_*(\Sigma,K)=\frac{\gamma(\Sigma_*K)}{\mu(K)},$$

$$\bar\delta(\Sigma,K)=\limsup_{\sigma\to\infty}\delta^*(\Sigma,\sigma K),$$

$$\underline{\delta}(\Sigma,K)=\liminf_{\sigma\to\infty}\delta_*(\Sigma,\sigma K).$$

If $\bar\delta(\Sigma,K)=\underline{\delta}(\Sigma,K)$ for all K, Σ is said to be "gleic mässig".

Let $\mathscr{S}$ be a collection of configurations. Let $\mathscr{K}$ be t set of all K in R^n. Various constants (densities) a defined as follows.

$$\delta^0(\mathscr{S},K)=\sup\bar\delta(\Sigma,K),\quad \delta_0(\mathscr{S},K)=\inf\underline{\delta}(\Sigma,K)\qquad(\Sigma\in\mathscr{S}$$

$$\delta^0(\mathscr{S})=\sup\delta^0(\mathscr{S},K),\quad \delta_0(\mathscr{S})=\inf\delta_0(\mathscr{S},K)\qquad(K\in\mathscr{K}$$

Theorem: If $\mathscr{S}$ is proper, then

$$\tilde\delta^0(\mathscr{S},K)=\lim_{\sigma\to\infty}\sup_{\Sigma\in\mathscr{S}}\delta^*(\Sigma,\sigma K),$$

$$\tilde\delta_0(\mathscr{S},K)=\lim_{\sigma\to\infty}\inf_{\Sigma\in\mathscr{S}}\delta_*(\Sigma,\sigma K)$$

exist. Furthermore, these limits, as well as $\delta^0(\mathscr{S},K)$ ar $\delta_0(\mathscr{S},K)$, are independent of K, and $\delta^0(\mathscr{S})=\tilde\delta^0(\mathscr{S}$ $\delta_0(\mathscr{S})=\tilde\delta_0(\mathscr{S})$. Furthermore, there exist "gleichmässig configurations Σ^0, Σ_0 in $\mathscr{S}$, such that $\delta^0(\mathscr{S})=\bar\delta(\Sigma^0)$ $\underline{\delta}(\Sigma^0)$, $\delta_0(\mathscr{S})=\bar\delta(\Sigma_0)=\underline{\delta}(\Sigma_0)$. Theorem (with an err corrected): If K is arbitrary and Σ is "gleichmässig", or K is starlike and Σ is arbitrary, then

$$\bar\delta(\Sigma,K)=\limsup_{\sigma\to\infty}\delta^*(\Sigma,\sigma K)=\limsup_{\sigma\to\infty}\delta_*(\Sigma,K),$$

$$\underline{\delta}(\Sigma,K)=\liminf_{\sigma\to\infty}\delta_*(\Sigma,\sigma K)=\liminf_{\sigma\to\infty}\delta^*(\Sigma,K).$$

Furthermore, if Σ' is a translation of Σ, then

$$\bar\delta(\Sigma',K)=\bar\delta(\Sigma,K),\quad \underline{\delta}(\Sigma',K)=\underline{\delta}(\Sigma,K).$$

Theorem: If $\mathscr{S}$ is an arbitrary collection of configuration and if K is starlike, then in the case where the limi exist,

$$\tilde\delta^0(\mathscr{S})=\lim_{\sigma\to\infty}\sup_{\Sigma}\delta^*(\Sigma,\sigma K)=\lim_{\sigma\to\infty}\sup_{\Sigma}\delta_*(\Sigma,\sigma K)\qquad(\Sigma\in\mathscr{S}$$

$$\tilde\delta_0(\mathscr{S})=\lim_{\sigma\to\infty}\inf_{\Sigma}\delta_*(\Sigma,\sigma K)=\lim_{\sigma\to\infty}\inf_{\Sigma}\delta^*(\Sigma,\sigma K)\qquad(\Sigma\in\mathscr{S}$$

W. Moser (Montreal, Que

DIFFERENTIAL GEOMETRY

See also 271, 317, 341, 623, 625, 631, 634, 800.

Bilinski, Stanko 52

Über eine Erweiterungsmöglichkeit der Kurventheorie

Monatsh. Math. **67** (1963), 289–304.

Through the formulas $\mathbf{t}_1=\mathbf{t}$, $\mathbf{n}_1=\mathbf{n}$, $\mathbf{b}_1=\mathbf{b}$, $\mathbf{t}_{i+1}=\mathbf{n}_i$, $\mathbf{n}_{i+1}=\mathbf{b}_{i+1}\times\mathbf{t}_{n+1}$, $\mathbf{b}_{i+1}=(\kappa_i/\kappa_{i+1})\mathbf{b}_i+(\tau_i/\kappa_{i+1})\mathbf{t}_i$; $\kappa_1=\kappa$, $\tau_1=\tau$, $\kappa_{i+1}=\sqrt{(\kappa_i^2+\tau_i^2)}$, $\tau_{i+1}=(\kappa_i\dot{\tau}_i-\dot{\kappa}_i\tau_i)/(\kappa_i^2+\tau_i^2)$, $i=1,2,3,\cdots$, an infinite set of moving trihedrals belonging to a space curve with unit tangent vector $\mathbf{t}$, curvature vector $\mathbf{n}$ and binormal vector $\mathbf{b}$ is defined. For all these trihedrals there exists a set of formulas analogous to those of Frenet [the author, Proc. Internat. Congr. Mathematicians (Amsterdam, 1954), pp. 200–201, Noordhoff, Groningen, 1954]. When between the points M and M^* of two curves C and C^* there exists a one-to-one correspondence such that the vectors $\mathbf{n}_i$ and $\mathbf{n}_i^*$ lie on the line MM^*, the curves C and C^* form a pair of Bertrand curves B_i of order i. The pair B_1 is the ordinary pair of Bertrand. The present paper contains a study of the pair B_2. One of the theorems is that the two curves of a B_2 pair are geodesics on the same ruled surface. The paper also contains an enumeration of the many papers on Bertrand curves and their generalisations.

D. J. Struik (Utrecht)

Backes, F. 527

Sur certaines sphères de Ribaucour menées par un cercle cyclique.

Acad. Roy. Belg. Bull. Cl. Sci. (5) **49** (1963), 1130–1134.

Author's summary: "Par le cercle générateur d'un système cyclique, et dont les foyers décrivent des surfaces isothermiques se correspondant dans la transformation de Darboux, on peut mener une infinité de sphères telles que les lignes de courbure se correspondent sur les deux nappes de leurs enveloppes et engendrent, par là-même, des congruences de sphères cycliques."

Fulton, Curtis M. 528

Spherical helices in n-space.

Tensor (*N.S.*) **15** (1964), 37–39.

Let C be a curve in an n-dimensional Euclidean space, s its arc length, κ_{r-1} $(1<r\leqq n)$ its least non-zero curvature, and $t_1(=t), t_2, \cdots, t_r$ its unit tangent, second normal, etc. We define the functions P_{ij}, Q_{ij} $(1\leqq i,j\leqq r)$ of s by the equations

$$t_j = \sum P_{ij}t^{(i-1)}, \quad t_j = \sum Q_{ij}t_r^{(i-1)},$$

where the derivatives are all taken with respect to s. The author calls C a helix of the first [second] kind if the vector $\sum P_{1j}t_j$ $[\sum Q_{1j}t_j]$ is constant.

It turns out that C is a helix of the first kind if and only if the vectors $t', t'', \cdots, t^{(r)}$ are linearly independent along the curve, a condition whose geometric significance has been considered by J. H. Butchart [Amer. J. Math. **55** (1933), 126–130] and the reviewer [J. London Math. Soc. **15** (1940), 168–172; MR **2**, 164]. The main result of this paper is that if C is a non-spherical helix of the first [second] kind, the locus of the center of its osculating sphere is a helix of the second [first] kind, and conversely.

Y.-C. Wong (Hong Kong)

Oniščuk, N. M. 529

A pair consisting of a congruence and a surface in a centro-affine space. (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. Geom. Sb. **161** (1962), 111–123.

Pairs $M=(K,P)$ consisting of a line congruence K and a surface P in centro-affine 3-space are considered. Canonical frames are found and 8 invariants obtained. These determine the pair M up to a centro-affine transformation. Geometrical interpretations of the canonical frames and invariants are given. The properties of the central quadrics of the surface are discussed. A classification of pairs is given.

A. Goetz (Wrocław)

Oniščuk, N. M. 530

The frame system of a subvariety in the theory of a pair consisting of a congruence and a surface in a centro-affine space. (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. Geom. Sb. **161** (1962), 124–130.

Subvarieties $H=(L,S)$ of pairs M of the preceding paper [#529], where L is a ruled surface of the congruence K and S is the line of intersection of L and P, are investigated. Semi-canonical frames for M are found in order to generate canonical frames for H. Invariants of H are found and expressed by invariant forms of M. The main classes of subvarieties H are determined.

A. Goetz (Wrocław)

Smirnova, V. V. 531

Stratifiable triples of congruences. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 147–165.

Three hyperbolic congruences (k), (l), (m) in P_3 are considered. Such a triple is called stratifiable if each pair of them is stratifiable from both sides. Stratifiable triples are always triples of T-congruences considered by the author earlier [same Ž. **3** (1962), 605–617; MR **25** #3450]. The author finds the solutions of the system of exterior differential equations determining stratifiable triples. Further triples with stratifications $(k)\to(l)\to(m)\to(k)$, triples with stratifications $(k)\to(l)$, $(k)\to(m)$ and $(l)\to(m)$ and triples with stratifications $(k)\leftrightarrow(l)$ and $(k)\leftrightarrow(m)$ are found. The author explains their geometrical meaning.

A. Goetz (Wrocław)

Mihăilescu, Tiberiu 532

Théorie de la correspondance entre deux variétés non-holonomes linéaires de l'espace projectif ordinaire. (Russian summary)

Czechoslovak Math. J. **13** (88) (1963), 435–472.

The author considers non-holonomic manifolds V_3^2 in a projective 3-dimensional space and a point-to-point correspondence between two such manifolds, proving that all concepts introduced by E. Čech [Acad. Tchèque Sci. Bull. Internat. Cl. Sci. Math. Natur. Médicine **23** (1923), 80] and E. Bompiani [Ann. Mat. Pura Appl. (4) **1** (1923), 259–284] for correspondences between two surfaces can be appropriately extended and usefully applied to the non-holonomic case.

E. Bompiani (Rome)

Porcu, Livio 533

Caratterizzazioni di coppie di curve piane omologiche e applicazioni agli ovali.

Ist. Lombardo Accad. Sci. Lett. Rend. A **97** (1963), 369–393.

L'auteur considère dans le plan projectif réel, des courbes élémentaires de Juel, dotées en tout point d'un élément

differentiel E_k régulier ou inflexionnel de premier espèce ou cuspidal de premier espèce d'un ordre k assez élevé, que l'on appelle autodual. Deux telles courbes sont homothétiques si l'on peut établir entre elles une correspondance biunivoque continue par laquelle deux points homologues sont alignés avec un point fixe et dont l'invariant projectif de Study et Bompiani des E_2 relatifs à ces points homologues sont constants. Deux courbes sont également homologiques si leurs tangentes en deux points correspondants se coupent sur une droite fixe, et si les invariants de Study et Bompiani sont encore constants. Il en résulte que deux courbes sont homothétiques s'il existe une correspondance biunivoque continue où les tangentes et normales affines aux points correspondants sont parallèles; si pour deux courbes à tangentes parallèles le rapport des courbures métriques aux points correspondants est constant, elles sont homothétiques. Ces propriétés appliquées aux ovales et aux coniques, donnent des caractérisations remarquables, en particulier, une ovale pour laquelle toute normale affine est une normale affine double, est une ellipse. *B. d'Orgeval* (Dijon)

Upadhyay, M. D. 534

Orthogonal correspondence in a φ-congruence.

Tensor (*N.S.*) **15** (1964), 29–36.

L'auteur a défini [J. Math. Soc. Japan **5** (1953), 95–104; MR **15**, 466] une φ-congruence associée à une congruence initiale λ, de façon que, en particulier, deux rayons correspondants se coupent sous l'angle constant φ. Il étudie ici la condition pour que les éléments linéaires des représentations sphériques des deux congruences se correspondent orthogonalement et il examine en particulier le cas où la congruence φ est une congruence de Ribaucour. *M. Decuyper* (Lille)

Amur, K. S. 535

On some curves on the director surface of a rectilinear congruence.

Tensor (*N.S.*) **15** (1964), 23–28.

Une courbe C sur la surface directrice S d'une congruence K est dite (a) courbe d'union, (b) hyperasymptotique, (c) hypernormale suivant que le rayon de K en un point quelconque de C appartient (a) au plan osculateur, (b) au plan rectifiant, (c) au plan normal de C, au point correspondant. Soit $\bar{S}$ la surface réglée formée par les rayons de K le long de C; on voit directement que dans les cas (a) et (b) la courbe C est respectivement asymptotique et géodésique de $\bar{S}$; on prouve aussi que dans le cas (c) la torsion géodésique de C relative à S est égale à la torsion du rayon correspondant de K. Enfin les courbures de C par rapport à $\bar{S}$ sont exprimées au moyen des courbures de C par rapport à S. *M. Decuyper* (Lille)

Amur, K. S. 536

On the curvatures of a vector field on a surface in relation to a congruence of curves.

Tensor (*N.S.*) **14** (1963), 6–17.

The reviewer's notion of normal curvature of a vector field [Amer. J. Math. **74** (1952), 955–966; MR **14**, 406] was generalized by R. N. Kaul for a congruence [Tensor (N.S.) **7** (1957), 110–116; MR **19**, 1074; E. K. Hsiao, ibid. **10** (1960), 73–80; MR **22** #4020]. In the same way the author of this paper extends the reviewer's notion of indicatrix torsion of a vector field [Proc. Amer. Math. Soc. **7** (1956), 449–457; MR **18**, 64; ibid. **8** (1957), 294–298; MR **18**, 930] for a congruence orthogonal to a vector field.

Some relations of associate curvature and normal curvature of a vector field with a congruence are derived and are given geometrical interpretations. *T. K. Pan* (Norman, Okla.)

Upadhyay, M. D. 537

Geodesic torsion of a congruence.

Tensor (*N.S.*) **15** (1964), 45–53.

Let $S: x^i = x^i(u^1, u^2)$ be a surface in E^3, $u^\alpha = u^\alpha(s)$ a curve C on S, and $\lambda^i = p^\alpha x^i_{,\alpha} + qX^i$ a unit vector along a line of a congruence λ with S as the surface of reference. This paper is a study of the following invariant

$$\tau_g = \frac{e^{\alpha\gamma} g_{\alpha\beta}(q_{,\gamma} + p^\delta d_{\gamma\delta}) du^\beta}{(g_{\alpha\beta} du^\alpha du^\beta g_{\gamma\delta} p^\gamma p^\delta)^{1/2}},$$

where $g_{\alpha\beta}$ and $d_{\gamma\delta}$ denote the fundamental tensors of S. Topics discussed consist of (1) the C along which $\tau_g = 0$, (2) the C along which τ_g assumes an extremum, (3) the extreme value of τ_g, and (4) special cases.

This τ_g is called the geodesic torsion of λ by a certain formal analogy as claimed by the author. It is essentially different from the geodesic torsion of C in the classical sense. Geometrically, the union torsion of C as defined by C. E. Springer [Amer. Math. Monthly **54** (1947), 259–262; MR **9**, 61] may be considered as the geodesic torsion of λ with respect to C. *T. K. Pan* (Norman, Okla.)

Upadhyay, M. D.; Nigam, I. S. 538

On congruence of Guichard.

Tensor (*N.S.*) **15** (1964), 40–44.

A congruence whose focal surfaces are met by the developables in the lines of curvature is called, with Bianchi, a congruence of Guichard. The lines of a congruence which pass through a curve on its surface of reference form a ruled surface of the congruence. Taking a focal surface as the surface of reference, the author derives the following equation of a ruled surface whose image on the spherical representation of a congruence of Guichard is a minimal line:

$$\frac{1}{\rho^2} + p^\gamma_{.\alpha} p^\delta_{.\beta} g_{\gamma\delta} du^\alpha du^\beta = 0,$$

where ρ denotes the distance between the focal points on a line of the congruence along the unit vector $p^\alpha x^i_{,\alpha}$ and $g_{\gamma\delta}$ the first fundamental tensor of the surface of reference $x^i = x^i(u^1, u^2)$.

Geometrical interpretations of the above equation constitute the main results of the paper. Some properties of a developable surface of a congruence of Guichard are included. *T. K. Pan* (Norman, Okla.)

Karapetjan, S. E. 539

Projective-differential geometry of families of hyperplanes. II. (Russian. Armenian summary)

Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk **16** (1963), *no.* 5, 3–22.

This is the second part of a paper whose first part was

published earlier [same Izv. **16** (1963), no. 3, 3–22; MR **27** #666] The paper deals with a-parametric families of d-dimensional planes in a projective n-space. The geometry of a first-order neighbourhood is developed, and some special classes of congruences or pseudocongruences of class r are introduced and investigated. Some properties of s-focal families are found. *A. Goetz* (Wrocław)

Karapetjan, S. E. 540

Projective-differential geometry of families of hyperplanes. III. (Russian. Armenian summary)

Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk **17** (1964), *no.* 1, 3–21.

This is a continuation of the preceding paper [#539]. An invariant trilinear form Φ is introduced. The results are applied to the theory of higher-dimensional surfaces in which new results are obtained or new interpretations of known results are given. Trilinear forms Φ_1 and Φ_2 belonging to second- and third-order neighbourhoods are introduced. Certain isomorphisms of geometries are given.

A. Goetz (Wrocław)

Ivlev, E. T. 541

The frame system of a subvariety in the theory of parabolic pairs of complexes in P_3. (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. Geom. Sb. **161** (1962), 54–64.

The author constructs the semi-canonical frame of a parabolic pair of complexes, gives its geometrical characteristics and establishes its connection with the canonical frame which was introduced in one of his previous papers [same Trudy **161** (1962), 42–53; MR **28** #5393]. Then he considers this frame as a canonical frame of a nonholonomic pair of congruences, which belongs to the given parabolic pair of complexes, and explains the geometrical sense of the corresponding invariants.

W. Wrona (Warsaw)

Rollero, Aldo 542

Contatti di superficie lungo rami curvilinei multipli.

Ist. Lombardo Accad. Sci. Lett. Rend. A **97** (1963), 115–128.

Un rameau curviligne multiple plan d'origine O est constitué d'un nombre fini de rameaux curvilignes irréductibles d'origine O, chacun compté une seule fois, se représentant sous la forme:

$$A(x, y) = y^k+y^{k-1}A_1(x)+\cdots+yA_{k-1}(x)+A_k(x) = 0,$$

k entier ≥ 2, A_i série formelle de degré $\geq i$ en x; A peut être le produit de plusieurs séries de ce type. L'entier k est l'ordre du rameau. La généralisation à l'espace conduit à la représentation du rameau curviligne multiple régulier sous la forme:

$$A(x, y) = y^k+y^{k-1}A_1(x)+\cdots+A_k(x) = 0,$$

$$z+f(x, y) = 0,$$

f série formelle de degré au moins 2 en x et y. L'auteur établit alors la représentation des surfaces contenant un tel rameau, et ayant en chacun de ses points un contact assigné d'ordre q. *B. d'Orgeval* (Dijon)

Hasin, G. B. 543

Stratifiable congruences of axes of focal congruences. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, *no.* 2, 23–29.

The present paper deals with the classification of nets, whose congruences of the axes are stratified together with the congruences of the axes of the neighbouring Laplace transforms, and mainly with focal properties of the congruence of nets in the general case $B_1B_2\neq 0$ [cf. the author, Dokl. Akad. Nauk SSSR **145** (1962), 1235–1238; ibid. **150** (1963), 54–57; MR **26** #6886]. *A. Kawaguchi* (Sapporo)

Bagautdinova, H. G. 544

On constructing a polarized normalization of a conformal space. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 2 (39), 3–6.

In a previous paper [same Izv. **1962**, no. 2 (27), 13–22; MR **26** #2975] the author considered the polarized normalization of conformal space. The connection G_{ij}^k which is induced by such a normalization is obtained from the Weyl connection $\mathring{\Gamma}_{ij}^k$ defined by the fundamental tensor g_{ij} and an additional vector l_k by a projective transformation: $G_{ij}^k=\mathring{\Gamma}_{ij}^k-m_i\delta_j^k-m_j\delta_i^k$. In the present paper the construction of such a normalization is treated, i.e., it is proved that there exists a solution of the fundamental differential equations of the normalization.

M. Kawaguchi (Sapporo)

Mihăileanu, N. 545

★Geometrie diferențială neeuclidiană [Non-Euclidean differential geometry].

Editura Academiei Republicii Populare Române, Bucharest, 1964. 305 *pp.* *Lei* 13.00.

The book deals with non-euclidean differential geometry, about to the extent of the treatment of euclidean differential geometry in the first volume of Bianchi's *Lezioni di geometria differenziale*, Vol. I [terza ed. interamente rifatta, Nicola Zanichelli, Bologna, 1927] to which it is also near in the spirit of the treatment (very much genuine geometry hidden behind a strict avoidance of vector or tensor notation). The book starts with a chapter on analytic non-euclidean geometry, which is a rather complete treatment of non-euclidean geometry à la Cayley-Klein, without too much motivation. The second chapter deals with curves; this is preceded by a study of spherical curves and a discussion of the use of models in the theory. Next are treated surfaces (with a nice collection of examples), line congruence, and an annex dealing with some problems important in the foundations of the subject, viz., Clifford parallelism, non-euclidean geometry as the inner differential geometry of spaces embedded in euclidean space, invariant operators (this section certainly is too rudimentary) and a comparison of euclidean and non-euclidean geometry. There is no index, but a discussion of non-euclidean work done in Romania, and a rather complete formulary of 32 pages. There are no exercises. Since it contains much more material than the one chapter in Coolidge [*The elements of non-Euclidean geometry*, Clarendon, Oxford, 1909], the book seems to be the only available ready reference to many detailed results of strictly local non-euclidean differential geometry in three-space.

H. W. Guggenheimer (Minneapolis, Minn.)

Bompiani, Enrico 546
Nuovi enti geometrici: Pseudoelementi differenziali. (English summary)
Rend. Sem. Mat. Fis. Milano **33** (1963), 236–255.

Author's summary: "The notion of pseudo (differential) element is given as equivalence class (with respect to an arbitrarily given equivalence relation) in a set of differential elements (of given order and dimensionality) with the same center; examples and applications thereof."

B. Cenkl (Prague)

Lehmann, Daniel 547
Extensions à courbure nulle d'une connexion.
C. R. Acad. Sci. Paris **258** (1964), 4903–4906.

The author makes the following two definitions. First, let G/H be a reductive homogeneous space with a fixed decomposition of the Lie algebra $\mathfrak{g}=\mathfrak{m}+\mathfrak{h}$, $\mathrm{ad}(H)\mathfrak{m}\subset\mathfrak{m}$. For a connection ω in a principal H-bundle P over V, a G-extension with zero curvature of (P,ω) relative to $\mathfrak{m}$ is, by definition, a connection $\tilde\omega$ with zero curvature on the principal G-bundle $\tilde P$ over V obtained by enlargement of the structure group, such that ω equals the $\mathfrak{h}$-component of the restriction of $\tilde\omega$ to P. Second, let E be a real vector bundle over V with a connection D (covariant differentiation operator for cross-sections). By a vectorial m-extension with zero curvature of (E, D) is meant the following set $(\tilde E, N, \tilde D)$: $\tilde E$ is a vector bundle with standard fiber R^m over V which contains E as a sub-bundle and another sub-bundle N such that $\tilde E$ is the Whitney sum of E and N, and $\tilde D$ is a connection with zero curvature in $\tilde E$ such that $D_X\sigma=p_E(\tilde D_X\sigma)$ for any tangent vector X to V and for any section σ of E, p_E being the projection of $\tilde E$ onto E.

Theorem 1 is concerned with the existence of a universal G-extension with zero curvature relative to $\mathfrak{m}$ for arbitrary connections on principal H-bundles over the base G/H, which is assumed simply connected. Theorem 2 gives local existence of a G-extension with zero curvature relative to $\mathfrak{m}$ for an arbitrary connection on a principal H-bundle P over V, where H is a compact Lie group $\subset O(k)$, G a Lie group containing $O(m)$, $m=2(2n+1)k^3$ with $n\geq\dim V$, and $\mathfrak{m}$ a suitable complement of $\mathfrak{h}$ in $\mathfrak{g}$. This result depends on that of Narasimhan and Ramanan on the universal connections [Amer. J. Math. **83** (1961), 563–572; MR **24** #A3597]. Theorem 3 discusses global existence of an analytic G-extension with zero curvature. Finally, Theorem 4 gives a vectorial m-extension with zero curvature of (E, D), where the holonomy group of D is a compact Lie group $\subset O(k)$ and $m=2(n+1)(2n+1)k^3$ with $\dim V\leq n$.

K. Nomizu (Providence, R.I.)

Morimoto, Akihiko 548
On normal almost contact structures.
J. Math. Soc. Japan **15** (1963), 420–436.

Let M be a C^∞ manifold of dimension $2n+1$ and suppose given an endomorphism ϕ on the tangent space, a vector field ξ and a 1-form η such that (1) $\eta(\xi)=1$, (2) rank $\phi=2n$, (3) $\phi(\xi)=0$, (4) $\eta\cdot\phi(X)=0$ and $\phi^2X=-X+\eta(X)\xi$ for all X. Then we say M has an almost-contact structure or (ϕ,ξ,η)-structure according to S. Sasaki [Tôhoku Math. J. (2) **12** (1960), 459–476; MR **23** #A591]. This is equivalent to reducibility of the tangent bundle to $U(n)$ and is implied by a contact structure, i.e., a 1-form η such that $\eta\wedge(d\eta)^n\neq 0$ on M.

If M_1 and M_2 both have (ϕ,ξ,η)-structure, then $M_1\times M_2$ has an associated almost complex structure J where $J(X_1,X_2)=(\phi_1(X_1)-\eta_2(X_2)\xi_1, \phi_2(X_2)+\eta_1(X_1)\xi_2)$ as noted by the author. If $M_2=\mathbf{R}$ and $(\phi,\xi,\eta)=(0, d/dt, dt)$ then, according to Sasaki and Hatekayama [J. Math. Soc. Japan **14** (1962), 249–271; MR **25** #4458], the (ϕ,ξ,η)-structure is called normal. The author proves that an almost-contact structure on M is normal if and only if the induced almost-complex structure on $M\times M$ is integrable and shows that this is equivalent to the vanishing of a certain tensor on M. In this paper and the following [#549, #550] the structures are assumed normal.

Next he studies the automorphisms of the structure and finds a homomorphism H of the cartesian product of the automorphism groups $A(M_1)\times A(M_2)$ of M_1 and M_2 into the automorphism group $A(M_1\times M_2)$ of the almost-complex structure of M_1 and M_2. Using this, he shows that if M has normal almost-contact structure, then its automorphism group is a Lie group in the compact-open topology. By way of adducing examples he proves a theorem which has as a consequence the fact that all the homogeneous contact manifolds studied by H. C. Wang and the reviewer [Ann. of Math. (2) **68** (1958), 721–734; MR **22** #3015] are normal contact manifolds. Finally he determines criteria for group manifolds to have left-invariant, normal, almost-contact structure, showing that if G is reductive (in particular, if it is compact) and odd-dimensional, then it will have such a structure.

W. M. Boothby (St. Louis, Mo.)

Morimoto, Akihiko 549
On normal almost contact structures with a regularity.
Tôhoku Math. J. (2) **16** (1964), 90–104.

This is a continuation of the paper reviewed above [#548]. In that paper, it was shown that a principal circle bundle over a complex manifold, if it has a connection satisfying certain conditions, admits a normal almost-contact structure. The first theorem here is a converse to this, i.e., given a circle bundle M over a manifold M_0 with M having a normal (ϕ,ξ,η)-structure, η defining a connection on M, and ξ tangent to the fibre, then M_0 has a unique complex structure J and 2-form Ω such that $d\eta=\pi^*\Omega$ and $\Omega(JX, JY)=\Omega(X, Y)$. Next it is shown that if (ϕ,ξ,η) is a normal almost-contact structure on M such that the integral curves of the vector field ξ are closed and ξ is regular, then the period function for the induced one-parameter group is a constant, and M is a circle bundle over a complex manifold M_0. Regularity means each point has a neighborhood whose intersection with a given integral curve consists of a single arc or is empty. It is also shown that if the period function is finite for one point, it is finite at each point of M. Finally the author considers families of contact (circle) bundles M over a given complex manifold M_0. He gives a criterion for equivalence (under diffeomorphism) based on equivalence of M_0 (and the 2-form Ω mentioned above).

W. M. Boothby (St. Louis, Mo.)

Tanno, Shûkichi 550
On fiberings of some non-compact contact manifolds.
Tôhoku Math. J. (2) **15** (1963), 289–297.

The author studies the structure of K-contact manifolds M which are defined as follows: M has dimension $2n+1$

and carries a 1-form η such that $\eta \wedge (d\eta)^n$ is never zero. Associated with such a form there is always an endomorphism ϕ of the tangent space and a vector ξ (see #548 above); but in addition, it is assumed that there is a metric g (giving thus a (ϕ, ξ, η, g)-structure) such that ξ is a Killing vector, i.e., an isometry of the structure.

Let L_ϕ be the vector space of infinitesimal transformations leaving ϕ invariant and A the subspace of automorphisms of the structure. Suppose it is a proper subspace (this implies M is not compact); then (Theorem 1) $L_\phi = A \oplus L$, where L is 1-dimensional, spanned by a vector Z such that $\mathfrak{L}(Z)\eta = -\eta$. The existence of such a Z being assumed, it is then shown that if M is simply-connected and ξ generates a global group of isometries, then it is a fibre bundle (product bundle) over an almost Kählerian manifold W with fibre the real line. The form η is shown to be a connection form in the bundle, and a necessary and sufficient condition for M to be Kählerian is given. *W. M. Boothby* (St. Louis, Mo.)

Legrand, Gilles 551

T-structures homogènes.

C. R. Acad. Sci. Paris **258** (1964), 4648–4650.

The theory of T-structures which was previously developed by the author [same C. R. **250** (1960), 3266–3268; MR **22** #8523] is discussed in the case of a homogeneous Lie space G/H. He performs the determination of a structure through a vector sub-space of complexified Lie algebra of G with the aid of his former papers [ibid. **250** (1960), 3441–3442; MR **23** #A2161; ibid. **256** (1963), 2087–2088; MR **27** #680]. One of the main results is the characterization of integrable homogeneous T-structures, that is, to obtain integrability conditions for the above structures. He also refers to reductive homogeneous spaces. *S. Ide* (Sapporo)

Krasnodębski, R. 552

Imbedding of a space with an affine connection in the affine space.

Ann. Polon. Math. **14** (1963/64), 303–309.

The author proves the theorem: "If an analytic space L^n with an affine connection is given, then L^n may, in general, be locally imbedded in the affine space A^N of dimension $N = \frac{1}{2}\{n^2+2n-\frac{1}{2}[1-(-1)^n]\}$ in such a way that the connection induced on the imbedded space in the sense of Galvani [J. Math. Pures Appl. (9) **25** (1946), 209–239; MR **9**, 380] coincides with the given connection." *B. Cenkl* (Prague)

Tallini, Giuseppe 553

Sulle connessioni proiettivamente equivalenti di una V_n compatta.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) **33** (1962), 244–252.

The author gives certain conditions under which two projectively equivalent connections Γ and $\bar\Gamma$ on a compact manifold V_n coincide. The following theorems are proved. (1) Let the first Betti number B_1 of V_n be equal to zero. If the connections Γ and $\bar\Gamma$ are projectively equivalent and if their Ricci tensors are equal, then the connections are identical. (2) Let B_1 be equal to zero. Let the projectively equivalent connections Γ and $\bar\Gamma$ have equal first contracted curvature tensors (R^i_{ijk}) and let the quadratic form $(R_{jk}-\bar R_{jk})\xi^j\xi^k$, R_{jk} and $\bar R_{jk}$ being the Ricci tensors of Γ and $\bar\Gamma$, respectively, be either positive definite or negative definite. Then the connections Γ and $\bar\Gamma$ coincide.

An analogous theorem for pseudometric connections is also proved. *B. Cenkl* (Prague)

Vodnev, V. T.; Fedenko, A. S. 554

On some symmetric, partially projective spaces. (Russian)

Dokl. Akad. Nauk BSSR **8** (1964), 213–216.

The authors study the groups of motions of the space given by

$$ds^2 = 2\sum_{p=1}^{m} dx^p\,dx^{\bar p} + (\varepsilon_q x^q\,dx^2)^2 + \left[\sum_{i=1}^{m}(x^{2i}\,dx^{2i-1} - x^{2i-1}\,dx^{2i})\right]^2,$$

$\bar p = p+n$; $p, q = 1, \cdots, n$; $\varepsilon_q = \pm 1$. *B. Cenkl* (Prague)

Nirmala, K. 555

Properties of the intrinsic derivatives of the first and higher orders of the unit vector tangential to a congruence of curves in V_m, relative to a curve in V_n.

Tensor (*N.S.*) **15** (1964), 120–127.

Let C be a curve in a subspace V_n of a Riemannian space V_m and (λ) a congruence of curves in V_m. With the same method used in deriving the Serret-Frenet formulae for a curve in V_n, the author supplies a set of analogous formulae for (λ) relative to C. From these formulae a few properties of (λ) relative to C are obtained. Example: A necessary and sufficient condition that (λ) should have constant absolute curvature with respect to C is that the second intrinsic derivative of λ with respect to C should be orthogonal to the principal normal vector to (λ).

T. K. Pan (Norman, Okla.)

Miron, R. 556

Les configurations de Myller $\mathfrak{M}(C, \xi_1{}^i, T^m)$ dans les espaces de Riemann V_n. II.

Tensor (*N.S.*) **15** (1964), 74–86.

Author's summary: Dans une note précédente [Tensor (N.S.) **12** (1962), 227–238; MR **26** #5483] nous nous sommes occupés des configurations de Myller $\mathfrak{M}(C, \xi_1{}^i, T^{n-1})$ des espaces riemanniens V_n. Nous présentons maintenant le cas général des configurations de Myller $\mathfrak{M}(C, \xi_1{}^i, T^m)$ $(1<m<n)$. La géométrie de ces configurations est un peu plus compliquée, mais elle aussi peut être construite par la méthode exposée [ibid.]. En appliquant cette méthode on obtient, pour $\mathfrak{M}(C, \xi_1{}^i, T^m)$, le repère invariant, le système complet d'invariants, les équations fondamentales et le théorème fondamental. On présente aussi des cas particuliers remarquables."

T. K. Pan (Norman, Okla.)

Pinl, M. 557

Über die Verwendung natürlicher Kurvenparameter als Parameter eingebetteter Riemannscher Mannigfaltigkeiten.

J. Reine Angew. Math. **214/215** (1964), 399–404.

This paper examines relations between isotropic curves and scale-transformations of parametric curves of n-dimensional manifolds which are locally embedded in N-dimensional real or complex euclidean space. A typical result is as follows. The second fundamental form of a minimal surface of R_3 with two non-linear series of isotropic curves is a euclidean differential form.
T. J. Willmore (Liverpool)

Chaki, M. C.; Gupta, Bandana 558
On conformally symmetric spaces.
Indian J. Math. **5** (1963), 113–122.

A χ-space is a Riemannian space which is conformally symmetric, i.e., whose conformal curvature tensor has zero covariant derivative. It is proved that a χ-space is symmetric in the sense of Cartan if it satisfies any of the following conditions: (a) it has a decomposable metric; (b) it is an Einstein space; (c) it is projective-symmetric, i.e., its Weyl projective curvature tensor has zero covariant derivative. Conditions are also found for a χ-space to be recurrent or Ricci-recurrent. *A. G. Walker* (Liverpool)

Rozenfel'd, B. A.; Karpova, L. M. 559
Symmetric semi-Riemannian spaces. **(Russian)**
Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 1 (38), 100–116.

In the present paper invariant metrics are defined for symmetric spaces and in particular for non-semisimple Lie groups. This metric is a generalization of the Riemannian metric, and it is the so-called semi-Riemannian metric. The author confines himself to the class of non-semisimple Lie groups such that the invariant semi-Riemannian metric is unique to within scale. This metric is studied in detail for groups of motions of euclidean, quasi-elliptic, Flagov and arbitrary semi-elliptic spaces, and also of manifolds of planes of various dimensions in these spaces. *A. Kawaguchi* (Sapporo)

Solodovnikov, A. S. 560
Spaces with common geodesics. **(Russian)**
Trudy Sem. Vektor. Tenzor. Anal. **11** (1961), 43–102.

The author determines all non-trivial geodesic classes of n-dimensional positive definite Riemannian metrics. A geodesic class of ds^2 is the class of all $d\bar{s}^2$ which have the same geodesic lines as ds^2. It is trivial if all $d\bar{s}^2$ determine the same connection.

The metric is represented in Levi-Civita form, $ds^2 = \sum_{\alpha=1}^{p} \prod_\beta' |f_\beta - f_\alpha|\, ds_\alpha^2$, where the variables $x^1, \cdots, x^n$ are divided into p disjoint sets $\{x^{i_\alpha}\}$, $\prod_\beta'$ denotes the product over $\beta = 1, \cdots, p$, $\beta \neq \alpha$, ds_α^2 is a metric form of variables x^{i_α}, and f_α is a function of the single variable x^{i_α} if ds_α^2 is one-dimensional and constant otherwise. If the curvature of the p-dimensional "adjoint" metric $ds^{*2} = \sum_{\alpha=1}^{p} \prod_\beta' |f_\beta - f_\alpha|\, dy_\alpha^2$ is not constant, ds^2 is said to be of principal type; if this curvature is constant, ds^2 is called an exceptional Levi-Civita metric.

If ds^2 is of principal type, its geodesic class is non-trivial, does not contain exceptional metrics, and consists of all metrics of the form

$$d\bar{s}^2 = \frac{1}{(af_1+b)\cdots(af_p+b)} \sum_{\alpha=1}^{p} \frac{1}{af_\alpha+b} \prod_\beta{}' |f_\beta - f_\alpha|\, ds_\alpha^2$$

(a, b constant).

If ds^2 is exceptional, the situation is more complicated, since the expansion of ds^2 in Levi-Civita form is not unique. The major part of the paper deals with this case. The author reduces the problem to the problem of determining all non-trivial geodesic classes of $V(K)$ spaces introduced in his earlier paper [Uspehi Mat. Nauk **11** (1956), no. 4 (70), 45–116; MR **18**, 930] and also solves the problem completely in this case. *A. Goetz* (Wrocław)

Anosov, D. V. 561
Ergodic properties of geodesic flows on closed Riemannian manifolds of negative curvature. **(Russian)**
Dokl. Akad. Nauk SSSR **151** (1963), 1250–1252.

Continuing his announcements of results in the theory of U-systems [same Dokl. **145** (1962), 707–709; MR **26** #716] the author states more theorems about this class of structurally stable systems which includes the geodesic flows mentioned in the title. A dynamical system (continuous = flow, discrete = cascade) on a manifold is a U-system if each tangent space splits into a direct sum $X^k + Y^l + R^1$, where R^1 is tangent to the flow (omitted in the discrete case) and X^k, Y^l are subspaces of positive and negative exponential asymptoticity for the variational equations. The subspaces X^k, Y^l induce foliations $\mathfrak{S}^k$, $\mathfrak{S}^l$. Theorem 1: The foliations $\mathfrak{S}^k$, $\mathfrak{S}^l$, are absolutely continuous. (This means that local mappings from one sheet to another have nice Jacobians.) Theorem 2: Every U-system is ergodic. The spectrum of any U-cascade is continuous, but since the flow built under a constant function over a U-cascade is a U-flow, the latter might have point spectrum. Surprisingly, this is the only way a point spectrum can arise: if a U-flow has an eigenfunction, that function is continuous and gives rise to a section of the flow on which the induced transformation is a U-cascade.

A foliation is called metrically transitive if the measure of any sheet is zero or that of the whole space. Theorem 4: For any U-cascade, the foliations $\mathfrak{S}^k$, $\mathfrak{S}^l$ are metrically transitive. Theorem 5: If a U-flow has continuous spectrum, the foliations are metrically transitive.
L. W. Green (Minneapolis, Minn.)

Nomizu, Katsumi; Yano, Kentaro 562
Une démonstration simple d'un théorème sur le groupe d'holonomie affine d'un espace de Riemann.
C. R. Acad. Sci. Paris **258** (1964), 5334–5335.

The theorem in question is due to Sasaki and Goto [Trans. Amer. Math. Soc. **80** (1955), 148–158; MR **17**, 659]. The proof is based on the following new lemma proved by the Kobayashi-Nomizu formalism: The affine holonomy group of a complete, simply-connected, irreducible, Riemann manifold has no fixed point in any tangent space. *H. W. Guggenheimer* (Minneapolis, Minn.)

Raghunathan, M. S. 563
Déformations des connexions linéaires et des métriques riemanniennes.
C. R. Acad. Sci. Paris **257** (1963), 2605–2606.

The content of this note is now included in the author's paper [J. Math. Mech. **13** (1964), 97–123; MR **28** #4484].
K. Nomizu (Providence, R.I.)

Freudenthal, Hans 564

Im Umkreis der sogenannten Raumprobleme.

Essays on the foundations of mathematics, pp. 322–327. *Magnes Press, Hebrew Univ., Jerusalem,* 1961.

Es wird davon ausgegangen, daß unter Helmholtzschem Raumproblem in der Literatur verschiedene Fragenkreise verstanden werden. Der Verfasser will einen Überblick über die einschlägigen Untersuchungen geben, soweit sie der Gruppentheorie nahestehen. Die Arbeit beginnt mit historisch interessanten Bemerkungen über Riemann, Helmholtz und Lie und einem Hinweis auf die wenig bekannte Arbeit von R. Lipschitz [J. Reine Angew. Math. **72** (1870), 1–56]. Das Helmholtzsche Problem wird als die Frage nach der Kennzeichnung derjenigen Gruppen aufgefaßt, welche die Beweglichkeit aufweisen, die man von der Drehungsgruppe in n Dimensionen her kennt. Es werden die wichtigsten Beiträge zu diesem rein algebraisch aufzufassenden Problem aufgeführt. Die Arbeit schließt mit einer kurzen Übersicht über den Fragenkreis, den Verfasser als Weyl-Cartansche Probleme bezeichnet [H. Weyl, *Mathematische Analyse des Raumproblems*, J. Springer, Berlin, 1923].

{Bemerkungen des Referenten: Während der Verfasser selbst sagt, daß erst Lie den bei Helmholtz "versteckten" Gruppenbegriff aufgezeigt habe, fasst er selbst die Helmholtzschen Intentionen rein gruppentheoretisch auf. Demgegenüber ist zu sagen, daß Helmholtz und auch Weyl eine Kennzeichnung der Metriken, und zwar der Riemannschen Metrik definiter bzw. indefiniter Signatur, erreichen wollten, wie aus den Arbeiten Helmholtz, [*Wissenschaftliche Abhandlungen*, Bd. II, pp. 618–639, J. A. Barth, Leipzig, 1883] und Weyls Anhang zur Ausgabe von 1919 (Berlin) von Riemanns Hypothesen [B. Reimann, *Über die Hypothesen, welche der Geometrie zu Grunde liegen*, J. Springer, Berlin, 1919] hervorgeht.}

D. Laugwitz (Darmstadt)

Wolf, Joseph A. 565

Homogeneity and bounded isometries in manifolds of negative curvature.

Illinois J. Math. **8** (1964), 14–18.

Corollary 1: Let N be a connected Riemannian homogeneous manifold with every sectional curvature nonpositive. Then (a) N is isometric to the product of a flat torus with a simply connected manifold. (c) N admits a transitive solvable group of isometries. (d) If there is no Euclidean factor in the de Rham decomposition of the universal Riemannian covering manifold of N, then (d_1) N is simply connected, (d_2) every bounded isometry of N is trivial, (d_3) every transitive group of isometries of N is centerless, (d_4) a transitive Lie group of isometries of N cannot admit a nontrivial bounded automorphism, and (d_5) N is Riemannian symmetric if it admits a transitive semisimple Lie group of isometries.

This extends and completes results of S. Kobayashi [Bull. Amer. Math. Soc. **68** (1962), 338–339; MR **25** #4468] and R. Hermann [Nederl. Akad. Wetensch. Proc. Ser. A **66** (1963), 47–56; MR **27** #2940]. The author's main tool is a theorem on isometries in general (not necessarily homogeneous) Riemannian manifolds of nonpositive curvature. The proper reference for the fact that (4) implies (3) of Theorem 1 (bounded isometries are Clifford translations) is not the reviewer's thesis, but assertion (6.5) of H. Busemann [Acta Math. **80** (1948), 259–310; MR **10**, 623].

(d_4) yields a result of J. Tits [Topology **3** (1964), suppl. 1, 97–107; MR **28** #2170]: A bounded automorphism of a connected semisimple Lie group induces the identity on every noncompact normal simple analytic subgroup.

L. W. Green (Minneapolis, Minn.)

Wu, H. 566

Decomposition of Riemannian manifolds.

Bull. Amer. Math. Soc. **70** (1964), 610–617.

In a previous paper [Illinois J. Math. **8** (1964), 291–311; MR **28** #4488], the author established the decomposition theorem of de Rham type for Riemannian manifolds whose metrics are not positive definite. The present paper is concerned with a related problem. The holonomy group of a Riemannian manifold M is said to be nondegenerately reducible if it leaves a proper nondegenerate subspace of the tangent space invariant. M is said to admit an affine decomposition if there exist affinely connected manifolds M_1 and M_2 and a connection-preserving diffeomorphism ϕ of M onto the direct product $M_1 \times M_2$. The most essential result proved here is that, for a simply connected complete Riemannian manifold M, the holonomy group of M is nondegenerately reducible if and only if M admits an affine decomposition. This is proved by a rather fine analysis of various subspaces arising from an affine decomposition and the action of the holonomy group. Corollary: If the maximal trivial space of the holonomy group of M is zero, then every affine decomposition of M is in fact an isometric decomposition (that is, ϕ is an isometry).

K. Nomizu (Providence, R.I.)

Walker, A. G. 567

Note on metrisable Lie groups and algebras.

Calcutta Math. Soc. Golden Jubilee Commemoration Vol. (1958/59), *Part I, pp.* 185–192. *Calcutta Math. Soc., Calcutta,* 1963.

A Lie group is called metrisable [S.-T. Tsou and A. G. Walker, Proc. Roy. Soc. Edinburgh Sect. A **64** (1955/57), 290–304; MR **20** #5821] if it admits a Riemannian metric (not necessarily positive definite) which is invariant under left and right translations. This is equivalent to the existence of a second-order covariant tensor $a(\lambda, \mu)$ in its Lie algebra satisfying the condition $a(\lambda \times \nu, \mu) + a(\lambda, \mu \times \nu) = 0$, where $\times$ denotes the Lie product. Such a Lie algebra is also called metrisable. The maximum number $m(R)$ of linearly independent invariant metrics admitted by the algebra R is called the metric index of R. A metrisable algebra is reduced if its derived algebra contains its centre.

The author considers special types of extensions G^+ of a Lie group G by a vector space V isomorphic with a group with underlying space $G \times V$ and multiplication $(x, \alpha)(y, \beta) = (xy, \alpha + h(x)\beta)$, h being a representation of G in the group V^a of automorphisms of V, and the corresponding extension of Lie algebras. In particular, $G^\times$ is considered, where $V = R$ is the Lie algebra of G, and h the adjoint representation. Conditions for the extension G^+ to be metrisable are derived. If R is metrisable, then $R^\times$ is also metrisable, and if R is reduced, $R^\times$ is also reduced. For R metrisable with trivial centre R^+ ($R^\times$) is a metrisable algebra with trivial centre which is not semi-simple.

If R has a unique metric (which is the case if and only if the complex form R^c of R is simple), then $m(R^\times)=2$. The metric index is 2 also for direct products of two complex simple algebras (real algebras with simple complex forms and for simple algebras with non-simple complex forms. A conjecture is stated that these three cases are the only cases when $m(R)=2$.

In the last section a metric of a group G^T on the tangent bundle of G is considered. *A. Goetz* (Wrocław)

Grotemeyer, K. P. 568

Über das Normalenbündel differenzierbarer Mannigfaltigkeiten.

Ann. Acad. Sci. Fenn. Ser. A I No. 336/15 (1963), 12 *pp.*

In this paper the author presents some known results of Chern, Lashof and Kuiper on total curvature of immersed manifolds, together with some new results of the following type: Let F be a closed surface in a three-dimensional euclidean space R^3, ξ the unit normal vector at a general point of F, K the Gaussian curvature of F, $\chi(F)$ the Euler characteristic of F, and n a fixed unit vector in R^3. Then

$$\iint_F K\langle n\xi\rangle^2\,do = \frac{2\pi}{3}\chi(F),$$

where $\langle\ \rangle$ denotes the scalar product of two vectors in R^3, and do the element of area of F.

C.-C. Hsiung (Bethlehem, Pa.)

Fomenko, V. T. 569

Bending and unique determination of surfaces of positive curvature with boundary. (Russian)

Mat. Sb. (*N.S.*) **63 (105)** (1964), 409–425.

As is known, a surface of positive curvature with boundary admits in general continuous bending. If the boundary satisfies certain conditions, then the surface may appear to be uniquely determined. The author has been investigating the isometric correspondence of a surface under the following boundary conditions: (a) Normal curvature k_{n_R} and geodesic torsion τ_{g_R} in the direction R satisfy along the boundary of the transformed surface the condition

$$a(s)k_{n_R}+b(s)\tau_{g_R}+c(s) = 0,$$

$a(s)$, $b(s)$, $c(s)$ being given functions of the arc of the boundary curve, R being a given field of directions along the boundary; (b) The principal directions of normal curvature or principal normal curvatures do not change along the boundary of the transformed surface.

The author shows that in case (a) the surface may be uniquely determined or admit continuous bending depending on the choice of $a(s)$, $b(s)$, $c(s)$ and R. In case (b) all the surfaces may be divided in two classes: (I) The surfaces uniquely determined by condition (b); (II) The surfaces admitting continuous bending under condition (b). The author applies the method of investigation given by I. N. Vekua in his book [*Generalized analytic functions* (Russian), Fizmatgiz, Moscow, 1959; MR **21** #7288].

W. Wrona (Warsaw)

Rembs, Eduard 570

Verbiegbarkeit der T-Flächen.

Math. Nachr. **26** (1963), 229–232.

This paper concerns itself with the problem of finding "non-trivial" isometric immersions into R^3 from a given surface embedded in R^3. The class of embedded surfaces which interest the author are the T-surfaces. A T-surface is that part of a toral surface with positive Gaussian curvature. A toral surface is an analytic surface homeomorphic to the torus which has a piece of positive and a piece of negative curvature which are separated by two closed parabolic curves k_1 and k_2. k_1 and k_2 should be convex plane curves, which lie in parallel planes, and which are tangent to the part of positive curvature along k_1 and k_2, respectively. The author's principal result is that a given T-surface admits at least a countable number of non-trivial isometric immersions into R^3. This result is suggested by the following theorem of Pogorelov. Any closed convex surface with positive Gaussian curvature, punctured at two arbitrary points, admits at least a countable number of non-trivial regular isometric immersions. The author shows that the obvious proof of completing the T-surface to a closed convex surface by adding the two plane discs that are bounded by k_1 and k_2, respectively, and then applying Pogorelov's theorem, fails for technical reasons. He then sketches a proof of his theorem independently of Pogorelov's ideas, which he bases on his own work on deformations (see Math. Z. **71** (1959), 89–93 [MR **21** #6009]; ibid. **73** (1960) 330–332 [MR **22** #5947]), on classical results on deformations, and on "scissor and paste" techniques.

E. A. Feldman (Princeton, N.J.)

Kručkovič, G. I. 571

On the uniqueness of a minimal semireducible decomposition. (Russian)

Dokl. Akad. Nauk SSSR **146** (1962), 999–1002.

In Verbindung mit seinen früheren Arbeiten [dieselben Dokl. **115** (1957), 862–865; MR **20** #2751; der Verfasser und C.-H. Ku, ibid. **120** (1958), 1183–1186; MR **21** #5221] und mit der Arbeit von A. S. Solodovnikov [Uspehi Mat. Nauk **13** (1958), no. 6 (84), 173–179; MR **21** #2266] der Verfasser untersucht weiter die sogenannten halbreduziblen Zerlegungen der riemannschen Metrik. In dieser Arbeit beweist er zwei Sätze, die sich aus der sogenannten Fortsetzung der Zerlegung nach rechts ergeben und sich auf die halbreduzible Zerlegung der Metrik eines nicht konstant gekrümmten Raumes und auf die minimale halbreduzible Zerlegung beziehen.

T. P. Anđelić (Belgrade)

Kručkovič, G. I. 572

Homogeneous spaces in the general theory of relativity. (Russian)

Trudy Sem. Vektor. Tenzor. Anal. **12** (1963), 71–95.

The paper deals with four-dimensional homogeneous pseudo-riemannian spaces $V_4{}^1$, i.e., with a metric form $ds^2=(\omega^1)^2+(\omega^2)^2+(\omega^3)^2-(\omega^4)^2$. In the first section all subgroups of the group $O^1(4)$ of rotations of the 4-dimensional pseudo-euclidean space $E_4{}^1$ are found. In the following three sections homogeneous $V_4{}^1$-spaces are determined whose groups of motions are at least 5-parametric. All $V_4{}^1$ with 4-parametric motion groups were found in a paper by Petrov, Kaĭgorodov and Abdullin [Izv. Vysš. Učebn. Zaved. Matematika **1960**, no. 4 (17), 158–169; MR **25** #5779]. 10-parametric motion groups

occur only for $V_4{}^1$-spaces of constant curvature. There is no homogeneous $V_4{}^1$-space with 8- and 9-parametric groups of motions. The $V_4{}^1$-spaces with 7-, 6- and 5-parametric motion groups, as well as the groups themselves, are found and described in the paper. A detailed summary of the results requires too much space for a short review. *A. Goetz* (Wrocław)

Bishop, R. L.; Goldberg, S. I. 573
On curvature and Euler-Poincaré characteristic.
Proc. Nat. Acad. Sci. U.S.A. **49** (1963), 814–817.

The authors announce many interesting results which strengthen known facts on curvature and Euler characteristic, in particular, holomorphic curvature which does not change sign and holomorphically pinched Kähler manifolds. A detailed paper by the authors has already appeared [see #574 below].
H. W. Guggenheimer (Minneapolis, Minn.)

Bishop, R. L.; Goldberg, S. I. 574
Some implications of the generalized Gauss-Bonnet theorem.
Trans. Amer. Math. Soc. **112** (1964), 508–535.

Die Autoren liefern Beiträge zu dem bekannten Problem, aus Annahmen über das Vorzeichen der Werte der riemannschen Krümmung auf einer kompakten riemannschen Mannigfaltigkeit Informationen über das Vorzeichen der Euler-Poincaré-Charakteristik herzuleiten. Für den Fall, dass die riemannsche Mannigfaltigkeit sogar eine kählersche Mannigfaltigkeit ist, wird die entsprechende Frage für die holomorphe Krümmung gestellt (d.h., für die Krümmung bezgl. der holomorphen Ebenen). Ausgangspunkt für die letzteren Ergebnisse ist eine sehr präzise Abschätzung des Wertebereichs der riemannschen Krümmung einer kählerschen Mannigfaltigkeit, wenn man für den Wertebereich ihrer holomorphen Krümmung Voraussetzungen macht. Dieses ist eine wesentliche Verbesserung von ursprünglich nicht korrekt bewiesenen Resultaten von Berger [Ann. Scuola Norm. Sup. Pisa (3) **14** (1960), 151–159; MR **25** #3477; ibid. (3) **16** (1962), 297; MR **28** #558], die der Referent schon in seiner Arbeit [Proc. Amer. Math. Soc. **12** (1961), 350–356; MR **23** #3535] benutzt hatte. Und zwar lautet das Hauptergebnis: "Wenn die Werte der holomorphen Krümmung in einem Intervall $[\lambda, 1]$ liegen, mit $\lambda \geqq 0$, dann liegen die Werte der riemannschen Krümmung in dem Intervall $[(3\lambda-2)/4, 1]$". Ein ähnliches Resultat ergibt sich für den Fall, dass die Werte der holomorphen Krümmung in einem negativen Intervall liegen. Zu den Konsequenzen dieses Theorems gehört unter anderem, dass eine kählersche Mannigfaltigkeit der reellen Dimension 6, deren holomorphen Krümmung in einem Intervall der Form $[0.42\cdots, 1]$ liegt und die noch eine weitere Bedingung an die algebraische Struktur des Krümmungstensors erfüllt, eine positive Euler-Poincaré-Charakteristik besitzt. Weitere, noch wichtigere Anwendungen finden sich in der Arbeit desselben Autoren [Tôhoku Math. J. (2) **15** (1963), 359–364; MR **28** #2511]. Zusammenfassend lässt sich sagen, dass die Autoren in der vorliegenden Arbeit wahrscheinlich alle möglichen Beiträge zu den eingangs erwähnten Problemen geben, die man überhaupt erhalten kann, wenn man sich auf das Studium der Algebra des Krümmungstensors beschränkt. Jeder weitere Fortschritt dürfte von der Einbeziehung der globalen Struktur abhängen, die in der Arbeit hier gar keine Rolle spielt.
W. Klingenberg (Mainz)

GENERAL TOPOLOGY

See also 19, 183, 605, 611, 621.

Lepage, Th. 575
Sur quelques aspects de la topologie.
Acad. Roy. Belg. Bull. Cl. Sci. (5) **49** (1963), 1328–1336.

An expository lecture to non-specialists.

Hammer, Preston C. 576
Extended topology: Structure of isotonic functions.
J. Reine Angew. Math. **213** (1963/64), 174–186.

A function mapping a class of sets into a class of sets is called isotonic if it preserves inclusions. Let X be a set and let F be the class of all maps of 2^X into 2^X. For $f \in F$ the author defines $\Sigma f \in F$ by $(\Sigma f)(Y) = \bigcup \{uZ | Z \subseteq Y\}$ and $\Delta f \in F$ by $(\Delta f)(Y) = fY \cap (X - \bigcup \{fZ | Z \subset Y\})$. Basic properties of the operators Σ, Δ on F are developed.

$T \subset 2^X$ is a union base for $f \in F$ if for each $Y \subseteq X$, $fY = \{fZ | Z \subseteq Y,\ Z \in T\}$. $u \in F$ is a union generator for $f \in F$ if $fY = \bigcup \{uZ | Z \subseteq Y\}$. A procedure is developed for obtaining bases and generators for isotonic functions by using the operators Σ, Δ. *M. L. Curtis* (Houston, Tex.)

Boël, Jacques 577
La séparation de l'espace $\Pi(A)$.
Ann. Soc. Sci. Bruxelles Sér. I **77** (1963), 61–68.

The author gives necessary and sufficient conditions for a ring A in order that its structure space $\Pi(A)$ satisfies one of the axioms T_F or T_{YS} of C. E. Aull and W. J. Thron [Nederl. Akad. Wetensch. Proc. Ser. A **65** (1962), 26–37; MR **25** #1529]. E.g., $\Pi(A)$ is a T_F-space if and only if every primitive ideal of A is either a minimal or a maximal primitive ideal. The presentation suffers through the lack of examples and the avoidance of topological terminology and notation. {Both assertions of the corollary following Proposition 9 are erroneous.} *P. C. Baayen* (Amsterdam)

Grimeisen, Gerhard 578
★Das Produkt topologischer Räume in einer Theorie der Limesräume.
Habilitationsschrift zur Erlangung der Lehrberechtigung (Venia Legendi) für das Fach Mathematik an der Technischen Hochschule Stuttgart.
Paul Jllg Photo-Offsetdruck, Stuttgart, 1962. 40 *pp.*

In an earlier paper [Math. Ann. **147** (1962), 95–109; MR **25** #3500], the author discussed axiomatically the concept of limit spaces. The present work is a systematic study of the cartesian product of limit spaces, which is analogous to, but more general than, the usual cartesian product of topological spaces. *Ky Fan* (Evanston, Ill.)

Berri, Manuel P. 579
Categories of certain minimal topological spaces.
J. Austral. Math. Soc. **4** (1964), 78–82.

In the lattice of topologies on a given set X, a topology is said to be minimal with respect to some property P if there exists no other weaker (coarser, smaller) topology on X with property P.

Using characterizations of minimal Hausdorff topologies and minimal regular topologies, the author obtains category theorems. Some of the results are as follows: Every countably infinite minimal Fréchet space is of first category. Every uncountably infinite minimal Fréchet space is of second category. Every denumerable minimal Hausdorff space is of second category. Every minimal regular space is of second category.

E. Duda (Coral Gables, Fla.)

Levine, Norman 580

Simple extensions of topologies.

Amer. Math. Monthly **71** (1964), 22–25.

Let $(X, \mathcal{T})$ be a topological space ($\mathcal{T}$ is the collection of all open subsets); if $A \subset X$, let $\mathcal{T}(A)$ consist of all $G \cup (G_1 \cap A)$, where $G, G_1 \in \mathcal{T}$. Some simple results concerning the topology $\mathcal{T}(A)$ are proved. For instance, if A is closed in $(X, \mathcal{T})$ and $(X, \mathcal{T})$ is [completely] regular, then $(X, \mathcal{T}(A))$ is [completely] regular; if A is closed and $(X, \mathcal{T})$ is normal, then $(X, \mathcal{T}(A))$ is normal if and only if $X - A$ is normal as a subspace of $(X, \mathcal{T})$.

M. Katětov (Prague)

Steiner, Eugene F. 581

The relation between quasi-proximities and topological spaces.

Math. Ann. **155** (1964), 194–195.

The author studies quasi-proximity spaces, as introduced by W. J. Pervin [same Ann. **150** (1963), 325–326; MR **27** #1917]. Those quasi-proximities which are equivalent to topogenic structures in the sense of A. Császár [cf. *Fondements de la topologie générale*, Akadémiai Kiadó, Budapest, 1960; MR **22** #4043; *Grundlagen der allgemeinen topologie*, Akadémiai Kiadó, Budapest, 1963; MR **26** #6917; *Foundations of general topology*, Macmillan, New York, 1963; MR **28** #575] are characterized, and also those which are equivalent to perfect topogenic structures, and hence to topological spaces.

P. C. Baayen (Amsterdam)

Ceder, J. G. 582

On maximally resolvable spaces.

Fund. Math. **55** (1964), 87–93.

Let X be a topological space. The question considered here is, how many pairwise disjoint dense subsets does X admit? The reviewer [Duke Math. J. **10** (1943), 309–333; MR **5**, 46] showed that various classes of spaces have two such sets if devoid of isolated points. To describe the problem further, let $\Delta(X) = \min\{\text{card}\,(U): U \text{ is open and } \varnothing \neq U \subset X\}$. Let $\chi(X)$ be the least cardinal number of a base of open sets of X. W. Sierpiński [Fund. Math. **36** (1949), 68–71; MR **11**, 120] showed that a metric space X such that $\Delta(X) \geqq \mathfrak{m} \geqq \aleph_0$ contains $\mathfrak{m}$ pairwise disjoint sets each intersecting every open set in at least $\mathfrak{m}$ points. The present paper extends the above results. The space X is called maximally resolvable if X has an isolated point or contains $\Delta(X)$ pairwise disjoint sets each of which intersects every nonvoid open set in at least $\Delta(X)$ points. The following theorems are proved. If $\aleph_0 \leqq \chi(X) \leqq \Delta(X)$, then X is maximally resolvable; this generalizes Sierpiński's theorem. A locally compact Hausdorff space is maximally resolvable. A T_0-space such that every point admits a complete family of open neighborhoods totally ordered by set inclusion is maximally resolvable. Any product of spaces, each of which is of one of the types mentioned in the two preceding sentences, is maximally resolvable. Some additional, rather technical, results are also obtained.

Edwin Hewitt (Seattle, Wash.)

Comfort, W. W.; Gordon, Hugh 583

Disjoint open subsets of $\beta X \setminus X$.

Trans. Amer. Math. Soc. **111** (1964), 513–520.

Let X be a completely regular space. Conditions are examined under which $\beta X \setminus X$ contains $\mathfrak{n}$ pairwise disjoint non-empty open subsets. If $\mathfrak{n} = \mathfrak{m}^{\aleph_0}$, $\mathfrak{m}$ infinite, then the following condition is sufficient: There is a locally finite collection of power $\mathfrak{m}$ consisting of compact sets with non-empty interiors. It is shown that the existence of $\mathfrak{n}$ co-zero sets (i.e., sets of the form $f^{-1}(H)$, f continuous and real-valued, H open) with certain special properties is necessary and sufficient. If X is discrete of infinite power $\mathfrak{m}$, then $\beta X \setminus X$ contains $\mathfrak{n}$ disjoint open subsets if and only if $\mathfrak{n} \leqq \mathfrak{m}^{\aleph_0}$.

M. Katětov (Prague)

Lorch, Edgar R. 584

Compactification, Baire functions, and Daniell integration.

Acta Sci. Math. (Szeged) **24** (1963), 204–218.

From the author's introduction: "The purpose of this paper is to expose precise relations within and between the domains of compactification, Baire functions, and Daniell integration. The principal method exploited to this end is a new topology, called the ι-topology, which we have developed on an earlier occasion [Bull. Soc. Math. France **88** (1960), 469–497; MR **23** #A2502], which when introduced into any compact space is finer than the compact or β-topology. Our interest in compactification centers upon the more delicate notions of pseudo-countable, and realcompactifications. It seems fair to say that up to the present time these have not always been defined and approached in such a way as to make their theories completely transparent. In terms of the ι-topology, all the essential results are quite elementary.

"In order to carry through this study, it is necessary to construct the standard compactification. We abandon the traditional notion of compactifying a completely regular space with respect to the set of all bounded continuous functions in favor of the much more precise setting made available by contemporary topological algebra of compactifying an arbitrary set of points with respect to a Banach algebra of functions defined over it. The compactification is of course the structure space. This procedure gives all compactifications, not merely the Stone-Čech compactification, a very real advantage related to the fact that the notion of an algebra is finer than that of a topology (distinct algebras correspond to the same topology). The standard compactification is carried through using the notion of maximal positive cones of functions. This method is essentially equivalent to that of maximal convex ideals. It may be pointed out in passing that the method of filters of zero-sets has the

very real disadvantage that a most interesting class of singular functions, precisely those defining the pseudocompactification, have no zeros and therefore are automatically excluded from any filter procedure." {Reviewer's remark: See. however, the notion of e-filter discussed by the reviewer and Jerison [*Rings of continuous functions*, Van Nostrand, Princeton, N.J., 1960; MR **22** #6994].}

"It should be mentioned in this connection that the question of obtaining all compactifications of a given completely regular space has been studied in detail by Yu. Smirnov who based his results on the theory of proximities. We shall not elaborate on the rapport between these two formulations of the problem.

"The compactification of a set $\mathscr{E}$ with respect to a Banach algebra of functions **B** is obtained by adjoining to $\mathscr{E}$ all maximal positive cones. These are of two types, the strong which contain functions (singular!) with empty zero-sets and the residual or weak cones. The realcompactification is obtained by adjoining to the base space $\mathscr{E}$ the set of points consisting of the weak cones. If one adjoins the strong cones, one obtains a pseudocompactification (equivalent to countable compactification for zero-sets). Adjoining to $\mathscr{E}$ both types, one obtains the compact space $\mathscr{E}^\wedge$ and the algebra $\mathbf{B}^\wedge$ of all continuous functions over $\mathscr{E}^\wedge$. The β-topology over $\mathscr{E}^\wedge$ is the weak topology induced by the functions of $\mathbf{B}^\wedge$.

"Starting with the functions in $\mathbf{B}^\wedge$, one constructs the algebra **I** of all bounded Baire functions over $\mathscr{E}^\wedge$. The ι-topology is the weak topology induced over $\mathscr{E}^\wedge$ by the functions of **I**. It is central to all the problems we consider. For example, the realcompactification of $\mathscr{E}$ introduced by E. Hewitt is the ι-closure in $\mathscr{E}^\wedge$ of $\mathscr{E}$. Or again, $\mathscr{E}$ is pseudocompact if and only if it is ι-dense in $\mathscr{E}^\wedge$. These topological facts explain why there is a unique realcompactification but no unique pseudocompactification. The latter is characterized in nine distinct ways.

"While on this subject, one or two remarks may be made. The points corresponding to weak maximal cones are topologically very complicated to describe and for all purposes of denumerable analysis seem superfluous. Thus it may be that for a large class of problems pseudocompactness rather than compactness is desirable. For another thing, it would seem that the common distinction in the literature of two algebras over $\mathscr{E}$, that of the bounded and that of the unbounded functions, seems largely unnecessary. A space admits an unbounded continuous function if and only if it admits a bounded singular function without zeros.

"Turning to integration, the nexus of questions concerning Daniell integrals can be elucidated in the frame of this investigation. Necessary and sufficient conditions are given on the measure distribution over $\mathscr{E}^\wedge$ of a linear functional F in order that F be a Daniell integral or an anti-integral over $\mathscr{E}$. Finally, it is shown that the space $\mathscr{E}^\wedge$ is realcompact with respect to the algebra **I**."

L. Gillman (Rochester, N.Y.)

Osmatesku, P. K. 585

Generalization of a one-point bicompactification theorem of P. S. Aleksandrov. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 274–275.

The author gives a characterization of the one-point compactification of a T_1-space.

J. R. Isbell (New Orleans, La.)

Leader, Solomon 586

On a problem of Alfsen and Fenstad.

Math. Scand. **13** (1963), 44–46.

Two uniform structures on a given set are p-equivalent according to Alfsen and Fenstad [Math. Scand. **8** (1960), 97–104; MR **23** #A3543] if they define the same uniform neighborhoods of subsets; every p-equivalence class has a minimal member. The author gives a criterion, in terms of pseudometrics, for the existence of a maximal member of a p-equivalence class, and shows by example that the criterion is not always satisfied, thus answering negatively a question raised by Alfsen and Fenstad. As the author points out in a note added in proof, an example to the same effect has recently been published by Alfsen and Njåstad [Fund. Math. **52** (1963), 235–252; MR **27** #4207a].

D. Bushaw (Pullman, Wash.)

Arhangel'skiĭ, A. 587

On factor maps of metric spaces. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 247–250.

Let X and Y be completely regular spaces, and let f be a factor map from X onto Y, i.e., $V \subset Y$ is open if and only if $f^{-1}(V)$ is open. In the first part of this paper the author returns [cf. same Dokl. **153** (1963), 743–746; MR **28** #1587] to the problem of finding conditions under which the metrizability of Y follows from that of X. A typical theorem: If X is metrizable and strongly paracompact, and f is pseudo-open, compact, and monotone, then Y is metrizable (Theorem 5). (Pseudo-openness means that for every $y \in Y$ and every open U containing $f^{-1}(y)$, $f(U)$ is a neighborhood of y; f is compact if every $f^{-1}(y)$ is compact, monotone if every $f^{-1}(y)$ is connected.) The second part of the paper introduces "covering" maps: A continuous map $f: X \to Y$ is covering if for every compact $K \subset Y$ there exists a compact $F \subset X$ such that $f(F) \supset K$. The basic theorem here is that f is covering if and only if the induced homomorphism $f^*: C(Y) \to C(X)$ is a homeomorphism (into) when $C(X)$ and $C(Y)$ have the compact-open topology. The paper contains no proofs. *D. Bushaw* (Pullman, Wash.)

Sandberg, V. Ju. 588

Metrization of Lipschitz spaces. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 1018–1021.

Having introduced the concept of a Lipschitz space in a previous work [same Dokl. **145** (1962), 284–286; MR **26** #1843], the author gives here several necessary and sufficient conditions for such a uniform space to be metrizable. The conditions and relevant definitions are too involved to be given here.

J. G. Ceder (Santa Barbara, Calif.)

Lutts, John A., S. J. 589

Topological spaces which admit unisolvent systems.

Trans. Amer. Math. Soc. **111** (1964), 440–448.

Let X be a topological space and let $C(X)$ denote the space of continuous functions from X into R, the space of reals. Let $\{f_1, \cdots, f_n\}$ be a set of n elements of $C(X)$ with the property that if $\{x_1, \cdots, x_n\}$ is a set of n distinct points of X, then the matrix $\|f_i(x_j)\|$ has an inverse. The author calls such a set an n-R-U system. Mairhuber's theorem [Proc. Amer. Math. Soc. **7** (1956), 609–615;

MR 18, 125] states that if X is compact Hausdorff, then X admits an n-R-U system if and only if X is topologically imbeddable in a one-sphere. The author studies conditions on X such that the existence of an n-R-U system will still imply the topological imbeddability of X into a one-sphere. He shows that one such condition is that X be connected, locally compact with at least n distinct points, $n \geqq 2$. For other conditions we refer to the paper.

I. J. Schoenberg (Princeton, N.J.)

Naimpally, S. A. 590

A note on contraction mappings.

Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 275–279.

From the author's introduction: "Let (M, d) be a metric space with metric d. Let f be a self-mapping of M. Then f is called λ-locally contractive if to each $x \in M$, there corresponds an $\varepsilon(x) > 0$ such that for all $p, q \in M$, $p \neq q$, lying in the sphere $S(x, \varepsilon(x))$, (1) $d(f(p), f(q)) < \lambda d(p, q)$, $0 \leqq \lambda \leqq 1$. Also we suppose that there exists $x_0 \in M$ such that the sequence of iterates $\{f^n(x_0)\}$ contains a subsequence $\{f^{n_i}(x_0)\}$ which converges to (2) $\xi \in M$ as $i \to \infty$. . . . Edelstein [J. London Math. Soc. **37** (1962), 74–79; MR **24** #A2936] has shown that if $\lambda = 1$ and $\varepsilon(x)$ is independent of x, then (1) and (2) imply that ξ is a periodic point of f. Further, in the same paper he has shown that if ε does depend on x, then ξ need not be a periodic point of f. The space in Edelstein's counter-example is not connected. Rakotch has constructed a complete and connected metric space and a $\frac{1}{2}$-locally contractive self-mapping of it which has no fixed point. In this example, however, condition (2) does not hold. It is the purpose of this note to show that there exists a connected metric space (M, d) satisfying (2) and (1) with $\lambda < 1$ without ξ being a periodic point of f."

K. W. Kuan (Tallahassee, Fla.)

Day, George W.; Schubert, S. Roy 591

On perfect mappings.

Amer. Math. Monthly **71** (1964), 411–412.

Let $\{G_i\}$ be a collection of open coverings of X so that for each x with $U(x)$ an open set containing x and for some n every element of G_n containing p is a subset of $U(x)$. A space X admitting such a sequence $\{G_i\}$ is developable. The authors show that in a developable, point-wise paracompact space every subset of a separable set is separable and, provided X is also a Moore space, closures of conditionally compact sets are conditionally compact.

D. G. Bourgin (Urbana, Ill.)

Younglove, J. N. 592

Separability and compactness in pointwise paracompact spaces.

Amer. Math. Monthly **71** (1964), 412–413.

Let X be locally connected with no point an open set, and suppose Y has a neighborhood base of closed sets. Let P denote the collection $\{X, A\}$ when A is a subset of X which is dense in itself. The main result is that if $f: X \to Y$ and if $f(A)$ is dense in itself for every $(X, A) \in P$, then f is continuous. The constant mapping shows the converse is not valid generally.

D. G. Bourgin (Urbana, Ill.)

Kuratowski, K. 593

Mappings of topological spaces into lattices and into Brouwerian algebras.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 9–16.

The paper contains fourteen theorems which seem to represent the foundation of a generalized study of multi-valued continuous functions. For instance: Let $f_j: Y \to L$ be upper semi-continuous at y_0. If L is normal (this means the T_4 separation axiom applied to the lattice of closed subsets), then $f_0 \cap f_1$ is upper semi-continuous at y_0.

J. R. Isbell (New Orleans, La.)

Cook, H. 594

On the most general plane closed point set through which it is possible to pass a pseudo-arc.

Fund. Math. **55** (1964), 11–22.

E. E. Moise proved that any subcontinuum of the pseudo-arc is either a point or a pseudo-arc. Hence a compact subset M of the plane can be contained in a pseudo-arc in the plane only if each component is a point or a pseudo-arc. The author proves that the converse is also true by constructing for each such continuum a pseudo-arc containing it. The construction is too complicated to outline here.

M. L. Curtis (Houston, Tex.)

Grace, Edward E. 595

A totally nonaposyndetic, compact, Hausdorff space with no cut point.

Proc. Amer. Math. Soc. **15** (1964), 281–283.

A space X is aposyndetic at $x \in X$ if it is T_1 at x relative to compact connected neighborhoods, that is, if for each $y \neq x$ there is a compact connected neighborhood of x not containing y. A space is totally non-aposyndetic if it fails at each point to be aposyndetic. The example described in the title is given, answering negatively a question asked by A. D. Wallace.

M. L. Curtis (Houston, Tex.)

Fearnley, Lawrence 596

Characterizations of the continuous images of the pseudo-arc.

Trans. Amer. Math. Soc. **111** (1964), 380–399.

It is a classical result of Hahn and Mazurkiewicz that a space X is a continuous image of an arc if and only if it is compact, connected, locally connected and metrizable. Bing posed the problem of finding an analogous theorem for the pseudo-arc. The author shows that in one sense a precise analogy is not possible by exhibiting two compact metric continua H and K which are locally homeomorphic, but H is a continuous image of the pseudo-arc, whereas K is not. Thus no local property can decide which metric continua are images of the pseudo-arc.

A chainability condition called p-chainability is developed and proved to be implied by chainability. It is proved that a continuum is the continuous image of a pseudo-arc if and only if it is p-chainable. Hence any chainable continuum is such an image.

M. L. Curtis (Houston, Tex.)

Pym, John S. 597

On almost periodic compactifications.

Math. Scand. **12** (1963), 189–198.

Following Alfsen and Holm [Math. Scand. **10** (1962), 127–136; MR **26** #5091], the author produces the almost periodic compactification of a semigroup using uniform structures and the semigroup operation, with no use of almost periodic functions. *I. Glicksberg* (Seattle, Wash.)

Burgess, C. E. 598

Properties of certain types of wild surfaces in E^3.

Amer. J. Math. **86** (1964), 325–338.

Let $\bar{V}$ be a 3-manifold contained in a closed 3-manifold M^3, V its interior and M its boundary, and let W be the set of points at which M is wild. Theorem 10: If M' is a surface, contained in $\bar{V}$, that separates M^3 and contains W, then M' is wild at each point of W. Theorem 12: If there is a tame arc in M^3 that contains W, then there is a tame 2-sphere S in $M^3 - V$ such that $S \cap M = W$.

Theorem 10 generalizes a result due to S. Kinoshita [*Topology of 3-manifolds and related topics*, pp. 223–226, Prentice-Hall, Englewood Cliffs, N.J., 1962; MR **25** #3116]. *R. H. Fox* (Princeton, N.J.)

Posey, E. E. 599

Proteus forms of wild and tame arcs.

Duke Math. J. **31** (1964), 63–72.

Arcs A and B in R^3 are equivalent if there exists a homeomorphism h of R^3 onto itself such that $h(A) = B$. This paper is devoted to finding the "nicest" arcs in each equivalence class. One of the main results is that any arc A may be moved slightly by a homeomorphism of R^3 to an arc B which is polygonal wherever A was tame and the wild points of A are left fixed. An arc which contains no tame subarc is said to be completely wild and a characterization of such arcs is given as follows. The arc A is completely wild if and only if for any representation $\phi: I \to R^3$ of any arc equivalent to A, the function ϕ has local maxima on an everywhere dense subset of I.

M. L. Curtis (Houston, Tex.)

Doyle, P. H. 600

Certain manifolds with boundary that are products.

Michigan Math. J. **11** (1964), 177–181.

The main result is Theorem 1: Let M^n be an n-manifold with boundary such that Int $M^n = E^n$ and Bd $M^n = E^{n-1}$. Then, if $n \neq 3$, $M^n = E^{n-1} \times [0, 1)$. Shortly after this paper was submitted, Cantrell [Bull. Amer. Math. Soc. **69** (1963), 716–718; MR **27** #4237] obtained a stronger result. The assumption that $n \neq 3$ is essential, as there are known counter-examples. *K. W. Kwun* (Tallahassee, Fla.)

Rosen, Ronald H. 601

The five dimensional polyhedral Schoenflies theorem.

Bull. Amer. Math. Soc. **70** (1964), 511–516.

The main result of this paper is Theorem 1: In the 5-sphere with its usual piecewise linear structure, a subpolyhedron homeomorphic to S^4 bounds two closed topological 5-cells.

The author also proves the following. Theorem 2: Let Σ be an $(n-1)$-sphere in S^n with a singular set K (i.e., the set of points at which Σ fails to be locally flat in S^n) which is at most countable, and let $n > 3$. Then $S^n - \Sigma$ consists of a pair of open n-cells. Theorem 3: Let T be a triangulation of S^4 and Σ a subpolyhedron of T which is homeomorphic to S^3. Then $S^4 - \Sigma$ consists of a pair of open 4-cells. Theorem 4: Let Σ be a k-sphere in S^n with a singular set which is at most countable. If $n - k \geqq 3$, then $S^n - \Sigma$ is homeomorphic to $S^{n-k-1} \times R^{k+1}$.

H. R. Gluck (Cambridge, Mass.)

ALGEBRAIC TOPOLOGY

See also 42, 188, 519, 628, 629, 633, 634.

Harary, Frank; Hill, Anthony 602

On the number of crossings in a complete graph.

Proc. Edinburgh Math. Soc. (2) **13** (1962/63), 333–338.

This is a brief survey article concerning the so-called crossing number $c(G)$ of a graph, and the rectilinear crossing number $\bar{c}(G)$. $c(G)$ is defined to be the minimum number of intersections of edges when G is drawn in the plane, only two edges being allowed to intersect in any one point. $\bar{c}(G)$ is defined analogously, but here edges must be represented by straight-line segments. Clearly, $c(G) \leqq \bar{c}(G)$. The authors claim to have an example (the complete 8-graph K_8) which shows that the inequality may be proper, but it would appear that this is actually only a very strong conjecture. A further conjecture is that the upper bound for $c(K_n)$ (where K_n denotes the complete n-graph) given by R. Guy [Bull. Malayan Math. Soc. **7** (1960), 68–72] is exact. *G. Sabidussi* (Hamilton, Ont.)

Moon, J. W. 603

Tournaments with a given automorphism group.

Canad. J. Math. **16** (1964), 485–489.

Frucht [Compositio Math. **6** (1938), 239–250] proved that every finite group is isomorphic to the group of some graph. The author considers the similar problem for complete oriented graphs.

A complete oriented graph (round-robin tournament) consists of a finite number of vertices (players) $p, q, \cdots$ such that each pair of distinct vertices is joined by an oriented arc, indicated by $p \to q$ if the orientation is from p to q (player p beats player q). An automorphism is a permutation α of the vertices which preserves orientation, i.e., $\alpha(p) \to \alpha(q)$ if and only if $p \to q$; all the automorphisms constitute the group of the graph. The author proves that if G is a group of odd order, then there exist infinitely many strongly connected tournaments whose groups are isomorphic to G. Briefly, the author's construction is as follows. Corresponding to a minimal set of h generators of G (of order n), let T^* be the Cayley color graph [see Coxeter and Moser, *Generators and relations for discrete groups*, pp. 21–23, Springer, Berlin, 1957; MR **19**, 527]. The author completes T^* and then adds new vertices joined by colored directed edges according to a rule which depends on a preassigned listing of the elements. The colors on the edges are then wiped out and the resulting graph, which is oriented and complete on $n\binom{h+1}{2} + n$ vertices, has its group isomorphic to G. An easy extension of the construction produces infinitely many graphs with groups isomorphic to the same group G.

W. Moser (Minneapolis, Minn.)

Brown, R. 604

Cohomology with chains as coefficients.

Proc. London Math. Soc. (3) **14** (1964), 545–565.

The author discusses homology properties of $\mathrm{Hom}(X, A)$ where X, A are chain complexes of abelian groups and X is free (in fact, abelian groups could be replaced by Λ-modules if hom dim $\Lambda \leq 1$). He writes $H\,\mathrm{Hom}(X, A) = H^*(X; A)$ = "cohomology of X with coefficients in A". The associativity formula

$$\mathrm{Hom}(X \otimes Y, A) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, A))$$

easily leads to a Künneth formula

$$H^*(X \otimes Y; A) \cong H^*(X: H^*(Y: A))$$

which is natural in X but not in Y, A. Because positive chain complexes and semisimplicial abelian groups form equivalent categories, the theory also applies to the generalized cohomology $\pi(K, A)$, where K is a semisimplicial set and A a semisimplicial abelian group (by J. C. Moore, $\pi(K, A) \cong \prod_i H^i(K; \pi_i A)$; this is functorial in K but not in A). Natural transformations $\pi(-, A) \to \pi(-, B)$ are considered; they form a group $\mathrm{Op}(A, B)$ which (as always for representable functors) is isomorphic with $\pi(A, B)$. The map $\pi(A, B) \to \pi(A^Y, B^Y)$, called a "generalized suspension", is discussed (it reduces to the ordinary suspension $\mathrm{Op}(A, B) \to \mathrm{Op}(\Omega A, \Omega B)$ if the semisimplicial set Y equals $\Delta_1/\dot{\Delta}_1$). *A. Dold* (Heidelberg)

Jaworowski, J. W. 605

The relative simplicial approximation theorem and its application to an elementary proof of the Poincaré-Brouwer theorem.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 5–8.

This paper begins by stating the "relative simplicial approximation theorem": Let K be a finite simplicial complex, and let L be a subcomplex of K. Let $\varphi: L \to M$ be a simplicial map of L into a complex M, and let $f: |K| \to |M|$ be a continuous map such that $f|\,|L| = |\varphi|$. Then there exist a subdivision K_1 of K such that L is a subcomplex of K_1 and a simplicial map $\psi: K_1 \to M$ such that $\psi|\,L = \varphi$.

Its proof is promised in a subsequent paper. According to the author, E. C. Zeeman has also obtained the result independently [Proc. Cambridge Philos. Soc. **60** (1964), 39–43; MR **28** #1626].

This paper is mainly concerned with an application of the theorem giving an elementary proof that no even-dimensional sphere admits a non-zero vector field. {There are some (fortunately rather insignificant) misprints on page 6.} *K. W. Kwun* (Tallahassee, Fla.)

Białynicki-Birula, A. 606

On Vietoris mapping theorem and its inverse.

Fund. Math. **53** (1963/64), 135–145.

Author's summary: "Let X, Y, T be compact Hausdorff spaces and let $f: X \to Y$, $g_1: X \to T$, $g_2: Y \to T$ be continuous onto maps such that $g_2 f = g_1$. In this paper we prove (Theorem 1) that if f induces isomorphisms of ith Čech cohomology groups for $i = 0, 1, \cdots, n$ and a monomorphism of the $(n+1)$st cohomology groups of fibres $g_2^{-1}(x)$, $g_1^{-1}(x)$, for all $x \in T$, then f induces isomorphisms of ith cohomology groups of spaces Y and X. The result generalizes the well-known Vietoris-Begle theorem. On the other hand, we show (Theorem 2) that if there exists a totally disconnected subset $T_1 \subset T$ such that the fibres $g_1^{-1}(x)$, $g_2^{-1}(x)$ are $(n+1)$-acyclic for $x \in T - T_1$ and f induces isomorphisms of ith cohomology groups of Y and X for $i = 0, 1, \cdots, n$, then f induces isomorphisms of ith cohomology groups of fibres $g_2^{-1}(x)$, $g_1^{-1}(x)$ for all $x \in T$ and $i = 0, 1, \cdots, n$. In the last part we give some applications of the theorems connected with an Eilenberg-Kuratowski theorem [Fund. Math. **50** (1961/62), 515–517; MR **26** #1882]."

Sheaf-theoretic and spectral sequence techniques are used in the proofs of both Theorems 1 and 2. The complications in the proof of Theorem 2 seem to stem from the fact that T_1 is not assumed to be a closed subset of T. *F. Raymond* (Ann Arbor, Mich.)

Ramabhadran, N. 607

On the cohomologies of a space with coefficients in a sheaf.

J. Madras Univ. B **32** (1962), 169–174 (1963).

Soient S un faisceau sur un espace topologique X, Φ une famille de supports. On appelle Φ-recouvrement $U = \{U_i\}_{i \in I}$ un recouvrement de X ouvert, localement fini, tel que $U_i \neq U_j$ pour $i \neq j$ et tel qu'il existe $* \in I$ avec (adhérence $(\bigcup_{i \neq *} U_i)) \in \Phi$. On appelle $H^p_{\Phi,D}(X, S)$ le groupe de cohomologie de dimension p, construit à partir des cochaînes de Čech satisfaisant à la condition suivante: pour un Φ-recouvrement U, les 0-cochaînes du nerf de U s'annulent sur $*$ (définition de C. H. Dowker [*Lectures on sheaf theory*, Tata Inst. Fundamental Res., Bombay, 1956; MR **19**, 301]). Les cochaînes considérées sont des cochaînes de Čech à support dans Φ, d'où une application: $H^p_{\Phi,D}(X, S) \to \check{H}_\Phi{}^p(X, S)$ ($p^{\text{ième}}$ groupe de cohomologie de Čech à supports dans Φ) que l'auteur prouve être un isomorphisme naturel. *P. Dolbeault* (Poitiers)

Meyer, Jean-Pierre 608

The characterization of Moore-Postnikov invariants.

Bol. Soc. Mat. Mexicana (2) **7** (1962), 92–94.

The k-invariants of the Moore-Postnikov system of a fibering pull back to the trivial class in the total space and restrict to the k-invariants of the fibre. The author improves on previous published remarks on these conditions by pointing out that they characterize the k-invariant if and only if both the k-invariant and its restriction are annihilated by precisely the same set of homomorphisms of the coefficient group G into itself, e.g., $G = Q$ of Z_p if the restriction is nontrivial. The necessity is illustrated by an example with coefficients in the integers. *J. Stasheff* (Notre Dame, Ind.)

Hsiang, Wu-Chung 609

On Wu's formula of Steenrod squares on Stiefel-Whitney classes.

Bol. Soc. Mat. Mexicana (2) **8** (1963), 20–25.

The author rederives Wu's formulae about the action of Steenrod squares on Stiefel-Whitney classes. The proof is valid for "bundles" which are only fiberings in a homological sense, for it relies simply on the Thom isomorphism

q and the Adem relations. In principle, it is thus elementary and direct. However, it involves computational work with the binomial coefficients mod 2. The most conceptual approach to this subject remains that of "left-right algebra", for which see Brown and Peterson [Bull. Amer. Math. Soc. **69** (1963), 228–230; MR **26** #4369]. With the notation of this reference, one has only to define a right action of A on $H^*(B)$ by $ha = \varphi^{-1}\chi(a)\varphi h$, check formula 2.2 and apply Theorem 2.3; the required result follows at once. This demonstrates the overlapping of the two papers.
J. F. Adams (Manchester)

Thomas, Emery 610
Exceptional Lie groups and Steenrod squares.
Michigan Math. J. **11** (1964), 151–156.
The author fills in the last hole in our knowledge of the action of the Steenrod squares in exceptional Lie groups: In the mod 2 cohomology algebras of E_6, E_7, E_8 there exists a generator of dimension 15 such that $S_q{}^2$ takes it to the 17-dimensional generator. The proof involves detailed knowledge of the cohomology of the projective plane and explicit computations with Steenrod squares.
J. Stasheff (Notre Dame, Ind.)

Bing, R. H. 611
Inequivalent families of periodic homeomorphisms of E^3.
Ann. of Math. (2) **80** (1964), 78–93.
This paper offers a full array of wild, periodic homeomorphisms of E^3 and S^3. The results may be categorized according to the dimension of the fixed-point set. We do this below, first recalling that homeomorphisms of a space are called equivalent if they are conjugate in the group of all homeomorphisms of the space, and second, recalling that, according to P. A. Smith, the fixed-point set of a periodic homeomorphism of S^3 is a sphere of dimension -1, 0, 1, 2, or 3.

Dimension -1: (Theorem 5.1) If n is composite, there are uncountably many inequivalent fixed-point-free autohomeomorphisms of S^3 of period n.

Dimension 0: (Theorem 6.1) For each $n > 1$, there are uncountably many inequivalent autohomeomorphisms of S^3 with period $2n$ and such that the fixed-point set of each is a 0-sphere.

Dimension 1: (Corollary 4.4) For each $n > 1$, there are uncountably many inequivalent autohomeomorphisms of S^3 of period n such that the fixed-point set of each is a 1-sphere.

Dimension 2: (Corollary 2.2) There are uncountably many inequivalent involutions of S^3 such that the fixed-point set of each is a 2-sphere. (This result was first announced by Alford [Amer. Math. Soc. Notices **9** (1962), 105].)

As far as the wildness of these homeomorphisms in the last two cases is concerned, the author also proves that the points of wildness on the fixed-point set may be chosen to be any perfect subset of the 1-sphere or 2-sphere. The methods are based on the construction of carefully chosen upper semi-continuous decompositions of E^3 such that the decomposition space is again E^3.
L. Neuwirth (Princeton, N.J.)

Su, J. C. 612
Periodic transformations on the product of two spheres.
Trans. Amer. Math. Soc. **112** (1964), 369–380.
The author's introduction describes well the content: "This paper is concerned with a transformation group $(Z_p; X)$, where Z_p is a cyclic group of prime order p and X is a space having the same mod p cohomology algebra of $S^m \times S^n$, the product of an m-sphere and an n-sphere. This problem has the obvious motivation. If we have a pair $(Z_p; X_1)$ and $(Z_p; X_2)$ of transformation groups, where X_1 is a mod p cohomology m-sphere and X_2 a mod p cohomology n-sphere, then the diagonal action $(Z_p; X_1 \times X_2)$ is a transformation group of the type just mentioned. According to the well-known theorem of P. A. Smith ["Fixed points of periodic transformations", Appendix B of S. Lefschetz, *Algebraic topology*, Amer. Math. Soc., New York, 1942; MR **4**, 84] the fixed-point set has the same mod p cohomology algebra of the product of two spheres. That is, the fixed-point set inherits the same cohomology characteristics of the space. For the general case, there is a theorem of R. G. Swan [Comment. Math. Helv. **34** (1960), 1–16; MR **22** #5978] in which sufficient conditions are given to insure that the fixed-point set does have the mod p cohomology algebra of the product of two spheres. Following the same technique devised by him, we succeeded in refining his result to the extent that the cohomology algebra of the fixed-point set is entirely determined. The process is a rather tedious case-by-case study. To state briefly, there are a variety of possibilities for the cohomology of components of the fixed-point set. The components can be like a point, a sphere, a product of two spheres, a projective space, or a Klein bottle. To justify the unpleasant nature of this long list, we give some examples showing that most of the cases do happen. Moreover, these examples are all nice differentiable actions on manifolds. Thus in a sense there is not much room left for further improvement. Finally, our result also indicates there is still something of the space that is passed over to the fixed-point set. Namely, the Poincaré duality is valid for every component of the fixed-point set. Whether this is true in general might be an interesting conjecture."
T. Stewart (Notre Dame, Ind.)

Conner, P. E.; Floyd, E. E. 613
Fixed point free involutions and equivariant maps.
Bull. Amer. Math. Soc. **66** (1960), 416–441.
Les auteurs considèrent la catégorie dont les objets sont les couples (X, T) formés d'un espace de Hausdorff X et d'une involution continue T sur X, sans point fixe, et les morphismes, les applications continues équivariantes. C'est aussi la catégorie des espaces fibrés de fibre S_0. Le problème général est l'étude de $\mathrm{Hom}[(X, T), (X', T')]$ et l'établissement de critères permettant d'affirmer que $\mathrm{Hom}[(X, T), (X', T')] \neq \varnothing$. Le cas où l'un des objets est la sphère S_n munie de la symétrie par rapport à son centre est particulièrement important. Une "Fonction Co-indice" sur la catégorie est la donnée pour chaque (X, T) d'un nombre entier: Co-ind(X, T), positif si $X \neq \varnothing$, satisfaisant aux axiomes: 1. Si $\mathrm{Hom}[(X, T), (X', T')] \neq \varnothing$: Co-ind$(X, T) \leqq$ Co-ind(X', T'). 2. Si A et B sont des fermés, stables par T, de X, et que $X = A \cup B$: Co-ind$(X, T) \leqq$ Co-ind(A, T) + Co-ind$(B, T) + 1$. 3. Co-ind$(S_0, T) = 0$. Exemples: (a) le co-indice proprement dit, écrit avec un c minuscule pour le distinguer (notion définie par C. T. Yang sous le nom de "B-indice"): co-ind (X, T) est le plus petit entier n tel que $\mathrm{Hom}[(X, T), (S_n, T)] \neq \varnothing$.

Quelle que soit la fonction co-indice, on a toujours: Co-ind $(X, T) \leqq$ co-ind (X, T). (b) Le co-indice homologique relativement à un anneau L: co-ind$_L$ qui, lorsque $L = Z_2$ est le plus grand entier n tel que la n-ième puissance de la classe de Whitney w_1 du fibré (X, T) soit non nulle. Dualement, l'indice: ind (X, T) est le plus grand entier n tel que Hom$[(S_n, T), (X, T)] \neq \varnothing$, mais les auteurs ne tentent pas de donner des axiomes pour une "Fonction indice" ni de définir un indice homologique. L'indice et le co-indice stables se définissent de façon naturelle. Ces notions sont étudiées et appliquées. On a ind $(X, T) \leqq$ co-ind $(X, T) \leqq \dim X/T$. Le théorème suivant est énoncé sans démonstration: $H^i(X; Z) = 0$ pour tout $i \geqq k$, entraîne co-ind $(X, T) \leqq k$. On peut suspendre une involution T sur X en involution sur la suspension $S(X)$ et l'étendre à une involution sur l'espace des chemins d'origine donnée sur X: $P(X)$. On a co-ind $S(X) \leqq$ co-ind $X + 1$ tandis que co-ind$_L$ $S(X) =$ co-ind$_L$ $X + 1$ pour $L = Z$ ou Z_2. Pour n impair $\geqq 3$: $n - 1 =$ ind $P(S_n) <$ co-ind $P(S_n) = n$. Problème ouvert: calculer co-ind $P(S_n)$ pour n pair. Si T est l'involution naturelle d'un espace fibré B en sphères S_n de base X, co-ind$_{Z_2}(B, T)$ est relié aux classes de Whitney de B. Si X est une variété différentiable compacte et B le fibré des vecteurs unités tangents, ind$(B, T) = \dim X - 1$, d'où il résulte que si f est une application continue d'une sphère S_n dans une variété riemannienne compacte X, ou bien il existe $x \in S_n$ tel que $f(x) = f(-x)$, ou bien il existe $x \in S_n$ tel que $f(x)$ et $f(-x)$ ne soient pas joints par une géodésique unique de longueur minimum.

R. Deheuvels (Zbl **106**, 163)

Gottlieb, Daniel H. 614

Homotopy and isotopy properties of topological spaces.

Canad. J. Math. **16** (1964), 561–571.

In this paper, the author continues the reviewer's study of homotopy and isotopy properties of topological spaces [same J. **13** (1961), 167–176; MR **25** #574]. First, by a further study of weakly hereditary properties, he establishes a theorem with a corollary that normality, compactness, the property of being a Lindelöf space, local compactness, and paracompactness are not isotopy properties. Next, by means of a notion of open properties of spaces, he establishes another theorem which implies that separability, local connectedness, and local pathwise connectedness are not isotopy properties. In the rest of the paper, he also investigates dimension, separating subsets, and branch points of spaces.

S.-T. Hu (Los Angeles, Calif.)

Zhang, Su-cheng [Chang, Su-Cheng] 615

On the classification of mappings of polyhedra into the n-sphere.

Acta Math. Sinica **13** (1963), 231–237 (*Chinese*); *translated as Chinese Math.* **4** (1963), 252–259.

Let K be a bounded $(n+r)$-dimensional polyhedron, and let S^n be the n-sphere. The totality of continuous maps $f: K^{n+r} \to S^n$ may be treated as a set in which one writes $f_0 \sim f_1$ if there exists a homotopy relation between the maps f_0 and f_1. Hopf, Whitney, Eilenberg, Pontrjagin, Steenrod, Adem, and others have discussed the cases $r = 0, 1, 2$; the present article introduces obstructions and uses exact homology operations to obtain a necessary and sufficient condition for $f_0 \sim f_1$ that is valid for all r.

S.-T. Hu (Los Angeles, Calif.)

Arkowitz, M.; Curjel, C. R. 616

The group of homotopy equivalences of a space.

Bull. Amer. Math. Soc. **70** (1964), 293–296.

Let X be a finite 1-connected CW-complex, and let $\mathscr{E}(X)$ be its group of homotopy classes of homotopy equivalences. The authors first state theorems giving necessary and sufficient conditions, in terms of the homotopy groups of X and the Hurewicz homomorphism, in order that $\mathscr{E}(X)$ have all its subgroups finitely generated, be a finite group, or be a solvable group. These theorems are then applied to the case of an associative H-space G having the homotopy type of X to obtain information on $\mathscr{E}(G)$ in terms of the ranks of the homology and homotopy groups of G.

W. D. Barcus (Stony Brook, N.Y.)

Kan, Daniel M. 617

On torsionfree, torsion and primary spectra.

Bol. Soc. Mat. Mexicana (2) **8** (1963), 14–19.

This note presupposes the notation, definitions and results of the author [Illinois J. Math. **7** (1963), 463–478; MR **27** #2986; ibid. **7** (1963), 479–491; MR **27** #5253]. If X and Y are spectra, (X, Y) denotes the homotopy classes of maps of X into Y. If C is a collection of abelian groups, Y is called a C-spectrum if $(X, Y) \in C$ for all finite X. For appropriate choices of C this defines torsion, torsion-free, and primary spectra. If Y is a spectrum satisfying the extension condition, unique (up to homotopy equivalence) torsion and torsion-free parts of Y are defined, and it is shown that Y is an extension of its torsion part by its torsion-free part. Furthermore, it is shown that a torsion spectrum has the homotopy type of the weak product of its p primary parts.

The above decompositions do not give rise to the corresponding group-theoretic decomposition of the homotopy groups of a spectrum. For example, if Z is the torsion-free part of Y, $\pi_n(Z) \approx \pi_n(Y) \otimes Q$, where Q denotes the rationals.

E. H. Brown (Waltham, Mass.)

Hilton, P. J. 618

Nilpotency and H-spaces.

Topology **3** (1964), *suppl.* 2, 16 *pp.*

The central problem here is the study of homotopy-abelian loop spaces ΩF on non-H-spaces F. In particular, the author is concerned with fiberings $F \to X \to Y$ such that ΩF has the same H-homotopy type as $\Omega X \times \Omega^2 Y$ and hence is led to consider the corresponding subset of the kernel of $\Omega_*: \pi(X, Y) \to \pi(\Omega X, \Omega Y)$. The members of this subset are called P-classes. If Y is a $K(G, n)$, then the P-classes form a natural ideal in $H^*(X, G)$ which is closed with respect to cohomology operations. A rich source of examples are m-fold cup products, $m \geqq 3$. {For $m = 2$, the equivalence $\Omega F \simeq \Omega X \times \Omega^2 Y$ may fail to be an H-map, while for $m = 3$, the equivalence is an H-map but may fail to respect the associativity.} Why F itself is often not an H-space is investigated via higher Whitehead products. The author also studies in varying detail other cohomology theories, dual problems for cofibrations and homotopy groups, and the nilpotence of F^K for spaces K of restricted weak category.

J. Stasheff (Notre Dame, Ind.)

Verdier, Jean-Louis 619

Le théorème de dualité de Poincaré.

C. R. Acad. Sci. Paris **256** (1963), 2084–2086.

This note is an announcement of the following theorem. Let X, Y be topological spaces and $f: X \to Y$ a euclidean mapping. For each open subset V of X the functor $f_{!V}$ has bounded cohomological dimension, independent of V. Assume that X has finite topological dimension. For each object $F^\cdot$ of $I^-(\underline{X})$ and each object $G^\cdot$ of $I^b(\underline{Y})$, the morphism $\Delta: R_0\ \underline{\mathrm{Hom}}(f_!(F^\cdot);\ G^\cdot) \to R_0\ \underline{\mathrm{Hom}}(F^\cdot; f^!(G^\cdot))$ is an isomorphism. *E. Matlis* (Evanston, Ill.)

Deleanu, A. 620

On a certain result of Leray.

Fund. Math. **55** (1964), 1–9.

In this note, a slightly more general result of Leray [J. Math. Pures Appl. (9) **24** (1945), 95–167, p. 121; MR 7, 468] is obtained by using a modern proof based on the theory of spectral sequences. *S.-T. Hu* (Los Angeles, Calif.)

Hudson, J. F. P.; Zeeman, E. C. 621

On combinatorial isotopy.

Inst. Hautes Études Sci. Publ. Math. No. 19 (1964), 69–94.

This paper deals with polyhedral manifolds and piecewise linear (PL) proper embeddings. The equivalence of four different types of isotopy is examined. A PL-embedding $F: M \times I \to Q \times I$, where $F(x, t) = (f_t(x), t)$, is called a (level-preserving) isotopy between f_0 and f_1. If $M = Q$ and the isotopy F is a homeomorphism with $f_0(x) = x$, then F is called an ambient isotopy. g_0 and $g_1: M \to Q$ are called ambient isotopic if there is an ambient isotopy F of Q such that $f_1 g_0 = g_1$. The ambient isotopy F is called a move in case $f_t(x) = x$ for x in the complement of some ball in the interior of Q or of one that meets the boundary Q in a face. (Recall that by Alexander's theorem there is an isotopy keeping the boundary (or one face) fixed between any PL-homeomorphism h of a ball keeping the boundary (or one face) fixed, and the identity map of the ball.) A move is called linear in case it is PL-equivalent to the linear join of the identical map of a barycentrically (or one-face-barycentrically) subdivided simplex and a simplexwise linear homeomorphism keeping the boundary fixed. The latter is determined by the image of the barycentre (or face-barycentre).

The four types of isotopy, which yield equivalence relations in the set of proper PL-embeddings of M into Q are (a) isotopy, (b) ambient isotopy, (c) (ambient) isotopy by moves, (d) isotopy by linear moves. Clearly, (a)$\Leftarrow$(b)$\Leftarrow$(c)$\Leftarrow$(d). The authors prove the following theorems for the case M compact. Theorem 1: (b)$\Rightarrow$(c). Theorem 2: (a$'$)$\Rightarrow$(b), where (a$'$) means (a) with the extra condition that the isotopies are locally unknotted, as is always the case [Zeeman, Ann. of Math. (2) **78** (1963), 501–526; MR **28** #3432] when $\dim Q - \dim M = \mathrm{codim}(M, Q) \geqq 3$ or $= 0$. Theorem 3: (c)&($\mathrm{codim}(M, Q) > 0$)$\Rightarrow$(d). In particular, for $\mathrm{codim}(M, Q) \geqq 3$, the four types of isotopy define the same equivalence relation. The case $\mathrm{codim}(M, Q) = 0$ in Theorem 3 represents an interesting open problem. Several corollaries are deduced from the theorem. A theory on collars is developed, and in particular it is proved that any two collars of a manifold with boundary are ambient isotopic keeping the boundary fixed.

The paper is extremely well written and well illustrated. Several of the standard errors in the subject are cleared up effectively by the authors. Moreover, some of the proofs are very elegant. In a subsequent paper non-proper embeddings will be considered.

N. H. Kuiper (Bennekom)

TOPOLOGY AND GEOMETRY OF DIFFERENTIABLE MANIFOLDS

See also 189, 194, 276, 277, 341, 561.

★Topologia differenziale. 622

1° Ciclo, Urbino, 2–12 luglio 1962. Centro Internazionale Matematico Estivo (C.I.M.E.).

Edizioni Cremonese, Rome. 129 *pp.* (*not consecutively numbered*). *L.* 2.000.

This collection contains four papers (by J. Cerf, A. Haefliger, M. Kervaire, and S. Smale) which will be reviewed individually.

Munkres, James R. 623

★Elementary differential topology.

Lectures given at Massachusetts Institute of Technology, Fall, 1961. Annals of Mathematics Studies, No. 54.

Princeton University Press, Princeton, N.J., 1963. xi + 107 *pp.* $2.75.

According to the preface, this "set of notes" has two objects: (1) to give students a feeling for at least the geometric aspects of differential topology; (2) to provide easily accessible proofs of a few often used facts about differentiable manifolds. In both of these the author has succeeded admirably.

The two main goals of Chapter I are: (1) to prove that every C^1 immersion, imbedding, or diffeomorphism between C^∞ manifolds can be C^1 approximated by a C^∞ map of the same kind, and (2) to prove that every C^1 differentiable structure contains a C^∞ differentiable structure. In addition, the imbeddability of manifolds in Euclidean space is proved.

Chapter II is devoted to proving the existence and uniqueness of smooth triangulations. Algebraic topology is used only once, to prove that a close approximation to a homeomorphism is a homeomorphism. It is also necessary to know that an n-manifold has a covering of dimension n. Aside from these facts, the book is indeed "elementary"; even the implicit function theorem is proved.

The exposition is exceptionally clear. Whatever is not proved is clearly stated as an exercise; as the author remarks, the exercises form an integral part of the book.

M. W. Hirsch (Berkeley, Calif.)

Cairns, Stewart S. 624

The smoothability problem for manifolds. **(Japanese)**

Sûgaku **15** (1963/64), 157–160.

An invited address given at the Annual Meeting of the Mathematical Society of Japan in May, 1963.

Feldman, E. A. 625

The geometry of immersions. II.

Bull. Amer. Math. Soc. **70** (1964), 600–607.

This is a sequel to Part I, which appeared in same Bull.

69 (1963), 693–698 [MR **27** #1967], and announces further similar results; no proofs are given. Some theorems of global projective differential geometry are stated concerning the non-immersibility of certain manifolds in certain projective spaces without projective singularities. A theory of singularities of maps of vector bundles carrying G-structures is developed in great generality. The main application of this is a result closely related to *Carathéodory's conjecture*: a C^∞ immersion of a compact surface in 3-dimensional euclidean space can be approximated arbitrarily closely (with derivatives) by an immersion with at least $2|\chi|$ umbilics, where χ is the Euler characteristic. *W. F. Pohl* (Stanford, Calif.)

Wall, C. T. C. 626
Diffeomorphisms of 4-manifolds.
J. London Math. Soc. **39** (1964), 131–140.

Every diffeomorphism f of a closed oriented 4-manifold M determines an automorphism of the quadratic form $Q(M): H_2(M)\to Z$ defined by intersection numbers. In this paper the author proves that every automorphism of $Q(M)$ can be so obtained if $M=N\ \#\ (S^2\times S^2)$ for some simply connected manifold N, and if (1) $Q(N)$ is indefinite, or (2) rank $H_2(N)\leq 8$. The proof is obtained from an ingenious construction of the required diffeomorphisms and uses the author's results on quadratic forms [Math. Ann. **147** (1962), 328–338; MR **25** #2009; J. Reine Angew. Math. **213** (1963), 122–136; MR **27** #5732].

The diffeomorphisms are used to obtain, in case (1) above, an embedded 2-sphere in every homology class which is a primitive ordinary element of $H_2(M)$.

Several related theorems and generalizations are given involving 4-manifolds with boundary and simply connected 4-manifolds. *N. H. Kuiper* (Bennekom)

Wall, C. T. C. 627
On simply-connected 4-manifolds.
J. London Math. Soc. **39** (1964), 141–149.

The author is concerned with the classification of closed oriented simply connected differential (or combinatorial, that is the same in dimension ≤ 6) 4-manifolds. A homotopy equivalence $f: M_1\to M_2$ of manifolds is called tangential if it induces the tangent bundle of M_1 from that of M_2. Milnor [Sympos. Internac. de Topología Algebraica, pp. 122–128, Univ. Nac. Autónoma de México, Mexico City, 1958; MR **21** #2240] proved that two 4-manifolds as above are (3) tangentially homotopy equivalent if (1) they have isomorphic quadratic forms, or (2) they are h-cobordant. The author proves that (1), (2) and (3) define the same equivalence relation, which then also equals the homotopy equivalence relation. Moreover, if this relation holds between M_1 and M_2, then for some k, $M_1\ \#\ k(S^2\times S^2)$ and $M_2\ \#\ k(S^2\times S^2)$ are diffeomorphic. Nothing is known about the possible values of k. As an application the following corollary is deduced. The Grothendieck group with respect to a connected sum (#) of simply connected, closed oriented [spin-] 4-manifolds is free abelian of rank 2. Generators are represented by the complex projective plane with the usual and another orientation [respectively, the two S^2-bundles over S^2]. The proofs rest on many known results in differential topology partly due to the author.
N. H. Kuiper (Bennekom)

Massey, W. S.; Peterson, F. P. 628
On the dual Stiefel-Whitney classes of a manifold.
Bol. Soc. Mat. Mexicana (2) **8** (1963), 1–13.

The authors give number-theoretic conditions on n and k under which the dual Stiefel-Whitney class $\bar{w}_{n-k}(M^n)$ vanishes for all closed n-manifolds M^n [respectively, all orientable M^n]. For instance, $\bar{w}_{n-k}(M^n)=0$ if $0\leqq k<\alpha(n)$, where $\alpha(n)$ is the number of occurrences of 1 in the dyadic expansion of n; this result is best possible. For orientable manifolds the condition is more complicated. It implies $\bar{w}_{n-1}=0$ if $n>1$, and $\bar{w}_{n-2}=0$ if $n>2$ is not of the form 2^r, 2^{r+1}. Results on integral (possibly twisted) classes $\bar{W}_i$ are also obtained by using $\bar{W}_{2i+1}=\delta\bar{w}_{2i}$. The proofs are based on Poincaré duality and Wu's formula

$$\{Sq(y)-V\smile y\}_n = 0$$

which with $y=Sq^{-1}(x)/V$ becomes $\{Sq^{-1}(x)-x\smile\bar{w}\}_n=0$ [notation of Brown and Peterson, #629]: If $\bar{w}_{n-k}\neq 0$, then by Poincaré duality $\{Sq^{-1}(x)\}_n=x\smile\bar{w}_{n-k}\neq 0$ for some $x\in H^k$, and to $\{Sq^{-1}(x)\}_n$ the Serre technique of admissible sequences Sq^I is then applied to prove the result.
A. Dold (Heidelberg)

Brown, Edgar H., Jr.; Peterson, Franklin P. 629
Relations among characteristic classes. I.
Topology **3** (1964), *suppl.* 1, 39–52.

Let $Sq=\sum_{i=0}^{\infty}Sq^i$ denote the Steenrod automorphism in cohomology mod 2, and Sq^{-1} its inverse. Let $W=\sum_{i=0}^{\infty}W_i$ denote the (universal) Stiefel-Whitney class, and $V=Sq^{-1}W$; the class V is a polynomial in $W_1, W_2, \cdots$ which can be computed explicitly by Wu's formulae. Let $p=p(W_1, W_2, \cdots)$ be an arbitrary polynomial. Consider closed (differentiable) manifolds M of a fixed dimension n. If $z\in H^*(M, \mathbf{Z}_2)$ is such that $2\dim(z)<n-\dim(p)$, then in H^nM we have $\{[V\smile Sq^{-1}(p)]\smile z\}_n=\{V\smile[Sq^{-1}(p)\smile z]\}_n$ $=\{Sq[Sq^{-1}(p)\smile z]\}_n=\{p\smile Sq(z)\}_n=0$, where $\{\ \}_\nu$ denotes the ν-dimensional component; the second equality holds by Wu's characterization of V, the last because

$$\dim Sq(z)\leqq 2\dim(z)<n-\dim(p).$$

Since z is arbitrary, it follows from Poincaré duality that $\{V\smile Sq^{-1}(p)\}_j=0$ if $2(n-j)<n-\dim(p)$, i.e., if $2j>n+\dim(p)$. Every such pair (p, j) therefore yields a relation $\{V\smile Sq^{-1}(p)\}_j=0$ between Stiefel-Whitney classes of n-manifolds. The main theorem of the paper under review is that these are all relations between Stiefel-Whitney classes of n-manifolds, and an analogous result for orientable n-manifolds.

The general problem of determining the relations between characteristic classes of G-manifolds (orientable, stably almost complex, . . .) is formulated, and the group of these relations is characterized in terms of cobordism theory. The proofs of the main theorems are based on interesting arguments about stable and unstable modules over the Steenrod algebra A. In particular, a right operation of A is introduced on H^*BO, H^*BSO, H^*BU, $\cdots$; this situation is formalized by the notion of "right-left algebra", and is studied in general. *A. Dold* (Heidelberg)

Haefliger, André 630
Variétés feuilletées.
Topologia Differenziale (*Centro Internaz. Mat. Estivo*, 1° *Ciclo*, *Urbino*, 1962), *Lezione* 2, 46 *pp. Edizioni Cremonese, Rome.*

This is a very clear review of the results in the topological theory of foliated manifolds obtained by Ehresmann, Reeb and the author. It is particularly valuable for its exposition of the theorems in the author's thesis [Comment. Math. Helv. **32** (1958), 248–329; MR **20** #6702] on foliations of codimension one. Combining a reading of this paper with R. Palais' thesis [Mem. Amer. Math. Soc. No. 22 (1957); MR **22** #12162]—which the author does not mention—would give a complete picture of the state of the subject today at this level of generality. The report does not touch on that aspect of the theory which considers additional compatible differential-geometric structures on the manifold, such as Riemannian metrics, contact and symplectic structures (which is related to the calculus of variations), or Lie transformation groups.
R. Hermann (Evanston, Ill.)

Conlon, Lawrence 631
The topology of certain spaces of paths on a compact symmetric space.
Trans. Amer. Math. Soc. **112** (1964), 228–248.
If a compact Lie group K acts in a variationally complete manner on a compact manifold M, Bott and Samelson have shown that the topology of the space of paths beginning at a point and ending on an orbit of K can, in principle, be described in terms of the "algebraic" properties of the action of K on M [Amer. J. Math. **80** (1958), 964–1029; MR **21** #4430]. They also showed that K is variationally complete if M is a symmetric space and K is the isotropy subgroup of the isometry group. In this case, everything could be reduced to a readily computable fundamental domain for the action of K on M, in terms of the algebraic structure of the symmetric space. Many important and striking applications to the topology of symmetric spaces were obtained using these ideas.

The reviewer showed that variational completeness held when M is a compact symmetric space and K is any symmetric subgroup of the isometry group, e.g., one not necessarily the isotropy group. Also, in this case, K acting on M admitted a fundamental domain with some of the properties that led to the successful applications in the Bott-Samelson case [the reviewer, Proc. Amer. Math. Soc. **11** (1960), 544–546; MR **23** #A1748; Nederl. Akad. Wetensch. Proc. Ser. A **65** (1962), 291–298; MR **25** #2554]. However, the algebraic problems seemed considerably more complicated; this paper very successfully and elegantly pushes on over the obstacles and gives new applications.

The author's first task is to develop the algebraic machinery for defining precisely the fundamental domain. A key role is played here by a theorem of de Siebenthal [Comment. Math. Helv. **31** (1956), 41–89; MR **20** #926]. As an application, the case where M is the Cayley projective plane is handled quite easily.

The next topic is a theorem that the Bott-Samelson cycles are iterated sphere bundles. Araki has proved this in the case where K is the isotropy group [J. Math. Osaka City Univ. **13** (1962), 87–133; MR **28** #1630], and some of the author's techniques are generalizations of Araki's. Bott and Samelson had remarked that the orientability of these cycles implied that the path spaces had no torsion in homology. Using a clever argument involving the Stiefel-Whitney classes, the author reduces this to an algebraic condition that he calls regularity. Finally, he completes the picture by showing that regularity is also necessary for no torsion.
R. Hermann (Evanston, Ill.)

Švarc, A. S. 632
Stability of stationary values.
Dokl. Akad. Nauk SSSR **131** (1960), 1276–1278 (*Russian); translated as Soviet Math. Dokl.* **1** (1960), 439–441.
The author announces the following analogue of a theorem in Krasnosel'skiĭ [*Topological methods in the theory of nonlinear integral equations* (Russian), GITTL, Moscow, 1956; MR **20** #3464]. Let $p: E \to B$ be a differentiable fibering, E and B smooth closed manifolds with fiber F. Let J^* be a differentiable function on E, constant on fibers, having stationary values corresponding to sets of stationary points of dimensions (in decreasing order) $k_1, \cdots, k_s$. The number of stable stationary values of J^* is then bounded below by the largest t such that

$$k_1 + \cdots + k_{t-1} < \text{cat } B + (t-1)(\dim F - 1).$$

The bound is best possible if cat B is (the maximum) $\dim B + 1$.

The author defines a stable dimension sdim for sets of stationary points (the English translation is incorrect here) and conjectures that corresponding values of sdim in E and in B differ just by dim F. He announces a proof in a very special case. The final result is a sharpening of a very special case of the main theorem of Anosov [Dokl. Akad. Nauk SSSR **131** (1960), 223–225; MR **22** #9881], namely, the case that E is a Hilbert space and the (cyclic) group acts freely on $E - \{0\}$: then F has infinitely many stable stationary values. *J. R. Isbell* (Seattle, Wash.)

Al'ber, S. I. 633
Homologies of a non-oriented loop space and their application to the calculus of variations in the large. (Russian)
Dokl. Akad. Nauk SSSR **155** (1964), 13–16.
The author announces various results on the homology of a space of curves on a manifold, as well as applications to questions about geodesics. If M is a manifold, denote by $\Omega(M)$ the space of closed curves (loops) on M. This space admits an involution obtained by replacing t by $1-t$. Identifying curves with their images under this involution, we get the space of non-oriented closed curves $\hat{\Omega}(M)$. The first main result is a calculation of the mod 2 Betti numbers of $\hat{\Omega}(M)$, when M is homeomorphic to a sphere. Second, the author studies the cohomology ring structure for such spaces. Finally, he announces some results concerning geodesics, which use these calculations.
D. W. Kahn (New York)

Milnor, J. 634
★Morse theory.
Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51.
Princeton University Press, Princeton, N.J., 1963. vi + 153 *pp.* $3.00.
This book is devoted to an exposition of Morse theory. Starting from scratch, it goes through the proofs of the periodicity theorems of Bott for the unitary and orthogonal groups. The path taken to these theorems is by no

means a minimal geodesic but the detours along the way serve to display the power of the theory as it is developed. The book is divided into four parts, the contents of which are outlined below.

I. The book begins by showing how the topology of a manifold is related to the critical points of a smooth real-valued function on the manifold with no degenerate critical points. A point p is critical for a function f if the differential of f vanishes there and is degenerate if the Hessian of f has non-zero nullity there. The index of f at a critical point p is the dimension of the maximal subspace of the tangent space at p on which the Hessian is negative definite. If f is a real-valued function on a manifold M with no degenerate critical points and if $M^a = f^{-1}(-\infty, a)$ is compact for all real a, then M has the homotopy type of a CW-complex with one cell of dimension λ for each critical point of f of index λ. The results leading to this theorem are used to prove that a compact manifold which admits a function with just two critical points (non-degenerate) is homeomorphic to a sphere (Reeb's Theorem), to compute the integral homology groups of complex projective space, and to prove the Morse inequalities.

In order to establish the existence of functions with no degenerate critical points whose preimages of compact sets are compact, manifolds embedded in $\mathbf{R}^n$ are considered. Let $p \in \mathbf{R}^n$ and let L_p be the function defined on M whose value at q is the square of the distance from p to q. For almost all $p \in \mathbf{R}^n$ it is shown that L_p has the desired properties. A point $q \in M$ is a critical point for L_p if and only if the line joining p to q is perpendicular to M at q. As a preview of things to come the index theorem for L_p is proved. This states that if q is a non-degenerate critical point of L_p, the index of q is the sum of the nullities of the Hessian of $L_{p'}$ at q for all p' on the segment from q to p. The final section of Part I presents the simple, elegant proof of the Lefschetz theorem on hyperplane sections due to Andreotti and Fraenkel.

II. This part gives the reader a self-contained treatment of the topics in Riemannian geometry needed in the rest of the book. Affine connection, covariant derivative, parallel displacement along curves, curvature tensor, geodesics, completeness and the exponential map are defined and discussed briefly.

III. Now a change of focus occurs and the main object of attention becomes the space of piecewise smooth paths from p to q on a manifold M, $\Omega(M; p, q)$. This is almost always written simply as Ω and no confusion results from this abbreviation. The tangent space to Ω at a path ω, $T\Omega_\omega$, is the space of all piecewise smooth vector fields along ω which vanish at the end points. The energy function E is defined on Ω and if M is complete, E assumes its minimum d^2 on the minimal geodesics from p to q, where the distance from p to q is d. In analogy with the finite-dimensional case of Part I, critical points of E are defined and the first variational formula is used to identify the critical points as the geodesics from p to q. Again motivated by the finite-dimensional case, $\gamma \in \Omega$ at a geodesic, the Hessian E_{**} of E is defined to be a symmetric bilinear map of $T\Omega_\gamma \times T\Omega_\gamma$ into the reals whose index at γ is the dimension of the maximal subspace of $T\Omega_\gamma$ on which E_{**} is negative definite. The index of E_{**} at a minimal geodesic is zero.

A Jacobi field along a geodesic is a vector field which satisfies a certain second-order differential equation. Two points $p = \gamma(a)$ and $q = \gamma(b)$ are conjugate along a geodesic γ if there exists a non-zero Jacobi field J along γ such that $J(a) = J(b) = 0$. The multiplicity of p and q as conjugate points along γ is the dimension of the space of all such Jacobi fields. The null space of E_{**} at γ is the space of Jacobi fields along γ in $T\Omega_\gamma$. Thus E_{**} has non-zero nullity at γ if and only if p and q, the end points of γ, are conjugate along γ and the nullity of E_{**} at γ is just the multiplicity of p and q as conjugate points along γ. A pair of antipodal points on an n-sphere is shown to be conjugate along any great circle with multiplicity $n-1$. The index of E_{**} at a geodesic γ is finite and is equal to the number of points $\gamma(t)$, with $0 < t < 1$, such that $\gamma(0)$ and $\gamma(t)$ are conjugate along γ and each such point is counted with its multiplicity (Morse's Index Theorem).

If M is a complete Riemannian manifold and p and q are non-conjugate along any geodesic, Ω has the homotopy type of a countable CW-complex with one cell of dimension λ for each geodesic from p to q at which E_{**} has index λ. This is proved by applying the theory of Part I to E restricted to a finite-dimensional compact manifold with boundary, which is a deformation retract of $\Omega^a = E^{-1}[0, a]$, and then taking the union of Ω^{a_i} for an unbounded monotone sequence $\{a_i\}$ of non-critical values of E. As an application, the loop space of the n-sphere is shown to have the homotopy type of a CW-complex with one cell in dimension $k(n-1)$ for all integers $k \geq 0$. The assumption that p and q are non-conjugate along any geodesic is not restrictive since in a complete Riemannian manifold M, given any $p \in M$, for almost any $q \subset M$, p and q are non-conjugate along any geodesic.

Next the relation between curvature and geodesics is investigated. In particular, the results just described are applied to prove Cartan's theorem that a simply connected complete Riemannian manifold with all sectional curvatures $\leqq 0$ is diffeomorphic to $\mathbf{R}^n$ and that any two points on such a manifold are joined by a unique geodesic. The important results for the last part of the book are about manifolds on which the Ricci curvature K is positive definite. The key result here is S. B. Myers's theorem which states that if $K(U, U) \geqq (n-1)/r^2$ for every unit vector U, then every geodesic of length $> \pi r$ contains conjugate points and so is not minimal. This result yields that Ω has the homotopy type of a CW-complex with finitely many cells in each dimension if M is compact and K is positive definite everywhere.

IV. The object of this last part of the book is the proof of Bott's periodicity theorems for the unitary and orthogonal groups. It begins with a short discussion of symmetric spaces in which the conjugate points to a given point $p = \gamma(0)$ along a geodesic γ are located. They are the points $\gamma(\pi k/e_i)$, where k is any non-zero integer and e_i is any positive eigenvalue of the linear endomorphism of the tangent space at p which takes a vector W to the vector $R(V, W)V$, where R is the curvature tensor and V is the velocity vector of γ at p. Next a Lie group with a left- and right-invariant metric is shown to be a symmetric space, and if it is simply connected it is seen via Myers's theorem to be a Cartesian product of a Euclidean space by a compact group on which all sectional curvatures are positive. Thus the Ricci tensor of a compact simply connected Lie group is everywhere positive definite, and so its loop space has the homotopy of a CW-complex with finitely many cells in each dimension. By a direct computation using the location of conjugate points described above, it is

shown that this CW-complex has no odd-dimensional cells (Bott).

Now a special situation is considered for a complete Riemannian manifold M. Suppose that the distance between p and q is d, and that the space of minimal geodesics Ω^d is a topological manifold. If every non-minimal geodesic from p to q has index $\geqq \lambda_0$, then $\pi_i(\Omega, \Omega^d)=0$ for $0 \leqq i < \lambda_0$. Letting $M=S^{n+1}$ and p and q be antipodal, this theorem has the Freudenthal suspension theorem as corollary, that $\pi_i(S^n)=\pi_{i+1}(S^{n+1})$ for $i \leqq 2n-2$. Letting $M=\mathbf{SU}(2m)$ it is shown that the space of minimal geodesics from I to $-I$ is homeomorphic to the complex Grassmann manifold of m-dimensional vector subspaces of $\mathbf{C}^{2m}$, $\mathbf{G}_m(\mathbf{C}^{2m})$ and that every non-minimal geodesic from I to $-I$ in $\mathbf{SU}(2m)$ has index $\geqq 2m+2$. Thus $\pi_i(\mathbf{G}_m(\mathbf{C}^{2m}))=\pi_{i+1}(\mathbf{SU}(2m))$ for $i \leqq 2m$. This set of isomorphisms is the contribution of Morse theory to the periodicity theorem for the unitary group. The remaining isomorphisms needed to obtain the periodicity for the stable groups, $\pi_{i-1}\mathbf{U}=\pi_{i+1}\mathbf{U}$ for $i \geqq 1$, are all consequences of the exactness of the homotopy sequences of bundles.

The proof of the periodicity for the infinite orthogonal group $\mathbf{O}$ is somewhat longer since the period here is eight rather than two. Thus iterated loop spaces of $\mathbf{O}(n)$ must be studied. To this end, for n divisible by a high power of 2, subsets $\Omega_k(n)$ of $\mathbf{O}(n)$ are defined such that $\Omega_k(n) \subset \Omega_{k-1}(n) \subset \cdots \subset \Omega_0(n)=\mathbf{O}(n)$. Each $\Omega_j(n)$ is a smooth totally geodesic submanifold of $\mathbf{O}(n)$ and each component of $\Omega_j(n)$ is a symmetric space. For each $0 \leqq j < k$, an element $J_j \in \Omega_j(n)$ is distinguished and the space of minimal geodesics from J_j to $-J_j$ in $\Omega_j(n)$ is homeomorphic to $\Omega_{j+1}(n)$. Further, in each component of the space of paths from J_j to $-J_j$ in $\Omega_j(n)$, the index of a non-minimal geodesic is bounded from below by a function of n which goes to infinity with n. Thus $\pi_{i+1}\Omega_j=\pi_i\Omega_{j+1}$ for all i, where Ω_j is the direct limit of $\Omega_j(n)$ as n goes to infinity. The periodicity $\pi_i\mathbf{O}=\pi_{i+8}\mathbf{O}$ is completed now by showing that Ω_8 is homeomorphic to $\mathbf{O}$.

H. I. Levine (Waltham, Mass.)

Kuranishi, Masatake 635

Sheaves defined by differential equations and application to deformation theory of pseudo-group structures.

Amer. J. Math. **86** (1964), 379–391.

This paper presents a method for obtaining (under suitable local regularity conditions) a resolution of the sheaf of solutions of a system of linear differential equations, using the Cartan-Kuranishi theory of prolongations of differential systems.

A sheaf of cross-sections of a vector bundle is called an S-sheaf if it locally consists of the set of solutions of a system of linear differential equations defined on sections of the bundle. The main theorem says, roughly, that the quotient of an S-sheaf by an S-subsheaf is again an S-sheaf, permitting a step-wise construction of a resolution of an S-sheaf. In the last section, the author sketches how he would deal with the deformation-of-pseudogroup problems associated with the work of Kodaira and Spencer.

The aim of the paper as a whole seems to be to express the author's approach to the sort of general problems already extensively treated by D. C. Spencer [Ann. of Math. (2) **76** (1962), 306–398; MR **27** #6287a; ibid. (2) **76** (1962), 399–445; MR **27** #6287b]. The author presents no new applications, so it is difficult to decide whether these generalities can be put to work more readily than Spencer's on the underlying problems of geometry and analysis. *R. Hermann* (Evanston, Ill.)

PROBABILITY

See also 116, 207, 209, 212, 214, 260, 365, 414, 473, 679, 917, 1098.

Doob, J. L. 636

A ratio operator limit theorem.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **1** (1962/63), 288–294.

A more probabilistic proof is given of the following theorem due to the reviewer [Bull. Amer. Math. Soc. **68** (1962), 95–102; MR **24** #A3671]: Let (X_0, μ_0), (X_1, μ_1), $\cdots$ be finite measure spaces, let $T_n: L_1(X_{n-1}, \mu_{n-1}) \rightarrow L_1(X_n, \mu_n)$, $n=1, 2, \cdots$, be a sequence of bistochastic operators, that is, positive operators such that $T1=1$, T is of norm at most 1, and $T^*1=1$, where T^* is the adjoint operator. Let $T_{1n}=T_nT_{n-1}\cdots T_1$, so that $T_{1n}{}^*=T_1{}^*\cdots T_n{}^*$. Let f_0 be a function on X_0 satisfying $\int_{X_0} |f_0| \log^+|f_0| \, d\mu_0 < \infty$. Then $\lim_{n\to\infty} T_{1n}{}^*T_{1n}f_0$ exists almost everywhere relative to the measure μ_0, as well as in the L_1 topology.

As a generalization, an analogous "ratio theorem" is obtained. *G.-C. Rota* (Cambridge, Mass.)

Sobol', I. M. 637

Periods of pseudo-random sequences. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 367–373.

Sequences of pseudo-random numbers are usually generated by recurrence formulas of the type $x_{i+1}=\varphi(x_i)$. In order to increase the length L of the nonperiodic part of a sequence, a "perturbed" sequence may be used:

$$x_{i+1} = \varphi(x_i) \quad \text{if } i \not\equiv 0 \pmod{M},$$
$$= \psi(x_i) \quad \text{if } i \equiv 0 \pmod{M},$$

for some M. The author sets up a simple probabilistic model according to which L may be expected to increase by the factor $\sqrt{M}$. Empirical results seem to support the theoretical findings. *G. E. Noether* (Boston, Mass.)

Bennett, G. 638

On the probability of large deviations from the expectation for sums of bounded, independent random variables.

Biometrika **50** (1963), 528–535.

Let $x_1, \cdots, x_n$ be mutually independent random variables, $\operatorname{var}(x_i) \leqq \sigma_i^2$, $x_i-Ex_i \leqq M$, $\sigma^2=\sum \sigma_i^2$, $S=\sum(x_i-Ex_i)$. The author [J. Amer. Statist. Assoc. **57** (1962), 33–45] and the reviewer [ibid. **58** (1963), 13–30; MR **26** #1908] have derived two upper bounds for $P(S \geqq t\sigma)$, where $t>0$, which involve only $(t, M/\sigma)$ and $(t, M/\sigma, n)$, respectively. Under the additional assumption that the x_i are symmetrically distributed about their means, the author here derives two analogous bounds (one of which is contained in his earlier paper). These four and other bounds are plotted in several figures.

W. Hoeffding (Chapel Hill, N.C.)

Guter, R. S. 639

On the probability of detecting a region by linear search. (Russian. German summary)

Teor. Verojatnost. i Primenen. **9** (1964), 365–367.

Consider a circle C of radius R which contains a convex integrable region K and which is randomly cut by a chord of length L. The author shows (assuming a uniform distribution of a particular cutting angle) that the probability the chord intersects the region is

$$P = [2LV(K)/\pi + W(K)] \times [\pi R^2 - L(R^2 - L^2/4)^{1/2} - 2R^2 \sin^{-1}(L/2R)]^{-1},$$

where $V(K)$ and $W(K)$ are the linear and planar variations, respectively, of the region K according to A. S. Kronrod and A. G. Vituškin [see A. G. Vituškin, *On multidimensional variations* (Russian), GITTL, Moscow, 1955; MR **17**, 718]. *H. P. Edmundson* (Pacific Palisades, Calif.)

Kraft, Charles H.; van Eeden, Constance 640

A fluctuation theorem and a distribution-free test.

Ann. Math. Statist. **35** (1964), 1359–1360.

Let $X_1, X_2, \cdots, X_n$ be independent random variables with continuous and symmetric distributions. Let N be the number of positive sums in $\{\sum_{i\in T} X_i: T \subset (1, 2, \cdots, n)\}$. The authors prove by a clever combinatorial argument that $P(N=k)=1/2^n$. There is also a discussion of the consistency of tests based on N.

G. Baxter (Minneapolis, Minn.)

Zolotarev, V. M. 641

On asymptotically correct constants in sharpenings of the global limit theorem. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 293–302.

Let $\mathfrak{B}$ denote the class of distribution functions (d.f.'s) F with $\int x\,dF=0$, $\sigma^2=\int x^2\,dF>0$, $\beta=\int |x|^3\,dF<\infty$. Let $\xi_1, \xi_2, \cdots$ be independent random variables with common d.f. F in $\mathfrak{B}$,

$$F_n(x) = P\{\xi_1 + \cdots + \xi_n < x\sigma n^{1/2}\},$$

$$\rho_3(F_n, \Phi) = \int |F_n(x) - \Phi(x)|\,dx,$$

where Φ is the standard normal d.f. Esseen [Kungl. Tekn. Högsk. Handl. Stockholm No. 121 (1958); MR **20** #3591] has shown that the limit $A_3(F)=\lim_{n\to\infty} n^{1/2}\rho_3(F_n, \Phi)$ exists for every $F\in\mathfrak{B}$ and evaluated it explicitly. Let $B_3=\sup_{F\in\mathfrak{B}} (\sigma^3/\beta)A_3(F)$. Let $\mathfrak{L}$ denote the class of normal d.f.'s,

$$\rho_3(F_n, \mathfrak{L}) = \inf_{H\in\mathfrak{L}} \rho_3(F_n, H) = \rho_3(F_n, \Phi(b_n x + a_n)),$$

$$\bar{A}_3(F) = \lim_{n\to\infty} n^{1/2}\rho_3(F_n, \mathfrak{L}), \quad \bar{B}_3 = \sup_{F\in\mathfrak{B}} (\sigma^3/\beta)\bar{A}_3(F).$$

The author evaluates $\bar{A}_3(F)$ and the limit $a=\lim n^{1/2}a_n$ and shows that $\lim n^{1/2}(b_n-1)=0$ and $B_3=\bar{B}_3=\frac{1}{2}$.

W. Hoeffding (Chapel Hill, N.C.)

Rjauba, B. 642

Limit theorems for weakly dependent random variables. (Russian. Latvian and English summaries)

Litovsk. Mat. Sb. **3** (1963), no. 1, 207–217.

Two theorems on asymptotic normality for partial sums of "weakly dependent random variables of Markov type" are given. Here, "weakly dependent" means approximately Markovian, ergodic, to a specified degree. Theorem 1 gives four sufficient conditions for limiting normality of such variables, extending a theorem of Dobrushin on Markov variables.

The proof of Theorem 1 follows closely an argument from one of the author's earlier papers [same Sb. **2** (1962), no. 1, 125–128; MR **27** #6297]. Theorem 2 is a local limit theorem for integer-valued random variables, giving sufficient conditions for convergence of densities.

R. F. Gundy (Ann Arbor, Mich.)

Hekendorf, H. [Heckendorff, Hartmut] 643

A higher-dimensional limit theorem for distribution functions. (Russian. German summary)

Ukrain. Mat. Ž. **16** (1964), 365–373.

Author's summary: "Im ersten Teil vorliegender Arbeit werden notwendige und hinreichende Bedingungen dafür angegeben daß die Dichten der normierten Summen unabhängiger identisch verteilter s-dimensionaler Zufallsvektoren gegen eine stabile Verteilungsdichte konvergieren. Der zweite Teil enthält eine asymptotische Zerlegung der Differenz der Dichte der normierten Summe von n s-dimensionalen unabhängigen Zufallsvektoren (sowohl nicht identisch als auch identisch verteilter) und der Dichte der s-dimensionalen Normalverteilung nach Potenzen von $1/\sqrt{n}$."

Petrov, V. V. 644

Local limit theorems for sums of independent random variables. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 343–352.

Let $X_1, X_2, \cdots$ be a sequence of independent, identically distributed, random variables with finite absolute moment of some order $k\geq 3$. If the random variable

$$\frac{1}{\sigma\sqrt{n}}\left(\sum_{i=1}^{n} X_i - nm\right),$$

where $\mathbf{E}X_1=m$, $\mathbf{D}X_1=\sigma^2>0$, possesses an absolutely continuous distribution with the bounded density function $p_n(x)$ for some $n=n_0$, then there is a function $\varepsilon(n)$ independent of x such that $\lim_n \varepsilon(n)=0$, and

$$\left|p_n(x) - \varphi(x) - \sum_{j=1}^{k-2} \frac{P_j(-\varphi)}{n^{j/2}}\right| \leq \frac{\varepsilon(n)}{(1+|x|^k)n^{(k-2)/2}}$$

for all x $(-\infty<x<\infty)$, where

$$\varphi(x) = \frac{1}{\sqrt{(2\pi)}}\,e^{-x^2/2}, \quad P_j(-\varphi) = q_{3j}(x)\varphi(x);$$

the polynomials $q_{3j}(x)$ are of degree $3j$ in x, and their coefficients depend only on moments up to order $j+2$ inclusive. (For these polynomials there is an explicit formula due to the same author [Vestnik Leningrad. Univ. **17** (1962), no. 19, 150–153; MR **26** #800]. This result sharpens the theorem of B. W. Gnedenko [B. W. Gnedenko and A. N. Kolmogorov, *Grenzverteilungen von Summen unabhängiger Zufallsgrössen*, Akademie-Verlag, Berlin, 1959; MR **21** #6020]. For the sequence of independent identically lattice-distributed random variables the author sharpens results of C.-G. Esseen [Acta Math. **77** (1945), 1–125; MR **7**, 312].

C. V. Stanojevic (Detroit, Mich.)

Borovkov, A. A. 645

Analysis of large deviations in boundary-value problems with arbitrary boundaries. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 253–289.

Let $\xi_1, \xi_2 \cdots$ be a sequence of mutually independent, identically distributed random variables with $\mathbf{E}\xi_i=0$, $\mathbf{E}\xi_i^2=1$, $S_k=\xi_1+\cdots+\xi_k$, and let be $g(t)$ a positive function on the interval $[0, 1]$, $x(n)$ a function on positive integers, and $x(n)=n(1+O(1))$. Further let η denote the first index $k\leq n$ for which $x(n)^{-1}S_k\geq g(k/n)$ and let $\chi=S_\eta-x(n)g(\eta)$. The paper contains a detailed study of the asymptotic behaviour for $n\to\infty$ of the probabilities of the events: $\{\eta=k, \chi<y\}$, $\{\eta\leq n, \chi<y\}$, $\{\eta\leq n, \chi<y, S_n<x(n)u\}$. If $\sqrt{n}$ is used instead of $x(n)$ as the normalizing factor (normal deviation), the limits of these probabilities are furnished by the invariance principle for sums of independent random variables. *Petr Mandl* (Prague)

Borovkov, A. A. 646

Remarks on the theorems of Wiener and Blackwell. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 331–343.

Author's summary: "The paper consists of two parts. In the first part conditions are discussed for which an analytical transformation on a function, presented as an absolutely convergent Fourier series, retains asymptotic behaviour of its coefficients. In the second part these conditions are used for seeking the asymptotic behaviour of the remainder term in Blackwell's limit theorem. Some statements are also made therein describing the behaviour of the conditional mean of the first passage time in the random walk with on the average negative jumps."

M. Loève (Berkeley, Calif.)

Mamatov, M.; Halikov, M. K. 647

Global limit theorems for distribution functions in the higher-dimensional case. (Russian. Uzbek summary)

Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk **1964**, *no.* 1, 13–21.

Let $\xi_1, \xi_2, \cdots, \xi_n, \cdots$ be a sequence of independent identically distributed s-dimensional random variables, $\xi_n=(\xi_{1n}, \cdots, \xi_{sn})$, $n=1, 2, \cdots$. Let $E(\xi_{jn})=0$ $(j=1, 2, \cdots, s)$, and define $\eta_n=(\xi_1+\cdots+\xi_n)/\sqrt{n}$. Let $p_n(x)$ denote the absolutely continuous part of the probability distribution of η_n and let $\varphi(x)$ be the density of the normal law with expectation 0 and the same covariances as ξ_n. The author studies some asymptotic properties of

$$C_n^{p,q}(s) = \int_{R_s} |p_n(x)-\varphi(x)|^p|x|^q\,dx,$$

where $|x|=\sqrt{(x_1^2+\cdots+x_s^2)}$. *J. E. Cigler* (Vienna)

Furstenberg, Harry 648

Noncommuting random products.

Trans. Amer. Math. Soc. **108** (1963), 377–428.

This is a profound memoir. We extract from the author's clear introduction: "Let $X_1, X_2, \cdots, X_n$ be a sequence of independent real-valued random variables with a common distribution function $F(x)$, and consider the sums $X_1+X_2+\cdots+X_n$. A fundamental theorem of classical probability theory is the strong law of large numbers which asserts that with probability one, $X_1+X_2+\cdots+X_n\sim n\int x\,dF(x)$, provided that $\int|x|\,dF(x)$ is finite. It is natural to inquire whether there exist laws governing the asymptotic behavior of products $X_nX_{n-1}\cdots X_1$, where the X_j are now identically distributed independent random variables with values in an arbitrary group. This type of question arises, for example, in considering solutions to differential or difference equations with random coefficients. To illustrate this, consider the problem of determining the asymptotic behavior of a random sequence $\{\xi_n\}$ satisfying $\xi_n=u_n\xi_{n-1}+v_n\xi_{n-2}$, where (u_n, v_n) forms a sequence of independent identically distributed random vectors. In this case we can write

$$\begin{pmatrix}\xi_{n+1}\\ \xi_n\end{pmatrix} = X_nX_{n-1}\cdots X_1\begin{pmatrix}\xi_1\\ \xi_0\end{pmatrix},\quad X_j = \begin{pmatrix}u_{j+1} & v_{j+1}\\ 1 & 0\end{pmatrix},$$

and so the rate of growth of the ξ_n is governed by the behavior of the matrix product $X_nX_{n-1}\cdots X_1$.

"The present investigation considers a noncompact semi-simple Lie group G and independent G-valued random variables $\{X_n\}$ with a common distribution μ on G. The law of large numbers here can be given the following form. One exhibits a finite-dimensional linear space $\mathscr{V}_G$ of functions $\psi(g)$ on G, such that to every probability measure μ that is well-behaved in a sense to be specified, there corresponds a linear functional α_μ on $\mathscr{V}_G$ and, with probability 1,

$$\psi(X_nX_{n-1}\cdots X_1) \sim n\alpha_\mu(\psi)$$

as $n\to\infty$, for each $\psi\in\mathscr{V}_G$. This may be illustrated by an example. Let G be the group of conformal transformations of the unit disc onto itself. For every $g\in G$, set $\psi'(g)$ equal to the non-euclidean distance of $g(0)$ from 0. ψ' itself does not belong to $\mathscr{V}_G$ but it differs from a function $\psi\in\mathscr{V}_G$ by a bounded function on G. Consequently, we will have a $n^{-1}\psi'(X_nX_{n-1}\cdots X_1)\to\alpha_\mu(\psi)$.

"In general, let G be any semi-simple group and Λ an irreducible representation of G by $m\times m$ matrices. Let u be a vector in R^m and let $\|\cdot\|$ denote a norm on R^m. There will be a $\psi\in\mathscr{V}_G$ such that for each distribution μ, $\log\|\Lambda(X_nX_{n-1}\cdots X_1)u\|\sim n\alpha_\mu(\psi)$. A similar relation holds, under more restricted conditions, for the angle between various column vectors of the product matrix $X_nX_{n-1}\cdots X_1$, where G is a matrix group. The logarithm of the angle is asymptotic to $n\alpha_\mu(\psi_0)$, and in certain cases one can show that $\alpha_\mu(\psi_0)<0$. This implies that the columns of the product $X_nX_{n-1}\cdots X_1$ are 'ultimately' proportional.

"In contrast to the classical situation, the computation of the limit $\alpha_\mu(\psi)$ of $n^{-1}\psi(X_nX_{n-1}\cdots X_1)$ cannot, in general, be made directly in terms of the measure μ. The expression for $\alpha_\mu(\psi)$ involves, in addition to μ, an auxiliary measure ν on a space $B(G)$ associated with the group G. It turns out that ν is determined by μ as the solution to a convolution equation $\mu*\nu=\nu$, the meaning of which will become clear presently.

"The main step in the proof that the limit of $n^{-1}\psi(X_nX_{n-1}\cdots X_1)$ exists for $\psi\in\mathscr{V}_G$ consists of showing that the above convolution equation determines a unique probability measure ν. This equation is natural in any case in view of the appearance of ν in the expression for $\alpha_\mu(\psi)$. This question is studied in some detail. If a group G acts on a space M, then the product gx is defined for $g\in G$, $x\in M$. It follows that if μ is a probability measure on G and ν a probability measure on M, then $\mu*\nu$ may be defined as the measure on M giving the distribution of gx, where g and x independently have distributions μ and

ν, respectively. One asks whether for given μ, there exists a probability measure ν on M satisfying $\mu * \nu = \nu$. Such a measure is called a stationary measure for μ. Moreover, if ν exists, one may ask if it is unique for a given space M. These two questions are related, for if we let $M^{(2)}$ denote the space of pairs of distinct points of M, then a stationary measure exists on $M^{(2)}$ if and only if there exist distinct stationary measures on M.

"As to the existence of a stationary measure for μ on M, it is shown that this is independent of the measure μ and depends on only the space M in question (Theorem 2.1). More precisely, the necessary and sufficient condition is that a certain subgroup $H(G)$ leave fixed some probability measure on M.

"The question of the existence of a stationary measure for μ on a G-space M has independent interest. A probability measure on G induces on every G-space M a 'random walk' $\{W_n\}$, where one lets W_{n+1} arise by applying a random g to W_n, g having distribution μ. The author then inquires as to the nature of this random walk. There are at least three possibilities that may be distinguished. The walk may be transient, that is, with probability 1, W_n may ultimately leave every compact set. It may be null-recurrent, that is, W_n may return with probability 1 infinitely often to some compact set, but the expected time for return to the set from a point of M will be infinite. Finally, the expected time may be finite; this is sometimes called the positively recurrent case. (These definitions are informal and only serve to give an idea of the possibilities.) It may be seen that the latter case will occur if and only if μ possesses a stationary measure on M.

"The question of the positive recurrence of W_n depends only on the state space M and not on the measure μ generating the random walk. When M is compact, there will always be positive recurrence. For certain groups G, under the assumption that $M = G/L$ and L is a connected subgroup, one can say conversely that if M is not compact, the walk is either null-recurrent or transient.

"For certain M one can also assert the transience of the random walk independent of μ. For example, if G is any noncompact semi-simple group, the induced random walk on G itself is always transient. However, there are homogeneous G-spaces for which the transience or recurrence of the W_n does depend on the measure μ.

"The paper is divided into two parts, the first of which is concerned with the question of stationary measures and the second with laws of large numbers. Part II depends upon Part I primarily in the application of the uniqueness criterion for a stationary measure."

G.-C. Rota (Cambridge, Mass.)

Rosenberg, Milton 649

The square-integrability of matrix-valued functions with respect to a non-negative Hermitian measure.

Duke Math. J. **31** (1964), 291–298.

Let $\mathbf{M} = [M_{ij}]$ be a non-negative Hermitian-matrix-valued countably additive set function on a σ-algebra $\mathfrak{B}$ of subsets of a space Ω, i.e., each element M_{ij} is a countably additive real-valued set function on $\mathfrak{B}$. The problem is to define the integral $\int_\Omega \mathbf{\Phi} \cdot d\mathbf{M} \cdot \mathbf{\Psi}^*$, where $\mathbf{\Phi}, \mathbf{\Psi}$ are matrix-valued functions, so that the space $\mathbf{L}_{2,\mathbf{M}}$ of functions $\mathbf{\Phi}$ for which $\int_\Omega \mathbf{\Phi} \cdot d\mathbf{M} \cdot \mathbf{\Phi}^*$ exists is a Hilbert space under the inner product $(\mathbf{\Phi}, \mathbf{\Psi})_\mathbf{M} = \operatorname{tr}\{\int_\Omega \mathbf{\Phi} \cdot d\mathbf{M} \cdot \mathbf{\Psi}^*\}$.

This problem has been discussed by Wiener and Masani [Acta Math. **98** (1957), 111–150; MR **20** #4323; ibid. **99** (1958), 93–137; MR **20** #4325] and Rozanov [Uspehi Mat. Nauk **13** (1958), no. 2 (80), 93–142; MR **22** #5076] in the case where Ω is either $(0, 2\pi]$ or $(-\infty, \infty)$ and the measure $\mathbf{M}$ is a Lebesgue-Stieltjes measure determined by a non-negative Hermitian-matrix-valued absolutely continuous function $\mathbf{F}$. The author solves the problem on an arbitrary $(\Omega, \mathfrak{B})$ without the requirement of absolute continuity. This is achieved by showing that $\mathbf{M}$ is always absolutely continuous with respect to the positive measure $\operatorname{tr} \mathbf{M}$. This means that there is a matrix-valued function $\mathbf{M}'(\omega)$ $(\omega \in \Omega)$ (the Radon-Nikodým derivative of $\mathbf{M}$ with respect to $\operatorname{tr} \mathbf{M}$) such that

$$\int_A \mathbf{M}'(\omega)\, d(\operatorname{tr} \mathbf{M}) = \mathbf{M}(A) \quad (A \in \mathfrak{B}).$$

Thus $\int_\Omega \mathbf{\Phi} \cdot d\mathbf{M} \cdot \mathbf{\Psi}^*$ can be defined as $\int_\Omega \mathbf{\Phi} \cdot \mathbf{M}' \cdot \mathbf{\Psi}^*\, d(\operatorname{tr} \mathbf{M})$. The author then goes on to prove his main result, that $\mathbf{L}_{2,\mathbf{M}}$ is a Hilbert space.

The applications to stationary stochastic processes, as discussed by Wiener and Masani [loc. cit.] are summarised in the final two sections. *G. G. Gould* (Syracuse, N.Y.)

Blackwell, David 650

Probability bounds via dynamic programming.

Proc. Sympos. Appl. Math., Vol. XVI, pp. 277–280. *Amer. Math. Soc., Providence, R.I.*, 1964.

The author's aim is to show how a dynamic programming idea can be applied to obtain certain interesting probability upper bounds. Let M be any set of probability measures on the Borel sets of n-space E_n. A sequence $\{X_n\}$ of n-dimensional random sequences is called an M-sequence if M contains (i) the distribution of X_1, and (ii) the conditional distribution of X_{k+1}, given $X_1, \cdots, X_k$, for every integer k. Fix $R \subset E_n$ and $r \in R$. Define $S_k = r + X_1 + \cdots + X_k$. Let N be the smallest k for which $S_k \notin R$. Define $v(r) = 0$ if $r \notin R$; otherwise $v(r) = \min EN$, where the minimum is taken as $\{X_k\}$ varies over all M-sequences. To find an upper bound for v (that is, a function f such that $v \leq f$), it suffices to find a non-negative function f such that $Tf \leq f$, where $Tf = \sup_{m \in M} T_m f$ and the transformation T_m is defined by

$$T_m f(r) = 1 + \int f(r+x)\, dm(x)$$

for $r \in R$ and is 0 for $r \notin R$.

E. Parzen (Stanford, Calif.)

Cramér, Harald 651

On the linear prediction problem for certain stochastic processes.

Ark. Mat **4**, 45–53 (1960).

The author recalls one of his earlier results that a stochastic process $\{x_n\}$ in discrete time can be represented by

$$x_n = \sum_{r<n} c_{nr_k} \xi_{r_k} + y_n,$$

where $\{y_n\}$ is linearly deterministic, the ξ_r are uncorrelated with one another or with $\{y_n\}$, and the r_k constitute a uniquely determined sequence of integers (the points of indetermination).

The author now considers a class of harmonisable

processes $x_n = \int_0^{2\pi} e^{inu}\, dz(u)$ for which $F(u, v) = E[z(u)\overline{z(v)}]$ satisfies some moderate regularity conditions, and determines a necessary and sufficient condition (in the form of a representation of F) that the process $\{x_n\}$ should have a given set of points of indetermination.

P. Whittle (Manchester)

Varga, László 652

On a stochastic process concerning the logarithmic counting rate meter. (Hungarian. English and Russian summaries)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **7** (1962), 479–485 (1963).

Author's summary: "This paper deals with the following stochastic process:

$$\eta_t = (2\xi_t - 1) \int_0^t A(t-x)\, d\xi_x,$$

where $A(t)$ is continuous and of bounded variation in $(0, \infty)$; ξ_t is a secondary process generated by means of a homogeneous Poisson process: $\xi_t = 0$ if the numbers of events are, occurring in the time interval $(0, t]$, even and $\xi_t = 1$ otherwise. In the present paper the mean value, the dispersion and the correlation function of the η_t is given. Finally an adjustment problem of the logarithmic counting rate meter is treated."

Malevič, T. L. 653

On the asymptotic behaviour of the estimate for the spectral function of a stationary Gaussian process. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 386–390.

Author's summary: "Let ξ_n $(n = 0, \pm 1, \pm 2, \cdots)$ be a real stationary Gaussian sequence with an absolutely continuous spectral function $F(\lambda)$, and let $F_N(\lambda)$ be the sample spectral function. We assume that $F(\lambda)$ has no interval of constancy, and that $f(\lambda) = F'(\lambda) \in L_2[0, \pi]$. Then the sequence of measures P_N generated by the process $\zeta_N(\lambda) = N^{1/2}[F_N(\lambda) - F(\lambda)]$ converges weakly to the measure which is generated by the Gaussian process $\zeta(\lambda)$ with $\mathrm{M}\zeta(\lambda) = 0$ and

$$\mathrm{M}\zeta(\lambda)\zeta(\mu) = 2\pi \int_0^{\min(\lambda,\mu)} f^2(x)\, dx.$$

A similar result holds for the process ξ_t with continuous time $(0 \le t < +\infty)$."

F. L. Spitzer (Ithaca, N.Y.)

Babkin, V. I. 654

On the distribution of maximum points of a diffusion process with variable parameter. (Russian. German summary)

Teor. Verojatnost. i Primenen. **9** (1964), 373–378.

The author studies diffusion processes which admit a change in parameter at one instant. The distribution function, the density function, the characteristic function and the first four moments of the maximum of such a process are determined.

E. Lukacs (Washington, D.C.)

Hobby, Charles; Ylvisaker, N. Donald 655

Some structure theorems for stationary probability measures on finite state sequences.

Ann. Math. Statist. **35** (1964), 550–556.

Let P be a stationary measure on the doubly infinite product space $\prod_{k=-\infty}^{\infty} X_k$, where every $X_k = X = \{0, 1, \cdots, q-1\}$. Let X^n be $\prod_{k=1}^{n} X_k$ and let P_n denote the measure induced on X^n by P. The authors study properties of measures P_n induced by some P and consider also the converse problem: Given P_n, find some extremal measures P which reduce to P_n. Let N_n denote the number of points $(X_1, \cdots, X_n)$ having positive P_n-measure. Obviously $N_n \le q^n$. Two kinds of extensions from P_n to P are considered: A Markov type extension which gives largest possible values of N_{n+1} and an extension with minimal N_{n+1}. Some theorems on the growth of N_{n+k} are derived.

J. E. Cigler (Vienna)

Georgiou, P. 656

Random variables with values in Banach spaces and their probability distributions. (Greek. German summary)

Bull. Soc. Math. Grèce (N.S.) **4** (1963), *no.* 2, 1–87.

This is a detailed exposition of the elements of an abstract theory of random variables on a Boolean algebra with values in a Banach space. The definitions and proofs are natural adaptations of the corresponding ideas for scalar-valued variables. The arguments are in some instances somewhat unclear, and at one point (Lemma 6.3.2) an independent assumption appears to have been made in the course of the proof. The reviewer has also experienced difficulty in finding some connection between the numerous references cited in the footnotes and the text. We proceed to outline.

A simple random variable over a non-atomic Boolean algebra F with values in a Banach space X is naturally defined as a mapping $\chi: a_i \to x_i \in X$, where $a_1, \cdots, a_n$ are disjoint and have the identity as union; algebraic operations are defined in the standard way. Assuming F equipped with a σ-additive probability measure w, convergence in probability (w-convergence) is introduced and the set of all random variables is defined as the completion E of the space E' of simple random variables. The various structures of E' are then extended to E, and expectation values are introduced through weighted sums and extension by continuity. For a simple random variable $\chi: a_i \to x_i$ let $\chi^{-1}(B) = \bigcup \{a_i | x_i \in B\}$ and $\mu_\chi^*(B) = w(\chi^{-1}(B))$; take any algebra $\mathscr{B}$ of subsets of X, χ a random variable to which the sequence (χ_n) of simple variables w-converges and consider the subalgebra $\mathscr{B}_\chi$ of $\mathscr{B}$ which consists of all B such that for any $\delta > 0$ there are ε, $N > 0$ such that $\mu_{\chi_n}^*(B_\varepsilon) < 0$ for all $n > N$ (here B_ε is the ε-neighborhood of B). Then $\lim \mu_{\chi_n}^*(B)$ exists, is independent of (χ_n) and defines a σ-additive probability measure on $\mathscr{B}_\chi$—the probability distribution of χ. For any B in $\mathscr{B}_\chi$ the sequence $(\chi_n^{-1}(B))$ converges in probability to an element $\chi^{-1}(B) \in F$ and the map χ^{-1} (the spectrum of χ) is a measure-preserving σ-homomorphism of $\mathscr{B}_\chi$ into F which determines χ uniquely. Using this, the concept of independence is defined and studied.

P. C. Deliyannis (Chicago, Ill.)

Benayoun, Raphaël 657

Sur un modèle stochastique utilisé dans la théorie mathématique des épidémies.

C. R. Acad. Sci. Paris **258** (1964), 5789–5791.

In this note, the author considers Bailey's [Biometrika

50 (1963), 235–240; MR **28** #5500] suggestion that the number of susceptibles n in the differential-difference equations for a simple epidemic be replaced by $N=n+\varepsilon$. This is done in order to derive more simply from their Laplace transforms, as $\varepsilon\to 0$, explicit expressions for the probabilities $p_r(\tau)$ that there are r susceptibles at time τ. The author points out the labor-saving advantages of the method. *J. Gani* (E. Lansing, Mich.)

Baumann, Wilhelm 658

★Über Lückenprozesse mit beschränkten Lücken.

Inauguraldissertation der Naturwissenschaftlichen Fakultät der Friedrich-Alexander-Universität Erlangen-Nürnberg zur Erlangung des Doktorgrades.

Naturwissenschaftliche Fakultät der Universität Erlangen-Nürnberg, Erlangen-Nürnberg, 1962. ii + 36 *pp.*

The author studies Markovian gap-processes which are defined as follows: Let $\{X_n : n\geqq 0\}$ be a sequence of random variables which take on the values 0 or 1. Let $X_0=1$, and let $\{L_k : k\geqq 1\}$ be the lengths of the successive gaps between occurrences of state 1. A Markovian gap-process is one for which $\{L_k : k\geqq 1\}$ is a homogeneous Markov chain. It is therefore the same as a two-state, discrete parameter Markov renewal process as studied for example by the reviewer [Ann. Math. Statist. **32** (1961), 1243–1259; MR **27** #4273]. The author's main interest is the spectral analysis of the process. It is shown that the covariance function of the process (given the natural initial measure to make it stationary) is absolutely convergent, and admits a spectral density function which is rational in $\exp(2\pi i\lambda)$. Several examples are given. *R. Pyke* (Seattle, Wash.)

Watanabe, Takesi 659

A remark on the strong Markov property.

Mem. Fac. Sci. Kyushu Univ. Ser. A **17** (1963), 176–180.

From the author's introduction: "The purpose of this note is to prove a theorem on the strong Markov property which has been essentially known but whose explicit proof could not be found in any references."

Let $X=(X_t, \zeta, M_t, P_x, \theta_t)$ be a measurable stationary Markov process in the notation of Dynkin [E. B. Dynkin, *Theory of Markov processes*, Prentice-Hall, Englewood Cliffs, N.J., 1961; MR **24** #A1747]. Set $\Omega_t=\{\zeta>t\}$. Consider the following properties of a family of Borel fields, $\{\mathcal{N}, \mathcal{N}_t : t\geqq 0\}$. (i) $\mathcal{N}_t$ is a Borel field in Ω_t for each $t\geqq 0$; (ii) if $t\geqq s$ and $A\in\mathcal{N}_s$, then $A\cap\Omega_t\in N_t$; (iii) $\mathcal{N}_t\subset\mathcal{N}$ for each $t\geqq 0$; (iv) $\mathcal{N}_t\subset\mathcal{N}_t^{(x)}$ and $\mathcal{N}\subset\mathcal{N}^{(x)}$ for a given x in the state space where $\mathcal{N}^{(x)}$ is the completed Borel field of $\mathcal{N}$ relative to the measure P_x and $\mathcal{N}_t^{(x)}$ is the Borel field of all subsets of Ω_t which are in $\mathcal{N}^{(x)}$ and which differ by at most a set of P_x-measure 0 from a set in $\mathcal{N}_t$. The theorem referred to in the above quotation states that if the process X is strongly Markovian relative to the family $\{\mathcal{M}, \mathcal{M}_t\}$ at x, it is also thus relative to any family $\{\mathcal{N}, \mathcal{N}_t\}$ which satisfies properties (i)–(iv) above. *R. Pyke* (Seattle, Wash.)

Rosenblatt-Roth, Millu 660

Sur la stabilité relative forte des sommes des variables aléatoires en chaîne de Markoff non homogène.

C. R. Acad. Sci. Paris **258** (1964), 3628–3630.

Four theorems concerning strong stability of sums of random variables are given without proofs. For example, if $\{\xi_n\}$ is a Markov chain such that $E\xi_n\geqq 0$,

$$1-\sup_{x,y,A}|P(\xi_{n+1}\in A|\xi_n=x)-P(\xi_{n+1}\in A|\xi_n=y)|>\lambda>0,$$

$E\zeta_n\to\infty$ and $\sum_{n=1}^{\infty}[E\zeta_n]^{-2}\operatorname{var}\zeta_n<\infty$, where $\zeta_n=\sum_{i=1}^{n}\xi_i$, then $d_n^{-1}\zeta_n\to 1$ with probability 1 for certain constants d_n. *J. Hájek* (Prague)

Kunita, Hiroshi; Watanabe, Takesi 661

Notes on transformations of Markov processes connected with multiplicative functionals.

Mem. Fac. Sci. Kyushu Univ. Ser. A **17** (1963), 181–191.

In this paper the authors "show that those regularity properties of Markov processes such as the right continuity of trajectories, the strictly Markov property and the quasi-continuity from the left are invariant under the transformation of a right continuous multiplicative functional whose expectation is less than or equal to 1". *R. Pyke* (Seattle, Wash.)

Nikol'skiĭ, M. S. 662

The probability of a Markov point hitting a planar region of small diameter. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 352–357.

Let the Markov process $x(t)$ in euclidean n-space E_n have the transition probability density $p(\sigma, x, \tau, y)$ which satisfies the Kolmogorov equation

$$\partial p/\partial\sigma+\sum_{i,j}a^{ij}(\sigma, x)\,\partial^2 p/\partial x^i\,\partial x^j+\sum_i b^i(\sigma, x)\,\partial p/\partial x^i=0.$$

Let $z(t)$ be a non-random path in E_n and S_t the sphere of radius ε centered at $z(t)$. Let $\phi(\sigma, x, \tau)$ denote the probability that $x(t)\in S_t$ for some $t\in[\sigma, \tau]$ under the condition $x(\sigma)=x$. The author determines under certain regularity assumptions the asymptotic value of $\phi(\sigma, x, \tau)$ as $\varepsilon\to 0$ in the case $n=2$. (For $n\geqq 3$ see Miščenko and Pontrjagin [Izv. Akad. Nauk SSSR Ser. Mat. **25** (1961), 477–498; MR **27** #6661] and Kolmogorov, Miščenko and Pontrjagin [Dokl. Akad. Nauk SSSR **145** (1962), 993–995].) *W. Hoeffding* (Chapel Hill, N.C.)

Wu, Li-de [Wu, Li-te] 663

On the distributions of integral functionals of homogeneous denumerable Markov processes.

Acta Math. Sinica **13** (1963), 86–93 *(Chinese); translated as Chinese Math.* **4** (1963), 97–104.

Let $x(t)$ be a denumerable homogeneous Markov chain with transition matrix $P(t)$, strongly continuous at zero, none of whose elements is identically zero. Further, assume that the backward Kolmogorov equation holds. Consider $\xi^n(\omega)=\int_0^{\tau^{(n)}}V(x(t))\,dt$, where V is a non-negative function from the positive integers to the real numbers and $\tau^{(n)}$ is the first entrance time of $x(t)$ to the set $\{n, n+1, \cdots\}$. The author shows the distribution of $\xi^n F_{kn}(t)=P\{\xi^n\leqq t|x(0)=k)$ can be identified with the transition probability $\bar{\bar{p}}_{kn}(t)$ of the process $\bar{\bar{x}}(t)$ obtained from $x(t)$ by first stopping it at $(n, n+1, \cdots)$ and then running the stopped process with a random clock determined by $V(x)$. Consequently $F_{kn}(t)$ satisfies the backward Kolmogorov equation of the $\bar{\bar{x}}$ process. Questions of

finiteness of the moments of ξ^n and whether $\lim_{n\to\infty} \xi^n$ is a random variable are considered.

R. S. Bucy (Baltimore, Md.)

Harris, Theodore E. 664

★The theory of branching processes.

Die Grundlehren der Mathematischen Wissenschaften, Bd. 119.

Springer-Verlag, Berlin; Prentice-Hall, Inc., Englewood Cliffs, N.J., 1963. xiv + 230 *pp.* $9.00.

This work is a comprehensive and rigorous account of branching (i.e., multiplicative) processes and their applications. Chapter I deals with the classical Galton-Watson process and its asymptotic behaviour. Chapter II considers the generalisation which arises when we suppose that the multiplying particles are of several different types with different multiplicative properties. The next chapter considers the further generalisation to point processes in which the "type" of the particle is defined by a point in a continuous region of Euclidean space (e.g., if it has an energy). Chapter IV uses the results of the previous chapter to consider neutron processes in the particular case where all have the same energy and the scattering is isotropic. Chapter V extends previous results to continuous time, and Chapter VI to processes in which the probability of multiplication is age-dependent. The last chapter deals with electron-photon cascades. A good bibliography completes a book which must now be essential reading for all those working in these fields.

P. A. P. Moran (Canberra)

Karlin, Samuel; McGregor, James 665

Direct product branching processes and related Markov chains.

Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 598–602.

The authors introduce a class of branching processes with individuals of p types, where each individual yields progeny of all types. The model allows for mutation, migration into the population, and selection. Associated Markov chains are set up by fixing the population size. In certain cases the eigenvalues of the transition probability matrix are characterized in terms of the parameters of the process and given a probabilistic interpretation. Limiting processes are obtained for these Markov chains by letting the population size tend to infinity.

D. R. Truax (Eugene, Ore.)

Adke, S. R. 666

A stochastic population diffusing on a finite interval.

J. Indian Statist. Assoc. **2** (1964), 32–40.

The author continues work started jointly with J. E. Moyal [J. Math. Anal. Appl. 7 (1963), 209–224; MR 28 #1664] on a model of a population diffusing on the real line according to Brownian motion and at the same time undergoing a birth and death process. In this paper the population is restricted to a finite interval I with reflecting endpoints, the individual birth rate λ and death rate μ are allowed to be functions of the time t, and the individuals in the population are assumed to be indistinguishable. Starting with a single ancestor at x at time τ, the problem is to obtain the distribution of the population size n at time t, as well as the conditional spatial distribution, given n. Generating functions are derived from which these quantities can be computed. If the integrals of $\lambda(t)$ and $\mu(t)$ from τ to ∞ are finite, then, as $t\to\infty$, n has a limiting distribution, and given n, the asymptotic spatial distribution is that of n independent random variables, each uniformly distributed on I. If λ and μ are constant, the asymptotic behavior is less simple, but in any case the asymptotic spatial distribution, given n, is independent of x.

R. A. Wijsman (Urbana, Ill.)

Ruben, Harold 667

Generalised concentration fluctuations under diffusion equilibrium.

J. Appl. Probability **1** (1964), 47–68.

From the author's summary: "Smoluchowski's classical analysis of the temporal fluctuation, under diffusion equilibrium, of the number of particles in a fixed region R of space is generalised to a set of disjoint regions: specifically, the single Smoluchowski region is divided into a finite number of non-intersecting subregions. The generalisation allows a more rigorous test of some of the consequences of the Einstein-Smoluchowski theory of Brownian motion to be carried out, and at the same time enables the Avogadro constant to be estimated with greater precision than is possible with the single region. In particular, the reversibility paradox of Loschmidt and the recurrence paradox of Zermelo are re-examined from the point of view of the fluctuations of configurations (a configuration being defined as the set of occupation numbers for the various subregions) rather than that of total concentration for the single region."

G. Baxter (Minneapolis, Minn.)

Kesten, Harry 668

Ratio theorems for random walks. II.

J. Analyse Math. **11** (1963), 323–379.

The author considers a homogeneous discrete random walk on a d-dimensional lattice R and with a slightly strengthened aperiodicity condition. Ω denotes any subset of R. In standard notation $P^n(x, y) = P\{\chi_n = y \mid \chi_0 = x\}$; $Q_\Omega{}^n(x, y) = P\{\chi_n = y', \ \chi_R \notin \Omega, \ 1 \leqq k \leqq n-1 \mid \chi_0 = x\}$; $u_n = P^n(0, 0)$; f_n is the probability of this being the first return; T_Ω is the least value of n such that $\chi_n \in \Omega$ (if none such exists, $T_\Omega = \infty$); $a(x) = \sum_{n=0}^\infty [P^n(0, 0) - P^n(0, x)]$; $\tilde{g}_\Omega(x, y) = \sum_{n=0}^\infty Q_\Omega{}^n(x, y)$; $\Pi_\Omega(x, y) = \sum_{n=1}^\infty Q_\Omega{}^{(n)}(x, y)$ if $y \in \Omega$ and is zero otherwise. He also defines $\tilde{g}_\Omega(x) = \lim_{|y|\to\infty} \tilde{g}_\Omega(x, y)$ if [case (a)] the walk is persistent and $d = 1$, or [case (b)] $d = 1$, $\sigma^2 = \sum x^2 P(0, x) = \infty$; $g_\Omega(x) = \lim_{y\to\infty} \frac{1}{2}\{\tilde{g}_\Omega(x, y) + \tilde{g}_\Omega(x, -y)\}$ if [case (c)] the walk is persistent, $d = 1$, $\sigma^2 < \infty$; and $\tilde{g}_\Omega(x) = P_x\{T_\Omega = \infty\} \sum_{n=0}^\infty u_n$ for a transient walk [case (d)], where, in general, $P_x\{E\}$ is used to denote the probability of some event E for walks beginning at x.

The following are a few examples out of an impressive wealth of results. In case (c), if $\sum x P(0, x) = 0$, $\lim_{n\to\infty} f_n{}^{-1} Q_{\{0\}}{}^n(x, y) = \tilde{g}_{\{0\}}(x)\tilde{g}_{\{0\}}(-y) + xy\sigma^{-4}$. If $\phi(t)$ is the characteristic function of the distribution of individual steps and if $0 < \lim_{t\to 0} |t|^{-\alpha}(1 - \phi(t)) = Q < \infty$ for some $1 \leqq \alpha \leqq 2$, then, in cases (b) and (c),

$$f_n \sim (\alpha - 1) \sin(\pi/\alpha)\{\Gamma(1/\alpha)\}^{-1} Q^{1/2} n^{1/\alpha - 2} \quad \text{for } \alpha > 1,$$

$$\sim \pi Q_n{}^{-1} (\log n)^{-2} \quad \text{for } \alpha = 1.$$

Under the same conditions

$$\lim_{n\to\infty} f_n^{-1} \sum_{u,v\in\Omega} Q_\Omega^n(u,v) = 1.$$

The proofs are based on classical function theory and use the same probability concepts as in Part I [the author and F. Spitzer, same J. **11**(1963), 285–322; MR **28** #5478]. An important connection between the probability and function-theoretic ideas is established by means of a theorem, of which the following is part: Let s be any positive real d-vector $(s_1, s_2, \cdots, s_d)$ and $P(x_1, x_2, \cdots, x_d)$ the probability of the step $(x_1, \cdots, x_d)$, let $f(s) = \sum_x P(x_1, \cdots, x_d) s_1^{x_1} \cdots s_d^{x_d}$. Then there exists a positive d-vector s_0 such that $f(s_0) = \mathrm{Min}_{s>0} f(s) < \infty$. Also $u_n^{1/n}$ and u_{n+1}/u_n both tend to $f(s_0)$ as n tends to infinity. This theorem yields substantial extensions and applications of some earlier results of Chung and Erdős [Mem. Amer. Math. Soc. No. 6 (1951); MR **12**, 722]. *J. Gillis* (Rehovot)

Lévy, Paul 669

Le déterminisme de la fonction brownienne dans l'espace de Hilbert. II.

Ann. Sci. École Norm. Sup. (3) **80** (1963), 193–212.

This paper is a continuation of the author's previous work on the Brownian motion $X(A)$ with a parameter A ranging over a Hilbert space Ω [same Ann. (3) **79** (1962), 377–398; MR **27** #830]. The probability that for any $O \in \Omega$, there exists a continuous path approaching O, along which as $A \to O$, $X(A)$ has any finite and infinite limiting values, is equal to 1. Let $S \subset \Omega$ be a spherical surface with center O and radius R, $\{O_{x_k}\}_1^\infty$ rectangular axes, s_n the intersection of the plane $O_{x_1 \cdots x_n}$ and S, and suppose there is given a function $f(A)$ over S; then, according to R. Gâteaux, the mean value of $f(A)$ over S is defined to be the limit, as $n \to \infty$, of the mean value of $f(A)$ over s_n. Suppose that Ω is the product of its subspaces Ω_1, Ω_2 with origin O in common, and let S_1, S_2 be spherical surfaces respectively in Ω_1, Ω_2, with center O and radius R in common. The author now proves that with probability 1 the mean values of $X(A)$ over S, S_1, S_2, exist and are equal. Suppose next that Ω is the product of subspaces Ω_1, Ω_2, orthogonal to each other, with center O in common. Given $A \in \Omega$, let x and y be the projections of OA, respectively, over Ω_1 and Ω_2. A mapping (1) $y = f(x)$ defines a surface S whose dimension is to be dim Ω_1. S is called an $(\omega - n)$-surface when dim $\Omega_1 = \infty$, dim $\Omega_2 = n < \infty$. The mapping (1) is analytic in an open set $V \subset \Omega_1$ when each of its components $y_h = f_h(x)$, $1 \leq h \leq n$, has a Taylor series $f_h(x) = \sum_1^\infty p_k(x-a)$, convergent in a sphere about any $a \in V$, where $p_k(x)$ is a homogeneous polynomial of degree k in x_m, the mth component of x. Given an $(\omega - n)$-surface analytic in V, we prolongate it to obtain an analytic $(\omega - n)$-surface S. The author proves that if $X(A)$ is given in an open subset $\mathscr{E}$ of S, then $X(A)$ over S is completely determined. *G. Maruyama* (Fukuoka)

Lévy, Paul 670

Le mouvement brownien fonction d'un ou de plusieurs paramètres.

Rend. Mat. e Appl. (5) **22** (1963), 24–101.

This is a survey of the theory of the Brownian motion process based on lectures given at the Mathematical Institute of the University of Rome in April 1962. The author has been a leading contributor to the development of this theory, and he reports on his own work as well as on that of others. Though the history of the subject is not neglected, the emphasis is on recent developments. Many proofs are at least indicated and, possibly due to the omission of details, their governing ideas are clearly brought forward. Many results are, of course, only mentioned, but their relation to previous work is indicated. Various open problems are stated.

Contents: (I) The function $X(t)$ of Bachelier-Wiener. Existence and continuity (20 pages). (II) Various properties of $X(t)$. Extension to Brownian motion in n-dimensional space (15 pages). (III) Brownian motion depending on N parameters (13 pages). (IV) General study of Gaussian functions (17 pages). (V) Determinism of $X(A)$ in Hilbert space (8 pages). (The results of this chapter were not covered by the author's lectures. He first announced them in C. R. Acad. Sci. Paris **254** (1962), 3962–3964 [MR **26** #820] and a fuller treatment is in Ann. Sci. École Norm. Sup. (3) **79** (1962), 377–398 [MR **27** #830].) There is a bibliography of 25 works of the author, as well as an extensive bibliography by other authors.

A. Dvoretzky (Jerusalem)

Prabhu, N. U. 671

Time-dependent results in storage theory.

J. Appl. Probability **1** (1964), 1–46.

This paper is the first article in the first issue of the Journal of Applied Probability. It is a review article of the current status of dam theory. Its contents are best described by its chapter and section headings which are: (I) The dam process in discrete time, (1) Moran's model for the dam and an extension to it, (2) Time-dependent solutions for the discrete dam, (3) The problem of emptiness; (II) The dam process in continuous time, (4) The input process, (5) Formulation of the dam process in continuous time, (6) The distribution of storage, (7) The wet period in a dam, (8) A dam with simple Poisson inputs, (9) The queue $M/G/1$ as a storage system, (10) A dam process with non-homogeneous Poisson inputs, (11) The case of continuous inputs; (III) Some further models, (12) Gani and Pyke's model, (13) Dams with ordered inputs, (14) A model with random linear inputs and outputs, (15) Some general comments: unsolved problems and directions for research. There are some 70 references given. This paper represents a good up-to-date survey which will be of considerable use to workers in the field. *R. Pyke* (Seattle, Wash.)

Conolly, B. W. 672

On the busy period in relation to the single server queueing process with Erlangian input and general service time. (Italian, French, Spanish, and German summaries)

Giorn. Ist. Ital. Attuari **26** (1963), 217–232.

From the author's summary: "The following features of busy periods are investigated: the duration in time; the number of customers who have received service, both at an arbitrary instant during the busy period and when it terminates; and the size of the queue at an arbitrary instant during the busy period. Some checks on the formulae are given." *S. C. Port* (Santa Monica, Calif.)

Breiman, Leo 673

On some probability distributions occurring in traffic flow. (French summary)

Bull. Inst. Internat. Statist. **39** (1962), *livraison* 4, 155–161.

This precedes and announces some results later published in the author's paper [Ann. Math. Statist. **34** (1963), 308–311; MR **26** #2326]. It also describes some stochastic properties of traffic for which velocities of cars are independent, identically distributed, random variables having a Poisson distribution in space. For example, it is shown that cars with velocities in some given range must also have a Poisson distribution in space. The author views as unfortunate his conclusion that most spatial distributions must in the course of time approach a Poisson distribution, but this conclusion clearly results from his assumption that the cars do not interact.

G. Newell (Providence, R.I.)

Pacioni, Goffredo 674

Sull'impiego della teoria delle code semplici per la risoluzione approssimata di problemi relativi a code multiple. (French, English, Spanish, and German summaries)

Giorn. Ist. Ital. Attuari **26** (1963), 306–316.

It is suggested that a stochastic service system with k like servers, each with average service time Y, may be considered for approximate purposes as a single-server system with average service time Y/k (the arrivals forced to wait are supposed to equalize the several waiting lines).

J. Riordan (Murray Hill, N.J.)

Kingman, J. F. C. 675

On continuous time models in the theory of dams.

J. Austral. Math. Soc. **3** (1963), 480–487.

Let $Y(t)$ be a right-continuous function of bounded variation in every finite subinterval of $t \geqq 0$, which has no downward jumps and satisfies $Y(0)=0$. Let $z_0 \geqq 0$ and let $Y_-(s)$ be the total negative variation of Y over $(0, s]$. The author proves that the unique non-negative measurable solution of the equation $Z(t)=z_0+Y(t)+\int_0^t \zeta(s)\,dY_-(s)$, where $\zeta(t)=1$ or 0 according as $Z(t)$ is zero or positive, is

$$(*) \qquad Z(t) = \max\left\{\sup_{0\leqq s\leqq t} [Y(t)-Y(s)],\ Y(t)+z_0\right\}.$$

In application to dam models, the function Y above represents the net input, whereas Z represents the content of the dam. The importance of the theorem is that it considers the input of a dam as an almost arbitrary function over time. It also justifies the use of (*) as the definition of the content of a dam as used by J. Gani and the reviewer [J. Math. Mech. **9** (1960), 639–651; MR **22** #12606].

R. Pyke (Seattle, Wash.)

Gel'fond, A. O. 676

An estimate for the remainder term in the limit theorem for recurrent events. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 327–331.

The author describes a new approach to the proof of the renewal theorem. Let $\{p_n\}$ be a given probability measure on the positive integers which is aperiodic (the g.c.d. of the set of n such that $p_n>0$ is one) and has finite νth moment $\sum n^\nu p_n$ for some integer $\nu \geqq 1$. Let a_n be the usual renewal sequence, uniquely determined by $a_0=1$ and $a_n=\sum_{k=1}^n p_k a_{n-k}$ for $n\geqq 1$. As usual, the first moment $\sum kp_k$ is denoted by μ. In this setting it is asserted that

$$a_n = \mu^{-1}+\mu^{-2}\sum_{k=n+1}^{\infty}\sum_{j=k+1}^{\infty} p_j + R_n,$$

where the error term R_n is of the order $O(n^{-\nu}\log n)$ as $n\to\infty$. This result is considerably stronger than the classical theorems, even in the cases $\nu=1$ and $\nu=2$. The proof, which proceeds by representing the error term R_n as an integral over the unit circle, is quite elementary. However, the estimate given for the integrand on page 330, line 8 from the bottom, seems to be false; in the case when $\nu=1$ it may be corrected by replacing the estimate by its square. With this modification the author's method will indeed yield the classical renewal theorem (i.e., the fact that $R_n\to 0$), but it is not clear to the reviewer whether the theorem is correct as stated above.

F. L. Spitzer (Ithaca, N.Y.)

Jewell, William S. 677

Markov-renewal programming. I. Formulation, finite return models.

Operations Res. **11** (1963), 938–948.

From the author's summary: "A special structure in dynamic programming which has been studied (by several authors) is the problem of programming over a Markov chain. This paper extends their results and solution algorithms to programming over a Markov-renewal process—in which the intervals between transitions of the system from state i to state j are independent samples from a distribution that may depend on both i and j. For these processes, a general reward structure and a decision mechanism are postulated; the problem is to make decisions at each transition to maximize the total expected reward at the end of the planning horizon. The paper is divided into two parts. This (paper) describes the properties of Markov-renewal processes, the reward structure, and the decision process. Algorithms for finite-horizon and infinite-horizon models with discounting are presented."

R. Pyke (Seattle, Wash.)

Jewell, William S. 678

Markov-renewal programming. II. Infinite return models, example.

Operations Res. **11** (1963), 949–971.

From the author's summary: "This paper is a continuation of a previous one [see #677 above] which investigates programming over a Markov-renewal process. Given a reward structure, and a decision mechanism that influences both the rewards and the Markov-renewal process, the problem is to select alternatives at each transition so as to maximize total expected reward. In this paper, we investigate the infinite-return models where it becomes necessary to consider only stationary policies that maximize the dominant term in the reward. It is then important to specify whether the limiting experiment is: (i) undiscounted, with the number of transitions $n\to\infty$, (ii) undiscounted, with a time horizon $t\to\infty$, or (iii) discounted, infinite n or t, with discount factor $\alpha\to 0$. In each

case, a limiting form for the total expected reward is shown, and an algorithm developed to maximize the rate of return. The problem of finding the optimal or near-optimal policies in the case of ties is still computationally unresolved. Extensions to nonergodic processes are indicated, and special results for the two-state process are presented. Finally, an example of machine maintenance and repair is used to illustrate the generality of the models and the special problems that may arise."

R. Pyke (Seattle, Wash.)

STATISTICS

See also 640, 657, 665, 1087.

Birnbaum, Z. W. 679

★Introduction to probability and mathematical statistics. *Harper & Brothers, Publishers, New York*, 1962. viii + 325 *pp*. $6.50.

This is an introductory text on probability and statistics. The first 163 pages provide a careful mathematical development of probability theory (axiomatic definition of probability, discrete and continuous n-dimensional random variables, transformations of random variables, expectation, characteristic functions, regression and correlation). The next 140 pages discuss some procedures of statistical inference (estimation of means, distribution theory of estimates in the normal case, hypothesis tests, multiple correlation and regression, goodness-of-fit tests using χ^2, some non-parametric techniques, decision theory). While the emphasis throughout the book is on the statement of theorems rather than the handling of data, a number of numerical examples and exercises are given. However, the author states that the book does not aim to teach the elements of statistical technique. The book satisfactorily achieves its aim as a text for students with good mathematical background being introduced to mathematical statistics.

E. Parzen (Stanford, Calif.)

Freeman, Harold 680

★Introduction to statistical inference. *Addison-Wesley Publishing Co., Inc., Reading, Mass.-London*, 1963. xiv + 445 *pp*. $7.75.

This book provides a concise and lucid description of the basic problems of statistical inference, sampling and distribution theory. It also gives an elementary treatment of probability. In a way this treatment should be very useful to the beginner because the danger of getting too involved algebraically is minimised.

The book is primarily intended for the undergraduates in mathematical statistics but it should prove useful to others also who are interested in making a wise use of the statistical tools in their respective fields of specialisation. Some elementary training in mathematics (especially calculus and coordinate geometry) is a prerequisite of the book.

The author does not discuss some important topics like decision theory, sequential analysis, time series analysis, etc., perhaps to save space and not to make things messy. Discussion of most of the topics is so short that the student has to look for other sources. As the author is quite aware of these drawbacks, he provides a detailed list of 211 references. He also provides simultaneously a list of selected text books.

The book is divided into five parts, viz., (1) probability and random variables, (2) specific probability distributions, (3) sampling theory, (4) statistical inference and (5) an application to regression. The section on probability and random variables contains six chapters covering the discussion on the fundamental rules of probability, an axiomatic approach to the probability concept of random variable, expectation, moment generating function, marginal and conditional distribution and conditional expectation, etc. The second section on specific probability distributions gives the binomial, Poisson, negative binomial, multinomial, and normal distributions, along with their properties. Gram-Charlier series and Hermite polynomials are given in the last chapter of this section. It is useful to put all these important distributions together in one section for a ready reference. It is, however, not understandable why gamma and beta distributions are omitted and the Pearson family is not introduced at all. The section on sampling theory introduces random sampling from general populations as well as from a normal population. It gives important tests of significance and the distributions of relevant statistics in this connection (like t, F, and chi-square). This section also includes a chapter on order statistics. The Bayesian inference along with the theory of point estimation, interval estimation and testing of statistical hypotheses are the subject matter of Section 4. Last of all in Section 5 the author presents regression analysis very briefly but covers its salient features.

To sum up, the order of presentation is very systematic and elegant. However, the omission of important topics and the brevity of discussion of the topics chosen are felt strongly while reading the book.

A. L. Nagar (Delhi)

Williamson, Eric; Bretherton, Michael H. 681

★Tables of the negative binomial probability distribution. *John Wiley & Sons, London-New York*, 1963. 275 *pp*. $13.50.

The terms of the negative binomial distribution are formed by the expansion of $p^k(1-q)^{-k}$. The probability of exactly n events ($n=0, 1, \cdots$) is given by the $(n+1)$st term in this expansion, that is, by $P(n)=\binom{n+k-1}{k-1}p^kq^n$. After the first term, $P(0)=p^k$, has been computed, successive terms can be computed from the recurrence relation $P(n)=q(n+k-1)P(n-1)/n$. The cumulative probability of n or fewer events is given by $F(n)=\sum_{r=0}^{n}\binom{r+k-1}{k-1}p^kq^r$, and is obtained by summing the first $(n+1)$ terms of the distribution. Tables are included in this volume for the following 1480 pairs of values of p and k:

$$\begin{aligned}
p &= 0.05, & k &= 0.1(0.1)0.5;\\
p &= 0.10, & k &= 0.1(0.1)1.0;\\
p &= 0.12(0.02)0.20, & k &= 0.1(0.1)2.5;\\
p &= 0.22(0.02)0.40, & k &= 0.1(0.1)2.5(0.5)5.0;\\
p &= 0.42(0.02)0.60, & k &= 0.1(0.1)2.5(0.5)10.0;\\
p &= 0.62(0.02)0.80, & k &= 0.2(0.2)5.0(1)20;\\
p &= 0.82(0.02)0.90, & k &= 0.5(0.5)10.0(2)50;\\
p &= 0.95, & k &= 2(2)50(10)200.
\end{aligned}$$

The computation of individual terms and cumultative

probabilities was continued until $F(n)$ exceeded 0.999900, then terminated. The tables were computed on a Ferranti "Pegasus" computer to nine decimal places, but were rounded to six decimal places before being printed.

The introductory material includes, besides the description of the tables summarized above, sections on history and applications, properties of the negative binomial, and parameter estimation. It is pointed out that most of the practical applications of the negative binomial distribution are due to its derivation as a generalized form of the Poisson distribution, $e^{-\lambda}\lambda^r/r!$, and it is shown that if λ, instead of being constant, has a Gamma distribution, the Poisson distribution generalizes to the negative binomial distribution. It is shown that the mean of the negative binomial distribution is kq/p and that the variance is kq/p^2. Since $p<1$, the variance is always greater than the mean. Expressions for third and fourth moments are also obtained, and it is shown that the negative binomial distribution is unimodal, with the frequency increasing until the highest integral value of n for which $n\leq q(n+k-1)$, after which it declines again. In the discussion of parameter estimation, it is shown that the sample mean m is a maximum likelihood estimator of the distribution mean M, so that the method of moments and the method of maximum likelihood give the same estimator. The two methods, however, yield different estimators of k, which can also be estimated from the proportion of zeros. The large-sample efficiencies of the method of moments and the method based on the proportion of zeros, relative to the method of maximum likelihood, are shown graphically.

H. L. Harter (Dayton, Ohio)

Bracken, Jerome; Schleifer, Arthur, Jr. 682

★**Tables for normal sampling with unknown variance. The Student distribution and economically optimal sampling plans.**

Division of Research, Graduate School of Business Administration, Harvard University, Boston, Mass., 1964. xii+193 *pp.* $8.00.

This book is the second of a series entitled Studies in Managerial Economics, the first being one by Raiffa and Schlaifer [*Applied statistical decision theory*, Div. Res., Graduate School of Business Administration, Harvard Univ., Boston, Mass., 1961; MR **22** #8618]. The contents of the present volume are summarized in the following quotation from the preface: "The tables contained in this book are essentially of two kinds: tables of the ordinary Student or 't' density and cumulative functions, and tables to facilitate Bayesian analysis of certain commonly occurring decision problems in which sampling may or may not be involved. The tables of the Student density and cumulative functions show values to four (occasionally three) significant figures for 20 values of the degrees of freedom ν and for values of the argument t in steps of .01 from 0 either to 8 or to a value such that the cumulative right-tail function is approximately .00003$\cdots$. These tables can be used for all of the ordinary applications of the Student distribution and, in conjunction with one of the Bayesian tables, for computation of the incomplete first moment of the Student distribution. The Bayesian tables are useful primarily for problems involving a choice between two or more acts when the losses depend linearly on a single Student-distributed random variable or on the difference between two such variables. ... Various conditions under which the loss-determining random variable is Student-distributed are discussed in the introductory text: the most common ones are those whose information on the population mean or treatment yield has been obtained by single sampling from a normal population with known or unknown variance or from several populations or strata of populations where the variances are known at least up to a single multiplicative factor. In problems of this sort the tables make it easy to evaluate (1) the expected value of the loss to be incurred by immediate choice of a terminal act, without (additional) sampling, and (2) the expected reduction in this loss to be obtained by a single-stage sampling procedure in which the sample has any predetermined size. In deciding whether or not an (additional) sample should be taken, it is of course necessary to look at the expected net gain or the sample, i.e., the amount by which the expected reduction in loss exceeds the cost of sampling; and in many situations it will be desired to determine the sample size which maximizes the net gain of a single-stage sampling procedure. The tables make it possible to do this quite easily in any problem in the category defined above, by simply tracing the behavior of the net gain as a function of sample size."

The 39-page introduction includes sections on description of tables; Student distributions: prior, posterior, and preposterior; two-action problems with linear utility; other applications of the tables; interpolation; and computation of the tables; also a list of eight references. The tables themselves, aggregating about 150 pages, were printed by photo-offset directly from the computer output in order to avoid typesetting errors.

H. L. Harter (Dayton, Ohio)

Tiago de Oliveira, J. 683

La représentation des distributions extrémales bivariées. (English summary)

Bull. Inst. Internat. Statist. **39** (1962), *livraison* 2, 477–480.

Author's summary: "In a previous paper [Univ. Lisboa Revista Fac. Ci. A (2) **7** (1959/60), 219–228] the author obtained the necessary and sufficient conditions for a bivariate distribution function to be a bivariate distribution function of extremal variates. In this paper the author obtains an unique representation depending on a distribution function and a bounded parameter; in the absolutely continuous case the representation is given by an almost arbitrary density of probability and a bounded parameter."

Roach, S. A. 684

The frequency distribution of the sample mean where each member of the sample is drawn from a different rectangular distribution.

Biometrika **50** (1963), 508–513.

Using n-dimensional geometry, the author derives the density of $\bar{X}=(X_1+X_2+\cdots+X_n)/n$, where the X_i's are independent random variables each having rectangular distributions on the intervals $(0, l_i)$, with the l_i's not equal to one another. The cases $n=2$ and $n=3$ are given detailed treatment; for the general case the central moments are derived.

J. Machek (Prague)

Tiku, M. L. 685

Approximation to Student's t distribution in terms of Hermite and Laguerre polynomials.

J. Indian Math. Soc. (*N.S.*) **27** (1963), 91–102.

Author's summary: "The sampling distribution of mean and variance has been formally expanded in terms of Hermite and Laguerre polynomials. The first few coefficients are worked out in terms of population cumulants up to order eight. The explicit form of Student's t distribution is given."

Tortorici, Paolo 686

Una dimostrazione elementare della tendenza, verso la legge normale, della legge di distribuzione degli scarti nel problema delle prove ripetute. (French, English, Spanish, and German summaries)

Giorn. Ist. Ital. Attuari **26** (1963), 52–63.

Author's summary: "L'auteur expose une démonstration élémentaire de la tendance, par $n\to\infty$, de la loi de distribution des écarts relatifs à n épreuves, sur un événement de probabilité constante en chaque épreuve, vers la loi de distribution normale."

Mott-Smith, J. C. 687

Two estimates of the binomial distribution.

Ann. Math. Statist. **35** (1964), 809–816.

Defining

$$E(n, s, p) = \sum_{r=s}^{n} \binom{n}{r} p^r(1-p)^{n-r},$$

$$R = E(n, s, p)\Big/\binom{n}{s} p^s(1-p)^{n-s},$$

two estimates of R are obtained. For $0 \leq s \leq n-1$, $n>1$, and $p \leq (s+1)/(n+1)$, an upper bound is obtained for R by bounding the tail of the distribution with a geometric series. The bound obtained is shown to be better than one given by R. R. Bahadur [same Ann. **31** (1960), 43–54; MR **22** #11424].

The second estimate R^* is obtained by an asymptotic expansion of the hypergeometric function and provides an estimate of R, R^* in terms of the normal distribution. It is shown that $(R^*-R)/R\to 0$ if s and $n-s\to\infty$, so that $\ln s/(n-s)^{1/3}\to 0$ and $0\leq p\leq (s-1)/(n-1)$.

{There is an obvious error above (25) on page 813, which in no way affects the correctness of the results. A correction will appear in the same Annals.}

Bernard Harris (Madison, Wis.)

Shah, S. M. 688

Use of a priori knowledge in the estimation of a parameter from double samples.

J. Indian Statist. Assoc. **2** (1964), 41–51.

The author generalizes an estimation method used by Katti [Biometrics **18** (1962), 139–147; MR **26** #3140]. It makes use of a guessed estimate, θ_0, of a real parameter θ. Also assumed available are uniformly minimum variance unbiased estimates (UMVUE) $\hat\theta_1$ and $\hat\theta_2$ of θ based respectively on a first sample and on a second sample, as well as an estimate $\hat\theta = a\hat\theta_1 + b\hat\theta_2$ $(a+b=1)$, UMVUE in the class of unbiased linear combinations of $\hat\theta_1$ and $\hat\theta_2$ with non-negative coefficients. An interval R on the real line is chosen as indicated below. If the observations in the first sample prove to be such that $\hat\theta_1\in R$, one sets $\theta^P=\hat\theta_1$. If $\hat\theta_1\notin R$, $\theta^P=\hat\theta$. The estimate θ^P is termed a preliminary test estimate (PTE). The interval R is to be chosen so as to minimize the expected mean square deviation of θ^P from θ_0, on the hypothesis that the true value of θ is θ_0. The method is applied to estimation of variance of a normal population. Efficiency is discussed, and shown to be greater than 1 if the guessed value is the true one. Some tabulated efficiencies are given. A table of expected percentage of overall sample saved if the PTE is used is also given. *H. D. Brunk* (Columbia, Mo.)

Kudô, Akio 689

A multivariate analogue of the one-sided test.

Biometrika **50** (1963), 403–418.

Let $X^{(1)}, X^{(2)}, \cdots, X^{(n)}$ be sample vectors from a multivariate normal population with unknown vector of means θ and known variance matrix Λ. Let $\bar X$ denote the sample mean vector. Let H_0 denote the null hypothesis that each component of θ is zero, while the alternative H_1 specifies that all are non-negative and at least one is positive. The author shows that X^0 is the maximum likelihood estimator of θ, subject to the condition that all components are non-negative, if and only if for each i one of the two vectors X^0 and $-\Lambda^{-1}(\bar X - X^0)$ has its ith component zero and the other has a non-negative ith component. He discusses the calculation of the maximum likelihood estimate and illustrates with an example. A formula for the distribution function of the log likelihood ratio is given. In an example involving a survey of child health, a test using the likelihood ratio statistic yields a result significant at the 5% level, while the usual χ^2 test does not. The results of the paper may be regarded as generalizing results of Bartholomew [Biometrika **46** (1959), 36–48; MR **21** #3067; ibid. **46** (1959), 328–335; MR **22** #3059], obtainable on particularizing to the case of independent components.

H. D. Brunk (Columbia, Mo.)

Takuchi, Hiraku 690

Theory of statistical estimates. I. (Japanese)

Sûgaku **14** (1962/63), 193–209.

This is the first part of an expository paper on the theory of statistical estimation developed recently (after about 1950). It deals with three topics: Sufficient statistics and unbiased estimates, the space of estimates of minimum variance, and the construction of unbiased estimates.

T. Kawata (Washington, D.C.)

Ewens, W. J. 691

Numerical results and diffusion approximations in a genetic process.

Biometrika **50** (1963), 241–249.

Suppose that in a population of fixed size M, the gene A has a selective advantage over the alternative allele a; then the number of genes A in any generation is a Markovian variate with transition matrix

$$P = \{p_{ij}\} = \left\{\binom{M}{j} p_i{}^j(1-p_i)^{M-j}\right\},$$

where $p_i=(1+s)i(M+si)^{-1}$. Exact expressions are not available in this model for (I) the probability of elimination of one or other gene, (II) the mean time to elimination, (III) the probability of elimination by the nth generation, or (IV) the value of the dominant non-unit latent root of P. For M small, the author compares the numerical values obtained for these by a high-speed computer with the results of diffusion approximations to the discrete process. He discusses their accuracy; considering the small values of M used, this is good.
J. Gani (E. Lansing, Mich.)

Bartko, John J.; Watterson, Geoffrey A. 692
Inference on a genetic model of the Markov chain type.
Biometrika **50** (1963), 251–264.
The authors discuss maximum likelihood estimation for the parameters in a certain positively regular Markov chain model of Moran [Proc. Cambridge Philos. Soc. **54** (1958), 60–71; MR **23** #B1034], which arises in genetics. The probabilities involved are linear in the parameters α_1, α_2, and such that the transition matrix is of the Jacobi form.

The parameters are estimated from the transition frequencies in a realization of the process, and some uniqueness theorems are proved for the estimators. The theorem on the spectral resolution of the transition matrix is given, and this is used to obtain the stationary probabilities and thence the expectation of the transition frequencies, and also the information matrix. A diffusion approximation is available for this genetic model, and maximum likelihood estimates for α_1, α_2 can also be obtained from this. The authors hope to extend their discussion of estimation to some chains in genetics having an absorbing state. *J. Gani* (E. Lansing, Mich.)

Iyer, P. V. K. [Krishna Iyer, P. V.]; Singh, Naunihal 693
Estimation of parameters from generalised censored normal samples.
J. Indian Soc. Agric. Statist. **14** (1962), 165–176.
Authors' summary: "The problem of estimating the parameters from censored samples has been considered by Hald [Skand. Aktuarietidskr. **32** (1949), 119–134; MR **12**, 193], Cohen [Ann. Math. Statist. **21** (1950), 557–569; MR **12**, 346], Gupta [Biometrika **39** (1952), 260–273; MR **14**, 487], Des Raj [J. Amer. Statist. Assoc. **48** (1953), 336–349; MR **14**, 1103] and others for a number of distributions. In actual practice the censoring of the samples may be different from that considered by these authors. For example, suppose we are interested in the study of the life of, say, clothing items. Let these items be issued to a military unit, on a particular date, out of which k failures are reported to the condemnation board after a fixed period. If now during this period a certain number of soldiers, say $n_1, n_2, \cdots, n_l$, is transferred to other units at different times $t_1', t_2', \cdots, t_l'$ for some strategic reasons or otherwise, it means that information regarding the life of the items used by these soldiers is not easily available from the records. The only information available is that these items have been in use up to the time of their transfer. We may treat such type of data as a sample from a normal population censored at various stages."

Iyer, P. V. Krishna [Krishna Iyer, P. V.]; Singh, Naunihal 694
Estimation of the mean and the standard deviation of a Type III population from censored samples.
J. Indian Statist. Assoc. **1** (1963), 161–166. (1 *insert*)
The authors give approximate maximum likelihood estimators of mean and standard deviation for censored sampling from a Type III population. A prescribed number of the largest values are assumed unknown. A table is provided for use with the method. A large sample approximation of the covariance matrix of the estimators is also given, and an example is worked out.
H. D. Brunk (Columbia, Mo.)

Allais, M. 695
Test de périodicité. Généralisation du test de Schuster au cas de séries temporelles autocorrelées dans l'hypothèse d'un processus de perturbations aléatoires d'un système stable. (English summary)
Bull. Inst. Internat. Statist. **39** (1962), *livraison* 2, 143–193. (3 *inserts*)
The author proposes a generalised test of periodicities in a stationary process, in which the periodogram is standardised by dividing through by a smoothed estimate of the spectral density based upon the low-lag autocorrelations, and the classical Schuster tests are then applied to this standardised periodogram. This is much the technique proposed and roughly justified some years ago by the reviewer [Trabajos Estadística **3** (1952), 43–57; MR **14**, 488].

The article is supported by some numerical studies, and an extensive bibliography. *P. Whittle* (Manchester)

Goodman, Leo A. 696
Simple methods for analyzing three-factor interaction in contingency tables.
J. Amer. Statist. Assoc. **59** (1964), 319–352.
The author gives a comprehensive review of methods for testing the hypothesis of no three-factor interaction in a three-dimensional contingency table, and constructs confidence intervals for parameters of interest. Detailed formulae and examples are given. The treatment is based largely on his recent contributions to this field [J. Roy. Statist. Soc. Ser. A **126** (1963), 94–108; MR **27** #3049; J. Roy. Statist. Soc. Ser. B **25** (1963), 179–188; Ann. Math. Statist. **35** (1964), 716–725; MR **28** #5515]. A new and simpler approximate test is given for a $2\times J\times K$ table, analogous to the approximate method of analysis for a two-way layout with unequal sub-class frequencies.
R. L. Plackett (Newcastle upon Tyne)

Villegas, C. 697
Confidence region for a linear relation.
Ann. Math. Statist. **35** (1964), 780–788.
Author's summary: "The analysis of a linear relation is considered when there are replications and all the variables involved are subject to errors or fluctuations. A test based on the F-distribution is derived for testing the hypothesis that the unknown relation is a given linear relation. From this test a joint confidence region for the coefficients of the linear relation is derived. A confidence region for the linear relation is then defined as the set

of all points which belong to hyperplanes not rejected by the test. This confidence region estimation procedure is compatible with maximum likelihood estimation of a linear relation, in the sense that the maximum likelihood hyperplane is contained in the confidence region for the linear relation, if this region is not empty."

E. Parzen (Stanford, Calif.)

Koller, Dieter 698

Prüfung der Normalität einer Verteilung.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2**, 147–166 (1963).

The author considers a (modified) chi-square test of normality where the computationally messy M.L. estimates of parameters (based on cell numbers) are replaced by $M = n^{-1} \sum_k' n_k \tau_k$, $s^2 = -h^2/12 + (n-1)^{-1} \sum_k' n_k(\tau_k - M)^2$. Here the sums are only over the finite intervals of length h; τ_k are the interval midpoints; n_k of the n observations fall in the kth interval. Means and variances of these estimators, and asymptotic properties of the resulting test under null and alternative hypotheses, are studied.

J. Kiefer (Ithaca, N.Y.)

Šalaevskiĭ, O. V. 699

On the existence of similar tests for the Behrens-Fisher problem. (Russian)

Dokl. Akad. Nauk SSSR **154** (1964), 795–797.

Let the statistics $\xi = (\bar{x} - \bar{y})/s_2$ and $\eta = s_1/s_2$ be computed from two independent normal samples of sizes m and n with identical means but generally different variances σ_1^2, σ_2^2. If $m = 4s + 3$ and $n = 2r$, the author proves that any similar critical function $\phi = \phi(\xi, \eta)$ equals 1 for at least one point in every neighbourhood of the point $\xi = 0$, $\eta = 1$. Now, since this point, as well as its close neighbourhood, represents typically "non-significant" results, we may conclude that no non-randomized test, depending on ξ and η only, is reasonable.

J. Hájek (Prague)

Agrawal, H. 700

A note on omission of variables in multiple regression analysis.

J. Indian Statist. Assoc. **1** (1963), 226–229.

Formulas are derived, using matrix methods, for the change in the least squares estimate of the vector of regression coefficients and the change in the error sum of squares when in a regression model a certain number of independent variates is ignored. {Note: The subscripts on the x's in (1) have to be interchanged in order to be consistent with the notation elsewhere in the paper.}

R. A. Wijsman (Urbana, Ill.)

Ghosh, M. N. 701

On the admissibility of some tests of MANOVA.

Ann. Math. Statist. **35** (1964), 789–794.

Independently normally distributed p-dimensional row vectors $z^{(1)}, \cdots, z^{(Q+m)}, y^{(1)}, \cdots, y^{(n)}$ with $n \geq p$ and with common non-singular covariance matrix Σ_1 are studied. The means are $E(z^{(r)}) = b^{(r)}$, $E(y^{(s)}) = 0$ for $r = 1, \cdots, Q+m$, and $s = 1, \cdots, n$, where $b^{(r)}$ are unknown p-vectors. The problem is to test the hypothesis H: $b^{(r)} = 0$ for $r = 1, \cdots, Q$. Consider the characteristic roots λ_1 of ZY^{-1} where $Z = \sum_{q=1}^{Q} z^{(q)'}z^{(q)}$ and $Y = \sum_{u=1}^{n} y^{(u)'}y^{(u)}$ with λ_1 the largest root. It is shown that the acceptance regions $\{x: \mathrm{Tr}(ZY^{-1}) \leq c\}$ and $\{X: \lambda_1 \leq c\}$ are admissible for the test H when the alternatives are unrestricted.

P. S. Dwyer (Ann Arbor, Mich.)

Richter, Wolfgang 702

Mehrdimensionale Grenzwertsätze für grosse Abweichungen und ihre Anwendung auf die Verteilung von χ^2. (Russian summary)

Teor. Verojatnost. i Primenen. **9** (1964), 31–42.

Theorem 2, a multidimensional local limit theorem for large deviations, is used in establishing two integral limit theorems. A generalization of the method of proof employed by Khintchine is featured. Theorem 1 results from applying these theorems to the distribution of χ^2.

P. S. Dwyer (Ann Arbor, Mich.)

Hoflund, Olle 703

Simulated distributions for small n of Kendall's partial rank correlation coefficient.

Biometrika **50** (1963), 520–522.

Author's summary: "The distribution of Kendall's (1948, 1955) partial rank correlation coefficient $t_{12 \cdot 3}$ has been simulated by a Monte Carlo method. The distribution studied is the distribution under complete null-hypothesis —all rankings are considered equally probable. Distributions have been derived for rankings of 4 up to 10 elements. 10,000 coefficients $t_{12 \cdot 3}$ have been calculated in each case. Certain results are tabulated . . . and certain comparisons are made with the normal distribution."

H. D. Brunk (Columbia, Mo.)

Nair, C. Ramankutty 704

A note on the analysis of serially balanced sequences.

J. Indian Statist. Assoc. **1** (1963), 93–96.

Sampford's (1957) analysis of a particular type of serially balanced sequence is here expressed in more formal matrix notation, and solution of the equations by means of characteristic roots is indicated. No simplification on the original is achieved. {There are a number of misprints and mis-usages both by printer and author.}

R. M. Cormack (Old Aberdeen)

Bhapkar, V. P. 705

The asymptotic power and efficiency of Mood's test for two-way classification.

J. Indian Statist. Assoc. **1** (1963), 24–31.

The author considers a non-parametric test (M) for the equality of treatment effects in a balanced incomplete block experiment, the test statistic being of the form $c\sum(m_i - b)^2$, where m_i is the number of observations for the ith treatment which exceed their block median and b and c are appropriate constants. The asymptotic relative efficiency (ARE) of this test as compared with the F test for treatment effects (expressed as a ratio of non-centrality parameters) is obtained and also the ARE as compared with a test (R) based on ranks. These are calculated to various initial distributions and various block sizes. In

all cases the ARE for the test M relative to the test R is less than unity, unless the block size is 2, when it is unity.
E. J. Hannan (Canberra)

Bánkövi, György; Sarkadi, Károly 706
5/9 replication of a 3^4 factorial experiment. (Hungarian. Russian and English summaries)
Magyar Tud. Akad. Mat. Kutató Int. Közl. **7** (1962), 509–536 (1963).

Shrikhande, S. S.; Singh, N. K. 707
A note on balanced incomplete block designs.
J. Indian Statist. Assoc. **1** (1963), 97–101.
The authors establish bounds for number of treatments common between some two blocks of BIBD satisfying some conditions. This generalizes the result of E. T. Parker [Proc. Amer. Math. Soc. **14** (1963), 729–730; MR **27** #4766]. However, Parker's result follows also from a result established previously by W. S. Connor [Ann. Math. Statist. **23** (1952), 57–71; correction, ibid. **24** (1953), 135; MR **13**, 617]. It is not established whether the bounds obtained by the authors are ever improvements on Connor's general bounds. The authors then use the idea of the reviewer [Proc. Amer. Math. Soc. **14** (1963), 731–732, MR **27** #4767] to obtain an upper bound to the number of treatments common between some blocks of BIBD in the general case. M. C. Chakrabarti points out in an answer to a query [J. Indian Statist. Assoc. **1** (1963), 230–234] that the bound obtained by the authors is no better than the known bound. Clearly, all the authors involved were unaware of the known bounds while working on the problem. The new ideas introduced may, however, aid future related research.
E. Seiden (E. Lansing, Mich.)

Trehan, A. M. 708
On the bounds of the number of common treatments between blocks of balanced incomplete block design.
J. Indian Statist. Assoc. **1** (1963), 102–103.
The author gives an upper and lower bound for the number of treatments common between some two blocks of BIBD hoping that there is an improvement over the previously established bounds obtained by W. S. Connor and Shrikhande and Singh [see #707 above]. M. C. Chakrabarti points out in an answer to a query [#707] that the bounds obtained by the author are no better than the old known bounds. Again it may be worthwhile noticing that M. C. Chakrabarti seemed not to be aware of these points before the paper was published since the author thanks him for suggesting the problem and his interest in the preparation of the note.
E. Seiden (E. Lansing, Mich.)

Elfving, G. 709
Decision-making under uncertainty. (Swedish)
Arkhimedes **1963**, *no.* 2, 19–26.
Although brief, this expository article combines broad coverage with lucid clarity in the presentation and discussion of the general foundation of decision theory. Voicing the much-debated difficulties about the empirical assessment of (a) the utility function U, and (b) the system of prior probabilities P, the author explains that if an individual's decisions about so-called conditional lotteries satisfy a general set of consistency postulates, the ensuing decisions will be as if they were based on existing and uniquely determined U and P.
H. Wold (Uppsala)

Robbins, Herbert 710
The empirical Bayes approach to statistical decision problems.
Ann. Math. Statist. **35** (1964), 1–20.
The paper represents the contents of the Rietz lecture of 1963 delivered by the author and is partly expository, describing the new subdiscipline of which Robbins is the initiator. Consider (a) a parameter space Λ with elements λ; (b) an action space A of elements a; (c) a loss function $L(a, \lambda) \geqq 0$; (d) an a priori distribution G of λ on Λ; (e) an observable random variable (r.v.) X with values x in a space $\mathfrak{X}$ on which a σ-finite measure μ is defined. For each $\lambda \in \Lambda$ the r.v. X has a specified probability density with respect to μ, say f_λ. The problem is to choose a decision function t from $\mathfrak{X}$ to A such that upon observing $X = x$ we shall take action $t(x)$ and incur loss $L[t(x), \lambda]$. Let the Bayes risk of t relative to G be $R(t, G)$. It is assumed that there exists a decision function t_G minimizing $R(\cdot, G)$, and $R(G)$ denotes the minimum. The novel element introduced by the author is the situation where G is unknown, except that it belongs to a given family Γ, and where the same decision problem is to be solved in a sequence of situations. Each of these situations, say the nth, is characterized by an independently determined pair (λ_n, X_n), where the parameter value is subjected to the same, though unknown, $G \in \Gamma$ and where the density of X_n is f_{λ_n}. The values x_n of the X_n are observable but not those of λ_n. Faced with the $(n+1)$st situation, the problem is to use the already observed $x_1, x_2, \cdots, x_n$ in order to define the decision function

$$t_{n+1}(\cdot) = t_{n+1}(x_1, x_2, \cdots, x_n; \cdot)$$

so that, after observing $X_{n+1} = x_{n+1}$, the action taken shall be $t_{n+1}(x_{n+1}) \in A$. It is said that the sequence $T = \{t_n\}$ is asymptotically optimal with respect to Γ if, for every $G \in \Gamma$, we have $R(t_n, G) \to R(G)$ as $n \to \infty$. The new results in the paper are concerned with conditions in which asymptotically optimal decision functions exist and with certain similar problems.
J. Neyman (Berkeley, Calif.)

Thompson, J. W. 711
A property of some symmetric two-stage sequential procedures.
Ann. Math. Statist. **35** (1964), 755–761.
It is desired to decide whether an unknown parameter θ is positive or negative using a two-stage sequential procedure. Assuming that the family of distributions is exponential with the parameter θ and symmetric in the sense $F_{-\theta}(x) = 1 - F_\theta(-x^-)$, several results are derived. For a symmetric prior distribution the Bayes solutions are such that the second sample size is non-increasing in $|s_1|$, where s_1 is the sum of the observations of the first sample. The expected number of observations for any admissible symmetric two-stage procedure is non-increasing in $|\theta|$. The class of Bayes solutions relative to

symmetric prior distributions and limits of such solutions form a complete class relative to the class of all symmetric procedures. *H. Chernoff* (Stanford, Calif.)

Rao, J. N. K.; Graham, Jack E. 712
Rotation designs for sampling on repeated occasions.
J. Amer. Statist. Assoc. **59** (1964), 492–509.
Authors' summary: "A unified finite population theory is developed for composite estimators of both the current level and change in level between consecutive occasions when a rotation sample design is employed. Explicit variance functions are given under the assumption that exponential and arithmetic correlation patterns hold over time for the characteristics of interest. Optimum values of the weight coefficients in the composite estimators, the optimum number of consecutive visits by a sampling unit, and the gain in efficiency relative to the use of simple estimators are all determined numerically. Moderate gains for level and appreciable gains for change are achieved when the correlation is high."
H. Chernoff (Stanford, Calif.)

Chakrabarti, M. C. 713
On the use of incidence matrices of designs in sampling from finite populations.
J. Indian Statist. Assoc. **1** (1963), 78–85.
Some results of finite population sampling theory are discussed within the framework of incomplete block designs under various restricted conditions.
M. Atiqullah (Dacca)

Des Raj 714
On double sampling for PPS estimation.
Ann. Math. Statist. **35** (1964), 900–902.
Elements of a finite population of size N are represented by two characteristics x and y. It is desired to estimate the sum of the y's by sampling the elements with probabilities proportional to the x's which are unknown. This type of problem appears in sampling agricultural holders where it is desired to sample with probability proportional to area. A double sample scheme is proposed. First a sample without replacement of size n' obtains observations $x_1, x_2, \cdots, x_n'$ summing to x'. From these n' elements, a subsample of size n is obtained with probability proportional to x_i and without replacement. The estimate $\hat{Y} = (N/n')(x'/n) \sum_1^n (y_i/x_i)$ is an unbiased estimate of the sum of the y's. Its variance is computed and an estimate of it is derived. *H. Chernoff* (Stanford, Calif.)

Pathak, P. K. 715
Sufficiency in sampling theory.
Ann. Math. Statist. **35** (1964), 795–808.
A finite population is considered, and a sample s is defined as a finite sequence of its elements, generally with replacement. A statistic T is called sufficient if $T(s) = T(s')$ for s and s' containing the same distinct elements. Sufficiency is postulated here for simplicity and not proved. Conditional mean values of estimators with respect to sufficient statistics are shown to improve these estimators for convex loss functions. These results are applied to improve estimators from interpenetrating subsamples and from two-stage sampling with replacement on the second stage.
J. Hájek (Prague)

Hillier, Frederick S.
Continuous sampling plans under destructive testing.
J. Amer. Statist. Assoc. **59** (1964), 376–401.
Author's summary: "Many existing continuous sampl plans are designed to provide a desired AOQL by requir 100% inspection whenever incoming quality is found to sufficiently poor. However, such extensive inspectio not practical for those cases where destructive testin required. The continuous sampling procedure develo here is especially designed for these situations where amount of inspection must be stringently limited. It u certain a priori information about the quality beha of the production process. It then improves upon information by a limited amount of sampling inspect in order to determine when a process adjustment needed. The objective in this formulation was to deve a procedure suitable for practical use which wo approximate the scheme which minimizes total expec cost." *T. Dalenius* (Stockho

Cramer, J. S.
Efficient grouping, regression and correlation in En curve analysis.
J. Amer. Statist. Assoc. **59** (1964), 233–250.
Clearly, the common practice of grouping the individ observations in household surveys according to inc does not introduce a bias in the least squares estimat the regression coefficient in Engel curve analysis. author shows that under realistic conditions of sur data the grouping increases the variance of the estim by a few per cent at most. Further, he illustrates the w known feature that the correlation coefficient is va overstated by the grouped data. *H. Wold* (Upps

Béthoux, Paul
Tests et estimations concernant certaines foncti aléatoires, en particulier laplaciennes.
Ann. Inst. H. Poincaré **17**, 255–322 (1962).
This paper considers various statistical problems ari when one observes continuously over a finite interv random function $X(t)$ which is usually assumed to normal with known covariance kernel. Chapter 1 c siders tests for the mean-value function, assuming it be one of a finite number of functions $\rho_k(t)$, and in part lar discusses existence of Radon-Nikodým derivatives band-limited noise, and the design of signals for grea discrimination. Chapter 2 discusses maximum likelih estimation of the mean-value function. Chapter 3 cusses estimation of the regression coefficients M_k regar as random variables in the model for the mean-va function $M(t) = \sum M_k g_k(t)$, where $g_k(t)$ are known functi Several examples (mainly stationary processes w rational spectral density functions) are given.
E. Parzen (Stanford, Ca

Ishii, Goro; Kudō, Hirokichi
Tolerance region for missing variables in linear statist model.
J. Math. Osaka City Univ. **14** (1963), 117–130.
For the case where there are missing observations in usual analysis of variance model, the authors constru

region such that the probability that the missing observations fall in this region is at least $1-\varepsilon$ identically in the unknown parameters, and the expected volume of the region is no greater than that of any other region with probability at least $1-\varepsilon$ of containing the missing observations. *L. Weiss* (Ithaca, N.Y.)

Akaike, Hirotugu; Yamanouchi, Yasufumi 720
On the statistical estimation of frequency response function.
Ann. Inst. Statist. Math. **14** (1962), 23–56.
One of the important uses of statistical spectral analysis techniques is in the study of the relationships between two time series. This paper is an extremely valuable contribution to the theory of cross-spectral techniques. It points out that the success of spectral density estimation arises from the reduction in variance afforded by smoothing. It seems easy to know how to cope with the fact that one is estimating "averaged power spectra". However, when estimating frequency response (or transfer) functions, dealing only with an "averaged frequency response function" is more dangerous, since phase variations may cause the frequency response function to "disappear" by averaging.

The authors obtain a representation for the bias in the smoothed frequency response function, and indicate a method of estimation which reduces bias. They then give a quick derivation of the distribution theory of estimates of coherency which they use to provide confidence intervals for the frequency response function. A practical computation scheme for estimation of the frequency response function is described. Several numerical examples are discussed. *E. Parzen* (Stanford, Calif.)

Daniels, H. E.; Ross, A. S. C. 721
The Hausa problem and some approximations to the required probability.
Biometrika **50** (1963), 514–517.
It was pointed out by F. W. Parsons in 1955 that the initial phoneme of abstract nouns of sensory quality in the West African language Hausa is never palatalized. The authors show, as an illustration of the saddlepoint method, that this is not statistically significant; see, for example, Daniels [Ann. Math. Statist. **25** (1954), 631–650; MR **16**, 603; the reviewer, ibid. **28** (1957), 861–881; MR **20** #386]. *I. J. Good* (Ashford)

Jain, J. P.; Amble, V. N. 722
Improvement through selection at successive stages.
J. Indian Soc. Agric. Statist. **14** (1962), 88–109.
Authors' summary: "Plant and animal breeders are often faced with the problem of finding the optimal values of intensities of selection at various stages, the optimal values considered being those which result in maximizing the annual genetic advance. With dairy cows under selection for milk yield, for example, each successive lactation provides new data on the milk yielding capacity of the animal. These lactation yields would form the basis for the successive stages of selection, selection at the rth stage being made on the evidence of yield in the latest lactation combined with the available information on previous lactation yields. A usual feature in selection problems is that we cannot assess directly the genetic value of character which we wish to improve. In the present case selection is aimed at securing cows with superior genotypic value for milk yield but it has to be based on the observed or phenotypic values in successive lactations. The basic approach, viz., of improving some character y which is not directly measurable, by means of indirect selection that is made from a group of tests or measurements $x_1, x_2, \cdots, x_r$ at successive stages, has been discussed by Cochran (1951). He has also derived the general form for gain in y expected after two-stage selection for the case when the variates y, x_1 and x_2 follow a multivariate normal distribution. This paper deals with the extension of the expression for selection gain to r stages. Approximate working methods for dealing with multiple stage selection are also given."

NUMERICAL METHODS

See also 60, 278, 291, 349, 400, 787.

Bulavko, I. G. [Булавко, И. Г.] 723
★Exact and approximate calculations [Точные и приближенные вычисления].
Izdat. Ministerstva Vysš., Sredn. Special. i Professional. Obraz. BSSR, Minsk, 1963. 107 *pp.* 0.17 *r.*
A semi-popular book explaining many of the elementary concepts used in numerical analysis.

Godunov, S. K. [Годунов, С. К.]; Rjaben'kiĭ, V. S. [Рябенький, В. С.] 724
★Introduction to the theory of difference schemes [Введение в теорию разностных схем].
Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1962. 340 *pp.* 0.99 *r.*
This book contains six chapters and has an appendix consisting of five papers on finite-difference methods, "On difference schemes for a solution of the heat equation", by Gel'fand and Lokucievskiĭ, "The sweep method for the solution of difference equations", by Gel'fand and Lokucievskiĭ, "Application of the energy method to difference equations", by P. D. Lax, "An estimate of the amount of computing work necessary for the approximate solution of a problem", by Bahvalov, and "Difference equations for parabolic equations and continual integrals", by Krylov.

The first chapter treats first- and second-order difference equations, and a discussion of difference schemes for the differential equation $u''+Au=0$. The second chapter contains methods of setting up difference schemes for ordinary differential equations and boundary conditions. Stability is defined, and its connection with convergence and round-off error is discussed. The third chapter treats the difference approximation of partial differential equations of parabolic and hyperbolic types. Difference methods are used to establish existence of a solution of the system $u_t-u_x=0$, $0\leq t$, $u(x,0)=\varphi(x)$ $(-\infty<x<\infty)$.

The fourth chapter deals with the solution of the equations arising from difference schemes and includes a proof that systems of the form $u_0=\varphi$, $a_nu_{n-1}-2b_nu_n+c_nu_{n+1}=g_n$, $u_N=\psi$, $a_n>\delta>0$, $c_n>\delta>0$, $b_n-\frac{1}{2}(a_n+c_n)>\delta>0$, not only have unique solutions but that small

errors in the coefficients of the difference equations induce correspondingly small errors in the solutions. The method of sweep for solving such equations is introduced and it is shown that small errors introduced in this method lead to small errors in the solution.

The fifth and sixth chapters treat stability as an operator property. It is shown by examples how a difference equation can be put in the form $u^{m+1} = Ru^m + \tau\rho^m$, where u^{m+1} represents the solution at some time strip t_{m+1}. It is shown that uniform boundedness of the norm of the powers of the operators R is necessary, and under certain other restrictions, also sufficient for stability. The spectrum of an operator is defined and it is shown that if an operator has an element of its spectrum outside the unit circle, it cannot be uniformly bounded. Methods for calculating the spectrum are discussed.

Raymond Johnson (Houston, Tex.)

Skoblja, N. [Скобля, Н.] 725

★Tables for the numerical inversion of the Laplace transform

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{xp}F(p)\,dp$$

[Таблицы для численного обращения преобразования Лапласа

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{xp}F(p)\,dp].$$

Izdat. "Nauka i Tehnika", Minsk, 1964. 44 *pp.* 0.13 *r.*

This can be regarded as an extension of tables given by H. E. Salzer [MTAC **9** (1955), 164–177; MR **17**, 1203]. There are given the A_k and p_k occurring in the quadrature formula

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{p}p^{-s}\varphi(p)\,dp \doteq \sum_{k=1}^{n} A_k\varphi(p_k)$$

which is exact for $\varphi(p)=p^{-j}$, $j=0, 1, \cdots, 2n-1$. If $P_n(p^{-1})=\prod(p^{-1}-p_k^{-1})$, an explicit expression and a differential equation for P_n is obtained, generalizing results obtained by Salzer in the case $s=1$. An explicit expression for A_k is also given.

Using the differential equation, a non-linear set of equations for the x_k is obtained which was solved using Newton's method. The A_k were then calculated from the explicit expression. The calculations were performed on the URAL-1. Values were given for $n=1(1)10$, $s=.1(.1)3$ and are to about 8S for the p_k and 7S for the A_k.

John Todd (Pasadena, Calif.)

Kruglikova, L. [Кругликова, Л.] 726

★Tables for the numerical Fourier transform [Таблицы для численного преобразования Фурье].

Izdat. "Nauka i Tehnika", Minsk, 1964. 31 *pp.* 0.13 *r.*

These are extensions of tables given in L. G. Kruglikova and V. I. Krylov [Dokl. Akad. Nauk BSSR **5** (1961), 279–283; MR **26** #886]. There are given the A_k and x_k occurring in the quadrature formula

$$\int_0^\infty (1+\sin x)f(x)\,dx \doteq \sum_{k=1}^{n} A_k f(x_k)$$

(and the companion cosine formula) which is exact for $f(x)=(1+x)^{-s-i}$, $i=0, 1, \cdots, 2n-1$, and applicable when $f(x)=\Phi(x)/(1+x)^s$ with $\Phi(x)$ bounded. Values are given for $s=r+\frac{1}{4}, r+\frac{1}{3}, r+\frac{1}{2}, r+\frac{2}{3}, r+\frac{3}{4}, r+1$, $r=1, 2, 3$ and for $n=1(1)8$. [The cited paper gives values for $s=1$, $n=1(1)5$.] The values of the x_k are given to 14-10S and those of A_k to 14-5S, the precision decreasing as n and x_k increase.

The computations were carried out on the computer MINSK-1 along the following lines. A system of linear equations for the coefficients a_k in the representation $\prod(x-x_k)=\sum a_k(1+x)^k$ is obtained and solved. Then the x_k are obtained by Newton's method. Finally, three methods are described to compute the A_k.

John Todd (Pasadena, Calif.)

Pietrzykowski, Tomasz 727

On a certain class of iteration methods for nonlinear equation.

Algorytmy **1**, *no.* 2, 21–27 (1963).

In a paper by the reviewer and F. L. Bauer [Numer. Math. **2** (1960), 55–59; MR **22** #7251] a class of iterative methods for solving linear systems of equations was developed. In the present paper the author adapts some of these to the iterative solution of nonlinear systems, and gives the necessary convergence proofs. Unfortunately, the printing is poor and there are several misprints.

A. S. Householder (Oak Ridge, Tenn.)

Jakovlev, M. N. 728

On the solution of non-linear equations by iterations. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 522–524.

The equation (*) $F(x)=0$ is considered under the assumption that $F(x)$ is the gradient of a functional $f(x)$ on a real Hilbert space H, that is,

$$(F(x)h, h) = \lim_{t\to 0}\frac{f(x+th)-f(x)}{t},$$

for real t and arbitrary $h \in H$. Several iterative processes are considered for the solution of (*), or, equivalently, for minimizing $f(x)$. If

$$M(h, h) \geqq (F'(x)h, h) \geqq m(h, h), \quad m > 0,$$

for x in some sphere about the initial approximation x_0 and arbitrary h, the sequence defined by

$$x_{n+1} = x_n - \frac{2}{M+m}F(x_n) \quad (n = 0, 1, 2, \cdots)$$

converges to a solution of (*) which is unique in S with the speed of a geometric progression.

If $(F'(x)h, h) \geqq m(h, h)$ on the whole space H, $m>0$, and α_n is the minimum of

$$\varphi(\alpha) = f(x_n - \alpha[F'(x_n)]^{-1}F(x_n)),$$

then the Newton-type process

$$x_{n+1} = x_n - \alpha_n[F'(x_n)]^{-1}F(x_n) \quad (n = 0, 1, 2, \cdots)$$

converges, starting from any $x_0 \in H$, to the unique solution of (*) which is the minimum of $f(x)$ on H. The rate of convergence is faster than a geometric progression.

Variants of the above processes are also considered, as ·ll as the case that the calculation of successive approxi- ıtions is subject to error. *L. B. Rall* (Madison, Wis.)

tryshyn, W. V. 729
On a general iterative method for the approximate solution of linear operator equations.
Math. Comp. **17** (1963), 1–10.
ıe author considers a general iteration procedure for lving the operator equation (1) $Au=f$, where f is a given ·ment of a Banach space X and where A is a given unded operator. He first shows that the equation (1) s a unique solution u^* for every f in X if and only if ere exists a continuously invertible operator C and a unded linear operator B with an inverse B^{-1} such that e series $\sum_{i=0}^{\infty} T^i g$, where $T=I-C^{-1}BA$, converges for ery g in X. (A continuously invertible operator on X s X as its range and has a bounded inverse.) Moreover, is given by the above series with $g=C^{-1}Bf$. The con- rgence requirement for the series is implied by any one the following successively more stringent requirements: n $\sup_{n\to\infty} \sqrt[n]{\|T^n g\|}<1$ for every g in X; $\lim_{n\to\infty} \sqrt[n]{\|T^n\|}$ 1; and $\|T\|<1$. The author next shows that if for every in X the equation (1) has a unique solution, then the quence of iterants defined by $Cu_{n+1}=Lu_n+Bf$, $n=$ 1, $\cdots$, converges to the solution u^* for any u_0 in X if ıd only if $\sum_{i=1}^{\infty} T^i g$ converges for any g in X. The author ves error estimates which apply in the case of con- rgence. The general results are applied to a number of ethods, many of which have been used for matrix prob- ms. Included among these methods are the following: $_{+1}=(I-\alpha A)u_n+\alpha f$, where α is a positive constant; $_{n+1}=(C-A)u_n+f$; $u_{n+1}=(I-BA)u_n+Bf$; the general- ·d successive overrelaxation method considered in an rlier paper by the author; the Gauss-Seidel method; and hers. Also included are methods applicable to the solu- on of integral equations.
{There are a number of misprints and minor errors. For stance, the phrase "with $g=C^{-1}Bf$" should be added the statement of Theorem 1. On page 2, the use of the ımber one instead of I in several places is distracting. ı page 3, line 14, replace λ_σ by λ and on page 4, line 5, place H by X.} *D. M. Young, Jr.* (Austin, Tex.)

ı, Li-zhe [Hsu, L. C.] 730a
Several iterative processes with relatively high rate of convergence and a method for estimating the multi- plicity of real roots.
Acta Math. Sinica **12** (1962), 331–340 (*Chinese*); *trans- lated as Chinese Math.* **3** (1963), 361–371.

u, Li-zhe [Hsu, L. C.] 730b
Correction and remark on the paper "Several iterative processes with relatively high rate of convergence . . .".
Acta Math. Sinica **14** (1964), 175–176 (*Chinese*); *trans- lated as Chinese Math.* **5** (1964), 194–196.

ıe first of these papers is divided into three parts. The cond paper is a correction to the first part of the first ıper. (i) Four Newton-type iterative procedures for ıding roots of functions are discussed. Three have order convergence greater than 2.4. This material is related that of Salehov and Mertvecova [Izv. Kazan. Fil. Akad. Nauk SSSR Ser. Fiz.-Mat. Tehn. Nauk **5** (1954), 77–108]. (ii) Conditions are given for uniform convergence of the "Čebyšev process" (third-order Newton). (iii) A method of estimating multiplicities of real roots is given. The problem of estimating the number of real roots of a function in an interval is transformed into one of deter- mining the number of solutions of a related function.
L. W. Ehrlich (Silver Spring, Md.)

Grossman, D. P. 731
Some sufficient conditions for convergence of the Seidel method. (Russian)
Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 136–138.
Consider a system of linear equations $x=Tx+d$, where x is an unknown n-dimensional vector, d a known vector and T a known matrix such that $|I-T|\neq 0$, I the n- dimensional identity matrix. The author proves that if T is indecomposable with non-zero diagonal elements, then the Gauss-Seidel process generated by this system con- verges, provided that the sum of the absolute values of the elements of each row is ≤ 1 with strict inequality for at least one row. It appears that this result is contained in one of the references listed by the author: L. Collatz [Math. Z. **53** (1950), 149–161; MR **12**, 361].
E. Seiden (E. Lansing, Mich.)

Golub, Gene H. 732
Bounds for eigenvalues of tridiagonal symmetric matrices computed by the LR method.
Math. Comp. **16** (1962), 438–445.
Let A be a real, symmetric matrix of order n, and let $x_0=0$, $x_1=x$, $x_{k+1}=Ax_k-\alpha_k x_k-\beta_{k-1}x_{k-1}$ $(k=1,2,\cdots,$ $n-1)$ be a sequence of vectors generated by the Lanczos algorithm, where

$$\alpha_k = (x_k{}^T A x_k)(x_k{}^T x_k)^{-1},$$

$$\beta_{k-1} = (x_k{}^T A x_{k-1})(x_{k-1}^T x_{k-1})^{-1}.$$

If $\sigma_k{}^2=\beta_k+\beta_{k-1}$, where $\beta_0=\beta_n=0$, it is shown that each interval $[\alpha_k-\sigma_k, \alpha_k+\sigma_k]$ contains at least one eigenvalue of A. It is pointed out how this result can be used in con- nection with Rutishauser's LR algorithm (both with or without acceleration) to obtain arbitrarily sharp bounds for the eigenvalues. A trick for avoiding the computation of square roots in Rutishauser's cubically convergent variant of the LR algorithm is indicated.
P. Henrici (Zürich)

Wang, Jin-ru [Wang, Chin-ju] 733
A gradient method for finding the eigenvalues and eigen- vectors of a self-adjoint operator.
Acta Math. Sinica **13** (1963), 23–28 (*Chinese*); *translated as Chinese Math.* **4** (1963), 24–30.
Let X be a real Hilbert space, A a bounded self-adjoint positive definite operator, λ_1 the largest eigenvalue of A, X_1 the subspace corresponding to λ_1. Suppose x_0 is not orthogonal to X_1. Define

$$x_{k+1} = x_k - 2\frac{(Ax_k, x_k)}{(Ax_k, Ax_k)} Ax_k, \quad k = 0, 1, 2, \cdots.$$

It is shown that as k increases, x_{2k} approaches e_1, x_{2k+1} approaches $-e_1$, and (Ax_k, x_k) approaches λ_1, where e_1 is

an eigenvector associated with λ_1. An application to an integral equation is also presented. Finally, an improvement on the bound in Weyl's theorem [Wielandt, Proc. Sympos. Appl. Math. Vol. VI, pp. 261–282, McGraw-Hill, New York, 1956; MR 19, 179] is given for λ_1. The method of this paper is claimed to be an improvement of the fourth method of Krasnosel'skiĭ [Uspehi Mat. Nauk 11 (1956), no. 3 (69), 151–158; MR 18, 676; Amer. Math. Soc. Transl. (2) 12 (1959), 155–162; MR 22 #3737].

L. W. Ehrlich (Silver Spring, Md.)

Rybashov, M. V. [Rybašov, M. V.] 734

On a certain method for solving the global problem of searching for the roots of finite equations by means of an electronic simulator.

Avtomat. i Telemeh. **23** (1962), 1396–1398 (*Russian. English summary*); *translated as Automat. Remote Control* **23** (1963), 1311–1313.

The main idea in this brief note is conceptually simple. To find the roots of the system of equations,

$$e_i : f_i(x_1, \cdots, x_n) = 0, \quad i = 1, \cdots, n,$$

consider the "curve" K in n-space defined by $n-1$ of these equations, say $e_1, \cdots, e_{n-1}$. Move along this curve until it intersects the "surface" $f_n(x_1, \cdots, x_n) = 0$. This point of intersection is a root. Continue in this way until all roots are found. To do this computationally, the author suggests integrating a set of differential equations for K, starting at some initial point. As points of K are computed, $f_n(x_1, \cdots, x_n)$ is evaluated. Zero values of f_n are roots of the system. Unfortunately, this method requires at least one root of the equations $e_1, \cdots, e_{n-1}$ to begin. Furthermore, if a digital computer is used, there are all the usual problems with stability, truncation, error, etc., about which the article says nothing.

Finally, we note the differential equations for K are given incorrectly. *E. K. Blum* (Middletown, Conn.)

Babuška, I.; Prager, M.; Vitasek, E. 735

Completion of numerical processes and the double-sweep method. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 351–353.

Modulo a generous sprinkling of misprints, the authors point out that the double-sweep method ("metod progonki") for the solution of the finite-difference analogue of the boundary-value problem (*) $(ay')' - by = f$, $y(0) = y(1) = 0$ (under appropriate conditions on a, b, and f) may be viewed as the finite-difference analogue of (1) $\{\varphi' = -b\varphi^2 + 1/a,\ \varphi(0) = 0;\ \varphi\xi' = \varphi f - \xi/a,\ \xi(0) = 0;\ y' = \xi + y/\varphi,\ y(1) = 0\}$ and that solving the discrete version of (*) by the "bordering method" leads in the limit to (2) $\{\varphi' = -b\varphi^2 + 1/a,\ \varphi(0) = 0;\ \varphi\xi' = \varphi f - \xi/a,\ \xi(0) = 0;\ q_x' = -q_x/a\varphi\ (t \geq x),\ q_x(x) = 1;\ w_x' = -q_x\xi/a\ (t \geq x),\ w_x(x) = 0;\ y(t) = w_t(1)\}$. Thus, each of the systems (1) and (2) (which are initial-value problems) gives the solution y of the boundary-value problem (*). *T. I. Seidman* (Seattle, Wash.)

Osborne, M. R. 736

A note on finite difference methods for solving the eigenvalue problems of second-order differential equations.

Math. Comp. **16** (1962), 338–346.

The author discusses two "marching" techniques for finding solutions of finite-difference approximations to the eigenvalue problem

$$\frac{d^2y}{dx_2} + p(x)\frac{dy}{dx} + (\lambda q(x) + r(x))y = 0, \quad 0 \leq x \leq l,$$

$$Ay(0) + B\frac{dy(0)}{dx} = 0, \quad y(l) = 0.$$

Corrections to tentative eigenvalues are found (a) by forming Rayleigh quotients of the corresponding tentative solutions, (b) by Newton's method. Connections with previous work by Cooley [Math. Comp. 15 (1961), 363–374; MR 23 #B2602] and Fox [*The numerical solution of two-point boundary value problems in ordinary differential equations*, Oxford Univ. Press, New York, 1957; MR 21 #972] are established. The crucial role of (mathematical) instability in the marching technique is examined. In case of instability, it is proposed to march from both endpoints. Several interesting numerical examples are given.

P. Henrici (Zürich)

Feldman, Karol; Moszyński, Krzysztof 737

Notes on the realization of the algorithm of finding the solution of the boundary problem for a system of ordinary linear differential equation. (Polish. English summary)

Algorytmy Zeszyt Specjalny No. 1 (1963), 61–69.

Authors' summary: "The paper suggests a rather simple method for computing the right-hand sides of differential equations used in another paper [Moszyński, Algorytmy 2 (1964), no. 3, 25–43]. This algorithm has been applied for a problem of the following form:

$$y_i(x) = \sum_{j=1}^{n} a_{ij}(x)y_j(x) + f_i(x), \quad i = 1, 2, \cdots, n,$$

$$\sum_{j=1}^{n} b_{ij}^{(1)}y_j(a) + \sum_{j=1}^{n} b_{ij}^{(2)}y_j(b) = C_i, \quad i = 1, 2, \cdots, n.$$

Flow diagrams of the proposed method's realization, as well as some notes on results obtained, are enclosed."

Chase, P. E. 738

Stability properties of predictor-corrector methods for ordinary differential equations.

J. Assoc. Comput. Mach. **9** (1962), 457–468.

The author conducts a thorough study of the quantitative stability properties of a number of special multi-step methods for the numerical solution of ordinary differential equations. In addition to the usual predictor-corrector methods (corrector applied an infinite number of times), he also considers the cases where the corrector formula is applied only once, and where the result of the corrector formula is averaged with that of the predictor formula in order to increase the order of the local discretization error. The definition of stability adopted in the paper depends on the step h and on the differential equation under consideration, which is assumed to be of the form $y' = Ay + g(x)$ ($A = \text{const}$). If $A \leq 0$, a method is called stable if the moduli of all zeros of the characteristic polynomial of the resulting difference equations are bounded by 1. If $A > 0$, no definition of stability is formulated explicitly; however, the condition that the moduli should be bounded by e^{Ah} seems to be implied. By computing the zeros

numerically and graphing their moduli the author obtains several interesting results, among which we mention the following: (i) Contrary to the asymptotic theory valid for $h\to 0$ (see the reviewer's book [*Discrete variable methods in ordinary differential equations*, pp. 259–262, Wiley, New York, 1962; MR **24** #B1772]), the stability of a method can, for moderate values of Ah (such as 0.3), depend very sensitively on the number of times the corrector formula is applied. (ii) A method proposed by Hamming [J. Assoc. Comput. Mach. **6** (1959), 37–47; MR **21** #973] enjoys particularly favorable stability properties, especially if the corrector is applied repeatedly. These conclusions are substantiated by experimental computations on the problem $y' = -100y+100$, $y(0)=0$. {Reviewer's remark: The author concentrates entirely on stability and does not consider at all the accuracy of the methods in question, which depends on how closely the principal root of the characteristic polynomial approximates e^{Ah}. The issue of accuracy is somewhat obscured by the numerical example, where both the exact and the approximate solution trivially tend to 1 as $x\to\infty$.}

P. Henrici (Zürich)

Gorbunov, A. D.; Šahov, Ju. A. 739

Approximate solution of the Cauchy problem for ordinary differential equations with a preassigned number of correct signs. I. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **3** (1963), 239–253.

(For terminology used in this review, see the reviewer's book [*Discrete variable methods in ordinary differential equations*, Wiley, New York, 1962; MR **24** #B1772].) For solving the initial-value problem for a single ordinary differential equation, the authors propose to simultaneously use two one-step methods whose orders p are the same, but whose principal error functions have the sum zero. The average of the two solutions thus obtained is more accurate than either of them. Several such pairs of methods of the orders $p=1, 2, 3$ are exhibited, and some numerical examples are given. {Reviewer's remarks: The proposed algorithm does not produce strict lower and upper bounds for the exact solution, as the analysis takes into account only the leading term in the expansion of the local discretization error. In view of this and of the rather complicated structure of some of the proposed formulas, it is not clear whether the same effect could not be obtained by working with a Runge-Kutta formula of order $p+1$.}

P. Henrici (Zürich)

Tihonov, A. N.; Gorbunov, A. D. 740

Error estimates for a Runge-Kutta type method and the choice of optimal meshes. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 232–241.

The authors extend in a direct way to the case of nonuniform step sizes their earlier works [same Ž. **2** (1962), 537–548; MR **28** #1774; ibid. **3** (1963), 195–197; MR **28** #1773] on the numerical solution in a bounded interval of the vector Cauchy problem for ordinary differential equations by methods of Runge-Kutta type. If h is here the maximum step taken, then the error estimates read essentially the same as before. The optimal choice of the "normal distribution function" (not in the usually probability sense) of the step size, defined to be a positive continuously differentiable function $\phi(x)$ such that the kth step $h_k=x_{k+1}-x_k=\lambda\phi(x_k)$, $k\leq n-2$, and $h_{n-1}\leq \lambda\phi(x_{n-1})$, λ a positive parameter, is also discussed [cf. Morrison, J. Assoc. Comput. Mach. **9** (1962), 98–103; MR **24** #B906]. Optimality here requires the conditional minimization of a function of $\phi^s(x)$, where s is the order of Runge-Kutta method used.

M. L. Juncosa (Santa Monica, Calif.)

Kurmit, A. A. [Kurmītis, A. A.] 741

Error theory for the Runge-Kutta method. (Russian. English summary)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **18** (1963), *no.* 4, 35–48.

The author presents a method to estimate the error of the numerical solution obtained by the one-step method for the Cauchy problem. Let $x'=f(t,x)$ $('=d/dt)$ be a given equation and $x(t_0)=x^0$ be a given initial condition. Let $J(t,x)$ be the Jacobi matrix of $f(t,x)$ with respect to x and put

$$Q(t,x) = U^{-1}(t)J(t,x)U(t)-U^{-1}(t)U'(t),$$

where $U(t)$ is an arbitrary non-singular matrix. For estimation of the error, the author uses the quantity

$$\gamma[Q(t,x)] = \lim_{h\to+0}\frac{\|I+hQ(t,x)\|-1}{h}$$

(the author calls this quantity the logarithmic norm of $Q(t,x)$), where I is the unit matrix and $\|\cdots\|$ denotes any norm. The author's method is based on the following lemma: Let $[t_k, t_{k+1}]$ $(k=0,1,2,\cdots; t_{k+1}-t_k=h)$ be the divided intervals and let d_k and r_k be, respectively, the truncation error and the round-off error at the kth step. If $\gamma[Q(t,x)]\leq\gamma(t)$, $\|J(t,x)\|\leq v(t)$ and

$$\|U^{-1}(t)\|\cdot\|d_k+r_k\|\cdot(1+hv(t))/h \leq \tau(t) \qquad (t_k\leq t\leq t_{k+1}),$$

then, for the error Δ_k of the numerical solution at $t=t_k$, $\|\Delta_k\|\leq\|U(t_k)\|\varepsilon(t_k)$, where $\varepsilon(t)$ is a solution of the equation $\varepsilon'=\gamma(t)\varepsilon+\tau(t)$ satisfying the initial condition $\varepsilon(t_0)=\varepsilon_0\geq\|U^{-1}(t_0)(x_0-x^0)\|$ (x_0 is the initial value used for numerical integration). The author states that $U(t)$ should be chosen so that $\|U^{-1}(t)\|$ and $\|U(t)\|$ may not become very large and $\gamma[Q(t,x)]$ should be as small as possible. For estimation of the truncation error d_k, the author gives two methods, one of which uses the truncation errors of the multistep formulas. The author also says that the region in which $\gamma[Q(t,x)]$ is estimated can be made narrower if one can find some constant D such that $\|U(t_k)\|\varepsilon(t_k)\leq D$ $(k=0,1,2,\cdots)$. Last, one numerical example is presented.

In a final remark, the author refers to the reviewer's paper [J. Sci. Hiroshima Univ. Ser. A-I Math. **25** (1961), 3–62; MR **25** #759]. However, the reviewer is not sure that his error estimate is essentially improved by the present author's method.

M. Urabe (Fukuoka)

Volkov, E. A. 742

Effective error bounds by the method of nets for solutions of the Dirichlet problem for the Laplace equation on polygons. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 735–738.

Concerned with the fact that, in general, bounds on the

kth derivatives of the boundary function are not enough to ensure the boundedness of the solution to the Dirichlet problem in regions with boundaries which are only piecewise smooth [see Fufaev, same Dokl. **131** (1960), 37–39; MR **22** #9750], the author extends to the case of a polygon (with some restrictions on the angles) some known estimates of the error between the solution of the Dirichlet problem and solutions of analogous difference equations resulting from 5- and 9-point formulas. Two types of estimates are given: analytical ones of Gershgorin type which depend upon radius of circles covering the region, and others, more precise, in the form of solutions of auxiliary systems of difference equations.

The author also gives estimates on the derivatives of harmonic functions of polygons and coefficients of asymptotic derivatives of them at the corners of the polygon expressed through bounds on the derivatives of the functions on the sides of the polygon.

M. L. Juncosa (Santa Monica, Calif.)

Stummel, F. 743

★Näherungsmethoden zur Lösung elliptischer partieller Differentialgleichungen.

Vortrag auf der Jahrestagung der Deutschen Mathematiker-Vereinigung in Frankfurt am Main, 1963. HMI-B 31.

Hahn-Meitner-Institut für Kernforschung Berlin, Berlin. 1963. iii + 22 *pp.*

Author's summary: "Dieser Vortrag beschäftigt sich mit Fragen aus dem Gebiet der Näherungsmethoden linearer elliptischer partieller Differentialgleichungen. Behandelt wird dabei das Problem der Konvergenz der Lösungen von Differenzengleichungen gegen Lösungen der zugehörigen Differentialgleichungen mit funktionalanalytischen Hilfsmitteln, die Frage der Fehlerabschätzung von Näherungslösungen im Zusammenhang mit der Methode der orthogonalen Projektion und schließlich die iterative Lösung von Randwertaufgaben für elliptische Differenzen- beziehungsweise Differentialgleichungen im Rahmen der Hilbertraumformulierung dieser Aufgabe."

Pierre, Donald A. 744

A rational fraction approximation formula for $\exp[-(sD)^{1/2}]$ with applications.

J. Soc. Indust. Appl. Math. **12** (1964), 93–104.

Consider the solution of $\partial^2 v(x,t)/\partial x^2 = K\partial v(x,t)/\partial t$ such that $v(\infty,t)=0$ and $v(0,t)=r(t)$. Then by use of the Laplace transform, (*) $d^2V(x,s)/dx^2 = KsV(x,s)$ and $V(L,s)=R(s)\exp[-(sD)^{1/2}]$ when V and R are the Laplace transforms of v and r with respect to t.

Now consider the finite-difference equivalent of (*) in the form $V_{m+1,n}-2V_{m,n}+V_{m-1,n}=qV_{m,n}$, where $q=Ks(\Delta x)^2$, $m\leq n$, $V_{0,n}=R(s)$ and $V_{n+1,n}=(1+q-q_e)V_{n,n}$. Here the boundary condition at $x=\infty$ is replaced by a new condition depending on q_e. If m, $n\to\infty$ and $\Delta x\to 0$ such that $m\Delta x\to L$ and $n\Delta x\to 0$, the solution of the difference equation approaches that of the differential equation (*). Under these limiting conditions with $G_{m,n}(s)=V_{m,n}(s)/R(s)$, then $G_{m,n}(s)\to\exp[-(sD)^{1/2}]$. This leads to rational approximations to the exponential function. The parameter q_e is introduced to provide a means of adjusting lower-order approximations for better fit.

The author shows how the rational approximations may be used to approximate the inverse transform of $\exp[-(sD)^{1/2}]/s$. Other applications are also given.

Y. L. Luke (Kansas City, Mo.)

Hersch, Joseph 745

Lower bounds for all eigenvalues by cell functions: A refined form of H. F. Weinberger's method.

Arch. Rational Mech. Anal. **12** (1963), 361–366.

The author is concerned with constructing, by means of finite-difference methods, lower bounds for all eigenvalues of a homogeneous membrane covering a plane domain G which is a union of squares of equal side h. He shows that the use of cell functions (introduced by him in an earlier paper [C. R. Acad. Sci. Paris **240** (1955), 1602–1604; MR **16**, 929]) permits a refinement of H. F. Weinberger's method [Pacific J. Math. **8** (1958), 339–368; ibid. **8** (1958), 941; MR **21** #6097], where "node" functions were used. The advantages of the author's method are the following: (a) the domain no longer needs to be augmented, thus sharper bounds are obtained; (b) any slit in G is taken into account in the auxiliary discrete problem, whereas it disappears in Weinberger's augmented domains.

P. Henrici (Zürich)

Thomée, Vidar 746

Elliptic difference operators and Dirichlet's problem.

Contributions to Differential Equations **3** (1964), 301–324.

The author considers a general class of difference methods for the Dirichlet problem for elliptic equations with constant coefficients of the form

$$(1)\qquad Lu = \sum_{|\beta|=|\gamma|=m} a_{\beta\gamma}D^{\beta+\gamma}u = F,\quad a_{\beta\gamma}=a_{\gamma\beta},$$

where $\sum_{|\beta|=|\gamma|=m} a_{\beta\gamma}\xi^{\beta+\gamma}>0$ for real $\xi\neq 0$. The difference approximation of Lu is of the form

$$(2)\qquad L_h u(x) = h^{-2m}\sum_\alpha c_\alpha u(x+\alpha h),$$

where h is the mesh-width. Such a difference operator is called elliptic if $\sum_\alpha c_\alpha e^{i\langle\alpha,\theta\rangle}>0$ for real $\theta\neq 0$, $|\theta_j|<\pi$. Under the assumption of ellipticity and of consistency of (2) with (1), it is shown that the solution of the discrete problem converges to the solution of the corresponding continuous problem as $h\to 0$. Moreover, the difference between the two is $O(h^{1/2})$ as $h\to 0$. An iterative method is suggested for solving the discrete problem, and its convergence rate is investigated.

W. Littman (Minneapolis, Minn.)

D'jakonov, E. G. 747

Difference schemes with a splitting operator for nonstationary equations. (Russian)

Dokl. Akad. Nauk SSSR **144** (1962), 29–32.

Let $u(x,t)$ be the solution of the linear parabolic equation

$$(1)\qquad D_0 u = \sum_{s=1}^{p}[D_s(a_s(x,t)D_s u)+b_s(x,t)D_s u] + d(x,t)u+f(x,t)$$

in $Q_T=\bar\Omega\times[0\leq t\leq T]$, where $\bar\Omega$ is a parallelepiped in p-dimensional x-space, $D_0=\partial/\partial t$, $D_s=\partial/\partial x_s$, and $u(x,0)=$

$\varphi(x)$, while $u = u_S = \psi(x, t)$ for $(x, t) \in S$ (the lateral surface of Q_T). Let Δ_s [or $\bar{\Delta}_s$] be the forward [backward] difference quotient operator in the x_s-direction with mesh length h_s, and let $\tilde{\Delta}_s = 0.5(\Delta_s + \bar{\Delta}_s)$. At time t_n define the operators $L_s^{(n)} = \bar{\Delta}_s a_s^{(n)} \Delta_s + b_s^{(n)} \tilde{\Delta}_s + d_s^{(n)} E$, and $A_s^{(n)} = E - \tau_n L_s^{(n)}$, where E is the identity operator, $\tau_n = t_n - t_{n-1}$, and $d_s^{(n)}$ are more fully described below. Then approximate (1) by the difference equation

$$(2) \qquad \left(\prod_{s=1}^{p} A_s^{(n)}\right) v^{(n+1)} = (E + \bar{d}^{(n)}) v^{(n)} + \tau_n f^{(n)},$$

$$v^{(0)} = \varphi, \quad v_S^{(n)} = \psi^{(n)},$$

where $d_s^{(n)}$ and $\bar{d}^{(n)}$ are chosen so that (i) for boundary conditions $\psi = 0$ we have $L_s^{(n)}$ negative-definite, and (ii) $\bar{d}^{(n)} + \sum_s d_s^{(n)} = d^{(n)}$. The split operator in the left member of (2) is presumably easier to invert than the non-splittable operator that would be obtained for more obvious difference analogues of (1). The author states without proof theorems which assure stability and convergence: $\|v^{(n+1)}\| \leq (1 + c\tau_n)\|v^{(n)}\|$ (for $\psi = f = 0$), and $\|u^{(n)} - v^{(n)}\| = O(\|\tau\|) + O(\|h\|^2)$, where $\|\tau\| = \max \tau_n$ and $\|h\| = \max h_s$. Similar theorems are stated for the linear wave equation obtained from (1) by replacing D_0 by D_0^2, and for other generalizations to periodic solutions (in x) of (1).

J. H. Giese (Aberdeen, Md.)

Kuropatenko, V. F. 748

A difference method for calculating shock waves. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **3** (1963), 201–204.

For the Lagrangian form of the equations of one-dimensional flow expressed in conservation form, the author uses a conventional difference scheme in shock-free regions of the xt-plane. All functions are evaluated at points $(i + \frac{1}{2})h$, $n\tau$. For any flow function f, $\partial f/\partial t$ is approximated by a forward difference, and $(\partial f/\partial x)_{i+1/2}$ by $(f_{i+1} - f_i)/h$. In regions judged to be shock-free by virtue of $\partial u/\partial x \geqq 0$, $2f_i = f_{i-1/2} + f_{i+1/2}$. When $\partial u/\partial x < 0$ the difference equations are supplemented by the conservation conditons at a shock with an appropriate interpretation of u_i and p_i in terms of velocity, pressure, etc., before and after the shock. Numerical results for the early stages of motion in a shock tube reveal the usual moderate broadening of the shock and much poorer representation of a rarefaction wave.

J. H. Giese (Aberdeen, Md.)

Moore, R. H. 749

Newton's method and variations.

Nonlinear Integral Equations (Proc. Advanced Seminar Conducted by Math. Research Center, U.S. Army, Univ. Wisconsin, Madison, Wis., 1963), pp. 65–98. *Univ. Wisconsin Press, Madison, Wis.,* 1964.

This is an excellent expository article on Newton's method in Banach spaces, its generalizations, extensions and applications to numerical computations, particularly to certain nonlinear integral equations. The presentation is based on results in the paper of L. V. Kantorovič [Uspehi Mat. Nauk **3** (1948), no. 6 (28), 89–185; MR **10**, 380; English transl., Nat. Bur. Standards Rep. 1509 (1952); MR **14**, 766].

H. D. Block (Ithaca, N.Y.)

COMPUTING MACHINES

See also 734, 1109.

Randell, B.; Russell, L. J. 750

★ALGOL 60 implementation. The translation and use of ALGOL 60 programs on a computer.

With a Foreword by E. W. Dijkstra.

Published for The Automatic Programming Information Centre, Brighton College of Technology, by Academic Press, London-New York, 1964. xiv + 418 *pp.* 84*s.*

This book presents a very complete documentation of an ALGOL 60 processor. Although the particular processor described in detail is the "Whetstone Compiler" for the KDF 9 computer, the authors have attempted to make their presentation as machine-independent as possible. The processor in question does not produce object code (i.e., machine language instructions) as most processors termed "compilers" do. Instead, it produces a sequence of intermediate-level pseudo-instructions, which are then interpreted to execute the intended algorithm. Thus, the authors are more concerned with the analysis of the input language (i.e., ALGOL 60) than the synthesis of the machine code which results. The book is written for experts, and for them it should be very valuable. At the beginning is a brief historical survey of compiling methods. These methods are quite definitely viewed through the eyes and interests of the present authors, and the survey should be read with this in mind. The foreword by Dijkstra contains a philosophy that the reviewer cannot agree with at all, e.g., ". . . For most of the difficulties met during translation can be traced down to a single source, viz. either an awkward feature of the language or an awkward feature of the machine. Now I hope that we all agree that there is no point in imposing unnecessary burdens on the translator or its composers. On the contrary, they make translators unnecessarily expensive to construct, unnecessarily expensive to run and, worst of all, they tend to affect the reliability and trustworthiness of the whole system. If we agree on this, the art of language designing becomes the art of designing a powerful and systematic language primarily from those elements which can be processed elegantly by a translator—and this book shows that this set is certainly not empty—and the art of machine designing becomes the art of building-in such features that can be used by a translator in a neat and orderly way—and the Control Routine described in this book shows some of these features." It is the opinion of the reviewer that translators need be written only once, but the languages must be used from then on. Obviously, constructions in the language may often be modified in natural ways so as to make translation easier, and ambiguities cannot be tolerated, but the convenience of the translator writer must not be the criterion against which language design is measured.

B. A. Galler (Ann Arbor, Mich.)

Baumann, R.; Feliciano, M.; Bauer, F. L.; Samelson, K. 751

★Introduction to ALGOL.

Prentice-Hall Series in Automatic Computation.

Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964. x + 142 *pp.* $6.00.

This book is a well-written description of the ALGOL 60

language. It is intended for the non-specialist, and the authors wisely refrain from belaboring some of the more subtle and controversial aspects of the ALGOL language. The book is organized into three major sections which present (1) the elementary, more-or-less standard constructions that one would expect in a general procedure-oriented language, (2) the more sophisticated constructions which have to do with block structure and scope of definitions, and (3) some of the most advanced and important innovations of the language. The Appendix contains a formal description of the language in what has become known as the Backus Normal Form. It is actually the Revised Report on the Algorithmic Language ALGOL 60, which came out of a meeting of several ALGOL 60 authors and others in 1962 [Naur (Editor), Numer. Math. **4** (1962/63), 420–453; MR **27** #4390]. This book should serve as a good introduction to the language.

B. A. Galler (Ann Arbor, Mich.)

Pierce, William H. 752

Interwoven redundant logic.

J. Franklin Inst. **277** (1964), 55–85.

Author's summary: "In interwoven redundant logic, each (redundant) gate receives B versions of each input, and makes its output from the redundant input information. Some (redundant) gates, such as threshold gates and AND-OR-NOT gates, correct errors in one layer. Other (redundant) gates correct errors in two alternating layers. Of these, all-NAND or all-NOR networks automatically have the necessary alternation of layers, while with AND, OR, and NOT networks the alternation must be designed into the network. Both single-layer and alternating-layer error correction schemes correct the most errors when the plan specifying which input signals are connected to which gates is systematic. Rules and reliability analyses for such plans are developed, plans stated, and simple, practical circuit implementations given. Restoring organs and quadding are both very special cases of interwoven redundant logic. Interwoven redundant logic is theoretically important because it unifies redundancy techniques, and practically important because it permits error correction in the same circuits which perform logic."

H. D. Block (Ithaca, N.Y.)

Ashenhurst, R. L. 753

Function evaluation in unnormalized arithmetic.

J. Assoc. Comput. Mach. **11** (1964), 168–187.

The paper is concerned with the evaluation of functions of a single variable when unnormalized number representation is used. In such cases the function evaluation is usually subjected to certain adjustment rules, such as transformation of an arbitrary number $(e,f)\equiv f\cdot 2^e$ into a form $(e-m,\ 2^m f)$ with $\frac{1}{2}\leq|2^m f|<1$. The particular problem discussed here is the establishment of an adjustment rule that is not too difficult to apply in the case of elementary functions, and that is also useful in estimating the error propagated in one stage of functional evaluation. The analysis carried out indicates that an appropriate adjustment of the functional value may be obtained by means of the rule $m_2\approx m_1+\log_2|xF'(x)/F(x)|$ when proceeding from argument x to functional value $F(x)$. This rule is applied to a number of elementary functional operations and found to be quite appropriate.

M. Lotkin (Cherry Hill, N.J.)

MECHANICS OF PARTICLES AND SYSTEMS

See also 324, 328, 329, 330, 335, 466, 893, 1027, 1041, 1099, 1100.

Yang, A. T.; Freudenstein, F. 754

Application of dual-number quaternion algebra to the analysis of spatial mechanisms.

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 300–308.

Authors' summary: "Dual-number quaternion algebra has been used to obtain explicit, closed-form algebraic expressions for the displacements, velocities, velocity ratios, forces, torques, mechanical advantages, transmission angle, and locking positions in spatial four-link mechanisms having one turning pair and three turn-slides. The results have been programmed for automatic digital computation."

Steigenberger, J. 755

On separation of variables in mechanical systems. (French and German summaries)

Acad. Roy. Belg. Bull. Cl. Sci. (5) **49** (1963), 990–1014.

Author's summary: "Some investigations concerning the properties of separability of mechanical systems whose Hamiltonians involve linear terms with respect to the moments. Using orthogonal coordinates, construction of separable systems by integration of the separability conditions. Application to the one-particle problem in special relativity."

de Franchis, Franco 756

Sui sistemi anolonomi il cui moto è definito da un sistema lagrangiano.

Boll. Un. Mat. Ital. (3) **18** (1963), 367–376.

The author discusses the case of a non-holonomic dynamical system where the motion may also be described by a Lagrangian. The relations which exist between the constraints and the quadratic form of kinetic energy such that a Lagrangian formulation is possible are derived. It is supplemented by a simple example of motion of a particle in space. *N. D. SenGupta* (Bombay)

Sagitov, M. N. 757

Certain cases of the motion of a rotating sphere of variable mass whose axis is horizontal. (Russian. Kazak summary)

Izv. Akad. Nauk Kazah. SSR Ser. Fiz.-Mat. Nauk **1963**, *no.* 3, 88–99.

The author considers the problem of a sphere rotating about its horizontal diameter, whose mass M changes according to the law $M=M_0/(1+\alpha t)$, M_0 denoting the initial mass of the sphere, and α some constant characterizing the change of the mass.

The author supposes that the change of the mass does not influence the position of the mass center with respect to the sphere, that the exterior dimensions of the sphere remain unaltered, and that the angular velocity of the sphere is of constant modulus and orthogonal to the initial velocity of the mass center (i.e., with only the sense variable).

He mentions as possible acting forces the gravity, the resistance force, the Magnus effect and the reactive force, but he solves only five simple elementary cases, e.g., the motion of a rotating sphere only under the action of the Magnus effect, etc. *T. P. Anđelić* (Belgrade)

Novoselov, V. S. 758

Extremal character of integral principles of non-holonomous mechanics in non-holonomous coordinates. (Russian. English summary)

Vestnik Leningrad. Univ. **17** (1962), *no.* 1, 124–133.

A variational apparatus to the second order in non-linear non-holonomic coordinates is worked out, and formulas for the second variations of integrals of Hamilton-Ostrogradsky and Euler-Lagrange principles in the above coordinates are given. *M. Kiveliovitch* (Paris)

Rumjancev, V. V. 759

On the motion of certain systems with non-ideal constraints. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1961**, *no.* 5, 67–76.

L'auteur étudie les systèmes mécaniques à liaisons idéales et non idéales (par exemple des servo-liaisons ou des liaisons à frottement, etc.).

On suppose que parmi tous les déplacements du système admissible avec les liaisons idéales, il existe un certain nombre de déplacements pour lesquels les réactions des liaisons non idéales ne fournissent pas de travail. Pour ces systèmes, l'auteur établit une analogie du principe des moindres contraintes de Gauss sans y introduire explicitement les réactions des liaisons. *M. Kiveliovitch* (Paris)

Slutskin, A. A. [Sluckin, A. A.] 760

Motion of a one-dimensional nonlinear oscillator under adiabatic conditions.

Ž. Èksper. Teoret. Fiz. **45** (1963), 978–988 (*Russian. English summary*); *translated as Soviet Physics JETP* **18** (1964), 676–682.

In this paper the author considers systems with one degree of freedom, having the following properties: the Hamiltonian of the system depends on an external parameter, which is a specified function of time; when the parameter is constant the motion in the system is periodic; the parameter changes very little during the characteristic period of the system. It is shown, by using methods of classical mechanics, that throughout the process the change of the adiabatic invariant is exponentially small.

Then the author proposes a method which permits the derivations of the change of the adiabatic integral in a closed form for many important cases.

Y. Kozai (Tokyo)

Rogers, E. H. 761

A minimax theory for overdamped systems.

Arch. Rational Mech. Anal. **16** (1964), 89–96.

Duffin [J. Rational Mech. Anal. 4 (1955), 221–233; MR 16, 979] successfully generalized Rayleigh's method to apply to systems of the form

$$A\ddot{q}+B\dot{q}+Cq = 0,$$

where A, B, C are semi-definite matrices satisfying an overdamping condition, by considering as functionals of u the roots of

$$x^2(Au, u)+x(Bu, u)+(Cu, u) = 0.$$

Here, similar results are shown to hold for a general functional $p(u)$ associated with a linear operator T_x depending on a parameter x, where $p(u)$ satisfies the conditions $p(\alpha u)=p(u)$ $(\alpha\neq 0)$, as for any Rayleigh quotient; $(T_{p(u)}u, u)=0$; $\partial(T_xu, u)/\partial x|_{x=p(u)}>0$, a generalized overdamping condition. *R. C. T. Smith* (Armidale)

Laval, G.; Pellat, R.; Cotsaftis, M.; Trocheris, M. 762

Marginal stability condition for stationary non-dissipative motions.

Nuclear Fusion **4** (1964), 25–29.

Authors' summary: "A method is proposed for discussing the linear stability of the stationary states of a Lagrangian system by considering the marginally stable modes. A necessary and sufficient condition for marginal stability is derived. Perturbation theory is used to compute the change in frequency of a marginally stable mode as a result of a small change in the system. It is then possible to determine the changes of stability of a system depending continuously on one parameter. The method is of practical interest as it enables one to discuss stability by considering only normal modes for real values of the frequency. It reduces to Poincaré's method of forms of bifurcation in the particular case of equilibria, for which the square of the frequency of a normal mode is always real."

Ševelo, V. N.; Štelik, V. G. 763

On the theory of the non-autonomous mathematical pendulum. (Russian. English summary)

Ukrain. Mat. Ž. **14** (1962), 372–382.

L'auteur étudie le mouvement d'un pendule de masse $m(t)$ et de longueur $l(t)$

$$(1)\qquad (ml^2x')'+mglf(x) = 0$$

pour différents cas: (a) $f(x)\sim x$; (b) $f(x)\sim x-x^3/3!$; (c) $f(x)\sim x-x^3/3!+x^5/5!$; (d) $f(x)=\sin x$; en posant pour $t\geqq t_0\geqq 0$ les conditions suivantes sont remplies: (α) $m(t)$ et $l(t)$ sont des fonctions infiniment dérivables, (β) $0<m_1\leqq m(t)\leqq m_2<\infty$; $0<l_1\leqq l(t)\leqq l_2<\infty$, ou ($\gamma$) $0<l_1\leqq l(t)\leqq l_2<\infty$; $m(t)l^2(t)=\exp(\int\alpha(t)\,dt)$; $|\alpha(t)|\leqq\alpha_2<\infty$; m_1, m_2, l_1, l_2 et α_2 sont des constantes.

Les conditions (α) et (β) ou (α) et (γ) sont suffisantes pour l'existence d'une solution unique de l'équation (1) pour tous les $t\geqq t_0\geqq 0$.

Le mouvement du système (1) est étudié par rapport aux positions d'équilibre qui sont les racines de l'équation $f(x)=0$. *M. Kiveliovitch* (Paris)

Osetinskii, Yu. V. [Osetinskiĭ, Ju. V.] 764

The stability of the motion of a shaft.

Dokl. Akad. Nauk SSSR **151** (1963), 295–298 (*Russian*); *translated as Soviet Physics Dokl.* **8** (1964), 669–672.

A cursory treatment is given of the motion of a disc on an overhung weightless vertical shaft. The principal moments of inertia of the disc about axes perpendicular to the undeformed shaft differ by a small amount determined by a parameter ε. The disc is centrally located on the shaft. A perturbation procedure is followed using ε as the small parameter. The zero-order solution is given which predicts the usual forward and reverse precession while the first-order solution gives the first approximation to the regions of instability.

P. R. Pasley (Houston, Tex.)

Bottema, O. 765

On the stability of circular motions in the field of two fixed centres.

Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 301–309.

Zwei feste Attraktionszentren M_1 und M_2 (Massen m_1 und m_2) wirken nach dem Newtonschen Gesetz auf einen Punkt P (Masse m), der mit konstanter Winkelgeschwindigkeit einen Kreis C (Radius R) in einer zu M_1M_2 normalen Ebene beschreibt, dessen Mittelpunkt S auf der Achse M_1M_2 liegt. Die Stabilitätsverhältnisse einer solchen Bewegung wurden von V. G. Degtjarev [Prikl. Mat. Meh. **26** (1962), 778–779; MR **27** #2688] untersucht. Die vorliegende Arbeit erstellt zuerst die analytischen Elemente der genannten Kreisbewegung. Hat dann der Mittelpunkt S der Kreisbahn vom Mittelpunkt 0 der Strecke M_1M_2 die Entfernung u, und ist der Bahnradius R, so kann der Zusammenhang zwischen R und u am einfachsten durch ein Diagramm in der (R, u)-Ebene veranschaulicht werden, das im Falle $m_1^2 > m_2^2$ eine algebraische Kurve 8-ter Ordnung und im Falle $m_1^2 = m_2^2$ eine Kurve 6-ter Ordnung (oder ein Teil dieser Diagramme) ist. Die Diskussion der Stabilitätsverhältnisse (in Zylinderkoordinaten) führt auf eine zu dem Kriterium von Degtjarev äquivalente Stabilitätsbedingung, die schließlich durch Heranziehung der Veränderlichen R und u bedeutend vereinfacht werden kann. Man erhält so auch einen einfachen graphischen Überblick über die Bereiche stabilen und unstabilen Verhaltens der Kreisbewegung, die in der (R, u)-Ebene durch eine algebraische Kurve 4-ter Ordnung voneinander geschieden werden.

K. Strubecker (Karlsruhe)

Rosenberg, R. M. 766

On normal mode vibrations.

Proc. Cambridge Philos. Soc. **60** (1964), 595–611.

This paper contains a summary of the work of the author and his co-workers on the study of the "normal modes" of vibration of discrete nonlinear dynamic systems [the author and C. P. Atkinson, J. Appl. Mech. **26** (1959), 377–385; MR **21** #6730; the author, ibid. **27** (1960), 263–268; MR **22** #2058; Trans. ASME Ser. E. J. Appl. Mech. **28** (1961), 275–283; MR **22** #11545; ibid. **29** (1962), 7–14; MR **25** #792; Quart. Appl. Math. **21** (1963), 37–47; MR **26** #1497; the author and C. S. Hsu, Analytic methods in the theory of non-linear vibrations (Proc. Internat. Sympos. Non-linear Vibrations, Vol. I, 1961), pp. 380–416, Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1963; MR **28** #2302].

T. K. Caughey (Pasadena, Calif.)

ELASTICITY, PLASTICITY

See also 349, 371, 466, 745, 911.

★Second-order effects in elasticity, plasticity and fluid dynamics. 767

International Symposium, Haifa, Israel, April 23–27, 1962. Edited by Markus Reiner and David Abir. International Union of Theoretical and Applied Mechanics.

Jerusalem Academic Press, Jerusalem; Pergamon Press, Oxford, 1964. xvi + 795 *pp.*

Papers of mathematical interest will be reviewed individually.

Arutjunjan, N. H. [Арутюнян, Н. Х.]; Abramjan, B. L. [Абрамян, Б. Л.] 768

★The torsion of elastic bodies [Кручение упругих тел].

Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1963. 686 *pp.* 2.50 *r.*

Gradually, individual chapters of A. E. H. Love's *A treatise on the mathematical theory of elasticity* [4th ed., Dover, New York, 1944; MR **6**, 79] are being replaced by monographs summarizing the relevant literature which has been growing rapidly. In this sense, this book replaces Chapters XIV, XVIII and XXI. Chapter I (76 pp.) with an exhaustive formulation of the problem touches on various analogies and photo-elasticity, and presents general solutions. Chapters II (120 pp.), III (70 pp.), IV (80 pp.), V (34 pp.) treat rods with solid polygonal, hollow polygonal, circular notched, and elliptical notched cross-sections, respectively, for homogeneous, isotropic media. Chapters VI (63 pp.), VII (58 pp.) relate to composite and thin-walled rods. Bodies of revolution are treated in Chapters VIII (85 pp.), IX (57 pp.) and X (34 pp.), devoted to circular and conical, stepped and general sections, respectively. Detailed presentation of formulae is responsible for the size of the monograph which is a long-needed addition to the literature. Most chapters are followed by summaries of the literature up to the date of publication. Special interest attaches to Chapter VIII with classical and new results on the torsion of the cone and hemisphere.

J. R. M. Radok (Adelaide)

Lomakin, V. A. 769

Statistical description of the stressed state of a body under deformation. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 1274–1277.

A continuous medium and the equations of static equilibrium are considered. It is assumed that the state of stress can be characterized by moments of various orders in the sense of the theory of probability. The differential equations satisfied by these moments are deduced, and the case of an isotropic statistical distribution is discussed.

M. Mişicu (Bucharest)

Novozhilov, V. V. [Novožilov, V. V.] 770

On the forms of the stress-strain relation for initially isotropic nonelastic bodies (geometric aspect of the question).

Prikl. Mat. Meh. **27** (1963), 794–812 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1219–1243.

A set of six linearly independent (three-dimensional second-order) symmetric tensors forms a basis for the space H_6 of symmetric tensors. A variety of results concerning bases for H_6 and its subspaces are given. For example, if two symmetric tensors **a** and **b** do not have a common principal direction, then **I**, **a**, **b**, $\mathbf{a}^2$, $\mathbf{b}^2$, $\mathbf{ab}+\mathbf{ba}$, $\mathbf{a}^2\mathbf{b}+\mathbf{ba}^2$, $\mathbf{b}^2\mathbf{a}+\mathbf{ab}^2$ suffice to form a basis for H_6. The theory of tensors depending on one scalar argument is also considered and the results applied to the determination of stress-strain relations for initially isotropic materials. *G. F. Smith* (New Haven, Conn.)

Koiter, W. T. 771

Couple-stresses in the theory of elasticity. I, II.

Nederl. Akad. Wetensch. Proc. Ser. B **67** (1964), 17–29; *ibid.* **67** (1964), 30–44.

In Part I, the author rederives the equations of equilibrium given previously by Mindlin and Tiersten [Arch. Rational Mech. Anal. **11** (1962), 415–448; MR **26** #2057] for a special class of linearly elastic materials with couple-stresses. In Part II, the uniqueness theorems and minimum principles of classical elasticity theory are extended to the more general theory considered here, and the equations of equilibrium for finite deformations are obtained. By examination of the solution of the linear equations for the bending of a flat plate, the author concludes that its deviation from the classical solution and experiment is much too great if the couple-stress terms are made sufficiently large to account for stress gradient effects observed in fatigue tests.

R. A. Toupin (Yorktown Heights, N.Y.)

Ziegler, Hans 772

Some extremum principles in irreversible thermodynamics with application to continuum mechanics.

Progress in Solid Mechanics, Vol. IV, pp. 91–193. North-Holland, Amsterdam, 1963.

Tant à cause de l'importance présentée par la thermodynamique et la présence d'un champ électromagnétique en diverses applications, dans le domaine de la mécanique des milieux continus, qu'à cause de l'interaction de phénomènes physiques qui sont d'ailleurs distincts, la mécanique des milieux continus devient de plus en plus une physique des milieux continus. Les effets électromagnétiques peuvent être souvent négligés, mais pour l'étude des propriétés des corps où apparaîssent des frottements dans le processus de la déformation, comme sont les corps plastiques, viscoélastiques, etc., on doit tenir compte du phénomène thermodynamique, surtout pour la déduction des lois constitutives; c'est ainsi qu'il se produit une croissance de l'entropie, ce qui nous conduit à la thermodynamique de certains processus irréversibles.

Les équations constitutives des milieux continus doivent être donc consistantes avec les théorèmes fondamentaux de la thermodynamique. Le premier et le troisième théorèmes n'imposent aucune restriction pour ces équations; le deuxième théorème exclut les processus qui donnent naissance à une entropie négative, ce qui restraint le champ des équations constitutives possibles. Le principe énonoé par L. Onsager en 1931 correspond à des processus irréversibles et permet de déduire les lois de la viscoélasticité linéaire, par exemple; mais les problèmes nonlinéaires (la plasticité, les fluides non-newtoniens, etc.) ne peuvent pas être étudiés par ce moyen. En 1961, le présent auteur énonça le principe de la moindre force irréversible, équivalent au principe du maximum de la vitesse de l'entropie; ces principes—qui font appel à une certaine intuition physique—sont valables aussi dans le cas nonlinéaire, généralisant ainsi la théorie d'Onsager et étant applicables à une classe plus large de milieux continus. Le principe de la moindre force irréversible peut être basé sur les résultats de la mécanique statistique, telle qu'elle a été fondée par J. W. Gibbs en 1902. L'auteur discute ces problèmes dans le cas quasistatique, en sept paragraphes, suivis d'une bibliographie de 49 titres.

Après une introduction où l'on précise l'état actuel du problème, on passe dans le § 2 à certaines considérations générales de nature statistique; on discute ainsi des problèmes énergétiques concernant les effets du système macroscopique, le microsystème—qui est considéré purement mécanique—étant conservatif, holonome et scléronome. On peut déduire ainsi les équations de Lagrange, en admettant aussi l'existence de forces gyroscopiques, qui dérivent d'un potentiel gyroscopique linéaire (et homogène) en vitesses généralisées; c'est ainsi que la théorie de Gibbs est généralisée. Une importante discussion se rapporte aux hypothèses restrictives ou aux généralisations faites; mentionnons spécialement l'introduction dans les fonctions utilisées d'un certain nombre de paramètres x_k ($k=1, 2, \cdots, n$), à côté des coordonnées généralisées q_k, paramètres qui caractérisent l'influence microscopiques des environs. C'est ainsi qu'on obtient les équations de Hamilton. Ces résultats permettent aussi une étude de nature statistique.

Le § 3 est dédié à une étude de la thermodynamique classique. Après avoir défini les états d'équilibre, l'énergie intrinsèque et l'entropie, on passe à une approximation dans le cadre de la mécanique quantique et l'on définit et l'on discute des processus quasistatiques. On passe ensuite en revue les trois théorèmes fondamentaux, en insistant surtout sur le deuxième théorème; ce théorème est écrit sous une forme qui correspond non seulement aux processus réversibles mais aussi aux processus irréversibles, le phénomène de l'écoulement plastique dans le cas du cisaillement simple étant expliqué avec son aide. Les processus thermodynamiques irréversibles sont présentés dans le § 4. On discute ainsi les principes d'extrême établis par l'auteur et mentionnés ci-dessus, le principe du maximum de la vitesse de l'entropie étant équivalent au principe du maximum de la vitesse du travail de dissipation; ces principes prennent une forme intéressante dans l'espace des phases. Ils sont discutés du point de vue de leur conséquences; par exemple, le principe de la moindre vitesse admet une interprétation intéressante dans l'espace des forces. On discute ensuite les fonctions de dissipation quasihomogènes et, particulièrement, le cas des fonctions homogènes de degré r. On énonce encore d'autres principes d'extrême, comme le principe de la moindre énergie de dissipation, le principe du minimum de la vitesse de l'entropie et le principe du minimum de l'entropie.

Dans le § 5 on fait des applications à la mécanique des milieux continus. On introduit les systèmes composés, formés d'un nombre quelconque de systèmes fondamentaux (caractérisés par les paramètres mécaniques et la température), afin que l'échange d'énergie soit possible. On introduit ensuite les déformations, les vitesses de déformation et les tensions à l'aide desquelles on écrit

les équations constitutives. En supposant qu'une partie finie d'un milieu continu est un système composé, on traite aussi du problème du flux de chaleur pour le cas général de l'anisotropie. Le problème des équations constitutives pour différents types de milieux continus est traité en détail dans le § 6. C'est ainsi qu'on présente, d'une manière unitaire, à l'aide du principe de la moindre force irréversible, les cas des solides élastiques, viscoélastiques (spécialement le corps Kelvin), viscoplastiques (avec de différents cas particuliers) et plastiques (avec différentes conditions de plasticité et la discussion de certains postulats); de même, l'auteur donne les lois constitutives pour les fluides visqueux ainsi que pour différents cas particuliers (fluides parfaits, incompressibles, etc.). Mentionnons aussi le cas des fluides à une vitesse de distorsion limitée. Dans le § 7 on fait quelques remarques finales, en discutant aussi les cas des corps Maxwell et élasticoplastiques.

La monographie de l'auteur présente d'une forme unitaire et élégante l'état actuel des recherches dans le domaine de la thermodynamique des processus irréversibles ainsi que les intéressants résultats de l'auteur dans cette direction, avec des applications à la mécanique des milieux continus. Elle représente ainsi une contribution de grande valeur à la littérature de spécialité.

P. P. Teodorescu (Bucharest)

Knops, R. J. 773

Further considerations of the elastic inclusion problem.

Proc. Edinburgh Math. Soc. (2) **14** (1964/65), 61–70.

Author's summary: "An equation is derived for the strains of an arbitrary elastic field in an infinite matrix perturbed by several inclusions. The equation is solved exactly when the shear moduli of the inclusions and the matrix are identical, and also when only a single ellipsoidal inclusion perturbs a field uniform at infinity. Mean values of the strains are then calculated for non-uniform fields perturbed either by an ellipsoid or by a system of weakly-interacting spheres."

In a note added in proof, the author calls attention to an earlier monograph by V. D. Kupradze [*Progress in Solid Mechanics*, Vol. III, North-Holland, Amsterdam, 1963; MR **29** #1785] in which more extensive results on this same problem are given.

R. A. Toupin (Yorktown Heights, N.Y.)

Payne, L. E. 774

Some isoperimetric inequalities in the torsion problem for multiply connected regions.

Studies in mathematical analysis and related topics, pp. 270–280. *Stanford Univ. Press, Stanford, Calif.*, 1962.

By ingenious applications of symmetrization techniques, the author derives many inequalities relating the torsional rigidity, the torsion function, and in one case even the fundamental frequency of a multiply connected domain to its area and the areas of its holes.

H. F. Weinberger (Minneapolis, Minn.)

Seth, B. R. 775

Generalized strain measure with applications to physical problems.

Second-order Effects in Elasticity, Plasticity and Fluid Dynamics (*Internat. Sympos., Haifa*, 1962), *pp.* 162–172. *Jerusalem Academic Press, Jerusalem; Pergamon, Oxford*, 1964.

Author's summary: "All the known strain measures can be represented by the generalized strain measure $(1-ne)^{-1/n}$, where n may be called the coefficient of strain measure. Any stress-strain tensor relation cannot give a complete quantitative picture. The result obtained by using any one of them will have to be generalized to conform to experimental data. The generalization may be justified by assuming the coefficients of the medium to be functions of the strain invariants. The introduction of new coefficients of the medium may therefore not be found necessary in many cases."

Seth, B. R. 776

Survey on second-order elasticity.

Second-order Effects in Elasticity, Plasticity and Fluid Dynamics (*Internat. Sympos., Haifa*, 1962), *pp.* 261–265. *Jerusalem Academic Press, Jerusalem; Pergamon, Oxford*, 1964.

An expository paper.

Tonojan, V. S. 777

A two-dimensional contact problem for an elastic quarter-plane with fixed vertical edge. (Russian. Armenian summary)

Akad. Nauk Armjan. SSR Dokl. **37** (1963), 249–258.

The title problem is investigated by means of Fourier integrals. No numerical results are given.

L. S. D. Morley (Farnborough)

Kotljar, S. M. 778

On the state of stress of an infinite strip. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 1 (38), 69–72.

For the strip $0 \leq y \leq h$, a mixed boundary-value problem is considered by prescribing zero stresses and displacements along the boundaries $y=0$ and $y=h$, but $u=f(x)$ for $y=0$ and $|x|<a$. One looks for a solution using the Airy stress function. The problem is reduced to a Fredholm integral equation of the second kind.

N. Cristescu (Bucharest)

Bose, S. C. 779

Stresses in a circular cylinder having a rigid spherical inclusion under tension. (Polish and Russian summaries)

Arch. Mech. Stos. **15** (1963), 635–643.

Expressions are derived to determine the stresses in a circular cylinder of finite cross-section which is under axial tension and has a symmetrically-located rigid spherical inclusion.

The method of solution of the equations is illustrated by numerical examples. *W. D. Kroll* (Washington, D.C.)

Vaĭnšteĭn, F. A. 780

On the state of stress of an unbounded orthotropic strip. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 1 (38), 10–13.

The problem of the stress distribution in an unbounded orthotropic strip is considered for the mixed boundary-value conditions at $y = \pm a$: $\sigma_y = 0$ for $0 < x < +\infty$, $\tau_{xy} = 0$ for $-\infty < x < +\infty$, $v = \mp v_0$ for $-\infty < x < 0$. The author looks for solutions of the fourth-order equation satisfied by the stress function. *N. Cristescu* (Bucharest)

Kalker, J. J. 781
The transmission of force and couple between two elastically similar rolling spheres. I, II, III.
Nederl. Akad. Wetensch. Proc. Ser. B **67** (1964), 135–149; *ibid.* **67** (1964), 150–164; *ibid.* **67** (1964), 165–177.

Two elastic spheres of the same material are rigidly mounted on two rigid shafts and the spheres pressed together. The shafts are rotated about their respective axes, the relative position of which remains the same, so that the spheres roll over each other. Owing to friction the spheres exert a tangential force and a torsional couple upon each other across the contact area. The author calculates the total force and couple transmitted between the spheres when these quantities are small, it being assumed that the contact area is circular and that slip takes place only on the trailing edge of this area, i.e., that adhesion takes place throughout the contact area. The spheres are approximated by half-spaces and the solution found by expressing the displacement components in terms of harmonic functions, suitable representations for these functions being found by introducing oblate spheroidal coordinates. The results obtained are compared with Johnson's experimental and theoretical results on the rolling of steel balls. *W. D. Collins* (Manchester)

Reiss, Edward L. 782
Extension of an infinite plate with a circular hole.
J. Soc. Indust. Appl. Math. **11** (1963), 840–854.

The "thick plate" theory for the above problem is treated very thoroughly using an asymptotic expansion in terms of the parameter h/R, where $2h$ is the thickness of the plate, R the radius of the hole. The two lowest-order "three-dimensional corrections" to the plane stress theory have been used in the numerical work, the results being of considerable interest. *R. C. T. Smith* (Armidale)

Washizu, K. 783
Some considerations on a naturally curved and twisted slender beam.
J. Math. and Phys. **43** (1964), 111–116.

An approximate theory, taking into account the effects of torsion without end-section warping and of transverse shear deformations, is presented for a naturally curved and twisted slender beam. One end of the beam is built-in and external forces are applied to the other with distributed forces and moments applied along the beam. Methods of solving the equations are indicated.
W. D. Kroll (Washington, D.C.)

Gallagher, Richard H. 784
★A correlation study of methods of matrix structural analysis.
With contributions by Ivan Rattinger and John S. Archer. Report to the 14th Meeting, Structures and Materials Panel, Advisory Group for Aeronautical Research and Development, NATO, Paris, France, July 6, 1962. A Pergamon Press Book.
The Macmillan Co., New York, 1964. xii + 113 *pp.* (3 *inserts*)

Le travail contient six chapitres, une bibliographie de 35 titres, un indice d'auteurs et un appendice où l'on présente l'état actuel dans le calcul des structures, à l'aide des machines électroniques, en Europe occidentale et aux États Unis.

Après une courte introduction, on passe aux méthodes de l'analyse matricielle des structures: (i) la méthode des forces (de la flexibilité) et (ii) la méthode des déplacements (de la rigidité). Dans le 3[ième] chapitre on donne des méthodes de calcul des propriétés (rigidité, flexibilité, etc.) des différents éléments composants des structures. Les relations de réciprocité jouent un rôle important dans cette direction.

L'analyse des structures multicellulaires est faite dans le 4[ième] chapitre, en utilisant différentes idéalisations analytiques. Dans le chapitre suivant on donne des résultats numériques et sous la forme de diagrammes. Le dernier chapitre contient une recommandation de programme pour les machines électroniques à calcul; on passe aussi en revue différents problèmes à résoudre (problèmes d'ordre mathématique, d'idéalisation des structures, de systèmes élastiques nonlinéaires, d'anélasticité).

Le volume est d'un grand intérêt tant pour établir les dimensions d'une structure aéronautique que pour la recherche scientifique. *P. P. Teodorescu* (Bucharest)

Fraeijs de Veubeke, B. (Editor) 785
★Matrix methods of structural analysis.
Published for and on behalf of Advisory Group for Aeronautical Research and Development, North Atlantic Treaty Organization. A Pergamon Press Book.
The Macmillan Co., New York, 1964. v + 343 *pp.* $15.00.

Le volume contient cinq travaux et un indice d'auteurs; certains de ces travaux ont été présentés aux sessions scientifiques de l'AGARD, "Structures and Material Panel".

J. H. Argyris, S. Kelsey and H. Kamel sont les auteurs d'une étude monographique (164 pp.) où l'on présente l'état actuel de l'application des méthodes matricielles dans l'analyse des structures. Les deux méthodes fondamentales: la méthode des forces et la méthodes des déplacements sont ainsi utilisées tant pour les systèmes hyperstatiques linéaires que pour les systèmes nonlinéaires (grandes déformations). C'est ainsi qu'on peut apprécier l'effet spatial dans l'assemblage de divers éléments (barres, plaques planes et courbes minces), tenant compte de leur rigidités. On donne aussi quelques résultats de nature expérimentale. Dans cinq appendices sont calculées les matrices de rigidité pour différents types d'éléments (triangulaires, quadrilatères plans quelconques, parallélogramme, etc.) et l'on donne quelques informations concernant les grands déplacements et la stabilité dans la théorie des systèmes élastiques. La bibliographie contient seize titres.

Les limites supérieures et inférieures dans l'analyse matricielle des structures sont discutées par B. M. Fraeijs de Veubeke dans une autre étude (38 pp.). On utilise la

méthode des forces et la méthode des déplacements, sous la forme de la méthode des rigidités. Des résultats sont donnés surtout pour les plaques planes triangulaires. Dans un appendice on donne les principes généraux à l'aide desquels on obtient les limites supérieures et inférieures des coefficients d'influence. Dans un autre appendice, écrit en collaboration avec G. Sander, on donne des exemples et résultats numériques.

La troisième étude (64 pp.) est écrite par M. J. Turner, H. C. Martin et R. C. Weikel et contient d'autres développements et applications concernant la méthode des rigidités. Après une introduction où l'on précise la méthode de calcul, on fait une analyse de l'état de tension ; le problème des tensions thermoélastiques est aussi pris en considération. On étudie aussi le problème de la stabilité élastique des plaques et des colonnes. Un exemple de calcul correspond au fuselage idéalisé d'un hélicoptère. La bibliographie contient treize titres.

Une autre étude (50 pp.) contient des résultats sur les structures multicellulaires, étant éditée par I. C. Taig et R. I. Kerr. On utilise tant la méthode des forces que la méthode des déplacements. L'influence de certaines idéalisations particulières de telles cellules est prise en considération. Différents résultats pratiques sont présentés sous forme de diagrammes. La bibliographie est formée de douze titres.

La dernière étude (26 pp.) est écrite par P. H. Denke, étant dédiée à une programme pour les machines digitales concernant le calcul des structures nonlinéaires à l'aide de la méthode des forces. On considère aussi le problème de la stabilité et l'on tient compte des propriétés plastiques des matériaux, dans la théorie de Prandtl-Reuss. Différents résultats utiles de la mécanique des structures sont donnés dans quatre appendices.

Le développement du calcul à l'aide des machines électroniques permet une application intensive des méthodes matricielles, même pour un grand volume de calcul numérique. D'un tel point de vue, ces travaux représentent une contribution importante à l'étude des structures aéronautiques. *P. P. Teodorescu* (Bucharest)

Galfajan, P. O. 786

Solution of a mixed problem in elasticity theory for a rectangle. (Russian. Armenian summary)

Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk **17** (1964), *no.* 1, 39–61.

The mixed problem in elasticity theory for a rectangle on one side (border) of which the displacements are given, and on three others the stresses are given, is treated.

In taking the stress function in the form of a Fourier series, the author obtains for the coefficients infinite systems of linear algebraic equations which are quasi-completely regular and have absolute terms bounded from above. Under these conditions he claims that the coefficients of the Fourier development can be determined with any desired accuracy. As an example, the problem of bending of a cantilever (not precisely defined) loaded at the free end is numerically solved. He determines the distributions of the normal and shearing stresses near the fixed end as well as the deflection of the free end.

It is very laborious to verify these statements, the calculations being very long and not easy to follow.

T. P. Anđelić (Belgrade)

Frisch-Fay, R. 787

Approximate solution of $d^2\theta/ds^2+as\sin\theta=0$. (Polish and Russian summaries)

Arch. Mech. Stos. **15** (1963), 607–610.

This is the governing equation for large angular deflection θ of a uniformly loaded beam. The approximation consists in considering n evenly spaced point loads and leads to n simultaneous implicit equations for the θ_n, containing incomplete elliptic integrals. The reviewer thinks that these equations can be programmed for a computer, but feels it would be easier numerically to program the original equation. *D. R. Bland* (Manchester)

Vodička, Václav 788

Longitudinal vibrations of a composite conical bar. (German summary)

Österreich. Ing.-Arch. **17** (1962/63), 249–254.

This paper is concerned with the free longitudinal vibrations of a truncated cone, which consists of a number of homogeneous slabs of differing material properties. Using classical methods the author derives the characteristic equation, and hence the eigenvalues for the problem. The author then constructs the eigenfunctions associated with each eigenvalue. Proceeding in the classical manner, the author derives the orthogonality relations amongst the eigenfunctions. These relationships are then used to solve the problem of the free vibrations of a composite conical bar with a prescribed set of initial conditions.

T. K. Caughey (Pasadena, Calif.)

Dhaliwal, R. S. 789

Bending of a clamped thin elastic plate normally loaded over or along a concentric circle.

J. Sci. Engrg. Res. 7 (1963), 267–274.

Complex variable methods are used to find closed expressions for the complex potentials for clamped thin plates normally loaded over or along a concentric circle. The loading is symmetrical with respect to the center of the circle. *W. D. Kroll* (Washington, D.C.)

Naghdi, P. M. 790

Foundations of elastic shell theory.

Progress in Solid Mechanics, Vol. IV, pp. 1–90. *North-Holland, Amsterdam*, 1963.

La théorie des plaques courbes minces a été l'objet de nombreuses recherches, dues tant aux problèmes posés par la pratique qu'à l'intérêt théorique présenté. En effet, c'est ainsi que les problèmes de la théorie de l'élasticité peuvent être réduits aux problèmes bidimensionnels pour une certaine catégorie de corps, un calcul exact de ceux-ci étant pour la plupart pratiquement impossible. Mais on doit mentionner que la déduction des équations fondamentales du problème et la description mathématique du phénomène diffèrent d'auteur à auteur, une possibilité satisfaisante de comparer les résultats obtenus avec les résultats exacts qui seraient donnés par la théorie de l'élasticité (cas tridimensionnel) n'étant pas encore mise au point ; c'est encore un problème ouvert.

Généralement, les équations fondamentales peuvent être obtenues par deux méthodes : (i) en partant des équations générales de la théorie de l'élasticité, ainsi que l'a fait pour la première fois par H. Aron en 1874, ou

(ii) en utilisant l'hypothèse simplificatrice supplémentaire de l'élément linéaire, énoncée en 1888 par A. E. H. Love, d'après le modèle de l'hypothèse analogue faite par G. R. Kirchhoff en 1850 pour les plaques planes minces. Le présent auteur utilise la première méthode dans sa monographie, qui contient sept paragraphes, une spécification des notations employées et une bibliographie de 77 titres.

Après une courte introduction, où l'on précise l'état actuel du problème, on passe dans le § 2 à quelques considérations de géométrie différentielle des surfaces. Quoique les résultats obtenus se réfèrent spécialement à l'espace euclidien, il y en a qui restent valables aussi pour un espace de Riemann; mentionnons surtout l'attention accordée aux lignes de courbure. Dans le § 3 on déduit certaines expressions explicites pour différentes fonctions spatiales, invariants et tenseurs, ainsi que leur dérivées, à l'aide des fonctions de surface correspondantes, invariants et tenseurs; on donne d'une telle manière des formules pour les symboles spatials de Christoffel.

Le § 4 est dédié à l'état de déformation des plaques courbes minces. Un problème important dans cette direction est de choisir les grandeurs géométriques caractérisant cet état, surtout pour les déformations finies; on passe ainsi à une description bidimensionnelle du phénomène et aux conséquences de l'introduction de l'hypothèse de l'élément linéaire. C'est ainsi qu'on donne aussi la liaison entre les déformations spécifiques et les déplacements. Mentionnons spécialement l'expression des déplacements dans le plan tangentiel comme fonctions linéaires de la troisième variable x^3; de même, le déplacement u_3 est considéré indépendant de x^3 où l'on prend trois termes dans un développement en série par rapport à cette variable. On discute ensuite les équations de compatibilité. Certains résultats obtenus sont valables dans le cas général, mais la plupart restent valables seulement pour la théorie linéarisée. Le problème de l'état de tension est discuté dans le § 5. Après avoir obtenu les efforts sur la section de la plaque, on précise leur caractère tensoriel et l'on déduit les équations d'équilibre (ou de mouvement), en partant des équations générales de la théorie de l'élasticité. Les résultats sont exprimés d'une manière tensorielle, en utilisant la dérivée covariante, dans un système de coordonnées curvilignes quelconques (jusqu'à présent ces équations ont été rapportées seulement aux lignes de courbure). On obtient ainsi trois équations d'équilibre (de mouvement) pour les pseudo-résultantes (six fonctions inconnues) qu'on introduit sur la section de la plaque. L'auteur discute aussi la méthode directe (étude d'un élément de plaque courbe mince) d'obtenir ces résultats.

Une grande attention est accordée dans le § 6 aux lois constitutives de la théorie linéaire. C'est un des problèmes les plus difficiles, ces lois devant vérifier certains principes d'invariance et de consistance. Après quelques considérations générales sur les plaques courbes minces anisotropes, on passe à la déduction de certaines équations constitutives approximatives; mentionnons ainsi les équations de Flügge-Lur'e-Byrne, la première approximation de Love, les équations de Novozhilov, Koiter et Sanders. Ces résultats sont présentés d'une manière critique (discussion, comparaison); mentionnons la critique faite du point de vue des principes énoncés. Dans le § 7 on discute le système d'équations du type elliptique du problème, les théorèmes généraux—analogues à ceux de la théorie de l'élasticité—qu'on peut déduire, l'analogie statico-géométrique, l'analogie entre les deux problèmes aux limites fondamentaux et le problème des théorèmes d'existence et d'unicité.

La monographie de l'auteur est écrite dans un style très clair, est bien documentée, utilise d'une manière extensive l'appareil tensoriel et se base sur multiples résultats originaux. Elle apporte une contribution de valeur à l'élucidation de certains problèmes fondamentaux dans la théorie des plaques courbes minces.

P. P. Teodorescu (Bucharest)

Naghdi, P. M.; Kalnins, A. 791

Errata: "On vibrations of elastic spherical shells".

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 331.

The authors call attention to a computational error in part of their earlier paper [same Trans. **29** (1962), 65–72; MR **25** #1701] and replace Table 2 by a more accurate table.

DeSilva, C. N. [De Silva, C. N.]; Tersteeg, G. E. 792

Axisymmetric vibrations of thin elastic shells.

J. Acoust. Soc. Amer. **36** (1964), 666–672.

Applying extensional (momentless) theory with longitudinal inertia neglected to a class of shells of revolution with a free edge, the authors obtain a simple expression for one natural frequency. For hemispherical shells and a range of shallow shells, this frequency is close to the lowest natural frequency according to bending theory.

The second part of the paper treats edge effects in free vibration of another class of shells of revolution.

R. P. Nordgren (Houston, Tex.)

Schmidt, Robert 793

Sandwich shells of arbitrary shape.

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 239–244.

Author's summary: "Differential equations are derived for sandwich shells of arbitrary shape. The weak core is assumed to be elastic, orthotropic, and deformable in all directions. The face layers obey the Kirchhoff hypothesis. The core and the face layers are rigidly connected to each other at their natural boundaries."

Hersch, Joseph 794

Contribution to the method of interior parallels applied to vibrating membranes.

Studies in mathematical analysis and related topics, pp. 132–139. *Stanford Univ. Press, Stanford, Calif.*, 1962.

Let D be a membrane which is exterior to one of its boundary components Γ_1. Let it be fixed on Γ_1 and free on all other boundary components. The author shows that among all such D with given length L_{Γ_1} of Γ_1 and given area the annular membrane has the lowest fundamental frequency. This gives the best upper bound for the fundamental frequency of D in terms of its area and L_{Γ_1}.

H. F. Weinberger (Minneapolis, Minn.)

Hersch, Joseph 795

The method of interior parallels applied to polygonal or multiply connected membranes.

Pacific J. Math. **13** (1963), 1229–1238.

Let D be a membrane whose outer boundary Γ_0 is fixed, and which may have one or more holes along whose boundaries it is free. L. E. Payne and the reviewer [J. Math. Anal. Appl. **2** (1961), 210–216; MR **26** #7220] have extended methods of E. Makai and G. Pólya to obtain the best possible upper bound for the fundamental frequency of D in terms of the area of D and the length of Γ_0 alone.

In the first part of the paper the author obtains an improved upper bound in terms of the additional information that Γ_0 is an n-sided polygon, and an even better bound when the polygon is convex and its interior angles are given.

The author has previously [#794 above] given an upper bound of the same kind as that of Payne and the reviewer in the case when D is fixed along an inner boundary Γ_1 and free along its outer boundary, and possibly on the boundaries of some holes.

In the second part of the present paper the author combines this result with that of Payne and the reviewer to construct an upper bound for the fundamental frequency of a multiply connected membrane D of given area which is fixed along two or more of its boundary components, whose lengths are given.

The bound has the physically correct property that the effect of a hole approaches zero as the length of its perimeter goes to zero. *H. F. Weinberger* (Minneapolis, Minn.)

Guo, Zhong-Heng; Solecki, Roman 796

Free and forced finite-amplitude oscillations of an elastic thick-walled hollow sphere made of incompressible material. (Polish and Russian summaries)

Arch. Mech. Stos. **15** (1963), 427–433.

The authors consider the problem of large-amplitude radial oscillations of a hollow sphere of isotropic, elastic, incompressible material along lines used by the reviewer for the hollow cylinder [Quart. Appl. Math. **18** (1960/61), 71–77; MR **22** #3187; Trans. ASME Ser. E. J. Appl. Mech. **29** (1962), 283–286; MR **25** #798]. The cases of free oscillations in the absence of surface pressure and oscillations due to sudden application of constant surface pressure are treated. Conditions on the strain energy which guarantee periodic motion are given, and formulas for the periods of oscillation are derived.

J. K. Knowles (Pasadena, Calif.)

Renton, J. D. 797

On the stability analysis of symmetrical frameworks.

Quart. J. Mech. Appl. Math. **17** (1964), 175–197.

Author's summary: "The buckling behaviour of symmetrical frameworks is investigated. The intuitive assumption that the buckling modes will exhibit some form of symmetry is shown to be generally correct and these modes are analysed. Symmetry of a framework is taken to be that property which permits the framework to be divided into regions which transform into one another during a transformation of an overall set of coordinate axes. By means of this approach, criteria for the stability of the framework are found in their simplest forms."

Krauklis, P. V.; Molotkov, L. A. 798

On low-frequency oscillations of a plate on elastic half-space.

Prikl. Mat. Meh. **27** (1963), 947–951 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1451–1458.

This paper deals with longitudinal and transverse waves in an elastic plate resting upon an elastic half-space. The waves are generated by a source on the free surface of the plate. Waves having wave lengths large compared to the thickness of the plate are considered. Transform type solutions are used to explore various properties of these waves. *W. E. Boyce* (Troy, N.Y.)

Peralta, L. A.; Raynor, S. 799

Initial response of a fluid-filled, elastic, circular, cylindrical shell to a shock wave in acoustic medium.

J. Acoust. Soc. Amer. **36** (1964), 476–488.

Authors' summary: "The initial response of a fluid-filled cylindrical shell immersed in an acoustic medium when struck by an acoustic shock moving in the external fluid is solved by means of integral transformations. Both the fluid contained in the shell and the exterior fluid are assumed to obey linear wave equations; the two fluids are assumed to possess dissimilar properties. The effects of transverse shear and rotatory inertia are included in the equations governing the shell motion. The solution is limited to the 'front' region of the shell."

J. W. Craggs (Leeds)

Pobedrja, B. E. 800

On the geometric interpretation of dislocation theory. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, no. 1, 69–75.

Different problems in the continual theory of dislocations are studied by the method of Cartan's exterior forms in non-Riemannian geometry. *P. P. Teodorescu* (Bucharest)

Morozov, E. M.; Polak, L. S.; Fridman, Ja. B. 801

Variational principles in the development of cracks in solids. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 537–540.

The authors give a physical foundation for their previously published theory which describes the mechanical breaking of a solid body by means of an integral variational principle. Variational principles of mechanics, as well as of thermodynamics, both of reversible and of irreversible processes, are used. *N. Cristescu* (Bucharest)

Biot, M. A. 802

Theory of buckling of a porous slab and its thermoelastic analogy.

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 194–198.

Author's summary: "A theory is developed for the buckling of a fluid-saturated porous slab under axial compression. The problem is discussed in the context of the thermodynamics of irreversible processes. It is shown that there is a range of compressive loads, between a lower and upper critical value, for which the slab exhibits creep buckling. The problem of folding instability of a porous layer embedded in a viscous or viscoelastic medium is also analyzed and the dominant wavelength is evaluated. Identical behavior is derived by analogy for a thermoelastic slab with a critical range between isothermal and adiabatic buckling. The theory is applicable to a large

class of two-phase materials obeying the same thermodynamics. It also provides a simple analysis of thermoelastic damping of plates."

Green, A. E. 803

A note on wave propagation in initially deformed bodies.

J. Mech. Phys. Solids **11** (1963), 119–126.

Several miscellaneous results are given concerning the propagation of waves in homogeneously deformed elastic bodies. (1) A formula derived by Truesdell [Arch. Rational Mech. Anal. 8 (1961), 263–296; MR **25** #1706] for the speed of infinitesimal longitudinal waves in the direction of a principal axis of the underlying homogeneous strain in an isotropic Cauchy-elastic body is shown to be valid for certain anisotropic bodies. (2) Conditions for the existence of a strain energy function are given in terms of speeds of infinitesimal waves in a homogeneously deformed isotropic body. (3) A partial differential equation is derived for a class of one-dimensional (not necessarily infinitesimal) waves in a homogeneously deformed body. These waves are propagated without change of form in a Mooney material. The above results are derived on the assumption that temperature changes can be neglected. The author finally considers infinitesimal longitudinal waves propagating at constant entropy in a body which has been previously deformed homogeneously and isothermally.

J. K. Knowles (Pasadena, Calif.)

Viktorov, I. A. 804

Effects of a second approximation in the propagation of waves through solids.

Akust. Ž. **9** (1963), 296–300 (*Russian*); *translated as Soviet Physics Acoust.* **9** (1964), 242–245.

Second-order effects in the propagation of elastic waves through a homogeneous, isotropic solid are considered. The possibility of the generation of secondary waves which grow with distance travelled is investigated for particular kinds of plane, cylindrical and transverse waves and also for Rayleigh surface waves.

R. T. Shield (Providence, R.I.)

Kaul, Raj K. 805

On the propagation of pressure pulses in circular elastic rods. (German summary)

Z. Angew. Math. Phys. **14** (1963), 704–713.

The author uses the approximate displacement equations of motion due to Mindlin and Herrmann to study the far-field, long-time response of a semi-infinite rod subjected to a pressure shock on its end. The author's main purpose is to assess the influence of the higher radial mode of motion on the extensional mode in this two-mode theory, particularly in the vicinity of the maximum response of the latter (head of pulse), but this has already been brought out by this reviewer's earlier work on the problem [Proc. Third U.S. Nat. Congr. Appl. Mech. (Brown Univ., Providence, R.I., 1958), pp. 215–224, Amer. Soc. Mech. Engrs., New York, 1958; MR **21** #2414], i.e., a $1/\sqrt{t}$ contribution. More important are the "tail" contributions of the radial mode, where from cutoff, as brought out by the reviewer and Nisewanger, the mode offers a good approximation to the predominant response of the rod [J. Appl. Mech. **24** (1957), 240–244; MR **19**, 596]. More complete analysis of upper mode contributions to transient response may be found in the work of Lloyd and the reviewer [Proc. Fourth U.S. Nat. Congr. Appl. Mech. (Univ. California, Berkeley, Calif., 1962), Vol. 1, pp. 255–267, Amer. Soc. Mech. Engrs., New York, 1962; MR **28** #1824]. The reader may find some interest in the author's alternate asymptotic analysis.

J. Miklowitz (Pasadena, Calif.)

Keller, Joseph B.; Karal, Frank C., Jr. 806

Geometrical theory of elastic surface-wave excitation and propagation.

J. Acoust. Soc. Amer. **36** (1964), 32–40.

The authors generalize their geometrical theory of scalar surface waves [J. Appl. Phys. **31** (1960), 1039–1046; MR **25** #2753] to the elastic solid. The essential feature of the theory lies in the introduction of complex rays which are needed to satisfy the law of refraction at a boundary, i.e., the law $\sin\theta = V(\mathbf{x})/V_R(\mathbf{x})$, where θ is the angle between the incident ray and the normal to the surface, and $V(\mathbf{x})$ and $V_R(\mathbf{x})$ are the body (compression or shear) and surface wave velocities, respectively, at a real point $\mathbf{x}$, can only be satisfied for complex θ, since $V(\mathbf{x})/V_R(\mathbf{x})>1$. The complex rays travel from the source to the surface, along the surface, and then to an interior point of the solid. The authors give geometrical formulas for the determination of phases and amplitudes associated with each point on each ray. These involve excitation and radiation coefficients which are also given. The total displacement at a point is the sum of the displacements on all the rays through the point, each of which is constructed from the corresponding phase and amplitude. The theory is applicable to high-frequency, short-period waves or to any rapidly varying wave. As an example here, the authors apply it to determine the Rayleigh wave in Lamb's problem for the buried line compressional source. The authors have contributed an interesting and important paper in elastic wave propagation.

J. Miklowitz (Pasadena, Calif.)

Knopoff, L.; Hudson, J. A. 807

Scattering of elastic waves by small inhomogeneities.

J. Acoust. Soc. Amer. **36** (1964), 338–343.

The title problem is solved on the assumptions that the spatial variations of the elastic parameters and the density are small and independent of time; the correlation functions of these variations are Gaussian and isotropic; the distance from the observer to the scattering volume is of the same order as, or larger than, the wavelength; the linear dimensions of the scattering volume are large, and the size of the individual scattering centers small, compared with the wavelength; there is no backscattering. Explicit results are given for P- and S-wave scattering. The most significant result is that the scattered S-waves dominate the scattered P-waves in the wakes of both P- and S-incident-waves but are relatively stronger for an incident S-wave. It also is found that there are phase shifts in the direct waves.

J. W. Miles (Canberra)

Kaliski, Sylwester 808

Attenuation of surface waves between perfectly conducting fluid and solid in a magnetic field normal to the contact surface. (Polish and Russian summaries)

Proc. Vibration Problems **4** (1963), 375–385.

This paper concerns the surface waves generated at the interface of a liquid and a solid by a magnetic field normal to the interface. According to the author, "the aim . . . is . . . to show the existence and the computation of the velocity and the decrement of attenuation of surface wave". To this end the field is assumed to be weak, and a perturbation method is employed. First-order terms are retained and approximations to the above-mentioned quantities are calculated. Whether the mathematical existence of such waves is demonstrated in this way, however, seems at least arguable to the reviewer.

W. E. Boyce (Troy, N.Y.)

Chadwick, P.; Powdrill, B. 809

Application of the Laplace transform method to wave motions involving strong discontinuities.

Proc. Cambridge Philos. Soc. **60** (1964), 313–324.

Authors' summary: "Formulae are obtained for the Laplace transforms of the first and second partial derivatives of a function of time and position which, together with its first partial derivatives, is discontinuous on a moving surface. These formulae are then used in a discussion of the application of the Laplace transform method to the solution of mixed initial- and boundary-value problems in linear thermoelasticity. Particular attention is given to thermoelastic disturbances involving plane shock waves, and it is pointed out that no properly posed formulation of problems of this type has yet been found."

R. Muki (Tokyo)

Aliev, Kh. M. [Aliev, H. M.] 810

Destructive shock waves in frangible media.

Dokl. Akad. Nauk SSSR **151** (1963), 80–83 (*Russian*); *translated as Soviet Physics Dokl.* **8** (1964), 673–675.

Based on some experimental results, the paper concludes that when an explosion occurs in a mass of frangible material, an expanding zone of disintegration arises close to the volume containing the gaseous products of explosion, and this zone of crushed and pulverized material is subject to the action of high pressures at the same time as the unbroken part of the mass experiences a deformation that is propagated with high velocity.

Describing the pulverized material as a perfect fluid and the uncrushed material as an elastic solid and assuming that the disintegration wave front is expressible by a linear relation of the principal stresses, the author reduces the stated problem to a Cauchy problem for a system of differential equations (of the fifth order). The paper concludes with a discussion on the possible existence of the disintegration wave.

L. N. Tao (Chicago, Ill.)

Sobotka, Zdeněk 811

Some problems of non-linear rheology.

Second-order Effects in Elasticity, Plasticity and Fluid Dynamics (*Internat. Sympos., Haifa*, 1962), *pp.* 362–390. *Jerusalem Academic Press, Jerusalem; Pergamon, Oxford*, 1964.

Author's summary: "The non-linear deformation of various bodies is treated from the rheological point of view. The author derives first the laws of long-time deformation of concrete using the variable rheological parameters. Further, after developing the general rheological relationships by means of tensor analysis, the formulae are established which represent the simple second-order effects as well as the combined higher-order phenomena depending on products of two different rheological variables."

M. E. Gurtin (Providence, R.I.)

Perzyna, Piotr 812

The constitutive equations for work-hardening and rate sensitive plastic materials. (Polish and Russian summaries)

Proc. Vibration Problems **4** (1963), 281–290.

The author's previous result for rate-sensitive perfectly-plastic materials is generalized to cover work-hardening. Plastic dissipation $\int \sigma_{ij}\, d\varepsilon_{ij}{}^{p}$ is supposed to control the amount of hardening. Special examples chosen include isotropic hardening forms of the J_2 and J_2, J_3 type, but a general normality relation is shown based on the reviewer's postulate of stability. One-dimensional predictions with different functional forms are compared with some experimental results of Campbell and of Dorn.

D. C. Drucker (Providence, R.I.)

Drucker, D. C. 813

Survey on second-order plasticity.

Second-order Effects in Elasticity, Plasticity and Fluid Dynamics (*Internat. Sympos., Haifa*, 1962), *pp.* 416–423. *Jerusalem Academic Press, Jerusalem; Pergamon, Oxford*, 1964.

An expository paper.

Olszak, W.; Rychlewski, J. 814

Geometrical properties of stress fields in plastically non-homogeneous bodies under conditions of plane strain.

Second-order Effects in Elasticity, Plasticity and Fluid Dynamics (*Internat. Sympos., Haifa*, 1962), *pp.* 269–294. *Jerusalem Academic Press, Jerusalem; Pergamon, Oxford*, 1964.

Authors' summary: "The plastic non-homogeneity of a body may strongly influence the process of its plasticization. For plane strain of an ideally plastic medium, the well-known properties of the slip lines cease to hold. In the present paper the limitations of the geometry of slip lines and trajectories of principal stresses in the general case in which the yield limit $K = K(x, y)$ are derived and discussed."

M. E. Gurtin (Providence, R.I.)

Swamy, N. V. Chandrasekhara 815

Similarity conditions for pseudo plastic fluid flow.

J. Phys. Soc. Japan **18** (1963), 1701.

It is found that the condition for similarity is, in the usual notation, $U(x) = Cx^m$ as in Falkner-Skan-Hartree flow; and the differential equation for the flow is a generalization of Hartree's equation. The flow is assumed to be incompressible.

J. C. Cooke (Farnborough)

Zadojan, M. A. 816

On a particular solution of the equations of the theory of ideal plasticity. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 38–39.

Equations of the flow theory of plasticity in three dimensions are integrated under the assumptions that the strain rate tensor and the transverse shear components depend upon a single variable (z, say). Possible applications to pure bending of plates and plastic flow between rough plates are indicated. *A. Sawczuk* (Warsaw)

Lokoščenko, A. M.; Šesterikov, S. A. 817

On the distribution of slip-line directions under plastic deformation. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Mch. **1964**, *no.* 2, 37–40.

The problem of the distribution of slip plane traces at the ground surface is solved on the basis of representations of a real material by a set of elements with ideal properties as commonly employed in the theory of plasticity. It is assumed that the perpendicular to a possible slip plane makes with the direction of the applied stress an angle that can take, with equal probability, values ranging from $\frac{1}{4}\pi-\delta$ to $\frac{1}{4}\pi+\delta$, where $0\leq\delta\leq\frac{1}{4}\pi$. Based on this hypothesis, the distribution of slip line directions at the ground surface is determined theoretically.

N. Cristescu (Bucharest)

Woźniak, Czesław 818

Stressless deformation of thin shells in the steady temperature field. (Polish and Russian summaries)

Arch. Mech. Stos. **15** (1963), 327–339.

Conditions for the existence of a stressless state of deformation (not necessarily small) in thin anisotropic or isotropic shells under the action of stationary temperature fields are derived. The strain tensor as defined in a previous paper by the author, and different from Cauchy-Green's tensor, is introduced. The usual hypothesis concerning normal to middle surface is used, and the temperature field is taken to be approximately linear across the thickness of the shell. Some particular cases are discussed.

A similar investigation using other measures of strain would be valuable. *H. Parkus* (Vienna)

Jahanshahi, A. 819

Thin plates and shallow cylindrical shells subjected to hot spots.

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 79–82.

Three cases are studied: a hot "bending spot" (temperature varying linearly across thickness) moving uniformly along a thin plate, a hot "plane spot" (temperature constant across thickness) moving arbitrarily along the plate, and "plane" and "bending" spots fixed in a shallow cylindrical shell subjected to compressive forces in the direction of the generatrix. In the first two cases inertia is taken into account. Plate and shell are considered "unlimited", i.e., boundary conditions are disregarded.

H. Parkus (Vienna)

Jordan, Neal F.; Eringen, A. Cemal 820

On the static nonlinear theory of electromagnetic thermoelastic solids. I. (French, German, Italian, and Russian summaries)

Internat. J. Engrg. Sci. **2** (1964), 59–95.

From the authors' summary: "The basic field equations, boundary conditions, and constitutive equations, necessary for the treatment of problems involving the nonlinear, steady-state behaviour of isotropic elastic solids subject to large deformations, electromagnetic fields and thermal gradients, are derived and employed to solve three specific problems." *R. A. Toupin* (Yorktown Heights, N.Y.)

Jordan, Neal F.; Eringen, A. Cemal 821

On the static nonlinear theory of electromagnetic thermoelastic solids. II. (French, German, Italian, and Russian summaries)

Internat. J. Engrg. Sci. **2** (1964), 97–114.

From the authors' summary: "The general constitutive equations developed in Part I [#820] are simplified for the special cases of rigid bodies, electroelasticity and magnetoelasticity. Various order approximate constitutive equations are obtained for small vector and tensor fields which include many hitherto known and unknown cross effects. Linear equations, second-degree equations, equations linear in vector fields, and equations linear in deformation are presented."

R. A. Toupin (Yorktown Heights, N.Y.)

Rüdiger, D. 822

Bemerkung zur Integration der thermo-elastischen Grundgleichungen.

Österreich. Ing.-Arch. **18** (1964/65), 121–122.

The author introduces stress functions for the coupled thermoelastic equations and shows that the integration of the coupled thermoelastic equations can be reduced to that of four differential equations, each of which may be integrated independently. *R. Muki* (Tokyo)

Lee, E. H.; Rogers, T. G. 823

Non-linear effects of temperature variation in stress analysis of isothermally linear viscoelastic materials.

Second-order Effects in Elasticity, Plasticity and Fluid Dynamics (*Internat. Sympos., Haifa*, 1962), *pp.* 77–92. *Jerusalem Academic Press, Jerusalem; Pergamon, Oxford*, 1964.

The non-linear effects are due to the behavior of "thermorheologically simple" viscoelastic materials where a change in temperature produces a shift of all viscoelastic characteristic functions plotted on a log (time) abscissa. This shift is a non-linear function of temperature.

Using a reduced time suggested by Morland and Lee, the authors derive an integral equation for the radial stress gradient in a hollow sphere with ablating inner and outer radii loaded uniformly over each surface and subject to a radially symmetrical temperature distribution. The influence of viscoelastic energy dissipation on temperature is neglected. Since classical methods (Laplace transform) are ruled out by the time-dependent boundaries, the authors propose a numerical integration approach.

As an example, thermal stresses in a solid sphere with temperature constant on the ablating surface are calculated and are compared with corresponding purely elastic stresses. *H. Parkus* (Vienna)

Ignaczak, Józef 824

On the stress equations of motion in the linear thermoelasticity. (Polish and Russian summaries)

Arch. Mech. Stos. **15** (1963), 691–695.

The question of direct determination of stresses from a properly formulated initial and boundary value problem discussed by the author in previous papers is again taken up. An example of plane stress is given, where the system of equations contains only such stress components as are prescribed on the boundary. *H. Parkus* (Vienna)

Drucker, D. C. 825

Stress-strain-time relations and irreversible thermodynamics.

Second-order Effects in Elasticity, Plasticity and Fluid Dynamics (*Internat. Sympos., Haifa*, 1962), *pp.* 331–351. *Jerusalem Academic Press, Jerusalem; Pergamon, Oxford*, 1964.

A discussion of the relationships between constitutive relations for elastic, viscous and plastic materials and several techniques currently used in irreversible thermodynamics is given. The central question is whether, for non-linear materials, one should attempt to extend the Onsager relations in terms of an entropy production function or start from a new set of postulates. An example of an idealized non-linear viscous material is given which shows that the surface of constant entropy production is not related to the constitutive equation of the material through a normality condition. An alternative suggested is that specified types of path independence be introduced with a stability postulate. Several advantages of the alternative procedure are cited.

P. R. Paslay (Houston, Tex.)

Suhubi, E. S. 826

Longitudinal vibrations of a circular cylinder coupled with a thermal field.

J. Mech. Phys. Solids **12** (1964), 69–75.

Author's summary: "In this paper the longitudinal vibration of a circular infinite cylinder in the presence of a temperature field has been studied. The frequency equation has been obtained for two particular cases, namely, constant temperature and zero heat flux on the surface of the cylinder. The resulting equations have been solved for small radii and weak coupling between fields. For the former case no first-order thermal effect has been observed. In the latter, expressions have been found for the velocity of propagation and damping factor, which indicate the effect of thermal field on the elastic waves and which coincide, if the lateral contraction is neglected, with those found earlier by Sneddon [Proc. Roy. Soc. Edinburgh Sect. A **65** (1959), 121–142; MR **20** #7455] for the very short and very long wave lengths." *R. Muki* (Tokyo)

FLUID MECHANICS, ACOUSTICS

See also 345, 372, 386, 748, 767, 785, 808, 815, 1052.

Green, A. E. 827

A continuum theory of anisotropic fluids.

Proc. Cambridge Philos. Soc. **60** (1964), 123–128.

In Ericksen's theory of anisotropic fluids [Arch. Rational Mech. Anal. **4** (1960), 231–237; MR **22** #1274], the fluid is visualized as a suspension of non-spherical particles, e.g., ellipsoidal particles whose boundaries are surfaces of revolution, the preferred direction being the axis of revolution. If n_i is the component of the vector indicating this direction, the stress t_{ij} at a particle at any time within an arbitrary isotropic pressure p is assumed to be uniquely determined by n_i and by the gradients of velocity v_i at the same particle and time. Using invariance requirements and some possible simplifications, one can put the constitutive equations of this theory as follows:

$$(1)\quad t_{ij} = -p\delta_{ij} + 2\mu d_{ij} + (\mu_1 + \mu_2 d_{km}n_k n_m)n_i n_j + 2\mu_3(d_{jk}n_k n_i + d_{ik}n_k n_j)$$

with

$$(2)\quad \dot{n}_i = w_{ij}n_j + \lambda(d_{ij}n_j - d_{km}n_k n_m n_i),$$

where $n_k n_k = 1$, the λ's and μ's are material constants, $2d_{ik} = v_{i,k} + v_{k,i}$, and $2w_{ik} = v_{i,k} - v_{k,i}$, v_i being the velocity vector.

Now, in the present paper, the author claims that equation (2) does not seem to lie within the framework of established ideas in continuum mechanics. He therefore attempts to derive a theory of anisotropic fluids without introducing extra equations of the type (2). In this new development, he takes the extra stress $S + pI$ as a tensor function of velocity gradients V and rotation matrix R. This idea was suggested by Noll [J. Rational Mech. Anal. **4** (1955), 3–81; MR **16**, 764]. The difference between the two approaches mentioned above lies only in replacing n_i by R. Ericksen's theory is useful because it is simple and properly invariant. It has been possible to apply it to a number of problems [e.g., the reviewer, Arch. Rational Mech. Anal. **10** (1962), 101–107; MR **26** #7247] that are solved in closed form. Instead of two sets of constitutive equations, as used by Oldroyd [Proc. Cambridge Philos. Soc. **43** (1947), 100–105; MR **8**, 240], only one set is used in Ericksen's theory to meet the same end.

{In the reviewer's opinion, the utility of the author's equation (2) above and his continuum theory of anisotropic fluids will better wait till some specific problems are solved by using that theory. It is too early to decide.}

P. D. S. Verma (Kharagpur)

Coleman, Bernard D.; Noll, Walter 828

Simple fluids with fading memory.

Second-order Effects in Elasticity, Plasticity and Fluid Dynamics (*Internat. Sympos., Haifa*, 1962), *pp.* 530–552. *Jerusalem Academic Press, Jerusalem; Pergamon, Oxford*, 1964.

An expository paper surveying certain aspects of the authors' work on the title subject. The basic concepts are reviewed; to avoid obscuring the treatment, proofs of theorems are omitted. Two limiting cases are discussed: for slow motion, the theory of Newtonian fluids provides a first-order approximation for flow of simple fluids with fading memory; the corresponding approximation for small-amplitude motion is the Boltzmann theory; the second-order result is set out for each of these limits.

W. Langlois (San Jose, Calif.)

Gurevich, M. I. [Gurevič, M. I.] 829

Vortex near a free surface.

Prikl. Mat. Meh. **27** (1963), 899–902 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1370–1376.

A solution is presented for the two-dimensional potential flow past a point vortex, with a free surface above and a solid horizontal wall below. The fluid is taken to be incompressible and weightless. The object is to study, by the use of a simplified model, the motion of an immersed wing close enough to the free surface for the surface to be appreciably displaced from the undisturbed horizontal plane. In these circumstances it seems to the reviewer that the assumption of a weightless fluid requires special justification, for disturbances to the free surface must be affected by gravity.

The author obtains the general solution by complex variable methods and derives expressions for the depth of the vortex and the lifting force on the vortex. He then considers the limiting case of infinitely large depth and the case where the circulation of the vortex is in the sense to give a downward instead of lifting force.

M. G. Hall (Farnborough)

Verlet, Loup; Levesque, Dominique 830

On the theory of classical fluids. II. (French summary)

Physica **28** (1962), 1124–1142.

Authors' summary: "Des équations intégrales provenant de la sommation des diagrammes d'Ursell-Mayer ont été proposées récemment par plusieurs auteurs [Van Leuven, Groeneveld, et De Boer, Physica **25** (1959), 792–808; MR **22** #1146; Meeron, J. Mathematical Phys. **1** (1960), 192–201, MR **22** #6125; Morita et Hiroike, Progr. Theoret. Phys. **23** (1960), 1003–1027; MR **22** #7329; Rushbrooke, Physica **26** (1960), 259–265; MR **22** #5178; Verlet, Nuovo Cimento (10) **18** (1960), 77–101; MR **22** #11555]. Nous présentons ici des calculs effectués dans le cas de l'Argon, en utilisant le potentiel de Lennard-Jones, pour d'assez nombreuses valeurs de la température et de la densité. Les résultats théoriques se comparent très favorablement avec l'expérience jusqu'à des densités légèrement supérieures à la densité critique. Au delà, le désaccord avec l'expérience peut être assez grand."

Yudovich, V. I. [Judovič, V. I.] 831

The flow of a perfect, incompressible liquid through a given region.

Dokl. Akad. Nauk SSSR **146** (1962), 561–564 (*Russian*); *translated as Soviet Physics Dokl.* **7** (1963), 789–791.

The initial-boundary value problem for the Euler equations

$$\text{(1)}\qquad \frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v},\nabla)\mathbf{v} = -\nabla P+\mathbf{F}(x,t),\qquad \operatorname{div}\mathbf{v} = 0$$

has long been known to admit a unique solution "in the small" in the region interior to a curve (or surface) S, under the conditions

$$\text{(2)}\qquad \mathbf{v}|_{t=0} = \mathbf{a}(x),$$

$$\text{(3)}\qquad v_n|_S = 0,$$

and in the two-dimensional case it is a result of the author [same Dokl. **136** (1961), 564–567] that the solution exists for all time.

If (3) is replaced by the more general condition

$$\text{(3')}\qquad v_n|_S = \gamma(x,t)$$

(exterior directed normal), the boundary conditions no longer suffice to determine the solution uniquely. The author—pursuing a proposal of Kočin [Prikl. Mat. Meh. **20** (1956), 153–172]—shows how the boundary conditions can be augmented in this more general case to obtain a properly posed problem.

Suppose S consists of $n+1$ contours $S_0, \cdots, S_n$, on a finite number of which $\gamma \leqq c < 0$ holds. Denote these contours by S^-. Under the additional condition $\operatorname{curl}\mathbf{v}|_{S^-} = \pi(x,t)$, where $\pi(x,t)$ is—modulo some necessary restrictions—arbitrary, the author shows the existence, in the two-dimensional case, of a unique solution for all time. The method is constructive, being based on successive approximations, and it brings out clearly the physical basis for the additional condition.

The author states that the boundaries S_i need not be stationary in time, and also that the same procedure yields in three dimensions the unique existence for a sufficiently small time interval. This paper is, in the opinion of the reviewer, an interesting and valuable new contribution to the unduly neglected theory of the Euler equations of hydrodynamics. *R. Finn* (Stanford, Calif.)

Leroy, Jean 832

Sur les vibrations de respiration des cylindres minces partiellement remplis de liquide.

C. R. Acad. Sci. Paris **257** (1963), 2607–2609.

Author's summary: "Les formes propres et fréquences propres des modes de respiration de cylindres minces partiellement remplis de liquide sont déterminés à partir de la théorie générale des coques, la pression de liquide étant calculée grâce à l'équation du potentiel des fluides incompressibles. Le choix de bases de fonctions représentant les formes propres du cylindre vide permet une explication complète du problème et le calcul effectif des caractéristiques vibratoires. Une comparaison avec des résultats expérimentaux s'avère très encourageante."

Lal, Krishna 833

Theoretical considerations of the temperature distribution in a channel bounded by two co-axial circular pipes.

Proc. Cambridge Philos. Soc. **60** (1964), 653–656.

The temperature distribution, T, in a viscous incompressible fluid flowing between two co-axial circular cylinders is investigated. The dissipation due to friction is neglected, and it is assumed that heat is being added at a rate which is an exponential function of time. In the first section of the paper the author derives and discusses the solution

$$T = \sum_{n=1}^{\infty}\left[\frac{\alpha_n}{n}+D_nJ_0(rpi)+E_nY_0(rpi)\right]e^{nt},$$

where α_n, D_n and E_n are constants, the last two being determined in terms of α_n by boundary conditions. Here $p=\sqrt{(n/k')}$, where k' is constant.

The second section begins with a discussion of the asymptotic representation of this series when p is small. {The author has overlooked the fact that p is not the same for each term in the series and, because it depends on $\sqrt{n}$, cannot be simultaneously small for all terms if the whole infinite series is to be considered.} The asymptotic expansion when p is large is then considered. The last sections deal with the corresponding results in terms of $\exp(-nt)$. *D. R. Breach* (Toronto, Ont.)

Benney, D. J. 834

The flow induced by a disk oscillating in its own plane.

J. Fluid Mech. **18** (1964), 385–391.

While the flow field near an infinite disk performing small torsional oscillations can be computed by a Taylor series expansion in powers of an appropriate small parameter, the procedure is not satisfactory for computing the flow field at large distances from the plate, as was pointed out by Rosenblat [same J. **6** (1959), 206–220; MR **22** #437]. The reason for this is that even though the inertia terms are small at large distances, they are still of the same order as terms retained; hence the problem is inherently nonlinear. The author shows how, by introducing two length scales and an additional independent variable, to obtain a single expansion valid both near and at large distances from the disk. *R. C. DiPrima* (Troy, N.Y.)

Moore, K. C.; Jones, J. G. 835

Quasi-cylindrical surfaces with prescribed thickness distributions.

Quart. J. Mech. Appl. Math. **16** (1963), 521–527.

This is an extension of Jones's work [same J. **15** (1962), 471–485; MR **26** #3336] on half-ring wings in supersonic flow. The wing is represented by a surface distribution of sources, and a formula is obtained for the supersonic velocity field. Since linearised theory is used, the results of the two papers can be added to give the camber shape for a surface with loading and thickness. As an example, a thickness distribution with parabolic sections is considered; the resulting mean wing shape is found in terms of elliptic integrals. *A. W. Babister* (Glasgow)

Woods, L. C. 836

On the theory of growing cavities behind hydrofoils.

J. Fluid Mech. **19** (1964), 124–136.

A general method is given for treating two-dimensional unsteady cavitating flows past obstacles in an ideal (inviscid, incompressible) fluid with prescribed points of flow separation (e.g., at given corners). A sink at infinity, omitted in the author's previous analysis [Proc. Roy. Soc. Ser. A **229** (1955), 152–180; MR **16**, 1061], is included for reasons discussed in the following review [#837]. The problem is shown to be equivalent to a Riemann-Hilbert problem, and is solved for a symmetric wedge and for a flat plate at incidence. The method is similar to that proposed by D. P. Wang and T. Y. Wu [Arch. Rational Mech. Anal. **14** (1963), 127–152; MR **27** #4440]. *G. Birkhoff* (Cambridge, Mass.)

Benjamin, T. Brooke 837

Note on the interpretation of two-dimensional theories of growing cavities.

J. Fluid Mech. **19** (1964), 137–144.

The author makes an asymptotic analysis of the nearly two-dimensional flow of an incompressible fluid around a growing nearly two-dimensional cavity (i.e., one with slowly varying cross-section), such as might be generated by a moving hydrofoil. His idea is to replace $\phi_{xx}+\phi_{yy}=0$ by $\phi_{xx}+\phi_{yy}+\varepsilon^2\phi_{zz}=0$ ($\varepsilon \ll 1$). In such applications, this is more reasonable physically than to take account of small compressibility, using $\phi_{xx}+\phi_{yy}+c^{-2}\phi_{tt}=0$, $c \gg 1$. {Reviewer's note: Use of reflection in the free surface is often even more reasonable physically, and simpler mathematically.} *G. Birkhoff* (Cambridge, Mass.)

Tulin, Marshall P. 838

Supercavitating flows—small perturbation theory.

J. Ship Res. **7** (1963/64), *no.* 3, 16–37.

From the author's summary: "This paper contains analytical results which are both old and new. It also contains an historical introduction briefly outlining events leading up to the formulation of linearized theory and indicating the scope of its subsequent development and engineering applications. A large number of references to the extensive literature are presented. The new analytical results include: The logical development of the linearized theory of steady supercavitating flows; the presentation of a general solution for the linearized problem; introduction of a new nonlinear model for finite cavity flows; the formulation of a simple second-order theory; and the application of this theory to the case of flat plates and symmetric wedges."

Grekova, N. O. 839

A non-linear formulation of the Cauchy-Poisson problem. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1963**, *no.* 5, 68–80.

Using Lagrange parameters, the author develops a formal expansion to describe periodic waves of finite amplitude in a channel. She obtains a solution up to terms of third order in a small parameter, which is free of secular terms. *R. Finn* (Stanford, Calif.)

Ursell, F. 840

The decay of the free motion of a floating body.

J. Fluid Mech. **19** (1964), 305–319.

A long circular cylinder is floating in equilibrium on the free surface of an infinitely deep ideal fluid, the axis of the cylinder lying in the plane of the undisturbed free surface. The cylinder is given a small prescribed force or displacement in the vertical direction and subsequently is unrestrained, so that a transient decaying oscillation results. This two-dimensional initial-value problem is solved in terms of the corresponding solution for harmonic oscillations by taking the Fourier transforms of the velocity potential and boundary conditions with respect to time. The asymptotic behavior for large time is found explicitly from the low-frequency expansion of the harmonic oscillation solution; it is shown that the decaying motion of the cylinder is ultimately monotonic, and inversely proportional to the second or third power of time according as the initial disturbance was a displacement or force. It follows that the number of free oscillations is finite, whereas it had been generally assumed that the motion would resemble damped sinusoidal oscillations. This conclusion is important as an illustration of the difference between a floating body and a more conventional linear oscillator representable by a second-order differential equation. The author suggests that similar conclusions will apply to bodies of more general shape in two and three dimensions, but generalization to include a fluid of finite depth may change the basic character of the results. *J. N. Newman* (Washington, D.C.)

Keller, Joseph B. 841
Viscous flow through a grating or lattice of cylinders.
J. Fluid Mech. **18** (1964), 94–96.

Author's summary: "Viscous flow perpendicular to a line (or 'grating') of evenly spaced identical cylinders is considered in the case when the spacing between the cylinders is much smaller than their cross-sectional dimensions. Lubrication theory is used to find the pressure drop across the grating and hence the force on each cylinder. A square array (or 'lattice') of closely packed cylinders is similarly treated." *H. Hasimoto* (Baltimore, Md.)

Bagaeva, N. Ja.; Moiseev, N. N. 842
Three problems on the oscillation of a viscous fluid. (Russian)
Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 317–326.

This paper contains three applications of the method given in Moiseev's earlier paper [same Ž. **1** (1961), 548–550; MR **25** #4266]. *V. N. Constantinescu* (Bucharest)

Tipei, N. 843a
Effects of the microgeometry of surfaces on lubrication. I. The two-dimensional problem.
Rev. Méc. Appl. **8** (1963), 981–996.

Tipei, N. 843b
Effects of the microgeometry of surfaces on lubrication. I. The two-dimensional problem. (Romanian. English summary)
Acad. R. P. Romîne Stud. Cerc. Mec. Apl. **14** (1963), 995–1010.

The author establishes solutions of the equations of hydrodynamic lubrication for the case in which the roughness of the walls is considered. He solves the two-dimensional problem and then treats a quasi-two-dimensional problem.
{Both papers are identical in content.}
A. J. Lohwater (Providence, R.I.)

Jones, J. R. 844
Flow of elastico-viscous liquids in pipes with cores. I.
J. Mécanique **3** (1964), 79–99.

The author considers the flow of certain idealized elastico-viscous liquids in a circular pipe with an eccentric circular core moving longitudinally with constant velocity. The method followed is the same as that used by Oldroyd [Proc. Cambridge Philos. Soc. **43** (1947), 396–405; MR **8**, 612] in considering the plastic flow of Bingham solids between the same boundaries. Several approximations are taken to avoid the possible mathematical difficulties. A special case is discussed when R_a (radius of the pipe) is 7, R_b (radius of the core) is 16, and h (distance between the two centres) is 3, 5, 7. Numerical demonstrations of the velocity pattern and the lateral force are given in this particular case under different external conditions: (i) $U>0$, $V=0$, (ii) $U>0$, $V>0$, (iii) $U>0$, $V<0$, (iv) $U=0$, $V\neq 0$, where U is the axial pressure gradient and V is the velocity of the core. The numerical values could not, of course, be checked. *P. D. S. Verma* (Kharagpur)

Rivlin, R. S. 845
Second and higher-order theories for the flow of a visco-elastic fluid in a non-circular pipe.
Second-order Effects in Elasticity, Plasticity and Fluid Dynamics (*Internat. Sympos., Haifa*, 1962), *pp.* 668–677. *Jerusalem Academic Press, Jerusalem; Pergamon, Oxford*, 1964.

The flow of a viscoelastic fluid along a straight non-circular pipe is discussed according to the first, second, third and fourth order theories. It is shown that rectilinear flow is possible according to the first, second, and third order theories, but not according to the fourth order theory. There seems to be some scope for solving simple problems on third and fourth order fluids. An assumption is, however, made: The motion must be a slow one.
P. D. S. Verma (Kharagpur)

Slibar, A.; Paslay, P. R. 846
On the analytical description of the flow of thixotropic materials.
Second-order Effects in Elasticity, Plasticity and Fluid Dynamics (*Internat. Sympos., Haifa*, 1962), *pp.* 314–330. *Jerusalem Academic Press, Jerusalem; Pergamon, Oxford*, 1964.

Materials considered are viewed as Bingham type materials of constant viscosity and variable critical shear stress. The dependence of the critical shear stress on the deformation rate history is established by use of a memory function. The analytical formulation of the constitutive equations for describing the flow of this thixotropic material is verified by solving a couple of problems like shear between parallel plates and Couette flow.
P. D. S. Verma (Kharagpur)

Libby, Paul A.; Baronti, Paolo O.; Napolitano, Luigi 847
Study of the incompressible turbulent boundary layer with pressure gradient.
AIAA J. **2** (1964), 445–452.

A method of calculating the growth of incompressible turbulent boundary layers in pressure gradients is developed for use on high-speed computers. The physical model is essentially that of Coles [J. Fluid Mech. **1** (1956), 191–226; MR **18**, 355]. The shear stress T_1 and its gradient ST_1 say, and the velocity, all calculated from the law of the wall, are matched at a point η_1 ($\eta = y/\delta$) to those in the outer part of the layer. The latter are found by approximating to the shear stress in this region by means of a cubic in η, and by assuming the eddy viscosity to depend only on x. To close the resulting set of equations, a relation is needed between S and η_1: ideas from Clauser's study of equilibrium layers [J. Aeronaut. Sci. **21** (1954), 91–108] are introduced to provide this.
D. A. Spence (Farnborough)

Nanda, R. S. 848
Similar solutions of laminar boundary layer equations for power law fluids in two dimensions.
J. Sci. Engrg. Res. **7** (1963), 251–258.

The author presents six similarity solutions to a system of equations which represent a two-parameter generalization of the Prandtl boundary-layer equations. These are

asserted to represent the boundary-layer equations for a class of non-Newtonian fluids, but they correspond to a class of stress-deformation relations which is not tensorially admissible. *W. Langlois* (San Jose, Calif.)

Peube, Jean-Laurent 849

Sur le transfert de chaleur dans l'écoulement d'Hele Shaw.

C. R. Acad. Sci. Paris **258** (1964), 433–435.

The thermal problem for Hele-Shaw flow is reduced by a simple transformation to the corresponding problem for Poiseuille flow between two parallel planes.

D. W. Dunn (Ottawa, Ont.)

Van Dyke, Milton 850

Higher approximations in boundary-layer theory. III. Parabola in uniform stream.

J. Fluid Mech. **19** (1964), 145–159.

From the author's summary: "The classical laminar boundary layer on a parabolic cylinder is calculated using the Blasius series, with modifications to improve convergence, and supplemented by an asymptotic expansion valid far downstream from the nose. The flow due to displacement thickness is thereby found with sufficient accuracy to permit evaluation of its second-order effect upon the boundary layer near the stagnation point, using the results of Parts I and II [same J. **14** (1962), 161–177; MR **26** #5824; ibid. **14** (1962), 481–495; MR **26** #5825]. The skin friction and heat transfer are found to be reduced there by both displacement and curvature. Some corrections to Parts I and II are included."

D. W. Dunn (Ottawa, Ont.)

Crocco, L. 851

Transformations of the compressible turbulent boundary layer with heat exchange.

AIAA J. **1** (1963), 2723–2731.

From the author's summary: "A mathematical transformation is presented which converts the integral turbulent boundary-layer equations at high speed into corresponding equations at low speed. Pressure gradients as well as heat exchange are allowed. Two transformations can be defined, one in which the low-speed fluid is incompressible, and another in which the low-speed fluid is a gas. The conditions under which the transformations operate correctly, not only on the integral equations but also on the whole system of partial differential equations, are discussed."

D. W. Dunn (Ottawa, Ont.)

Amsden, Anthony A.; Harlow, Francis H. 852

Slip instability.

Phys. Fluids **7** (1964), 327–334.

The authors first analyze approximately the effect of viscosity and large amplitude on Helmholtz ("slip") instability, applying earlier criteria of Carrier and Esch to the former. They then use the numerical method of Fromm [J. E. Fromm and F. H. Harlow, Phys. Fluids **6** (1963), 975–982] to calculate the evolution of periodic Helmholtz instability in time, for various initial conditions and Reynolds numbers. Their calculations, presented graphically, improve substantially earlier numerical treatments of the same problem which neglected viscosity—and are compared with the latter. The relevant literature is reviewed thoughtfully. *G. Birkhoff* (Cambridge, Mass.)

DiPrima, R. C. 853

Stability of curved flows.

Trans. ASME Ser. E. J. Appl. Mech. **30** (1963), 486–492.

Author's summary: "In the present paper the general formulation of the hydrodynamic-stability problem is discussed. The problem of the stability of flow between rotating cylinders is considered in some detail. Several related problems which have been investigated in a complete or preliminary manner are mentioned. Finally, the recent work on the nonlinear aspects of the hydrodynamic-stability problem, with particular reference to the stability of flow between rotating cylinders, is discussed."

W. H. Reid (Chicago, Ill.)

Fejer, J. A. 854

Hydromagnetic stability at a fluid velocity discontinuity between compressible fluids.

Phys. Fluids **7** (1964), 499–503.

The stability of a vortex sheet between two perfectly conducting, inviscid, compressible fluids in the presence of uniform, arbitrarily oriented magnetic fields is investigated. The disturbances are three-dimensional. Certain special cases are discussed in detail. The author concludes that the introduction of a slight compressibility always reduces the stabilizing effect of magnetic fields. However, it has been noticed that, when reduced to the two-dimensional disturbance case, this conclusion, as well as equation (13) of this paper, does not check with the results obtained by this reviewer and Maslen in a paper to appear.

K. C. Wang (Baltimore, Md.)

Kurzweg, Ulrich H. 855

The stability of dissipative Couette flow between rotating cylinders in the presence of an axial magnetic field. (German summary)

Z. Angew. Math. Phys. **15** (1964), 39–45.

The effect of an axial magnetic field on the stability of a viscous, conducting fluid between rotating cylinders which are insulators is investigated. The small-gap approximation is made, and the resulting eigenvalue problem is solved approximately by using the Galerkin method for a range of Hartmann numbers and for a range of rotation rates $|\Omega_2/\Omega_1|<1$. Here Ω_1 and Ω_2 are the angular velocities of the inner cylinder and outer cylinder respectively. The results thus lie between the cases $\Omega_2/\Omega_1=1$ considered by Niblett [Canad. J. Phys. **36** (1958), 1509–1525; MR **21** #3193] and $\Omega_2/\Omega_1=-1$ considered by Chandrasekhar and Elbert [Proc. Roy. Soc. Ser. A **262** (1961), 443–454; MR **23** #B1820]. *R. C. DiPrima* (Troy, N.Y.)

Yang, Kwang-tzu [Yang, Kwang-Tzu] 856

Laminar free-convection wake above a heated vertical plate.

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 131–138.

From the author's summary: "Velocity and temperature profiles near the trailing edge of the plate are calculated for two Prandtl numbers by means of asymptotic series

expansions. These results are then continued to the rest of the wake region by an integral solution. It is found that the spreading of the boundary layer into the wake region is rather gradual." *D. W. Dunn* (Ottawa, Ont.)

Jones, J. G. 857
On the near-equilibrium and near-frozen regions in an expansion wave in a relaxing gas.
J. Fluid Mech. **19** (1964), 81–102.
The "near-equilibrium" and "near-frozen" solutions for the propagation of a relaxing expansion (or rarefaction) wave generated by an impulsively receding plane piston are discussed. In contrast to a relaxing acoustic wave, the non-linear convection effects are partially taken into account in this paper. In the near-equilibrium case, Burger's equation is used with the relaxation entering in the context of a bulk viscosity. Extensive study of the Burger's equation by Lighthill then immediately gives the solutions needed. The author's conclusion on the diffusive effects near the first and last rays of the asymptotic equilibrium simple wave is just what one would expect; his discussion on the connection between Chu's asymptotic acoustic solution and the present solution is interesting. In the near-frozen case, an entirely different approach—the characteristic perturbation method—is used, and analytic results are obtained to describe the flow near the wave front only. *K. C. Wang* (Baltimore, Md.)

Fonarev, A. S. 858
On the problem of adiabatic escape of a plane gas layer under arbitrary initial conditions. (Russian)
Inž. Ž. **3** (1963), *no.* 1, 18–26.
Consider a slab of ideal gas bounded at $r=-r_0$ by a fixed rigid wall, and initially at $r=0$ by a vacuum. To approximate the subsequent one-dimensional flow into a vacuum, starting from arbitrary initial conditions, the author perturbs a centered simple rarefaction wave and uses the early terms of power series expansions for the perturbations. Computed results are presented for constant initial density and linear initial pressure, and some comparisons are made with results of characteristic calculations.
J. H. Giese (Aberdeen, Md.)

Claus, Alfons J. 859
Large-amplitude motion of a compressible fluid in the atmosphere.
J. Fluid Mech. **19** (1964), 267–289.
The author finds solutions for the flow of a compressible fluid between two horizontal planes over finite ridges. As one might expect, the effect of compressibility is rather small if one substitutes for the compressible fluid an incompressible one, provided one has the same stratification of (potential) density in the two cases.

As regards the problem of air flow over mountains, this investigation does not add significantly to our understanding because the author has imposed a rigid upper boundary. It is possible to show, in fact, that if compressibility is of major importance for the flow of air over mountains, it is because the real atmosphere extends far above the tropopause, whereas incompressible model atmospheres end abruptly at a depth of ten kilometers or so.

As a basic fluid mechanics investigation, however, the author is to be commended for contributing exact perfect fluid solutions to the problem under consideration.
R. R. Long (Baltimore, Md.)

Dyment, Arthur; Gontier, Gérard 860
Sur la détermination des conditions de part et d'autre d'une onde de choc, au moyen d'une sonde d'arrêt.
C. R. Acad. Sci. Paris **258** (1964), 5156–5159.
Les auteurs décrivent d'abord la sonde d'arrêt qu'ils utilisent pour déterminer les conditions de l'écoulement de part et d'autre d'une onde de choc. Après avoir vérifié la précision de leur appareillage, celui-ci est appliqué à l'étude de l'onde de choc dans un écoulement transsonique. Les mesures faites permettent le tracé de l'image de l'onde de choc dans le plan de l'hodographe. D'après les auteurs, leurs mesures mettent en évidence le changement de concavité de l'onde de choc prévue par la théorie développée par l'auteur de cette analyse.
P. Germain (Paris)

Stewartson, K. 861
Viscous hypersonic flow past a slender cone.
Phys. Fluids **7** (1964), 667–675.
The general procedure for analyzing the problem of boundary-layer flow at hypersonic speeds has been well enough treated by dividing the flow into two parts—an inviscid zone bounded on one side by a shock and on the other by a streamline, and a viscous region between the body in question and the streamline. All of the flow properties can subsequently be found from this model. An exception to this technique, however, is the strong interaction region past a slender cone, for example. The work here considers the slender cone in detail, and expressions for the pressure, shock, skin friction, and heat transfer are derived. The results offered are for the case of an axisymmetric cone and with a fluid having a Prandtl number of unity and the viscosity given as a linear function of the temperature. The arguments have extended validity in that they can be generalized to include bodies whose shapes are not conical nor even axisymmetric. A more complicated fluid is also permissible. Finally, an approximate method by which the whole interaction region can be computed is suggested. *W. O. Criminale, Jr.* (Princeton, N.J.)

Dżygadło, Zbigniew 862
Self-excited vibrations of a pointed conical shell in supersonic flow. (Polish and Russian summaries)
Proc. Vibration Problems **4** (1963), 265–280.
The results of second-order theory are used to determine the pressures on a slender pointed cone in a supersonic flow. The equations of motion (with damping included) are reduced to a system of integral equations. Four linearly independent particular solutions are determined which satisfy the appropriate boundary conditions. The present paper deals with shells having simply supported or clamped edges.

The characteristic equation for steady motion is derived, and the critical Mach number (for undamped motion) is calculated for a cone of 20° vertex angle.
A. W. Babister (Glasgow)

Ladyzhenskii, M. D. [Ladyženskii, M. D.] 863
Hypersonic viscous flow over slender bodies.
Prikl. Mat. Meh. **27** (1963), 765–778 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1179–1196.
Hypersonic viscous flow over slender pointed or slightly blunted bodies of arbitrary cross-section is investigated by a boundary-layer type analysis. Consideration is given to flows where the viscous layer is of the same order as or greater than the body thickness, and where there exists moderate or strong pressure interaction.
M. H. Steiger (Freeport, N.Y.)

Mikhailov, V. N. [Mihaĭlov, V. N.] 864
Calculation of three-dimensional rotational supersonic flow of gas in the vicinity of a curve at which occurs a break in streamline.
Prikl. Mat. Meh. **27** (1963), 1083–1089 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1658–1667.
The problem of determining the three-dimensional supersonic flow of an ideal gas in the vicinity of a given curve L is considered. The solution is sought in the form of an expansion in one of the variables, which is proportional to the distance from the curve L. The formulation admits solutions with discontinuity (in slopes of streamlines) at the curve L. The leading terms of the development yield two solutions; one corresponds to a compression and the other to an expansion in the vicinity of the curve L.
H. K. Cheng (Palo Alto, Calif.)

Kotake, Susumu; Okazaki, Takuro 865
Jet noise (far noise field of subsonic free air jet).
Bull. JSME 7 (1964), 153–163.
The paper is a re-examination of Lighthill's classical theory in the light of experimental results. The main conclusions are: (1) The noise propagated in the axial direction is mainly contributed by the component of the velocity fluctuation in the same direction u and is responsible for the low-frequency component of the total noise; (2) The noise generation is most effective in the mixing region; (3) The convection of turbulence causes a shift in the spectrum with Mach number; (4) The eight power law is roughly valid. *L. S. G. Kovasznay* (Baltimore, Md.)

Giesekus, Hanswalter 866
Statistical rheology of suspensions and solutions with special reference to normal stress effects.
Second-order Effects in Elasticity, Plasticity and Fluid Dynamics (*Internat. Sympos., Haifa*, 1962), *pp.* 553–584. *Jerusalem Academic Press, Jerusalem; Pergamon, Oxford*, 1964.
Author's summary: "After formulating the rheological equation of state for that class of fluids in which normal stresses in steady shear flow occur in flow direction only (Weissenberg fluids), two examples of suspensions are presented to show that, in the general case, normal stresses in the direction of flow gradient do not vanish. In suspensions of particles with permanent anisotropy of shape, in which the Brownian movement is negligible, normal stress effects appear as first-order effects. As is the case with shear stress, these are a pronounced function of the type of flow. A very general formulation of the theory of dilute suspensions and solutions is possible. This is done specifically in the case of rigid particles. A deduction of an integral equation is offered for which solutions are stated and discussed."

Oldroyd, J. G. 867
Survey on second-order fluid dynamics.
Second-order Effects in Elasticity, Plasticity and Fluid Dynamics (*Internat. Sympos., Haifa*, 1962), *pp.* 699–711. *Jerusalem Academic Press, Jerusalem; Pergamon, Oxford*, 1964.

Bhatnagar, R. K. 868
Steady laminar flow of visco-elastic fluid through a pipe and through an annulus with suction or injection at the walls.
J. Indian Inst. Sci. **45** (1963), 126–151.
Author's summary: "The problem of steady laminar flow of visco-elastic fluids through a pipe and through a cylindrical annulus in the presence of suction and injection have been considered in Part A and Part B of the paper, respectively. The rate of suction at one wall has been taken to be equal to the rate of injection at the other wall in the case of annulus. For the pipe flow the effect of the visco-elasticity of the fluid on the axial velocity, axial pressure and the shearing stress have been depicted graphically. For the annulus, the perturbation method has been used for the axial velocity regarding KR as the perturbation parameter. For rapid convergence of the series solution, KR is taken to be equal to 0.1. It has been found that for Reynolds number $R=1$ the axial velocity profile is not symmetrical regardless of the radius ratio of the two boundaries of the annulus. This is in contrast to the Newtonian fluids where the profile is symmetrical for $R=1$. It has also been found that the points of inflexion in the axial velocity profiles occur only after a critical value of Reynolds number, viz., $R_c = 1.7$. The radial pressure variation depends on the direction of cross flow. In this respect, visco-elastic fluids have been found to differ from Newtonian fluids."

Jain, M. K. 869
Collocation method to study problems of cross-viscosity.
Second-order Effects in Elasticity, Plasticity and Fluid Dynamics (*Internat. Sympos., Haifa*, 1962), *pp.* 623–635. *Jerusalem Academic Press, Jerusalem; Pergamon, Oxford*, 1964.
The expressed purpose of this paper is to demonstrate the use of the collocation method to study problems of cross-viscosity. Following a brief formal review of the basic mathematics involved, the method is applied to two problems involving cross-viscosity, namely, non-Newtonian flow near a stagnation point with suction and flow near a rotating disk.

The numerical method of solution is thereby clearly and adequately illustrated. A question is raised, however, in the accompanying discussion by H. Giesekus regarding the practical value of the Reiner-Rivlin characterization of the fluid and its use in evaluating the non-Newtonian effect in practical applications. It appears to this reviewer that the use of this generally accepted formulation has enabled the author to demonstrate the mathematical procedure and derive several informative results of interest.

These results are in themselves instructive although it would certainly be desirable to extend the method to elastico-viscous fluids, as suggested by H. Giesekus.
R. R. Gold (Los Angeles, Calif.)

Nanda, R. S. 870
Visco-elastic flow due to a vibrating plane. (Polish and Russian summaries)
Arch. Mech. Stos. **15** (1963), 599–606.
The problem of a vibrating plane in a body of incompressible viscoelastic fluid is solved in the case when the plane is rotating with respect to an inertial frame of reference. The constitutive equation of the fluid is a special case of the Oldroyd equation. The choice of the particular constitutive equation and geometry make the hydrodynamic equations linear in the velocity fields, and these exhibit solutions analogous to the case of a Newtonian fluid. It is found that normal stresses, in addition to shear stresses, are required to maintain the flow near the oscillating plane. *P. P. Niiler* (Cambridge, Mass.)

Pratelli, Aldo M. 871
Analogia tra "viscosità magnetica" e "conducibilità termica" nelle piccole perturbazioni di fluidi comprimibili. I, II.
Ist. Lombardo Accad. Sci. Lett. Rend. A **97** (1963), 3–22; *ibid.* **97** (1963), 139–154.
These two papers are based on the remark that the linearized oscillations of two different fluids, to be described below, obey the same dynamical equations. One of them is a perfect gas, viscous and heat conducting; the other is an ionized gas in an external, uniform magnetic field, the perturbation being uniform along the latter; one must also assume a barotropic equation of state. The correspondence identifies the longitudinal magnetic field with the temperature and, roughly speaking, the magnetic viscosity with the thermal conductivity. The analogy is dealt with at great length and illustrated by many examples; in particular, the propagation and the damping of waves are calculated. Special attention is given in the second part to the propagation of perturbations depending on one space variable alone. *B. Bertotti* (Frascati)

Muhuri, P. K. 872
Flow formation in Couette motion in magnetohydrodynamics with suction.
J. Phys. Soc. Japan **18** (1963), 1671–1675.
The formation of Couette flow of an electrically-conducting, viscous, incompressible fluid between two parallel walls due to impulsively or uniformly accelerated motion of one wall under a constant uniform transverse magnetic field is treated by using the magneto-hydrodynamic equations. A uniform suction or blowing is applied at the moving wall. Assumptions are made that the magnetic Reynolds number is small so that the induced magnetic field may be neglected, and the electric field is zero everywhere. The problem is solved by using the Laplace transform. Solutions for the velocity profile are given in the form of infinite series of error functions. Numerical values of the skin friction coefficient on the moving wall are tabulated for both cases: with and without suction. These numerical results show that the effects of the magnetic field and the suction are to increase the skin friction on the moving wall. It appears, however, that the use of non-dimensional skin friction coefficients $\tau_1 h/\mu U$ and $\tau_a h/\mu At$ may be more desirable. The statement that "the skin friction coefficient increases as the Hartmann number increases, and increases as the distance between the plates decreases", should not be taken literally, since the Hartmann number decreases as this distance decreases. It is noted that the impulsively moving case has been treated by L. N. Tao (no suction) [J. Aerospace Sci. **27** (1960), 334–338; MR **22** #5312] and P. Ramamoorthy (with suction) [ibid. **29** (1962), 111–112]. *K. T. Yen* (Philadelphia, Pa.)

Davis, A. M. J. 873
Small disturbances in a conducting fluid in the presence of a current-carrying conductor.
Proc. Cambridge Philos. Soc. **60** (1964), 325–339.
This paper is based on one by Friedlander [same Proc. **55** (1959), 341–367; MR **22** #1289] on the propagation of small disturbances in a compressible conducting fluid in the presence of a magnetic field. Friedlander considered a uniform magnetic field, whereas the present author assumes the field to be produced by an electric current flowing axially along a solid conducting circular cylinder. The disturbance is taken to be impulsive at a single point, since more general cases can be built up by integration. The corresponding problem in an incompressible fluid is treated in greater detail, as it is more amenable analytically, and the forms of the disturbance are given for small and large times. *E. J. Watson* (Manchester)

Sears, W. R.; Mori, Y. 874
Studies of the inviscid boundary layer of magnetohydrodynamics.
Progress in Applied Mechanics, pp. 3–16. *Macmillan, New York*, 1963.
Similarity solutions of the inviscid boundary-layer equations for the aligned-field magnetohydrodynamic flow past a wedge are investigated on the assumption that the tangential magnetic field along the boundary is zero [W. R. Sears, Astronaut. Acta **7** (1961), 223–236; MR **26** #2131]. The presence of an algebraic singularity at the inner edge is concluded. An asymptotic solution for the outer part of the boundary layer indicates that the solution exists only at a trailing edge if the flow is sub-Alfvénic and only at a leading edge if the flow is super-Alfvénic. A numerical solution is performed for a super-Alfvénic stagnation flow, but it is not successful for the sub-Alfvénic case.
H. Hasimoto (Baltimore, Md.)

Glauert, M. B. 875
On magnetohydrodynamic flows with aligned magnetic fields.
J. Fluid Mech. **19** (1964), 49–59.
Two boundary-layer problems are considered, each derived from the Carrier-Greenspan equations [same J. **6** (1959), 77–96; MR **21** #6907] and associated with uniform parallel main streams and magnetic fields. First the fluid is moving past a fixed body in which the magnetic field is zero. It is shown that solutions are only possible if

$$\beta = \mu H_0^2/\rho U_0^2 < 1/\log \frac{1}{\sigma\mu\nu},$$

where μ is the permeability, H_0 the magnetic field outside the body, U_0 the main stream velocity, σ the conductivity, ν the kinematic viscosity, and MKS units are used. (A similar result has also been given by the reviewer and D. H. Wilson [ibid. **18** (1964), 337–349; MR **29** #3104].)

The author discusses the relevance of this result to the flow model for a perfectly conducting inviscid fluid in motion past a fixed body, in which the magnetic field is zero in the body [Sears and Resler, ibid. **5** (1959), 257–273; MR **21** #551]. He concludes that in most physical applications (when $\sigma\mu\nu \ll 1$) such a model is likely only to be possible for a limited range of magnetic fluid strengths.

In the second problem there is a shear layer between a moving stream and a stagnant region, while outside the layer the magnetic field is everywhere constant. An expansion of the solution in descending powers of $\beta^{1/2}$ leads to a contradiction at the second stage. The author uses this result to raise misgivings about the solution obtained by Chester [Chester and Moore, ibid. **10** (1961), 466–472; MR **22** #12937] for the flow past a fixed blunt body. In Chester's solution stagnant wakes, adjacent to undisturbed fluid, extend to infinity upstream and downstream of body and the magnetic field is undisturbed both inside and outside the body.

These misgivings are not, however, shared by this reviewer. Chester's solution is a limiting case ($\beta \to \infty$) and the author's argument is only relevant if, at finite β, the wakes have a discontinuous structure. It is much more likely in fact that, when β (>1) is finite and the fluid perfectly conducting and inviscid, the velocity at infinity is a continuous function of position, in which case the author's argument cannot be directly applied.

K. Stewartson (London)

Plato, Guenther 876

One- and two-dimensional propagation of Alfvén waves in inhomogeneous magnetic fields.

Phys. Fluids **7** (1964), 715–722.

Si considera il moto di un fluido non viscoso, compressibile, dotato di conducibilità elettrica infinita, soggetto ad un campo magnetico permanente non omogeneo B_0; sotto l'ipotesi che sia possibile linearizzare le equazioni che reggono il fenomeno, si introduce un sistema di coordinate curvilinee ortogonali, una delle quali nella direzione di B_0. Dopo aver dimostrato che rispetto alle linee B_0 non sono possibili modi longitudinali strettamente unidimensionali, si studiano i modi trasversali, mono- e bidimensionali; è da rilevare il fatto che, se le linee B_0 hanno curvatura finita, la propagazione monodimensionale è possibile solo se la velocità di Alfvén è stazionaria lungo la normale prinipale alle dette linee. Si ricavano infine condizione che per le onde studiate assicurano l'assenza di attenuazione e si sviluppano alcuni esempi.

R. Nardini (Bologna)

Chu, Chong-Wei 877

Nonlinear wave motion in a linear pinch.

Phys. Fluids **7** (1964), 455–462.

The motion of a linear pinch due to finite axisymmetric disturbances is studied. The plasma is considered as a single, inviscid, perfectly conducting, compressible fluid, and the usual magneto-hydrodynamic assumptions are made. The theory of characteristics is applied to study simple waves. A compression simple wave is found to steepen with time and is expected to develop into a shock with area change. It is found that the increase of entropy across such a shock is greater than that across an ordinary normal shock with the same pressure ratio. With respect to a single finite axisymmetric concave or convex disturbance with small area gradient in the axial direction, the pinch is stable in the sense that it will nowhere touch the coaxial wall and its physical quantities will remain finite.

T. Morse (Providence, R.I.)

Chushkin, P. I. [Čuškin, P. I.] 878

Hypersonic flow around a circle and a sphere in a magnetic field.

Prikl. Mat. Meh. **27** (1963), 1089–1094 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1668–1676.

Suppose a magnetised circle or sphere is placed in a uniform stream of perfectly inviscid, thermally non-conducting gas flowing along its axis of symmetry with a highly supersonic value of the Mach number. The gas in front of the detached shock wave is assumed to be non-conducting electrically, while behind the shock wave the coefficient of electrical conductivity is considered to be either constant or proportional to the fifth power of the temperature. The magnetic Reynolds number is assumed to be small and the electric field is taken to be everywhere zero. The magnetic field is supposed to be created by a plane or axi-symmetric magnetic dipole at the centre of the circle or sphere, respectively, and having its axis parallel to the direction of the incident stream. The method of integral relations is used to discuss the flow in the regions of influence and, in particular, in the construction of the detached shock wave and the calculation of the flow variables on the body surface. The region between the body surface and shock surface is divided into n subregions by similar surfaces, but the numerical calculations have been carried out only to the first approximation ($n=1$) and only for the region of subsonic and mixed flow, as it is believed that the calculations beyond the sonic point can be carried out more conveniently by the method of characteristics. It is found that stronger magnetic fields cause a significant increase in the pressure on the body and a pronounced movement of the sonic point away from the stagnation point, but in the case when the coefficient of electrical conductivity is variable, the changes are appreciably smaller. A method of extending the calculations to the case of any symmetrical blunt body and an alternative scheme of integration are also suggested.

J. N. Kapur (Kanpur)

Infeld, E. 879

Transverse magnetohydrodynamic waves in the magnetic field of a dipole.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 161–166.

Author's introduction: "In this paper we solve numerically the linearized equation for the fluid velocity of the Alfvén waves propagating in the field of a magnetic dipole. A comparison with the two-dimensional dipole case, for which we have the exact solution in analytical form, is made. Some possible applications for the Earth's magnetic field are indicated."

Sozou, C. 880
A two-dimensional dipole under a uniform plasma pressure.
Quart. J. Mech. Appl. Math. **17** (1964), 247–252.
Author's summary: "The magnetic field and the shape of the cavity in a uniform plasma under uniform pressure in the presence of a two-dimensional dipole are determined. The solution shows that there are cusps in the plasma over the poles of the dipole as expected; these, however, do not run down to the origin."

Warsi, N. A. 881
On curvature and torsion of stream line and magnetic line of force behind magnetogasdynamic shock wave.
Tensor (N.S.) **15** (1964), 128–141.
This paper deals with the differential effects of shock waves. By differentiating the Rankine-Hugoniot difference equations, one can calculate the jumps in derivatives of flow and field quantities across a shock front [the reviewer, J. Math. Mech. **9** (1960), 681–695; MR **22** #7572]. By means of these derivatives, the author finds the values of curvature and torsion of field and stream lines behind the shock. *R. P. Kanwal* (University Park, Pa.)

Kumar, Prem; Jain, P. C. 882
Pressure fluctuations within isotropic hydromagnetic turbulence.
Tensor (N.S.) **15** (1964), 1–4.
This paper deals with the incompressible, viscous and electrically conducting fluids. Following Chandrasekhar [Proc. Roy. Soc. London Ser. A **229** (1955), 1–19; MR **16**, 968], the authors discuss the pressure fluctuations within isotropic hydromagnetic turbulence. They express pressure fluctuation function $\overline{p'p''}$ in terms of certain scalars Q, H which define the second order correlation tensors Q_{ij} and H_{ij}. *R. P. Kanwal* (University Park, Pa.)

Šikin, I. S. 883
On the general theory of steady motion in relativistic hydrodynamics.
Dokl. Akad. Nauk SSSR **142** (1962), 296–298 (*Russian*); *translated as Soviet Physics Dokl.* **7** (1962), 13–14.
The author investigates the steady adiabatic motion of an ideal gas in relativistic hydrodynamics (within the framework of the special theory of relativity). He considers the entropy to be, in general, different from one streamline to another and shows that each such motion can be reduced to the non-relativistic motion of some special gas (called by the author auxiliary), the same having been already shown by F. I. Frankl' [same Dokl. **123** (1958), 47–48; MR **21** #2465] for the potential plane-parallel motion of an ultrarelativistic gas. *T. P. Anđelić* (Belgrade)

Šikin, I. S. 884
The interpretation of nonstationary relativistic hydrodynamics in Minkowski space.
Dokl. Akad. Nauk SSSR **142** (1962), 564–567 (*Russian*); *translated as Soviet Physics Dokl.* **7** (1962), 15–17.
In this paper the author, pursuing his investigations in relativistic hydrodynamics, considers the adiabatic, but this time, non-steady motion of an ideal gas (again without taking gravitational fields into account). He shows that the non-steady motion in relativistic hydrodynamics can be treated, in the same quantities as in four-dimensional space-time, as nonrelativistic steady motion, assuming that the constant in the Bernoulli equation is zero.
T. P. Anđelić (Belgrade)

Bachmat, Y.; Bear, J. 885
The general equations of hydrodynamic dispersion in homogeneous, isotropic, porous mediums.
J. Geophys. Res. **69** (1964), 2561–2567.
Authors' summary: "The general equations governing the hydrodynamic dispersion of a homogeneous fluid in a homogeneous and isotropic porous medium are described. The flow is saturated and steady, but not necessarily uniform. The equations are first stated in Cartesian coordinates. A general equation in tensorial form is then derived for nonuniform flow in any coordinate system. Finally, the equation is expressed in a simplified form in a curvilinear orthogonal coordinate system of which one family of coordinate curves coincides with the family of streamlines. Some two- and three-dimensional cases serve as illustrations."

Weiss, George H. 886
Simple derivation of the Faxén solution to the Lamm equation.
J. Mathematical Phys. **5** (1964), 675–676.
The problem to which the title refers describes the sedimentation of a substance in an infinite wedge rotating at a constant speed about an axis through the vertex. If the sedimentation coefficient is constant, the concentration satisfies a linear partial differential equation which is solved by use of Hankel transforms. Solutions are also obtained for the non-linear equations when the sedimentation coefficient varies linearly with concentration.
G. Newell (Providence, R.I.)

Lavrik, V. I. 887
Solution of problems of free filtration by the method of successive approximations. (Russian)
Ukrain. Mat. Ž. **15** (1963), 427–431.
This brief note indicates a method for the numerical solution by successive approximations of the problem of free filtration for a domain of a permeable medium bounded by an arbitrary curve. A simple numerical example is also given. *V. N. Constantinescu* (Bucharest)

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 386, 913, 947.

Peřina, J. 888
Une théorie covariante générale des images optiques avec emploi des groupes de Lie. (English and German summaries)
Optica Acta **10** (1963), 333–335.
The coefficients of partial coherence in an optical image are given an interpretation as the metric tensor of a non-Euclidean representation space, this "optical space"

being taken as finite-dimensional by virtue of Gabor's expansion theorem. This makes available the theory of non-Euclidean spaces for the discussion of coherence problems in optical imaging. A few simple examples are given, and the relevance of the theory of Lie groups in a more systematic discussion is briefly explained.
E. H. Linfoot (Cambridge, England)

O'Neill, Edward L.; Walther, Adriaan 889
The question of phase in image formation. (French and German summaries)
Optica Acta **10** (1963), 33–40.
In optical image formation, a one-dimensional analysis involves the functions $F(\nu)$, the aperture function; $f(x)$, its Fourier transform; $\phi(x)=|f(x)|^2$, the image intensity function; and $\Phi(\nu)$, its Fourier transform. The following three questions are important for optical theory. (1) Given a band-limited function of (x), how does one determine all other band-limited distributions of (x) yielding the same intensity distribution $\phi(x)=|f(x)|^2$? "Band-limited" means that $F(\nu)=0$ for $|\nu|>1$. (2) Given a real non-negative function $\phi(x)$, what are the conditions that there exist an amplitude function $f(x)$ such that $\phi(x)=|f(x)|^2$? (3) Given a real non-negative band-limited function $\phi(x)$, does there exist a relation between the real and imaginary part of $\Phi(\nu)$?
The questions are discussed but not answered. However, for diffraction at a slit a unique solution for (1) is demonstrated. *E. W. Marchand* (Rochester, N.Y.)

Walther, Adriaan 890
The question of phase retrieval in optics. (French and German summaries)
Optica Acta **10** (1963), 41–49.
This paper is a sequel to the paper above [#889]. In the present paper it is shown that if $\phi(x)=|f(x)|^2$ with $f(x)$ known to be band-limited, in some cases the function $f(x)$ is unique, but in other cases there is a finite or denumerably infinite number of distinct solutions depending on the nature of the given function $\phi(x)$.
It is also shown that for every non-negative intensity distribution function $\phi(x)$ which is integrable and band-limited, there exists in principle an optical system which will produce this distribution as the image of a star.
E. W. Marchand (Rochester, N.Y.)

Carter, G. W.; Loh, S. C.; Po, C. Y. K. 891
The field of current in a thin wire ring.
Proc. Cambridge Philos. Soc. **60** (1964), 613–619.
Authors' summary: "A simple expression is derived for the magnetic vector potential of current in a thin ring, in terms of the first derivative of toroidal functions of zero order. The axial and radial field components and the mutual inductance between two wire rings are obtained. These expressions are evaluated on a digital computer and the results are summarized in a series of graphs."

de Broglie, Louis 892
Ondes électromagnétiques et photons.
C. R. Acad. Sci. Paris **258** (1964), 6345–6347.
Author's summary: "L'auteur partant de l'action d'une onde électromagnétique sur un récepteur expose comment il conçoit la coexistence des ondes électromagnétiques et des photons dans le domaine hertzien et dans le domaine optique."

Baker, Don A. 893
Trapping of a charged particle in a static magnetic field.
Amer. J. Phys. **32** (1964), 347–354.
The aim of this paper is to dispel the "belief" that the Lorentz equation of motion excludes orbits which correspond to the trapping of charged particles injected externally into a region of time-independent magnetic field. Two such examples are given with very special types of field. In both these cases the orbits discussed are only for the particular choice of the initial velocity. Although in the author's opinion there is "enough freedom in the field specification" that the fields may be made to satisfy Maxwell equations, yet it is questionable so long as it is not shown that this is so. The argument put forward for the "trapping theorem" in the text is by no means convincing. Further, the fact that "Poincaré's recurrence theorem" excludes at most a denumerable set of orbits from its validity does not lead to the conclusion that the theorem "allows" a set of orbits for which it is not valid.
N. D. SenGupta (Bombay)

Hämeen-Anttila, K. A. 894
On the relativistic formulation of the Maxwell equations.
Ann. Acad. Sci. Fenn. Ser. A VI No. 120 (1963), 20 *pp.*
The author proposes a system of notation which allows one to write four-dimensional tensor equations without displaying explicitly all the indices. This method is applied to the Maxwell equations in moving matter and in a gravitational field. The resulting equations are claimed to be "mathematically more elegant" than the usual ones (to which they are, of course, equivalent). {The reviewer disagrees: Mathematical elegance should not be confused with mere typographical simplicity. Actually, in any practical calculation, the author's equations would prove more complicated than the usual ones.} *A. Peres* (Haifa)

Le Naour, Roger 895
Le principe de Huygens-Kirchhoff et la formulation de Kottler. (English summary)
Rev. Optique **43** (1964), 105–115.
Author's summary: "Les formules de Kottler peuvent être établies soit à partir de l'identité vectorielle de Green, soit par la méthode des potentiels vecteurs. Nous analysons dans chaque cas les hypothèses introduites au cours du calcul et montrons qu'en fait elles sont identiques. Si leur justification physique est acceptable, la base logique du calcul semble encore discutable."

Nyborg, P. 896
Approximate relativistic equations of motion for an extended charged particle in an inhomogeneous external electromagnetic field. (Italian summary)
Nuovo Cimento (10) **31** (1964), 1209–1228; *errata, ibid.* (10) **32** (1964), 1131.
The dynamical equations for electric and magnetic dipole moments of an extended charge distribution in an external

inhomogeneous electromagnetic field are studied. An extension from the non-relativistic to relativistic equations of motion is carried out with the use of a projection operator that separates the proper space parts of four-vectors and tensor densities from their time components. A generalization of the non-relativistic description of electric and magnetic moments in the Lorentz-covariant formalism leads to a "covariant electromagnetic moment tensor". The resulting equation of motion then refers to the rate of change of this tensor with proper time. The latter quantity is determined, in turn, by postulating that it must be linearly proportional to a tensor which is taken to be the relativistic generalization of the torque exerted on the charge distribution.

In the errata a correction is given for the linear relationship between the time derivative of the covariant electromagnetic moment tensor and the torque that acts on an extended charge distribution in an inhomogeneous electromagnetic field. *M. Sachs* (Boston, Mass.)

Parr, Wallace E. 897

Upper and lower bounds for the capacitance of the regular solids.

J. Soc. Indust. Appl. Math. **9** (1961), 334–386.

Pólya and Szegö [*Isoperimetric inequalities in mathematical physics*, Princeton Univ. Press, Princeton, N.J., 1951; MR 13, 270] have given an upper bound for the capacity of a closed surface S in terms of an arbitrarily chosen family of level surfaces covering the exterior of S. In the case when S is convex, the author chooses a particular family of such surfaces depending upon an arbitrary function, and minimizes the bound with respect to the arbitrary function. Explicit computations are done for the regular polyhedra. In particular, the upper bound 1.3351 is obtained for the capacity of a cube of side 2. This is a new record low.

H. F. Weinberger (Minneapolis, Minn.)

Rubinowicz, A. 898

Beugungswellen verschiedener Felder im Falle beliebiger einfallender Wellen.

Acta Phys. Polon. **23** (1963), 727–744.

G. Kirchhoff ist es in 1882 gelungen zu zeigen, dass man die Lösung der skalaren Schwingungsgleichung in einem Punkt P in der Gestalt

$$(1)\qquad u(P) = \int_F \mathbf{V}(P, Q)\mathbf{n}\, df_q,$$

wo

$$(2)\qquad \mathbf{V}(P, Q) = \frac{1}{4\pi}\left[\frac{e^{ikr_q}}{r_q}\operatorname{grad}_q u(Q) - u(Q)\operatorname{grad}_q \frac{e^{ikr_q}}{r_q}\right]$$

ist, darstellen kann. Hier bedeutet F eine geschlossene Fläche und Q einen Punkt auf dieser Fläche, r_q ist die Entfernung $\overline{PQ}$, P ist der Beobachtungspunkt und alle anderen Symbole haben die gewohnte Bedeutung. (1) und (2) folgen aus dem Greenschen Satze und sind eigentlich eine Verallgemeinerung des Huygensschen Prinzips auf den Fall, dass F mehr keine Wellenfläche ist. Der Verfasser hat weiter gezeigt [Phys. Rev. (2) **54** (1938), 931–936] dass man im Falle von divergierenden und konvergierenden Kugelwellen die Lösung dieses Kirchhoffschen Ansatzes in eine geometrisch-optische und eine Beugungswelle zerlegen kann. K. Miyamoto und E. Wolf [J. Opt. Soc. Amer. **52** (1962), 615–625; MR **25** #2784a; ibid. **52** (1962), 626–637; MR **25** #2784b] konnten dann weiter beweisen, dass das auch bei ganz beliebigen einfallenden Wellen möglich ist. Zu diesem Zwecke haben die genannten Verfasser die einfallende Welle in eine Fouriersche Reihe zerlegt. Der Verfasser hat ausserdem noch früher gezeigt, dass man das quellenfreie Vektorfeld **V**, das in (2) angegeben ist, mit Hilfe eines Vektorpotentials **W** ausdrücken kann. Ist **W** bekannt, so ist damit das Beugungsproblem gelöst. Die Ausdrücke von Miyamoto und Wolf für **W** können auch nach einer ganz einfachen geometrischen Methode hergeleitet werden, wie das ebenfalls der Verfasser [Acta Phys. Polon. **21** (1962), 451–468] gezeigt hat, der mit dieser Methode das Problem des elektromagnetischen Feldes gelöst hat. Da es sich in diesem Falle um Vektorfelder handelt, so werden die V zu Tensoren und die W zu Tensorpotentialen.

Ziel der vorliegenden Untersuchung ist diese Methode auf andere in der mathematischen Physik auftretende lineare Felder anzuwenden. Untersucht werden diesbezüglich die zweidimensionale Schwingungsgleichung, die dreidimensionale Wellengleichung und die feldfreie Diracgleichung. Weiter betrachtet der Verfasser nach dieser Methode die zwei- und dreidimensionale Potentialtheorie. Hier handelt es sich selbstverständlich nicht mehr um ein Diffraktionsproblem, die Laplacesche Differentialgleichung erhält man jedoch aus der Schwingungsgleichung $\Delta u + k^2 u = 0$, wenn man in der $k = 0$ setzt. Die vorher erhaltenen Resultate werden deshalb für den Fall besprochen, in dem man den Grenzübergang $k \to 0$ vollführt. Zuletzt wird noch das rein mathematische Problem kurz besprochen, dass man die Kirchhoffsche Methode auf eine analytische Funktion $f(z)$ der komplexen Veränderlichen z anwendet. Reel- und Imaginärteil sind dann Lösungen der ebenen Potentialgleichung $\Delta u = 0$.

T. Neugebauer (Budapest)

Rubinowicz, A. 899

Ist die Bezeichnung elektrische oder magnetische Multipolstrahlung gerechtfertigt?

Acta Phys. Polon. **24** (1963), 519–554.

The author introduces first the concepts of G-multipoles and K-multipoles. The G-multipoles are constructed by Maxwell's method of successive differentiation. Physically speaking, each differentiation generates a dipole source, and the sequence of l differentiations generates a source of 2^l poles in a definite arrangement, giving rise to the field of a G-multipole of order l. The K-multipoles of order l are spherical harmonic multipoles which can be written as $Y_l(\vartheta, \varphi)R_l(r)$. The latter can be expressed as a linear combination of G-multipoles of order l. For the monochromatic solutions of the scalar wave equation a K multipole of order l can be constructed as a linear superposition of G-multipoles of order l, $l-2$, $l-4$, $\cdots$; thus the field of the K-multipoles can be thought of as being generated from source systems which contain 2, 2^{l-2}, 2^{l-4}, $\cdots$ point sources each, ordered in a definite fashion. This is the physical content of the usual statement that a scalar field is generated by multipoles.

For the Maxwell field two complications arise: (1) we need to characterize two coupled fields; (2) the two fields are vector fields. Here the construction of an electric or magnetic G-multipole field by differentiation can be visualized, not as generating a distribution of poles, but

a distribution of electric and magnetic current elements. An electric and magnetic K-multipole can again be constructed as a superposition of G-multipoles. The author discusses with great clarity the geometry of the current systems underlying the different order multipoles. Since in this case currents are distributed, the specification of the sources alone is not sufficient, unless a convention is established according to which the sources are connected to form current systems. The author gives such a convention, and stresses the fact that without this convention a Maxwell field is not specified uniquely by a mutipole distribution; the latter can only be accomplished by a multicurrent distribution, and thus, generally speaking, the expression multipole radiation is inappropriate.

N. L. Balazs (Stony Brook, N.Y.)

Marziani, Marziano 900

Sull'energia irradiata da un dipolo in presenza di una terra non omogenea.

Boll. Un. Mat. Ital. (3) **19** (1964), 3–11.

A discussion of the propagation of radio waves, generated by a vertical electric dipole, over an inhomogeneous flat earth. The electric properties of the latter may vary across the surface of the earth according to any given distribution of the associated refractive index n. The author starts from the rigorous expression obtained for the amplitude Π of the Hertzian vector in the air (determining the field components there) when the distribution of the normal derivative $\partial\Pi/\partial z$ is known all over the earth's surface. An application of Leontovich's approximate boundary condition (connecting Π and $\partial\Pi/\partial z$ at this surface, taking account of the local values of n) enables one to pass to a corresponding representation depending on the distribution of Π itself. This latter function is next approximated by its value holding in the case of a perfectly conducting earth. The final expression (13), then arrived at, agrees with results obtained by other authors.

H. Bremmer (Eindhoven)

Streifer, William 901

Completely coherent radiation.

Ann. Physics **27** (1964), 72–78.

Mandel and Wolf [J. Opt. Soc. Amer. **51** (1961), 815–819] and Parrent [ibid. **49** (1959), 787–793; Opt. Acta **6** (1959), 285–296; MR **22** #8998] have proposed certain definitions of, and criteria for, the concept of coherence as applied to radiation. These involve the use of an autocorrelation function or "mutual coherence function". The author points out a disturbing situation: two waveforms may be perfectly coherent according to the criterion of Mandel and Wolf, and yet their observed waveforms might be quite different in appearance. He also considers the decomposition or factoring of the mutual coherence functions, with application to coherent radiation as distinguished from strictly monochromatic radiation.

The author derives a general form for completely coherent fields and shows that these fields satisfy the equations of geometrical optics, when arbitrary source fluctuations are permitted. Among the solutions only the spherical waves are physically allowable for sources confined to a finite volume. The decomposition of an incoherent source into infinitesimal sources producing coherent fields is equivalent to stating that these sources produce spherical waves, $r^{-1}f(t-r/c)$.

Wallace L. Anderson (New York)

Holliday, D. 902

Statistical properties of the black body radiation field.

Phys. Lett. **8** (1964), 250–252.

The black-body radiation field is considered as a quantized field excited by a c-number source which is assumed to be a gaussian random process. The N-point correlation function for the field is evaluated. For N odd it vanishes and for N even is a sum over products of two-point correlation functions. *K. J. Le Couteur* (Canberra)

Borovikov, V. A. 903

The Green's function for a diffraction problem on a polyhedral angle. (Russian)

Dokl. Akad. Nauk SSSR **151** (1963), 251–254.

In several earlier papers [same Dokl. **144** (1962), 527–530, ibid. **144** (1962), 743–746; ibid. **148** (1963), 545–548] the author has developed a theory of the Green's function for the diffraction problem in certain regions having corners. Starting from results of Petrašen' and his students [Leningrad. Gos. Univ. Učen. Zap. Ser. Mat. Nauk **32** (1958), 5–70; MR **22** #1321] on a plane angle, he successively treats the cases of a convex polygon, a dihedral angle, a prism and, in the present work, a polyhedral angle. His immediate object here is to study the asymptotic behavior of the Green's function, i.e., a solution of the wave equation $(\Delta+k^2)G(\mathbf{x}, \mathbf{a}, k)=\delta(\mathbf{x}-\mathbf{a})$, as $k\to\infty$, where $\mathbf{x}$ and $\mathbf{a}$ are the observation and source points, respectively, in a spherical system of coordinates with center at the apex of the polyhedral angle. For this he needs to investigate the singularities of the Green's function for the plane-wave case of the corresponding nonstationary problem. These singularities are determined by the "ray decomposition" of the non-analytic part of the latter function. The determination turns out to be possible in the case of Dirichlet or Neumann boundary conditions. Hence, under the limitations noted, the results are applicable to diffraction by a polyhedron.

R. N. Goss (San Diego, Calif.)

Arnush, Donald 904

Electromagnetic scattering from a spherical nonuniform medium. I. General theory.

IEEE Trans. Antennas and Propagation **AP-12** (1964), 86–90.

This paper consists primarily of a review of the formulation of the electromagnetic scattering problem in spherical coordinates. The field is represented in terms of two scalar functions (essentially radial components of the Hertz vectors), and the scattered amplitude is written both in terms of phase shifts and in terms of the solutions to two scalar scattering problems. It is pointed out that the radar (backscattering) cross-section may be considerably different even at short wavelengths from that calculated by a scalar wave approximation if the scattering object contains a surface of vanishing dielectric constant.

E. T. Kornhauser (Providence, R.I.)

Margulies, R. S.; Scarf, F. L. 905
Electromagnetic scattering from a spherical nonuniform medium. II. The radar cross section of a flare.
IEEE Trans. Antennas and Propagation **AP-12** (1964), 91–96.

The formalism developed in the preceding paper [Arnush, #904] is applied to the calculation of the radar cross-section of a spherically symmetric plasma whose density varies as r^{-2}. Thus there exists a radius at which the effective dielectric constant vanishes. The exact result is calculated and compared with several approximate results, including the cross-section for scalar wave scattering, showing discrepancies which may be orders of magnitude. A refined Born approximation technique is shown to yield answers of useful accuracy for this case, however. *E. T. Kornhauser* (Providence, R.I.)

Twersky, Victor 906
On propagation in random media of discrete scatterers.
Proc. Sympos. Appl. Math., Vol. XVI, pp. 84–116. *Amer. Math. Soc., Providence, R.I.*, 1964.

This paper deals with the propagation of plane waves through a random configuration of scatterers. The wave function Ψ describing the scattered field is a random function and methods for finding $\langle\Psi\rangle$, the expected value of Ψ, and $\langle|\Psi|^2\rangle$ are discussed. Two kinds of representations for Ψ are used. The first, called the "compact representation", is defined by a set of simultaneous integral equations for Ψ. The second, called the "expanded" series representation, is defined by the formal solution of these integral equations by iteration. These representations suggest two approaches to calculating $\langle\Psi\rangle$. The first is to average the compact representation and then simplify the functional equations for $\langle\Psi\rangle$, usually by heuristic arguments, to a form that can be solved. The second is to average the expanded series representation and attempt to sum the (simpler) averaged series. The author discusses a number of approximate solutions of these two types. In a final section the methods are illustrated by application to a simple special case. *C. H. Wilcox* (Madison, Wis.)

Bazer, J.; Hochstadt, H. 907
Diffraction of scalar waves by a circular aperture. II.
Comm. Pure Appl. Math. **15** (1962), 1–33.

Part I (by J. Bazer and A. Brown) appeared in New York Univ. Inst. Math. Sci. Div. Electromagnetic Res. Res. Rep. No. EM-144 (1959); IRE Trans. Antennas and Propagation **AP-7** (1959), special suppl., S12–S20.

The authors are concerned with diffraction problems associated with a circular aperture in a "soft" and a "rigid" plane infinite screen. Emphasis is put on the case of non-normal plane wave excitation. Such an excitation may be decomposed into an infinite series of angular harmonics about the cylindrical axis of symmetry of the aperture. The authors derive regular Fredholm integral equations of the second kind for each amplitude of these harmonics and find it possible to calculate the associated truncated Neumann series for these amplitudes in the case that wave number times radius of aperture is small in some sense. The Fredholm integral equations are similar to those given by D. S. Jones for normal incidence [Comm. Pure Appl. Math. **9** (1956), 713–746; MR **20** #4411] and by R. C. MacCamy and the reviewer [Z. Angew. Math. Phys. **11** (1960), 249–264; MR **23** #B2301] for non-normal incidence. The various different non-homogeneous terms in the Fredholm equations to which the authors allude are due to the form to which one chooses to convert the basic integral equations.
A. E. Heins (Ann Arbor, Mich.)

Chen, Yung Ming 908
Diffraction by a smooth transparent object.
J. Mathematical Phys. **5** (1964), 820–832.

The author extends the geometric theory of diffraction of J. B. Keller [Proc. Sympos. Appl. Math., Vol. 8, pp. 27–52, McGraw-Hill, New York, 1958; MR **20** #840] to problems involving dielectric diffracting bodies. The analysis is restricted to two-dimensional scalar fields and to smooth objects of large size but arbitrary shape. The author discusses two types of rays: geometric and diffracted. The geometric rays are those reflected and refracted by the body, plus the direct ray from the source, while the diffracted rays are those which give rise to surface rays on the object. The amplitudes of each type are determined from a knowledge of local reflection, transmission and diffraction coefficients and the principle of conservation of energy; the phases are found from a consideration of the local phase velocities. The complete solution is the sum of rays of all types at the field point. The solution for a circular cylinder (of dielectric constant greater than that of the surrounding medium) is compared to the asymptotic solution derived from an exact analysis. Complete agreement in form is obtained. The various "canonical" reflection, transmission and diffraction coefficients are found by a comparison of terms.

The author says that the method can be extended easily to 3-dimensional and vector field problems.
R. A. Hurd (Ottawa, Ont.)

Chen, Yung Ming 909
Wave propagation in stratified random media.
J. Res. Nat. Bur. Standards Sect. D **68D** (1964), 743–748.

This is an application of the author's recent paper [J. Mathematical Phys., in press] to a few of the problems of wave propagation in stratified random media. For a small inhomogeneity, the average value of the wave function, $\langle u\rangle$, is investigated by utilizing the effective refractive index, and also two illustrations are given using the special profiles of refractive index given by Landau and Lifshitz [*Quantum mechanics*, pp. 72–77, Addison-Wesley, Reading, Mass., 1958; MR **19**, 1230]. For a large inhomogeneity, the problem is treated by subdividing the medium into many parallel homogeneous layers of random media and utilizing the reflection and transmission coefficients previously derived by the author. It is finally concluded that, in each case, the small randomness has no drastic and unexpected effects on the behavior of the wave motion in the medium considered, and therefore that our reliance on the results obtained by a non-random approach is increased. It may be noticed, however, that the statistical problem of wave propagation in random media is not solved by obtaining the average value of the wave function $\langle u\rangle$ alone; indeed, there is also much interest in $\langle|u|\rangle$, $\langle|u^2|\rangle$ and $\langle u^*(x)u(x')\rangle$.
K. Furutsu (Boulder, Colo.)

Malmstadt, H. V.; Enke, C. G. 910

★Electronics for scientists. Principles and experiments for those who use instruments.

With the assistance of E. C. Toren, Jr.

W. A. Benjamin, Inc., New York, 1963. xi + 619 *pp.* $11.50.

According to the introduction, this book is designed for use at the junior or senior college level, or for self-study by practicing scientists who need an understanding of electronic circuits in their work.

The book consists of chapters on electrical measurements, power supplies, amplifiers, oscillators, servo systems, operational amplifiers, and electronic switching. In presenting these subjects, the authors rely on a qualitative and descriptive approach, using a minimum of formulas, and a maximum of illustrations. The principles of physics, or in particular of the theory of electricity, underlying all electronic measurements are not explained. Instead, the subject is developed strictly from a user's point of view, a user, moreover, who has had not much more than a high school physics education.

In these respects the present book differs from almost all other textbooks on electronics. Perhaps it is significant that many illustrations are taken from circuit diagrams of a well-known manufacturer of do-it-yourself electronic kits. The assembly of such kits requires no knowledge of electronics, but simply the ability to follow instructions, frequently in the form of three-dimensional perspective drawings.

Another unusual practice followed by the authors is the introduction and definition of basic concepts and quantities in footnotes. Thus in a long footnote on page 8, the authors try to explain what appears to them to be a difference between the concepts of "voltage" and "potential difference". Similarly, such concepts as resistance, current, shunt, diode, short circuit, parallel circuit, capacitance, impedance, phase, transformer, direct current, and many others, are introduced in the form of footnotes, frequently inaccurately, or in the form of synonyms which do not explain.

Each chapter is followed by directions for relatively standard experiments. In the absence of a real body of physical explanation, it is difficult to imagine the degree of understanding that can be gained by performing them. In the opinion of this reviewer, the title of this book is highly misleading. It is not suitable for use by scientists interested in acquiring a modicum of basic understanding.

W. Franzen (Boston, Mass.)

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See also 833, 849, 851.

Heaslet, Max A.; Fuller, Franklyn B. 911

Temperature distribution on conducting cylindrical shells including the effects of thermal radiation.

J. Soc. Indust. Appl. Math. **12** (1964), 136–155.

A thin cylindrical shell with cross-section specified as a circular arc is exposed to external incident heat radiation. Using the Stefan-Boltzmann law, the exact integro-differential equation for the steady-state temperature distribution T is derived and is immediately linearized assuming small temperature variations. The solution is given in closed form and is used to obtain some numerical results. The accuracy of the approximation is discussed and an improved approach (linear in T^4) is suggested.

H. Parkus (Vienna)

Dicker, Daniel; Friedman, M. B. 912

Solutions of heat-conduction problems with nonseparable domains.

Trans. ASME Ser. E. J. Appl. Mech. **30** (1963), 493–499.

Authors' summary: "A method is presented for obtaining eigenfunctions of and solutions to the transient heat-conduction equation for a wide class of three-dimensional convex hexahedral domains and two-dimensional convex quadrilateral domains having straight or curved boundaries for which separation of variables cannot be applied. The method is employed to solve for the temperature distribution in a trapezoidal domain, initially at zero temperature, the boundaries of which are subjected to suddenly applied values at the initial constant. The solution is obtained in the form of a series and an examination of successive terms indicates fairly rapid convergence; it is found that the one-term solution yields almost as good values as a four-term solution, which is significant since the former is obtained with little effort. An independent method is utilized for obtaining the steady-state solution, i.e., $t\to\infty$, and it is found that all approximations by the former method are substantially equal to the correct value for this case."

QUANTUM MECHANICS

See also 282a–b, 301, 468, 486.

Trammel, G. T. 913

Aharonov-Bohm paradox.

Phys. Rev. (2) **134** (1964), B1183–B1184.

This paper deals with the Aharonov-Bohm paradox to the extent that a classical particle moving in a field free region outside a long solenoid can affect the solenoid whilst not experiencing any effects itself. This is an example of the well-known fact that action and reaction are not equal and opposite when electromagnetic fields are present. The paradoxical nature of this effect does not seem to be in the same category as that of the Aharonov-Bohm paradox.

C. A. Hurst (Adelaide)

Aharonov, Y.; Bohm, D. 914

Answer to Fock concerning the time energy indeterminacy relation.

Phys. Rev. (2) **134** (1964), B1417–B1418.

In a previous paper [same Rev. (2) **122** (1961), 1649–1658; MR **22** #11832] the authors have shown that, contrary to a widespread view, the energy of a physical system can be accurately measured within an arbitrarily short time interval. Their argument was criticized by Fok [Ž. Èksper. Teoret. Fiz. **42** (1962), 1135–1139; MR **26** #2174] on the ground that their experimental setup included an external field whose time behavior had to be sharply defined. This, according to Fok, was a logical error (petitio principii) because the time-energy uncertainty of the external field was ignored at the outset.

In the present paper, it is shown that Fok's criticism was unjustified, because the over-all energy transfer from the external field to the measured system can be made arbitrarily small. *A. Peres* (Haifa)

Aharonov, Yakir; Bergmann, Peter G.; Lebowitz, Joel L. 915

Time symmetry in the quantum process of measurement.

Phys. Rev. (2) **134** (1964), B1410–B1416.

It is often stated that the measurement process in quantum theory is not time-symmetric, contrary to the dynamical laws of motion, which are entirely symmetric with respect to the direction of time. It has even been suggested [J. von Neumann, *Mathematical foundations of quantum mechanics*, Ch. 5, Princeton Univ. Press, Princeton, N.J., 1955; MR **16**, 654] that thermodynamic irreversibility enters into quantum theory through the process of measurement.

This widespread view is challenged in the present paper. The authors consider sequences of observations which are characterized by both the initial and the final states of the physical system, and formulate the theory of measurement in a conspicuously time-symmetric manner. They thus show that the basic laws of quantum physics, including those referring to measurements, are as completely time-symmetric as those of classical physics.

The usual time-asymmetry of quantum measurements enters only through "coherence destroying" manipulations, which eliminate the effects of post-selection in most experimental setups. However, it is in principle possible to have experiments based on post-selection only (the pre-selection coherence having been destroyed). In that case, the usual assignment of a state vector to a physical system on the basis of the most recent preceding observation is no longer justified. One should rather use the next following observation. This indicates that the quantum state of a system, though undoubtedly containing some elements of "reality" independent of any observer, also has subjective aspects.

Finally, the authors note that, in our universe, the origins of all kinds of radiations are spatially and temporally concentrated, and their destinations are not. As a consequence, it is extremely difficult to avoid preselection completely, while the absence of post-selection is the most common situation. Thus, in most cases, measurements in quantum theory are indeed irreversible. However, this is not a basic microscopic law, but only a corollary of the general irreversible behavior of our whole universe. *A. Peres* (Haifa)

Machado, W. M.; Schützer, W. 916

Bopp's formulation of quantum mechanics & the Einstein-Podolski-Rosen paradox.

An. Acad. Brasil. Ci. **35** (1963), 27–35.

This article starts with a helpful exposition and summary of Bopp's [Ann. Inst. H. Poincaré **15** (1956), 81–112; MR **19**, 215] method of constructing positive probability densities capable of representing pure or mixed quantum-mechanical states. This is extended to give basic results for a two-body system, and some analysis of the Bopp formalism corresponding to eigenstates of observables. The application to the Einstein, Podolsky and Rosen paradox leads to the observation that the paradox disappears in the Bopp description because an ensemble description is not a "complete" one. This resolution could, of course, have been achieved equally well through any other probabilistic description, such as that by N. Wiener and the reviewer [Nuovo Cimento (10) **2** (1955), suppl., 982–1003; MR **20** #681a]. *A. Siegel* (Boston, Mass.)

Varadarajan, V. S. 917

Probability in physics and a theorem on simultaneous observability.

Comm. Pure Appl. Math. **15** (1962), 189–217.

Almost thirty years ago Kolmogorov set down the axioms of probability theory [A. N. Kolmogorov, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, J. Springer, Berlin, 1933]. In spite of their tremendous successes, there remains the problem of understanding the phenomenological background of these axioms. We now have a sound explanation of the motivation behind them. The explanation consists essentially in noting that the set of events associated with an experiment is a Boolean σ-algebra, and may be roughly thought of (with the aid of the Loomis representation theorem) as a Boolean σ-algebra of subsets of some space (indeed the space Ω in probability theory). However, the probability so formulated is not adequate to handle situations in quantum physics. There is nothing in the theory that corresponds to the uncertainty principle of Heisenberg. The interaction of probability and quantum theory is fundamentally different from that in statistical mechanics. In recent years such generalized probability theories were considered, e.g., Segal [Amer. J. Math. **76** (1954), 721–732; MR **16**, 149] and Mackey [Amer. Math. Monthly **64** (1957), no. 8, part II, 45–57; MR **20** #2608; *The mathematical foundations of quantum mechanics*, Benjamin, New York, 1963; MR **27** #5501]. Mackey has taken the decisive step toward a systematic exposition of these ideas. The present paper contains two features that make it more complete than Mackey's: (1) Results dealing with simultaneous observability are established here in a more general context; (2) This paper emphasizes more strongly the implication of these ideas toward a more general formulation of the basic concepts of probability theory. It is pointed out that there is a more general way of formulating the fundamental axioms which subsumes the probabilistic aspects of quantum physics and that there arise some interesting mathematical problems. More specifically, the author obtains results which, roughly speaking [cf. Theorem 3.4], say that joint distribution of several observables may be defined only when the observables discussed are simultaneously observable [cf. also von Neumann, *Mathematical foundations of quantum mechanics*, pp. 211–230, Princeton Univ. Press, Princeton, N.J., 1955; MR **16**, 654]. Other interesting results are also obtained. *H. Tong* (Middletown, Conn.)

Motz, Lloyd; Selzer, Adolph 918

Quantum mechanics and the relativistic Hamilton-Jacobi equation.

Phys. Rev. (2) **133** (1964), B1622–B1624.

The authors' attempt to show that the relativistic Klein-Gordon operator, applied to a wave function $e^{iS/\hbar}$, is equivalent either to quantum mechanics or, provided in addition the divergence of an energy-momentum 4-vector is

equated to zero, to the classical relativistic Hamilton-Jacobi equation. However, in the latter case the consistency of the two postulated equations is not established except for the trivial case of free particles.

K. J. Le Couteur (Canberra)

Edwards, S. F.; Gulyaev, Y. V. 919

Path integrals in polar co-ordinates.

Proc. Roy. Soc. Ser. A **279** (1964), 229–235.

Authors' summary: "Functional integrals of the usual diffusion type in $x(t)$, $y(t)$, $z(t)$ are discussed when transformed into polar co-ordinates $r(t)$, $\varphi(t)$, $\vartheta(t)$. It is found that the functional integration can be performed directly but the limiting process of taking $\Delta t \to dt$ is more complicated than that encountered in normal integral and differential calculus; in particular, terms of order $(\Delta t)^2$ cannot be neglected relative to terms of order (Δt)."

Smith, F. J. 920

The numerical evaluation of the classical angle of deflection and of the J.W.K.B. phase shift.

Physica **30** (1964), 497–504.

Author's summary: "Numerical formulae are derived for the classical angle of deflection, χ, and for the J.W.K.B. approximation to the phase shift, η. Based on Gauss-Mehler quadratures, these formulae are in general more accurate and simpler to use than other formulae for χ and η. For example, adopting the usual notation of atomic scattering theory, χ is given by

$$\chi = \pi\left\{1 - \frac{2b}{nr_m}\sum_{j=1}^{n/2} a_{n/2-j+1}g(a_j)\right\}$$

in which

$$a_j = \cos\frac{2j-1}{2n}\pi \text{ and } g(x) = \left[1 - \frac{V(r_m/x)}{E} - \frac{b^2x^2}{r_m^2}\right]^{-1/2}.$$

In the case of a repulsive potential this formula is often accurate to five and more decimal places when n is no larger than 6. Similar formulae and results are obtained for the phase shift."

Dollard, John D. 921

Asymptotic convergence and the Coulomb interaction.

J. Mathematical Phys. **5** (1964), 729–738.

The usual asymptotic conditions for scattering systems is not valid for a Coulomb potential because of the long range of this potential. Nevertheless, a differential scattering cross-section exists even for such a potential. In this paper one proposes a modified asymptotic condition which is satisfied for Coulomb-type interactions even for many-body systems. *J. M. Jauch* (Geneva)

Misra, B.; Speiser, D.; Targonski, G. 922

Integral operators in the theory of scattering.

Helv. Phys. Acta **36** (1963), 963–980.

von Neumann's spectral characterization [*Charakterisierung des Spektrums eines Integraloperators*, Actualités Sci. Indust., No. 229, Hermann, Paris, 1935] of a self-adjoint integral operator of Carleman type is extended to a wider class of operators including all normal operators, and related topics are discussed. The connection of such a mathematical problem with scattering theory is illustrated. In particular, it is shown that the scattering operator minus the identity is an operator of Hilbert-Schmidt class on the energy shell if and only if the total cross-section is finite. This is shown to be actually the case if the potential satisfies $\int_0^\infty |rV(r)|\, dr < \infty$.

S. T. Kuroda (Tokyo)

Coester, F. 923

Systematic approximations for the single-channel scattering amplitude.

Phys. Rev. (2) **133** (1964), B1516–B1519.

Author's summary: "The scattering amplitude for two-particle single-channel scattering may be computed by inverting a Hilbert-space operator $1-K$, where K is completely continuous and depends on the real energy as a parameter. This inversion can always be accomplished by obtaining a finite number of eigenvalues and eigenfunctions of K together with a modified Born expansion which is guaranteed to converge. Another method consists in the inversion of a finite matrix, the elements of which are computed by a convergent perturbation series."

J. M. Cook (Argonne, Ill.)

Kronfli, N. S. 924

Unitarity and analyticity in the complex energy plane of general scattering amplitudes. (Italian summary)

Nuovo Cimento (10) **30** (1963), 1465–1477.

Assuming the analyticity of scattering amplitudes as functions of the barycentric energy squared and the unitarity condition, the author analyzes the types of singularities arising at physical thresholds. This problem reduces, in the framework of this formulation, to the study of the analytical properties of the relativistic phase space. The author concludes that the amplitudes develop branch points at physical thresholds, and that the branch points are of the square root type or of the logarithmic type according to whether the number of particles in the intermediate states is even or odd, respectively. This conclusion is then shown to be consistent with the threshold behavior of the Feynman amplitudes studied by Nakanishi, Shimamoto, and also by Landau.

K. Nishijima (Champaign, Ill.)

de Alfaro, V.; Predazzi, E.; Rossetti, C. 925

A perturbative expansion for large angular momenta. (Italian summary)

Nuovo Cimento (10) **30** (1963), 522–535.

Authors' summary: "A perturbative formalism is presented, in the framework of potential scattering, in which the usual energy denominators are substituted by angular-momentum denominators. The trick is accomplished by expanding the partial wave function over free wave functions of different angular momenta at the same energy. A perturbative expansion is also given for $f_0(\lambda, -k)/f(\lambda, -k)$, f_0 and f being the free and the exact Jost functions, respectively. The convergence of the solution improves for high angular momenta."

C. G. Bollini (Buenos Aires)

Favella, L. 926
On the connection between S-matrix and the Neumann series expansion of the wave function. (Italian summary)
Nuovo Cimento (10) **31** (1964), 890–903.

The partial wave S-matrix in potential scattering is expressed as a quotient of two series involving the coefficients a_n of the Neumann series expansion of the regular wave function. Besides the usual conditions, the potential satisfies $V(r)=U(r)/r$, where $U(r)$ is an entire function of r. A Kontorovich-Lebedev transform of the same wave function yields the same expansion. The techniques and results are similar to those obtained by other authors (e.g., the reviewer, E. Predazzi and C. Rossetti [#925 above; Nuovo Cimento (10) **31** (1964), 42–55; MR **28** #4801]). *V. de Alfaro* (Princeton, N.J.)

Blažek, Mikuláš 927
Determination of the potential by the analytic properties of the scattering amplitude. (Russian. English summary)
Mat.-Fyz. Časopis Sloven. Akad. Vied **13** (1963), 147–175.

A method for solving the "inverse problem" of nonrelativistic scattering theory is presented. It is shown that if there are no bound states, then for any physical value of the angular momentum l, one can determine the potential V as

$$V(r) = -2\,dK_l(r,r)/dr,$$

where $K_l(r,\rho)$ is the solution of the linear integral equation

$$K_l(r,\rho)+\int_r^\infty K_l(r,t)F_l(t,\rho)\,dt+F_l(r,\rho) = 0.$$

The kernel is given by

$$F_l(r,\rho) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} dk[1-S_l(k)] \times[-krh_l^{(2)}(-kr)][-k\rho h_l^{(2)}(-k\rho)]$$

provided that the typical threshold condition for scattering by a short-range potential holds, i.e.,

$$\lim_{k\to 0}[1-S_l(k)]/k^{2l+1} < \infty.$$

Here $S_l=\exp(2i\eta_l)$, where η_l is the scattering phase shift, and $h_l^{(2)}$ is the spherical Hankel function of the second kind. For $l=0$ the integral equation coincides with one employed by Agranovič and Marčenko [Z. S. Agranovič and V. A. Marčenko, *The inverse problem in the quantum theory of scattering* (Russian), Izdat. Har'kov. Gos. Univ., Kharkov, 1960; English transl., Gordon and Breach, New York, 1963; MR **28** #5696]. The equivalence of the Gel'fand-Levitan equation and the Agranovič-Marčenko equation was proved by Faddeev [L. D. Faddeev, Uspehi Mat. Nauk **14** (1959), no. 4 (88), 57–119; MR **22** #1344]. *R. L. Warnock* (Chicago, Ill.)

Levintov, I. I. 928
Determination of the real part of the scattering amplitude for an asymptotic power law behavior of the imaginary part.
Ž. Èksper. Teoret. Fiz. **45** (1963), 1275–1277 (*Russian. English summary*); *translated as Soviet Physics JETP* **18** (1964), 876–877.

Some considerations are made regarding the possibility of fitting the experimental data of the scattering of particles and antiparticles. To this aim the high-energy behaviour is approximated by a Regge-like formula for the imaginary part of the scattering amplitude, and a dispersion relation with one subtraction is written down to approximately give the real part of the scattering amplitude. *E. Predazzi* (Turin)

Patashinskiĭ, A. Z. [Patašinskiĭ, A. Z.]; 929
Pokrovskiĭ, V. L.; Khalatnikov, I. M. [Halatnikov, I. M.]
An investigation of the S matrix in the complex angular momentum plane in the quasi-classical case.
Ž. Èksper. Teoret. Fiz. **45** (1963), 760–771 (*Russian. English summary*); *translated as Soviet Physics JETP* **18** (1964), 522–529.

Authors' summary: "An effective method is developed for the study of the scattering matrix $S(\nu)$ in the complex angular momentum plane $l=\nu-\frac{1}{2}$ in nonrelativistic quantum mechanics. The method is based on a study of properties of the quasi-classical solutions of the radial Schrödinger equation in the complex radius-vector plane." *E. Predazzi* (Turin)

Stichel, P. 930
Bethe-Salpeter structure of the non-relativistic three-nucleon problem.
Nuclear Phys. **52** (1964), 189–212.

An exact integral equation for the off-mass shell nucleon-deuteron scattering amplitude in the framework of the non-relativistic Schrödinger theory with only two-nucleon forces is derived. In order to be able to give a graphical representation of the equations (Feynman diagrams), the author has worked with matrix elements of time-ordered products of nucleon field operators (Bethe-Salpeter (BS) amplitudes). The BS equation for the elastic nucleon-deuteron scattering amplitude is derived by starting with an appropriate continuation of the scattering amplitude for the energies of the outgoing particles being off the mass shell, and by making use of a reduction formula. The kernel of the integral equation can be expressed in terms of the deuteron vertex function and that part of the off-mass shell two-nucleon scattering amplitude which is free from one-particle singularities in the energy. The integral kernel has therefore no one- or two-particle singularities in the total energy.

The inelastic nucleon-deuteron scattering amplitude is also studied and is found to consist of three parts: the deuteron break-up off the mass shell, initial state scattering due to the elastic nucleon-deuteron scattering amplitude and the final state scattering due to multiple two-nucleon scattering which is free from one- and two-particle singularities in the total energy.

A scheme of unitary approximations is constructed by making a kind of impulse approximation for the kernel of the BS equation, fulfilling unitarity in the elastic channel and introducing corresponding approximation for the inelastic scattering amplitude. The approximation discussed is for high energies and low momentum transfer. These approximations may be considered as a unitary generalization of the impulse approximation. Finally an approximate three-nucleon scattering amplitude is constructed. *B. P. Nigam* (Buffalo, N.Y.)

Hartle, James B. 931

Watson-Sommerfeld transformation for many-particle scattering amplitudes.

Phys. Rev. (2) **134** (1964), B612–B619.

In the framework of potential scattering, it is shown that if the partial wave scattering amplitude can be continued as an analytic function of the total angular momentum in such a way that a Watson-Sommerfeld transformation of the full amplitude is possible, then this is the only continuation with this property; moreover, the leading singularity determines the behaviour of the amplitude for large values of any single scattering angle.

The total angular momentum L not only determines the value of a parameter in the coupled equations for the partial wave amplitudes, but also determines how many equations there are, and so cannot be made complex at first sight. The author overcomes this difficulty by introducing an infinity of equations which become decoupled into physical and unphysical amplitudes when L is an integer, but which retain a meaning when L is complex. The author does not attempt to prove that with complex L his equations possess solutions, however. The treatment can be generalized to the case where exchange forces are present, and is illustrated by showing how sequences of nuclear rotational levels can be described by Regge trajectories. *R. F. Streater* (London)

Hartle, James B. 932

Complex angular momentum in three-particle potential scattering.

Phys. Rev. (2) **134** (1964), B620–B629.

From the author's summary: "In this paper we discuss the continuation of a class of three-particle potential scattering amplitudes to complex values of the total angular momentum. The class of amplitudes considered are those which describe a scattering in which a given pair of the particles is bound in the initial and final states. A nonrigorous discussion indicates that, except for simple kinematic factors, the only singularities present in the complex angular momentum plane are dynamical poles and possible isolated essential singularities. The Watson-Sommerfeld transformation of the full amplitude can be performed to display its large momentum-transfer behavior." *R. F. Streater* (London)

Kawai, T.; Masuda, N. 933

Trajectories of the resonance poles in the multi-channel scattering amplitudes.

Nuovo Cimento (10) **32** (1964), 243–247.

This is one more paper in the many that have been written on problems raised by Oakes and Yang [Phys. Rev. Lett. **11** (1963), 174–177] concerning the octet theory of baryon isobars. The present authors use the theory of p-wave chain approximation for two-channel reactions to study the existence of additional poles in the neighbourhood of the original resonance poles (used by Oakes and Yang to analyse the baryon resonances) on different sheets. *S. F. Tuan* (Lafayette, Ind.)

Hahn, Yukap; O'Malley, Thomas F.; Spruch, Larry 934

Static approximation and bounds on single-channel phase shifts.

Phys. Rev. (2) **128** (1962), 932–942.

Authors' summary: "A single-channel scattering process with the set of quantum numbers C is completely characterized by a phase shift, η_C. A common approximation in the determination of η_C for the scattering of a particle by a compound system is to assume that, apart from the possibility of an exchange of the incident particle with an identical target particle, the target is unaffected by the incident particle. The incident particle is then scattered by the static potential generated by the target in its ground state. The phase shift determined in this approximation, to be called $\eta_C{}^P$, can be calculated for a number of scattering processes. Let H represent the Hamiltonian of the entire system, incident particle plus target, let E_{T0} be the ground-state energy of the target, and let $E'_C{}^Q$ be the smallest energy for which $E'_C{}^Q + E_{T0} - H$ is a negative definite operator in the space in which the given quantum numbers are C and in which the ground state of the target is projected out. Utilizing the generalized optical potential formalism due to Feshbach and others, it can then be shown that $\eta_C > \eta_C{}^P$ if the incident energy is less than $E'_C{}^Q$. (The bound is probably valid for higher energies, perhaps for all energies for which the process remains a single-channel process. If so, however, the difference between η_C and $\eta_C{}^P$ for these higher energies will generally be large, of the order of a multiple of π, and the bound will not be immediately useful. We will, therefore, be concerned primarily with incident energies E' less than $E'_C{}^Q$.) Furthermore, as one allows for more and more virtual excitation of the target system, the approximate phase shift is guaranteed to improve if E' is less than $E'_C{}^Q$, and $E'_C{}^Q$ will itself increase. Applications are given to the scattering of electrons and of positrons by hydrogen atoms." *S. Rosendorff* (Haifa)

De Tollis, B. 935

Dispersive approach to photon-photon scattering. (Italian summary)

Nuovo Cimento (10) **32** (1964), 757–768.

Author's summary: "The scattering of light by light is considered with the aid of double dispersion relations. The spectral functions are easily obtained, in the lowest perturbative order, by means of the Cutkosky rules, and a Mandelstam representation is explicitly written. This approach is very suitable for the symmetry properties of this scattering reaction and allows one to obtain the differential cross-section in a straightforward way for all the values of the energy and of the angle. In order to give a detailed exposition of the method we consider in this paper only that amplitude which describes the scattering of linearly polarized photons in a direction perpendicular to the scattering plane." *F. R. Halpern* (La Jolla, Calif.)

Peierls, Ronald F.; Trueman, T. L. 936

Restrictions imposed by the optical theorem on exchanged quantum numbers.

Phys. Rev. (2) **134** (1964), B1365–B1367.

Authors' summary: "Crossing relations for particles of arbitrary spin are used to express the amplitudes for $A+B \to A+B$ in terms of amplitudes for the crossed reaction $A+\bar{A} \to B+\bar{B}$. Using the optical theorem, it is demonstrated that crossed amplitudes with positive parity, positive G parity, and positive signature cannot be

negligible for forward scattering. Implications of this result for the spin dependences of high-energy scattering amplitudes are discussed." *V. de Alfaro* (Princeton, N.J.)

Hassoun, G. Q.; Yennie, D. R. 937
Infrared divergence of the angular momentum of bremsstrahlung and the physical structure of the electron.
Phys. Rev. (2) **134** (1964), B436–B444.

The authors consider a problem pointed out by Biedenharn: The statistical independence of photon emission in the infrared limit in a Bremsstrahlung scattering process may lead to a logarithmically divergent angular momentum for the radiated photons and the scattering system. Straightforward calculations lead them to the conclusion that "... the problem of the infrared divergence is a mathematical, rather than a physical one". Still more precise would have been the wording that the flaw lies in the choice of idealization rather than in any inconsistency of mathematical principle.
T. Erber (Chicago, Ill.)

Kimura, Toshiei 938
On the commutation relation for the canonical energy density.
Progr. Theoret. Phys. **31** (1964), 311–321.

Author's summary: "The interaction Hamiltonian in the interaction representation for an arbitrary spin field coupled to the external gravitational field is obtained by using the method which is the extension of Yang-Feldman's. The equal time commutator for the canonical energy density is derived directly from the integrability condition of the Schrödinger equation in the interaction representation. It is shown that the presence of ordinary interactions among fields does not alter drastically the form of commutator for free fields, and also the relation between the present method and Schwinger's based on the extended conservation law is discussed [cf. J. Schwinger, Phys. Rev. (2) **130** (1963), 406–409; MR **27** #4545]."
Y. Takahashi (Dublin)

Joos, Hans 939
Group-theoretical models of local-field theories.
J. Mathematical Phys. **5** (1964), 155–164.

The author studies the axioms of local quantum field theory in a simplified situation. The simplification consists in replacing the 4-dimensional space-time continuum of physics by a set of q^2 points (q being some prime number), namely, the 2-dimensional vector space over the field of integer numbers modulo q. In this space one can define the analogue of the inhomogeneous Lorentz group and distinguish proper Lorentz transformations from space or time reflections. Also the distinction between space-like, time-like or light-like separation of two points can be made in a natural fashion. Therefore all the customary axioms of local field theory can be formulated for this geometry. The "quantum field" consists of q^2 operators (one operator for each point) which are transformed into each other by the covariance group, which is of finite order.

In spite of this drastic simplification, a complete classification of the possible solutions is a formidable task which is not attempted by the author. He gives, however, several examples of quantum fields which satisfy all the axioms in this simplified geometry.
R. Haag (Urbana, Ill.)

Prugovečki, Eduard 940
Multiplicative symmetries in axiomatic quantum field theory.
J. Mathematical Phys. **5** (1964), 442–455.

A quantum field theory satisfying Wightman's axioms and using n different fields $\varphi_j(x)$, $j=1, \cdots, n$, is said to have a "multiplicative symmetry" if there exists a unitary operator U_M in the Hilbert space such that $U_M\varphi_j(x)U_M^{-1}=\lambda_j\varphi_j(x)$ for all x, and $j=1, \cdots, n$, the λ_j being complex numbers (of modulus one). The author analyzes the possible structure for the group of multiplicative symmetries and discusses the connection of such symmetries to superselection rules.
R. Haag (Urbana, Ill.)

Streater, R. F. 941
Intensive observables in quantum theory.
J. Mathematical Phys. **5** (1964), 581–590.

In conventional field theory the total energy, momentum, etc., are written as space integrals over local densities. This feature has not been incorporated in the present axiomatic schemes for local field theories because it is difficult to give a precise and usable mathematical definition of it. The author tries to give such a definition starting from the general principle of locality which assigns an "algebra of observables" $\mathfrak{A}(\mathcal{O})$ to every (open) region $\mathcal{O}$ of space-time. To simplify the discussion let us consider only those special regions which are the interior of a double cone whose base is a volume $\mathcal{V}$ in 3-dimensional space. Since such a region is characterized by the space volume $\mathcal{V}$, we shall replace the symbol $\mathcal{O}$ everywhere by $\mathcal{V}$ from now on. An obvious but untenable assumption would be to associate with every $\mathcal{V}$ operators $H(\mathcal{V})$, $\mathbf{P}(\mathcal{V})$, etc., corresponding to the energy content, momentum content, etc., of the region $\mathcal{V}$. These operators should be affiliated with $\mathfrak{A}(\mathcal{V})$ and should act on all the operators in $\mathfrak{A}(\mathcal{V})$ in the same manner as the global energy, momentum, etc. For example,

$$(1) \qquad [H(\mathcal{V}), S] = [H, S] \text{ for all } S \in \mathfrak{A}(\mathcal{V}).$$

This situation (called "strictly intensive" by the author) is impossible because, roughly speaking, the integral of the Hamiltonian density $h(x)$ over a sharply defined region turns out to be no operator; to get well-defined operators one must at least average $h(x)$ with smooth weight functions $f(x)$, i.e., consider quantities of the form $\int h(x)f(x)\,dx$, where f is a continuous, several times differentiable function.

The author remarks incidentally that the non-existence of operators $\mathbf{P}(\mathcal{V})$ (momentum in a finite volume) is directly related to the fact (first proved by H. Araki) that the operator rings $\mathfrak{A}(\mathcal{V})$ cannot be factors of type I.

To avoid the boundary effects the author gives a more sophisticated definition of objects like $H(\mathcal{V})$ and $h(x)$. Concerning the former there is presumably no difficulty in defining an operator $H(\mathcal{V}; \mathcal{V}_1)$ which satisfies (1) and is affiliated with $\mathfrak{A}(\mathcal{V}_1)$ as long as $\mathcal{V}_1$ is any open region containing the closure of $\mathcal{V}$. Selecting one such operator

for every $\mathscr{V}_1$, $\mathscr{V}$ corresponds to choosing one test function with support in $\mathscr{V}_1$ which has the constant value 1 in $\mathscr{V}$ for every $\mathscr{V}_1, \mathscr{V}$. Considering then a set of such $H(\mathscr{V}; \mathscr{V}_1)$ and defining an equivalence relation between such sets to get rid of the arbitrariness in the choice of the test functions, one may define $H(\mathscr{V})$ as an equivalence class of sets $H(\mathscr{V}; \mathscr{V}_1)$. By similar methods one may give a meaning to $h(x)$ in its relation to the algebras of local observables. *R. Haag* (Urbana, Ill.)

Mayer, M. E. 942

Note on the formulation of gauge-invariance conditions in S-matrix theories. (Italian summary)

Nuovo Cimento (10) **32** (1964), 317–322.

Formulation of the gauge-invariance condition in the S-matrix theory offers a conceptual difficulty. Usually this invariance is defined as the invariance of the Lagrangian, or field equations, but neither of them is assumed in the S-matrix theory. An alternative approach is to look for those conditions that are the consequences of gauge invariance but are independent of the specific form of the Lagrangian. The resulting relationships are, however, not expressible in terms of the S-matrix alone which is defined only on the mass shell. Under these circumstances the author tries to formulate this condition in terms of functional derivatives of the S-matrix with respect to an infinitesimal classical 4-vector field depending on macroscopic space-time coordinates. This condition can be used to derive identities of the Ward-Takahashi type for the expansion coefficients of the S-matrix in terms of asymptotic fields. The possible connection of the identities of this kind with the high-energy behavior of the S-matrix is suggested. *K. Nishijima* (Champaign, Ill.)

Filippov, A. T. 943

On the elimination of divergences in the quasipotential equations.

Phys. Lett. **9** (1964), 78–80.

From the author's introduction: "It is shown in this note how to avoid the divergences which arise when constructing quasipotential equations [A. A. Logunov and A. N. Tavkhelidze, Nuovo Cimento (10) **29** (1963), 380–399; MR **28** #866; A. A. Logunov, A. N. Tavkhelidze, I. T. Todorov and O. A. Khrustalev, ibid. (10) **30** (1963), 134–142; MR **28** #1895] in field theory. For the sake of simplicity we restrict ourselves to considerations of a scalar field φ with $\lambda\varphi^3$ or $g\varphi^4$ interaction. The extension of the results to more complicated cases is obvious."

F. Calogero (Rome)

Høgaasen, H. [Høgåsen, H.] 944

Polarization operators for $s=1$.

Z. Physik **178** (1964), 123–126.

For the case of a relativistic one-particle Hamiltonian for a particle with spin 1, one constructs the three spin projection operators which commute with the Hamiltonian.

J. M. Jauch (Geneva)

Smith, Charles Ray 945

Bound states in a Debye-Hückel potential.

Phys. Rev. (2) **134** (1964), A1235–A1237.

The Schrödinger equation for an electron in an exponentially shielded Coulomb potential (Debye-Hückel potential) is solved by using the perturbation method. It is shown that there is a finite number of bound states and that these states are non-degenerate with respect to the orbital angular momentum. First-order perturbation is used to obtain expressions for the energy-levels in terms of the Debye length, the principal quantum number and the orbital quantum number. The matrix elements of the potential are calculated in the representation of the hydrogen atom wave functions. *F. C. Auluck* (Delhi)

Bros, J.; Epstein, H.; Glaser, V. 946

Some rigorous analyticity properties of the four-point function in momentum space. (Italian summary)

Nuovo Cimento (10) **31** (1964), 1265–1302.

The analyticity domain of the 4-point function in p-space is studied by geometrical methods of analytic completion. The reviewer believes that this paper gives very significant results in this field.

On the basis of "linear properties" of Wightman functions, the 4-point function has been known to have the following three basic properties: (a) Analyticity in the union of two kinds of cones $\mathscr{T}_j^{\pm}$ (8 in total) and $\mathscr{T}_{jk}^{\pm}$ (24 in total); (b) Relations of coincidence which imply analyticity at the intersection of closures of any pair of cones except on any one of static cuts Γ_i (mass cuts) and Γ_{ij} (energy cuts); (c) The Steinmann relations.

On the basis of these three properties, the authors obtain the following results: (1) (Theorems 1 and 2) The holomorphy envelope of the union of two cones $\mathscr{T}_j^+$, $\mathscr{T}_{jk}^+$ and relevant analyticity points in (b) is the convex hull Λ_{jk} of $\mathscr{T}_j^+ \cup \mathscr{T}_{jk}^+$ minus Γ_k. A simple parametrization of Λ_{jk} is also given. (2) (Theorem 3) The holomorphy envelope of the union of 4 tubes $\mathscr{T}_{jk}^-$, $\mathscr{T}_{kj}^-$, $\mathscr{T}_{mn}^+$, $\mathscr{T}_{nm}^+$ (the Steinmann quartet) and relevant analyticity points in (b) relative to the class of functions satisfying (c) is the convex hull Ξ_{mn} of the union of 4 tubes minus the two cuts Γ_{jm} and Γ_{jn}. (3) (Theorem 4) The 4-point function is analytic at all the boundary points of tubes $\mathscr{T}_j^{\pm}$ and $\mathscr{T}_{jk}^{\pm}$ except the seven sets Γ_k and Γ_{jm}. (4) (Theorem 5) Let $\mathscr{R}_{jk}$ be real points not on the sets Γ_i, $i=0, 1, 2, 3$, and the set Γ_{jk}. There exists a neighbourhood $N(\mathscr{R}_{jk})$ of it such that the 4-point function is analytic in $N(\mathscr{R}_{jk})$ minus 2 cuts Γ_{jn} and Γ_{jm} $(n, m \neq j, k)$. Therefore the scattering amplitude is analytic in a complex cut neighbourhood of physical points (on the mass shell). (5) Analyticity of partial-wave amplitudes in a cut neighbourhood of the real physical values of s. (6) An application to a step in the proof of forward dispersion relations. *H. Araki* (Kyoto)

Moses, H. E. 947

Generalizations of the Jost functions.

J. Mathematical Phys. **5** (1964), 833–840.

Author's summary: "The Jost functions have proved valuable in the study of the analytic properties of the scattering phase for the radial Schrödinger equation. In the present paper we shall present an alternative definition of the Jost functions, prove the equivalence of the new definition to the usual one, and generalize the new definition to the one-dimensional Schrödinger equation $(-\infty < x < \infty)$, the three-dimensional non-separated

Schrödinger equation, and the three-dimensional non-separated Dirac equation. It is hoped that these generalizations lead to a better understanding of the analytic properties of the scattering operator for these and related dynamical systems. The generalized Jost functions are shown to be operators in the variables which label the degeneracy of the continuous spectrum of the Hamiltonians which are considered." *I. Kay* (Ann Arbor, Mich.)

Wu, Alfred C. T. 948

Perturbation analyticity and axiomatic analyticity. I. Connection of the Landau singularity manifold with Källén $\Xi_n(t)$ manifold and Jost DANAD manifold.

Phys. Rev. (2) **135** (1964), B222–B229.

A famous problem in the theory of quantized fields is the calculation of the domain of analyticity of the configuration space function (Wightman function). The size of the domain one obtains from general principles depends on how much information is used. If we assume only Lorentz invariance and no negative energy states, we obtain analyticity in the "extended tube". If we assume local commutativity as well, we get a much larger domain, the "envelope of holomorphy". It is known that in certain cases a part of the boundary of the envelope of holomorphy has the same parametric form (called the Jost DANAD curve) as part of the boundary of the extended tube, except that the signs of certain parameters are reversed.

If, instead of the Wightman functions, we study the analyticity of the retarded functions in momentum space, rather similar problems present themselves; in this case some guide to the answer can be sought from the singularities of Feynman graphs, the possible singularities of which are known to lie on "Landau curves".

In the present paper the author finds the geometric envelope of the Landau curves for certain graphs as the internal masses vary over all positive values. This is relevant for the momentum space envelope of holomorphy since no knowledge of intermediate states is used in its derivation. It is found that there is a very close correspondence with Källén's method of getting the extended tube using the envelope of the singularities of the generalized singular functions Δ^+, except that, again, the signs of certain parameters are reversed. The resulting curve, called (DANAD)′, is a specific conjecture for part of the boundary of the envelope of holomorphy.

{Reviewer's opinion: The results found here can be understood as follows: We might hope that perturbation theory gives at least part of the boundary of the envelope of holomorphy in p-space and this is formally similar to the envelope of holomorphy in x-space. That the boundary of the envelope of holomorphy is formally similar (in parts) to the boundary of the extended tube is not really so surprising in view of the above-mentioned results on analytic completion.} *R. F. Streater* (London)

Wu, Alfred C. T. 949

Perturbation analyticity and axiomatic analyticity. II. Hankel transforms.

Phys. Rev. (2) **135** (1964), B230–B232.

In this paper the connection between the Landau surfaces and the boundary of the extended tube [see the previous review, #948] is further elucidated by explicit study of the Hankel transform of a Feynman graph. It is shown that this transform can be written as an integral over the generalized singular functions Δ^+ for certain ranges of the parameters, clearly showing the connection with the extended tube. *R. F. Streater* (London)

DeWitt, Bryce S. 950

Theory of radiative corrections for non-Abelian gauge fields.

Phys. Rev. Lett. **12** (1964), 742–746.

Author's summary: "Verfasser deutet an, wie man für die genannten Typen von Feldern Strahlungskorrekturen zu berechnen hat, wobei explizite Kovarianz, konsistente Renormierbarkeit und Unitarität der S-Matrix gefordert werden. Vor allem muß man zur Kompensation unerwünschter Terme ein 'Untergrundfeld' einführen, welches eine ähnliche Rolle spielt, wie die Hilfsfelder in der üblichen Theorie der Regularisierung." *G. Heber* (Leipzig)

Hiida, K. 951

Pseudoscalar charge density of spin-$\frac{1}{2}$ particles. I. Existence.

Phys. Rev. (2) **134** (1964), B174–B181.

In this paper, the author studies the electromagnetic properties of spin-$\frac{1}{2}$ particles with finite masses in a theory where the interactions are renormalizable and invariant under time reversal but not invariant under space reflection [the author, same Rev. (2) **132** (1963), 1239–1248; MR **28** #860]. The importance of the parity-nonconserving counter terms is stressed in connection with the existence of the renormalized S-matrix. As a consequence of this, the pseudoscalar charge density is introduced in addition to the usual scalar charge density. An example is given for the photon vertex of the spin-$\frac{1}{2}$ particles with finite masses on their mass shells and an estimation of the coefficient of the parity-nonconserving counter term is carried out. Finally, the author gives a short argument on the observability of the pseudoscalar charge density. *S. Aramaki* (Urbana, Ill.)

Rafanelli, Kenneth; Schiller, Ralph 952

Classical motions of spin-$\frac{1}{2}$ particles.

Phys. Rev. (2) **135** (1964), B279–B281.

Authors' summary: "From the WKB solutions to the squared Dirac equation we derive the classical trajectories and spin precessions originally postulated by Bargmann, Michel, and Telegdi for charged spin-$\frac{1}{2}$ particles. Identical equations of motion are obtained from the WKB solutions to the Dirac equation."

Prosser, Reese T. 953

Segal's quantization procedure.

J. Mathematical Phys. **5** (1964), 701–707.

As I. E. Segal has explicitly remarked [same J. **1** (1960), 468–488, p. 477; MR **24** #B1144], his quantization procedure differs from the (not always defined) conventional one when the Hamiltonian is more than quadratic in the canonical variables (though the difference may be only $O(h^3)$ in every realistic case). The author verifies this remark in detail with a class of concrete examples.

It should have been emphasized more clearly that Segal's procedure aims at a fundamental relativistic field theory, not the simpler one-particle models which can be viewed as its approximations.

In conclusion, the author recommends further study of the possible role of Segal's procedure in a divergenceless reformulation of the foundations of quantum field theory.

J. M. Cook (Argonne, Ill.)

Prosser, Reese T. 954

Convergent perturbation expansions for certain wave operators.

J. Mathematical Phys. 5 (1964), 708–713.

Let H_0 be an essentially self-adjoint operator, and $H=H_0+V$ with $V=AB$ self-adjoint. A and B are to be bounded so that $\|Be^{iH_0}Af\|\le K(\tau)\|f\|$ for all f and some $K(\tau)$ independent of f with $\int_{-\infty}^{+\infty}K(\tau)\,d\tau<1$. (For example, let $H_0=-\nabla^2$ and $(Vf)(x)=V(x)f(x)$ on $L_2(E_3)$, $\|V\|_1<\frac{8}{3}\pi$, $\|V\|_\infty<\frac{8}{3}\pi$.) Under these conditions the author establishes rigorously the validity of Dyson's perturbation expansion for the Møller wave operators, and proves that the transition operators $U(s,t)=e^{-iH_0s}e^{iH(s-t)}e^{iH_0t}$ converge strongly to unitary operators $U(s,\pm\infty)$, $U(\pm\infty,t)$, and $U(\pm\infty,\pm\infty)$ as t and $s\to\pm\infty$. The limits satisfy $U(0,\pm\infty)H_0=HU(0,\pm\infty)$, $U(\pm\infty,0)H=H_0U(\pm\infty,0)$, and $U(\pm\infty,\pm\infty)H_0=H_0U(\pm\infty,\pm\infty)$.

Under the same conditions on V, an alternate form of the Dyson expansion due to Møller is rigorized.

J. M. Cook (Argonne, Ill.)

Curtius, Peter 955

★Zur Störungsrechnung in der axiomatischen Feldtheorie.

Eidgenössischen Technischen Hochschule in Zürich zur Erlangung der Würde eines Doktors der Naturwissenschaften genehmigte Promotionsarbeit.

Buchdruckerei Birkhäuser AG, Basel, 1963. 40 *pp.*

After a very detailed review of axiomatic quantum field theory the author offers a "proof" that every perturbation theory solution of a Lagrangian field theory fulfils the axiomatic formalism term by term. It appears to the reviewer that the argument is somewhat formal at this point. In particular, the discussion of the important asymptotic properties of the field operators could have been considerably amplified. It is, e.g., well known that the asymptotic condition as conventionally formulated [H. Lehmann, K. Symanzik and W. Zimmermann, Nuovo Cimento (10) 1 (1955), 205–225; MR 17, 219] is not fulfilled even in second-order quantum electrodynamics [cf., e.g., the reviewer, Helv. Phys. Acta 25 (1952), 417–434; MR 14, 435]. Similar complications could very well occur in sufficiently high orders also in other theories, perhaps even for a self-interacting scalar field. Therefore, the statement made by the author in the summary that perturbation-theoretic solutions derived from arbitrary renormalized Lagrange formalisms satisfy the axiomatic perturbation theoretic τ-equations in every order can only be accepted with some reservations. The author ought to be aware of these complications as he illustrates his general discussion with a second-order calculation for electrodynamics for scalar particles where exactly the same problem occurs. This is also pointed out in its correct place (after equations (4.17) and (4.18) in the paper), but the author does not seem to feel that his general discussion is influenced by this circumstance.

G. Källén (Lund)

McKenna, James; Klauder, John R. 956

Continuous-representation theory. IV. Structure of a class of function spaces arising from quantum mechanics.

J. Mathematical Phys. 5 (1964), 878–896.

Parts I, II and III appeared in same J. 4 (1963), 1055–1058 [MR 27 #2279]; ibid. 4 (1963), 1058–1073 [MR 27 #2280]; ibid. 5 (1964), 177–187 [MR 28 #3708]. Authors' summary: "A rigorous development of the continuous representation of Hilbert space by bounded, continuous, multidimensional phase-space functions $\psi(\mathbf{p},\mathbf{q})$ is presented. It is shown that these functions form a closed subspace of $L^2(\mathbf{p},\mathbf{q})$ whose elements are functions and not equivalence classes. Differential properties are investigated and it is pointed out that there are a multitude of definitions whereby $\psi(\mathbf{p},\mathbf{q})$ possesses continuous derivatives of all orders. In one of these definitions, each $\psi(\mathbf{p},\mathbf{q})$ is proportional to a multidimensional, entire function $f(\mathbf{q}-i\mathbf{p})$, establishing a connection between Bargmann's Hilbert space of entire functions and one example of a continuous representation. Attention is devoted to the purely functional characterization of the continuous representation by means of the reproducing kernel as a special case of Aronszajn's general theory. Properties of various operators in a continuous representation are carefully defined."

H. Wakita (Hiroshima)

Hanus, W. 957

The physical meaning of a non-unitary transformation connecting the Dirac and Pauli formalisms.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 12 (1964), 109–112.

The direct elimination of the small components in the Dirac equation for an electron coupled to a classical electromagnetic field leads to a non-Hermitian Hamiltonian for the large components in the nonrelativistic approximation. By a further two-real-parametric family of non-unitary transformations one can reduce this non-relativistic Hamiltonian to Hermitian form and thus ensure the normalizability of the non-relativistic wave function [E. Schmutzer and G. Weber, Annuaire Univ. Sofia Fac. Sci. Phys. Math. Livre 2 Phys. 55 (1960/61), 105–115; MR 29 #4493]. The author presents arguments for regarding a particular one of these transformations as being the physically correct one.

R. Ingraham (University Park, N.M.)

Petrina, D. Ja. 958

Complex singular points of contributions of Feynman's diagrams and the continuity theorem. (Russian. English summary)

Ukrain. Mat. Ž. 16 (1964), 31–40.

A method is proposed for determining which points of the Landau surface of a Feynman graph are singular points lying on the physical sheet. The method is based on Bremermann's continuity theorem [H. J. Bremermann, Math. Ann. 127 (1954), 406–423; MR 16, 27].

R. L. Warnock (Chicago, Ill.)

Galzenati, E. 959

Spectral representation in potential scattering. (Italian summary)

Nuovo Cimento (10) **31** (1964), 1197–1208.

Author's summary: "Potential scattering in momentum space is analysed in an approach which makes use of a spectral representation for the wave function. We show that one can obtain solutions for the scattering, as well as for bound states, starting from the solutions of integral equations of the Volterra type. The procedure is discussed in detail and is compared with similar methods which have been developed in configuration space."

V. de Alfaro (Princeton, N.J.)

Calogero, Francesco; Charap, John M. 960

Analyticity properties in energy of the complex J partial wave amplitudes in the scattering of particles with spin.

Ann. Physics **26** (1964), 44–54.

Authors' summary: "The energy dispersion relations for the complex J partial wave amplitudes for the scattering of particles with arbitrary spin are derived. A discussion is given of the Gribov-Pomeranchuk-Azimov essential singularities, and of a mechanism which might prevent their propagation to indefinitely large values of J. Some properties of Jacobi functions are collected in an appendix."

V. de Alfaro (Princeton, N.J.)

Calogero, Francesco; Charap, John M. 961

On the Born approximation in the potential scattering of particles with arbitrary spins.

Ann Physics **26** (1964), 55–71.

Authors' summary: "The Born approximation to the scattering amplitude is discussed for the potential scattering problem with many channels and arbitrary spin. This is done for the full amplitude in both the representation in which magnetic quantum numbers are diagonal and in the helicity representation, and for the partial wave amplitude in the helicity representation. Spin-orbit interactions are also treated. A discussion of the asymptotic behaviour for large momentum transfer is given, on the basis of which it is conjectured that for a specified class of potentials the Regge poles will retreat to the left-hand J-plane for high energies."

V. de Alfaro (Princeton, N.J.)

Paul, Harry 962

Die Kohärenz der induzierten Strahlung.

Ann. Physik (7) **12** (1963), 290–301.

The author defines the quantum-mechanical dispersion (the square root of the expectation value of the square deviation from expectation value) of the field as a measure of coherence of the field, and derives the change in dispersion due to the induced emission of a single photon. He finds it to be equal to one-half the dispersion divided by the expectation value of the number of photons in the field. Since this change is a minimum when the dispersion is a minimum, induced emission can be said to be coherent with the inducing field, in this sense.

The mathematical description of the field is handled by the author in an interesting fashion. A weakness of the discussion, however, is the absence of the consideration of phase relationship between emitted field and inducing field, since correlation with the inducing field is usually of interest in induced emission.

I. R. Senitzky (Fort Monmouth, N.J.)

Logunov, A. A.; Lju, I-čen' [Liu, I-Ch'en]; Todorov, I. T.; Černikov, N. A. 963

Dispersion relations and analytic properties of partial wave amplitudes in perturbation theory. (Russian. English summary)

Ukrain. Mat. Ž. **15** (1963), 250–276.

The analyticity properties of the Feynman amplitudes of an arbitrary order of perturbation theory are studied. A majorization technique developed by the authors in some earlier papers is used [A. A. Logunov, A. N. Tavhelidze, I. T. Todorov and N. A. Černikov, Dokl. Akad. Nauk SSSR **135** (1960), 801–804; A. A. Logunov, I. T. Todorov and N. A. Černikov, Ž. Èksper. Teoret. Fiz. **42** (1962), 1285–1293; MR **26** #3428]. The π-π, N-N, and π-N elastic amplitudes are discussed. Dispersion relations for fixed t, fixed s, and fixed $\cos\theta$ are proved for an arbitrary order of the perturbation series. Partial-wave dispersion relations for N-N and π-π scattering follow. The analyticity properties of the partial-wave amplitudes for π-N scattering are also discussed. The paper begins with a brief survey of work on analyticity in perturbation theory.

R. L. Warnock (Chicago, Ill.)

do Amaral, C. Marcio; Srivastava, Prem P. 964

Regge trajectories for Coulomb potential modified by inverse square potential.

Nuovo Cimento (10) **32** (1964), 237–238.

The Regge trajectories for an $1/r^2$ potential modified by a Coulomb interaction are considered.

E. Predazzi (Turin)

Vaks, V. G.; Larkin, A. I. 965

Regge poles in the nonrelativistic problem with nonlocal and singular interaction.

Ž. Èksper. Teoret. Fiz. **45** (1963), 800–809 (*Russian. English summary*); *translated as Soviet Physics JETP* **18** (1964), 548–554.

An integral equation in the momentum space is written by using standard techniques for the off-shell scattering amplitude corresponding to a nonlocal Schrödinger equation. The singularities of the amplitude in the angular momentum variable are then investigated by considering the various possibilities: the interaction is local or non-local, and the interaction is "regular" or "singular" at the origin. In particular, it is shown that for singular non-local potentials there exists a pole condensation at $l = -1$.

E. Predazzi (Turin)

Amati, D. 966

Poli di Regge e diffusione ad alta energia.

Nuovo Cimento Suppl. (1) **1** (1963), 135–150.

The author gives a very simple and clear summary of the theory of continuation to complex angular momenta in potential theory and of the different problems arising in this connection in field-theoretical models. A survey is given of the most interesting points in the theory of

complex angular momenta and of the various developments of that theory toward an understanding of the open problems in elementary particle physics. Although the recent experimental evidence has severely reduced the original interest in this matter, the present paper is a very useful one for those people who want to get acquainted with the most important features of Regge's theory and of its subsequent developments. *E. Predazzi* (Turin)

Kawai, T.; Masuda, N. 967
Nonexisting region of the resonance poles in the scattering amplitudes with unstable particles.
Nuovo Cimento (10) **32** (1964), 248–252.

In this paper the authors investigate the analytic properties of the scattering amplitudes for processes involving production (including the unstable particles), with the method of multi-channel reactions in the p-wave chain approximation. The conclusions tend to favour virtual bound state models for higher resonances, because the isobar mechanism model [R. F. Peierls, Phys. Rev. Lett. **6** (1961), 641–643] cannot have a smaller decay width for the derived state as compared with the width of the primary unstable particle forming the derived state. It is argued that the virtual bound state model [J. S. Ball and W. R. Frazer, ibid. **7** (1961), 204–207] can have smaller decay width than that of the unstable particle which forms a virtual bound state with another stable particle. No attempt is made to synthesize the two approaches, and the reader is reminded of the paper by the reviewer on this topic [I. P. Gyuk and the reviewer, Nuovo Cimento (10) **32** (1964), 227–233]. *S. F. Tuan* (Lafayette, Ind.)

Bertocchi, L.; McMillan, M.; Predazzi, E.; Tonin, M. 968
Vertex renormalization in potential theory. (Italian summary)
Nuovo Cimento (10) **31** (1964), 1352–1363.

In an investigation dealing with the vertex renormalization, the authors examine a model obtained from the potential bound-state partial wave equation by the addition of an inhomogeneous term representing the interaction between the "bare" bound particle and the two particles forming the bound state. Two approaches are developed, the so-called perturbative and spectral, and the results for the vertex renormalization constant are compared. *V. de Alfaro* (Princeton, N.J.)

Martin, A. 969
On the behaviour of the partial-wave amplitudes for large angular momenta in potential scattering. (Italian summary)
Nuovo Cimento (10) **31** (1964), 1229–1245.

It is proved that the partial wave scattering amplitude, analytically continued in the angular momentum l, vanishes asymptotically as l diverges in the right-hand half-plane (including vertical directions). This result holds for potentials which may be represented as superpositions of Yukawa functions. In two appendices bounds and asymptotic expansions are given for Bessel functions and for certain integrals involving Bessel functions.
F. Calogero (Rome)

Zav'yalov, O. I. [Zav'jalov, O. I.] 970
A majoration technique in quantum mechanics.
Dokl. Akad. Nauk SSSR **151** (1963), 539–542 (*Russian*); *translated as Soviet Physics Dokl.* **8** (1964), 691–693.

An inequality is established which relates the Nth term of the Born series for the scattering of a non-relativistic particle by a Yukawa potential to the $(N+1)$st term. This relation shows that if the Nth term is regular as a function of the incident momentum, then the $(N+1)$st term is also. Hence the lowest singularity in the scattering amplitude can be determined from the second-order term. The particular case of a two-channel problem is considered and it is shown that there is a left-hand cut in addition to the kinematical right-hand cut.
C. A. Hurst (Adelaide)

Chang, N. P.; Mani, H. S. 971
Possible effects of strong interactions in Feinberg-Pais theory of weak interactions. II.
Phys. Rev. (2) **134** (1964), B896–B900.

In a previous paper by one of the authors [N. P. Chang, same Rev. (2) **133** (1964), B454–B456; MR **28** #2837], it was shown in a simplified model that the peratization theory of Feinberg and Pais leads to a conclusion inconsistent with the conserved vector current theory, namely, that the vector coupling constants in beta decay and muon decay are not equal as a consequence of the peratization effects. The authors reconfirmed the above conclusion in this paper by using more general arguments based on the modification of the Ward-Takahashi identities derived by the present reviewer in the case of quantum electrodynamics.

The essential assumption in deriving these identities is the gauge property of the electromagnetic field, however, and this assumption is not satisfied by the intermediate bosons considered by these authors.
K. Nishijima (Champaign, Ill.)

Polkinghorne, J. C.; Taylor, J. C. 972a
Feinberg-Pais equation for the trace of ladder diagrams.
Phys. Rev. (2) **134** (1964), B420–B423.

Bég, Mirza A. Baqi 972b
On the peratization theory of Feinberg and Pais.
Ann. Physics **27** (1964), 183–192.

In a paper published in 1963 [the reviewer and Pais, Phys. Rev. (2) **131** (1963), 2724–2761; MR **27** #6537] the possibility was discussed of obtaining finite results, in the unrenormalizable theory of vector mesons interacting with leptons, by a new technique of summing divergent graphs. In this first paper, the sum of the uncrossed ladder graphs for lepton-lepton scattering was carried out under certain approximations whose validity was suggestive but uncertain. In a second paper, these workers have shown that the same results may be obtained by a somewhat more rigorous procedure involving the solution of the corresponding Bethe-Salpeter equation. In this method, one splits the B-S equation into two parts, one of which is responsible for the divergence of the power series expansion, and is solved exactly, the other part being treated as a remainder and solved by iteration.

Since the appearance of these papers, there have been a number of new derivations of the fundamental result of Feinberg and Pais, of which the present papers are particularly transparent examples. In both of them there is a reliance on the technique of splitting the Bethe-Salpeter amplitude into two parts, one singular which is solved exactly, and the other iterable. In the paper of Bég, the singular B-S equation is solved by noticing that it is very similar to the corresponding equation for the four-Fermion interaction where the solution is well known. In the paper of Polkinghorne and Taylor, the separation is carried out by an ingenious graphical technique, and the graphs evaluated by a method analogous to that used previously by Polkinghorne for the high-energy behavior of Feynman graphs. In a separate section, Polkinghorne and Taylor study the equation for the Fourier transform of the B-S amplitude where the singular part can also be extracted. The most important result to emerge from these treatments is a more rigorous justification of the assumption that when the singular part of the B-S equation is separated, the remainder can be treated by iteration.

G. Feinberg (New York)

Khuri, N. N.; Pais, A. 973

Singular potentials and peratization. I.

Rev. Modern Phys. **36** (1964), 590–595.

It has been the fashion in recent years for workers investigating the analytic properties of scattering amplitudes to derive results in the case of scattering by fixed potentials, and then to conjecture that their results are true for general Feynman graphs. In this paper this procedure is reversed. In a number of recent papers [the reviewer and Pais, Phys. Rev. (2) **131** (1963), 2724–2761; MR **27** #6537; ibid. (2) **133** (1964), B477–B486; MR **28** #3730] it has been shown that a certain set of graphs which are individually infinite can nevertheless be summed to give finite results. It was furthermore conjectured that this phenomenon is more general than the set of graphs actually summed. The procedure used to sum the graphs involved the introduction of certain "regularizations" in order to make the individual graphs finite in an intermediate stage of the calculations. It was also conjectured that the answer was independent of the specific way in which these regularizations were carried out.

In the present paper, it is shown that these conjectures are in fact correct for the case of scattering by a singular repulsive potential. The cases treated are potentials which behave as gr^{-m} $(m>3)$ for small r. For these potentials, the individual terms of the Born series are divergent. Nevertheless, by regularizing the potentials by factors such as $(r/(r+\alpha))^m$, one obtains an equation which can be separated into two parts. One of these involves a constant which is the sum of the terms of the Born series that were previously divergent. This constant is, however, now finite in the limit $\alpha\to 0$ (no regularization), and independent of how the limit is approached. The other part of the equation is of the Volterra type and the limit $\alpha\to 0$ can be taken immediately, again giving a finite answer. Both of these phenomena are exactly parallel to the previous treatment of Feynman graphs in unrenormalizable field theory. One is therefore more hopeful that procedures used there, where one has no guarantee of the existence of solutions, really make sense.

G. Feinberg (New York)

Pais, A.; Wu, Tai Tsun 974

Singular potentials and peratization. II.

J. Mathematical Phys. **5** (1964), 799–804.

Authors' summary: "In a field theory with more than one unrenormalizable interaction (like the W-theory with weak and electromagnetic coupling), the important problem arises of resumming multiple series of individually divergent terms. In order to get a first insight in the new questions which arise for multiple as compared to single series, we study the analogous question for a superposition of two singular potentials, using a family of exactly soluble cases. We ask whether one can expand resummed series for the zero-energy scattering amplitude in powers of one coupling constant with coefficients depending on the other. The answer depends both on the relative magnitude of the coupling constants and on the relative degree of singularity of the interactions. Depending on these two conditions one finds three regimes, one where a convergent power expansion holds, another where an asymptotic expansion obtains, and a third where it is impossible to expand in powers of either single constant separately. It is conjectured that a similar situation will be true in a field theory with leading power singularities only (and no logarithmic ones), if such a theory has meaning."

G. Barton (Brighton)

Pais, A. 975

Notes on W_3 and related symmetries.

Phys. Rev. Lett. **12** (1964), 632–634.

Some implications are considered of the hypothesis that strong interactions are invariant under the group $W_3=U^{(1)}(3)\times U^{(2)}(3)$; most of the detailed discussion refers to the more specialized case of invariance under $SW_3=SU^{(1)}(3)\times SU^{(2)}(3)$. Applications are confined to baryons and mesons, hypothetical "fundamental triplets" not being considered.

G. Barton (Brighton)

McGlinn, W. D. 976

Problem of combining interaction symmetries and relativistic invariance.

Phys. Rev. Lett. **12** (1964), 467–469.

The following theorem is proved: Given a group whose generators consist of those of the inhomogeneous Lorentz group (Poincaré group) L and a semisimple Lie group I, and no more. Assume that the generators of I all commute with those of the homogeneous Lorentz group, then they also commute with the generators of translations and hence the group is a direct product of L and I.

A. O. Barut (Boulder, Colo.)

Whippman, M. L.; Gerstein, I. S. 977

Bound states and elementary particles in the limit $Z_3=0$.

Phys. Rev. (2) **134** (1964), B1123–B1127.

Authors' summary: "The limit $Z_3=0$ is studied in the Zachariasen model using dispersion theory techniques. The connection between bound states and elementary particles is demonstrated in this limit and it is shown how Castillejo-Dalitz-Dyson ambiguities are removed."

P. W. Higgs (Edinburgh)

Kotański, Andrzej; Namysłowski, Józef 978

The CDD poles and the agreement of a solution of the Low equation with the experimental results.

Acta Phys. Polon. **23** (1963), 557–566.

The agreement of the Salzman solution of the static meson theory Low equation [G. Salzman and F. Salzman, Phys. Rev. (2) **108** (1957), 1619–1628; MR **22** #11888] for P-wave π^+-p scattering with experiment holds only up to $E_{KIN} \approx 140$ Mev. Despite the limitations of the Low equation, agreement should be expected over a larger range. It is shown that the inclusion of one CDD pole [L. Castillejo, R. H. Dalitz and F. J. Dyson, ibid. (2) **101** (1956), 453–458] in the solution gives very good agreement up to 670 Mev, and a better location of the resonance. This pole violates Low's uniqueness condition for the solution [see L. Castillejo, R. H. Dalitz and F. J. Dyson, loc. cit., footnote 9]. *W. M. Frank* (Silver Spring, Md.)

Esteve, A.; Sona, P. G. 979

Conformal group in Minkowsky space. Unitary irreducible representations. (Italian summary)

Nuovo Cimento (10) **32** (1964), 473–485.

Authors' summary: "The three fundamental series of unitary irreducible representations of the Lie algebra of the conformal group in Minkowsky space are described. We use the homomorphism between the conformal group and the group $G_{2,2}$ of linear transformations which leave invariant the quadratic form $-|z_1|^2-|z_2|^2+|z_3|^2+|z_4|^2$." *C. Fronsdal* (Los Angeles, Calif.)

Kantorovich, A. 980

Reduction of triple product of septets in the G_2 symmetry model. (Italian summary)

Nuovo Cimento (10) **32** (1964), 633–653.

Author's summary: "The reduction coefficients of the triple product of G_2 seven-dimensional irreducible representations, $7\otimes 7\otimes 7$, have been calculated. The total hypercharge quantum number is introduced, in order to characterize mutually orthogonal vectors having the same total isospin and corresponding to the same weight." *C. Fronsdal* (Los Angeles, Calif.)

Englert, F.; Brout, R. 981

A possible theoretical argument which eliminates the (hitherto) unrealized representations of SU(3).

Phys. Rev. Lett. **12** (1964), 682–683.

Instead of adopting the direct product of SU(3) and the baryon gauge group $e^{i\phi}$, as the symmetry group for strong interactions, it is pointed out that one can make do with the smaller group U(3), which is in any case contained in $SU(3)\times e^{i\phi}$. It may then be postulated that baryon conservation nevertheless gives rise to a superselection rule, i.e., that in nature only those representations of U(3) are realized which factor into a representation of $e^{i\phi}$ and a representation of another group. It is shown that this other group automatically realizes the regular (adjoint) representation of SU(3), but never the representations 3, 3*, etc. In other words, the above postulate eliminates the unobserved triplet and/or fractional-charge representations of SU(3) which have been the subject of much recent speculation. *G. Barton* (Brighton)

Tati, Takao 982

The broken symmetry and a symmetrical set of charges in octet theory.

Progr. Theoret. Phys. **31** (1964), 511–513.

The author suggests that the SU(3) breaking terms of a Yang-Mills Lagrangian should contain an interaction with an X-meson coupled to a Y-Q, aside from a Y-interaction (coupled to hypercharge and giving the mass spectrum) and the electromagnetic interactions (coupled to Q). This would provide a semi-symmetric picture of the breaking in terms of I-, U-, V-spin subgroups of SU(3). *Y. Ne'eman* (Pasadena, Calif.)

de Swart, J. J. 983

The octet model and its Clebsch-Gordan coefficients.

Rev. Modern Phys. **35** (1963), 916–939.

The strongest interactions of elementary particles seem—at least approximately—to be invariant under a symmetry group isomorphic to SU(3) [M. Gell-Mann, California Inst. Tech. Synchrotron Lab. Rep. No. 20 (1961); Y. Ne'eman, Nuclear Phys. **26** (1961), 222–229; MR **23** #B2856]. This article is devoted in part to a straightforward exposition of those properties of this group that are relevant to its use in elementary particle physics: tensors, generators and their commutation relations, irreducible representations, Clebsch-Gordan coefficients. The point of view is very much that of the physicist, for the relevance of the mathematics to the physical model is always clear. The remainder of the article presents convenient and useful tables of the Clebsch-Gordan coefficients corresponding to the simpler representations of SU(3). *S. L. Glashow* (Berkeley, Calif.)

Yamaguchi, Y. 984

A new symmetry principle in particle physics—trio invariance.

Phys. Lett. **9** (1964), 281–283.

In this interesting letter a permutation-invariant theory of three fields is proposed. The fields are reduced into a singlet and a doublet with respect to the permutation group. Each of the reduced fields is supposed to carry a conserved charge, and these charges are related to strangeness, electric charge, and baryon number as in the Sakata model. The non-commutation between permutations and electromagnetic gauge transformations of the first kind then leads to invariance under the isospin group SU_2. The model thus motivates SU_2, rather than SU_3, in the context of three-field theories. While the author considers this to be a very desirable feature, he does not offer any comments concerning the logical role of SU_3 supermultiplets and the mass formula in his scheme. *G. R. Allcock* (Liverpool)

Freund, Peter G. O.; Nambu, Yoichiro 985

Broken $SU(3)\otimes SU(3)\otimes SU(3)\otimes SU(3)$ symmetry of strong interactions.

Phys. Rev. Lett. **12** (1964), 714–717.

This short note announces a number of new symmetries which might be expected to hold (approximately) at high energies, where the baryon masses become negligible. At present energies its main prediction is the existence of

sets of sixteen $0^{\pm}$, $1^{\pm}$ mesons. Some tentative identification of the known resonances with the mass-levels of the theory is made. The theory has nine baryons, similar to that of Schwinger [same Lett. **12** (1964), 237–239].
R. F. Streater (London)

Dothan, Y.; Harari, H. 986
Processes forbidden by the G_2 symmetry.
Nuovo Cimento (10) **32** (1964), 498–499.
Symmetries of the strong interactions result in definite predictions for electromagnetic and weak reactions involving strongly interacting particles. This paper shows that $G(2)$ symmetry [R. E. Behrends and A. Sirlin, Phys. Rev. (2) **121** (1961), 324–336; MR **23** #B303] is contradicted by the experimental evidence of electromagnetic interactions. *Y. Ne'eman* (Pasadena, Calif.)

Kompaneets, A. S. [Kompaneec, A. S.] 987
Classical model of a charged particle with angular momentum.
Ž. Èksper. Teoret. Fiz. **44** (1963), 2169–2172 (*Russian. English summary*); *translated as Soviet Physics JETP* **17** (1963), 1455–1457.
Author's summary: "Starting from the equations proposed by the author in a previous paper [same Ž. **43** (1962), 2185–2187; MR **28** #1905], it is shown that by balancing the repulsive electromagnetic field by an attractive vector meson field, a stable classical model of a charged particle can be constructed which possesses a mechanical angular momentum."

Treat, Richard P. 988
Possible link between isospace and the geometry of space-time.
Phys. Rev. Lett. **12** (1964), 407–408.
The six-dimensional space of the generators of local Lorentz transformations is here proposed as a carrier for internal degrees of freedom of elementary particles. Parallel transfer of a six-vector of this internal space from one space-time point to another is described by a set of generalized Yang-Mills gauge fields, whose essential properties are assumed to derive entirely from the gravitational space-time curvature. *G. R. Allcock* (Liverpool)

Martin, J. L. 989
Quantum theory of unstable particles.
Proc. Cambridge Philos. Soc. **60** (1964), 587–593.
A particle moving in a delta-function potential in three dimensions coupled to a discrete degree of freedom that represents absorption is used to discuss unstable particles. The usual violations of the exponential decay law are found. An operator representing the probability of absorption is introduced and a related eigen-value problem is formulated. *F. R. Halpern* (La Jolla, Calif.)

Masterson, Kleber S., Jr.; Sawada, Katuro 990
Energy of the many-fermion normal system.
Phys. Rev. (2) **133** (1964), A1234–A1244.
The authors study the Brueckner-Goldstone linked-cluster expansion for a normal fermion system at zero temperature, and re-derive a number of properties for it. A technique is described to express the interaction energy in terms of the true momentum density instead of the unperturbed Fermi distribution. *K. K. Gupta* (Bombay)

Balazs, Nandor L. 991
Thomas-Fermi theory of the atom as a solution of the density-matrix hierarchy.
Phys. Rev. (2) **134** (1964), A841–A852.
The author shows that the Thomas-Fermi theory of a neutral atom is a zero-order solution of the complete many-body problem provided the number of particles is large and the diagonal elements of the single particle density matrix are fast-oscillating functions of the distance from the diagonal. The higher-order corrections are considered and it is shown that these contain the Dirac and Weizsäcker corrections. *D. ter Haar* (Oxford)

Frost, Arthur A. 992
Approximate series solutions of nonseparable Schrödinger equations. I. Mathematical formulation.
J. Chem. Phys. **41** (1964), 478–482.
Author's summary: "A series solution method is developed that is a generalization of the method used by Pekeris for helium-atom calculations. It is equivalent to the linear variation method but requires few, if any, matrix component integrals to be evaluated. The method is illustrated by an application to the *s* states of the hydrogen atom."

Frost, Arthur A.; Inokuti, Mitio; Lowe, John P. 993
Approximate series solutions of nonseparable Schrödinger equations. II. General three-particle system with Coulomb interaction.
J. Chem. Phys. **41** (1964), 482–489.
Authors' summary: "The series solution method developed by Pekeris for the Schrödinger wave equation of two-electron atoms has been generalized to handle any three particles with Coulomb interaction. Calculations have been carried out with wavefunctions through the sixth degree which result in a linear combination of 84 terms for the unsymmetrical case. Systems for which numerical results are given are: the hydride ion with nuclear motion, the trielectron ($e^+e^-e^-$) or positronium ion, the positron-hydrogen atom interaction ($e^+e^-p^+$), and the mu-mesonic isotopic hydrogen molecule-ion ($p^+\mu^-d^+$)."

Frost, Arthur A.; Harriss, Donald K.; Scargle, Jeffrey D. 994
Approximate series solutions of nonseparable Schrödinger equations. III. B matrix method.
J. Chem. Phys. **41** (1964), 489–494.
Authors' summary: "The series solution **B** matrix method, developed in Paper I [#992] is applied to systems of one and two electrons with wavefunctions expressed in inter-particle coordinates. The application to more complicated systems is considered."

Faddeev, L. D. 995
Mathematical questions in the quantum theory of scattering for a system of three particles. (Russian)
Trudy Mat. Inst. Steklov. **69** (1963), 122 *pp.*

In this long paper the author treats some of the main mathematical questions in the quantum theory of scattering for the three-particle system. The central part of the work is an investigation of the properties of the resolvent operator $\mathbf{R}(z)=(\mathbf{H}-z\mathbf{E})^{-1}$; here $\mathbf{H}$ is the Hamiltonian operator, $\mathbf{E}$ is the unit operator, and z is a complex variable. Resu'ts on the behavior of $\mathbf{R}(z)$ for z near the positive real axis are applied to prove theorems on the expansion of an arbitrary function in eigenfunctions of $\mathbf{H}$, to substantiate the time-dependent formulation of scattering theory, and to construct the scattering operator. The study of the resolvent is by means of integral equations in the momentum representation. The author believes that these equations are more convenient in the three-particle problem than the corresponding equations in the coordinate representation. He also believes that the time-dependent formulation of the scattering problem is more appropriate than the stationary-state formulation when there are more than two particles involved. The particles are assumed to interact pair-wise, with a potential $\hat{v}_{ij}(x_i-x_j)$ describing the interaction of two particles with three-vector coordinates x_i and x_j. Conditions are imposed on the Fourier transform $v(k)$ of each two-particle potential $\hat{v}(x)$, specifically, a condition of boundedness and rate of decrease

$$(1) \qquad |v(k)| \leqq C(1+|k|)^{-1-\theta_0} \quad (\theta_0>0),$$

a continuity condition

$$(2) \qquad |v(k)-v(k+h)| \leqq C(1+|k|)^{-1-\theta_0}|h|^{\mu_0} \quad (|h|\leqq 1,\ \mu_0>0),$$

and a reality condition

$$(3) \qquad v(-k)=\overline{v(k)}.$$

Throughout most of the work it is also assumed that the two-particle Hamiltonian constructed from v has just one discrete eigenvalue (i.e., one bound state), non-degenerate and not equal to zero. The generalization to a finite number of non-zero discrete eigenvalues of finite multiplicity is trivial, and is avoided only for notational convenience.

The two-particle scattering problem is discussed in detail as a preliminary to the three-particle problem. The treatment of the two-particle case serves to illustrate the methods used in the relatively complicated three-particle problem, and also provides some results needed in the latter. The reviewer will attempt to summarize the methods and results for the two-particle case, indicate the main device which allows the same general procedure in the three-particle case, and then be content with the statement that the theorems for the three-particle case are close analogues of those for two particles. A detailed statement of the three-particle results is too cumbersome for a short review.

In the two-particle case the transition operator $\mathbf{t}(z)=\mathbf{v}-\mathbf{v}\mathbf{r}(z)\mathbf{v}$ is studied. Here $\mathbf{v}$ is the interaction operator and $\mathbf{r}(z)=(\mathbf{h}-z\mathbf{e})^{-1}$ is the resolvent of the Hamiltonian $\mathbf{h}$. The resolvent may be constructed from $\mathbf{t}(z)$. In the momentum representation $\mathbf{t}$ may be represented by a kernel $t(k,k',z)$ which satisfies the integral equation

$$(4) \quad t(k,k',z)= v(k-k')-\int v(k-k'')\left(\frac{k''^2}{2m}-z\right)^{-1} t(k'',k',z)\,dk''.$$

Conditions on $v(k)$ as mentioned above are assumed, with $\theta_0>\frac{1}{2}$, $\mu_0>\frac{1}{2}$. After detailed analysis of the singular integral involved, it is shown that the operator of equation (4) is completely continuous. A finite number of iterations of equation (4) results in an equation with the same kernel as equation (4), but with an inhomogeneous term having better asymptotic behavior than $v(k-k')$. The Fredholm alternative applies to this new equation. Investigation of the homogeneous equation and application of the Fredholm alternative leads to the following theorem. Theorem: For all complex $z\neq-\kappa^2$, where $-\kappa^2$ is the discrete eigenvalue of $\mathbf{h}$, (4) has a unique solution which satisfies the conditions

$$(5) \quad \begin{aligned} &|t(k,k',z)| \leqq C(1+|k-k'|)^{-1-\theta};\\ &|t(k+h,k'+h',z+\Delta)-t(k,k',z)| \leqq C(1+|k-k'|)^{-1-\theta}(|h|^{\mu}+|h'|^{\mu}+|\Delta|^{\nu});\\ &|t(k,k',z)-v(k-k')| \leqq C(1+|k-k'|)^{-1-\theta}(1+|z|)^{-\theta/2}. \end{aligned}$$

Here θ, μ, and ν may be chosen arbitrarily close to min $(1,\theta)$, μ_0, and $\frac{1}{2}$, respectively. Another theorem gives a representation for $t(k,k',z)$ displaying a pole at $z=-\kappa^2$.

Let $\mathfrak{h}$ be the Hilbert space of functions $f(k)$ with the scalar product $(f,f')=\int f(k)f'(k)\,dk$, where the integration is over all of three-space. Define the operator $\mathbf{k}(\varepsilon_1,\varepsilon_2)$ by

$$(6) \quad \mathbf{k}(\varepsilon_1,\varepsilon_2)f(k)=\int\frac{t(k,k',k'^2/2m+i\varepsilon_1)}{k^2/2m-k'^2/2m-i\varepsilon_2}f(k')\,dk'$$

and the operators $\mathbf{u}^{(+)}$ and $\mathbf{u}^{(-)}$ by $\mathbf{u}^{(\pm)}=\mathbf{e}-\mathbf{k}(\mp 0,\mp 0)$. The results on the two-particle resolvent are used to prove the following theorem on expansion of functions in eigenfunctions of the Hamiltonian $\mathbf{h}$. Theorem: An arbitrary $f\in\mathfrak{h}$ has a unique representation of the form

$$(7) \qquad f(k)=c\psi(k)+\mathbf{u}^{(\pm)}f^{(\pm)}(k),$$

and for an arbitrary bounded function $\phi(x)$, $-\infty<x<\infty$,

$$\phi(\mathbf{h})f=\phi(-\kappa^2)c\psi+\mathbf{u}^{(\pm)}\phi(\mathbf{h}_0)f^{(\pm)}.$$

ψ is the eigenfunction of $\mathbf{h}$ corresponding to the discrete eigenvalue $-\kappa^2$, and $c=(f,\psi)$. $f^{(\pm)}$ is defined by $f^{(\pm)}=\mathbf{u}^{(\pm)*}f$, where $*$ denotes Hermitian conjugation. The generalized Parseval relation holds: $\|f\|^2=|c|^2+\|f^{(\pm)}\|^2$. The sense in which (7) is an expansion in eigenfunctions of $\mathbf{h}$ is clarified by noting that the generalized function

$$u^{(\pm)}(k,k^{(0)})=\delta(k-k^{(0)})-\frac{t(k,k^{(0)},k^{(0)2}/2m\mp i0)}{k^2/2m-k^{(0)2}/2m\pm i0}$$

is a (formal) eigenfunction of $\mathbf{h}$ in the momentum representation with eigenvalue $k^{(0)2}/2m$.

On the subject of two-body scattering theory a theorem on the existence of wave operators is established. Theorem: The limit of $\mathbf{u}(t)=\exp(i\mathbf{h}t)\exp(-i\mathbf{h}_0t)$ as $t\to\pm\infty$ exists in the strong sense. Furthermore, $\lim_{t\to\pm\infty}\mathbf{u}(t)=\mathbf{u}^{(\pm)}$. This theorem may be applied to show that the time-dependent scattering problem makes sense. The latter is as follows. Given an element f_- of $\mathfrak{h}$, find a solution $f(t)$ of the Schrödinger equation $i\partial f/\partial t=hf$ satisfying the condition $\lim_{t\to-\infty}\|f(t)-\exp(-i\mathbf{h}_0t)f_-\|=0$. Show that there exists an element f_+ of $\mathfrak{h}$ such that

$$\lim_{t\to+\infty}\|f(t)-\exp(-i\mathbf{h}_0t)f_+\|=0.$$

Theorem: The time-dependent scattering problem has a unique solution. Also, $f_+=\mathbf{s}f_-$, where $\mathbf{s}=\mathbf{u}^{(+)*}\mathbf{u}^{(-)}$.

The analysis of the three-body problem follows the same general lines as the two-body problem, and results in completely analogous theorems. However, there is the initial difficulty that the three-body analogue of (4) cannot be handled by the Fredholm theory. This difficulty is overcome by working with the momentum representation of the following system of equations:

$$(8) \qquad \mathbf{W}_{\alpha\beta}(z) = \mathbf{W}_{\alpha\beta}^{(0)}(z) - \mathbf{T}_\alpha(z)\mathbf{R}_0(z) \sum_{\gamma \neq \alpha} \mathbf{W}_{\gamma\beta}(z).$$

The indices run over three values, corresponding to the three possible pairings of the particles (12, 23, 31). $\mathbf{R}_0$ is the resolvent for the free-particle Hamiltonian, and $T_\alpha(z)$ is defined by

$$T_\alpha(z) f(k_\alpha, p_\alpha) = \int t_\alpha(k_\alpha, k_\alpha', z - p_\alpha^2/2n_\alpha) f(k_\alpha', p_\alpha)\, dk_\alpha',$$

where t_α is the solution of (4) for potential v_α and a certain mass m_α; m_α and n_α are given functions of the three particle masses. The inhomogeneous term $\mathbf{W}_{\alpha\beta}^{(0)}$ is given by $\mathbf{W}_{\alpha\alpha}^{(0)} = 0$, $\mathbf{W}_{\alpha\beta}^{(0)} = -\mathbf{T}_\alpha \mathbf{R}_0 \mathbf{T}_\beta$. The equations (8) are subject to the Fredholm theory after iterations, and the resolvent $\mathbf{R}(z)$ may be constructed from their solution. Possible advantages of somewhat similar equations in the practical solution of three-body problems have been pointed out recently; see, for example, S. Weinberg [Phys. Rev. (2) **133** (1964), B232–B256; MR **29** #3163].

R. L. Warnock (Chicago, Ill.)

Blatt, John M. 996

★Theory of superconductivity.

Pure and Applied Physics, Vol. 17.

Academic Press, New York-London, 1964. xii + 486 *pp.* \$12.50.

The phenomenon of superconductivity is remarkable in that it resisted all theoretical explanation, in terms of known properties of nature, for almost forty years after its original discovery by Kammerlingh Onnes. However, since 1950 remarkable progress has been made in achieving a basic understanding of superconductivity. The author of this book played an important role in the development of the modern theory of superconductivity, a fact which the reader cannot fail to miss.

This is perhaps the first relatively complete monograph in English on the modern theory of superconductivity. It is written with considerable charm and skill, gradually leading the reader from phenomenology and elementary theory to the more complex and subtle parts of the theory. The foundation for the subsequent discussion is laid by a careful account of the experimental facts of superconductivity, including its two principal manifestations, the Meissner effect (exclusion of an external magnetic field by a superconductor), and the occurrence of persistent currents (currents may circulate forever without diminution in the superconducting state), as well as flux quantization, the observation of an energy gap in the electron energy distribution of a superconductor, and the isotope effect. He then passes on to Bose-Einstein condensation and to those theoretical models that preceded the modern theories of Bardeen, Bogoliubov, and their collaborators. After a thorough and critical presentation of these theories, he then goes on to apply them to an interpretation of the phenomena actually observed in the laboratory. It is clear from these parts of the book that many unsolved problems remain.

Altogether, the author's book provides a fascinating view of a rapidly developing subject and should be of interest to anyone active in low-temperature physics, or in theoretical solid-state physics. From the nature of the subject, this book will probably become obsolete in a comparatively few number of years. However, it will always provide a basic foundation for the subject.

W. Franzen (Boston, Mass.)

STATISTICAL PHYSICS, STRUCTURE OF MATTER

See also 31, 772, 879, 909, 1047.

Levine, D.; Meijer, P. H. E. 997

Solution of two master equations coupled by particle exchange.

Physica **30** (1964), 615–624.

Authors' summary: "The steady state is calculated in approximation for two linearly coupled master equations representing Gaussian Markoff processes. The solution is used to calculate the perturbation of particle exchange on the temperature and stream velocity of two adjacent gas layers moving tangentially past each other."

Ludwig, G.; Müller, W. J. C.; Schröter, J. 998

A derivation of Boltzmann's equation with an assumption of determinacy. I.

Physica **30** (1964), 479–492.

Authors' summary: "A derivation of the classical Boltzmann collision equation is given, proceeding from Boltzmann's original interpretation of the density function $f(\mathbf{r}, \mathbf{v}, t)$. The essential assumptions together with well-known ones, already formulated in other derivations, are that $f(\mathbf{r}, \mathbf{v}, t)$ is determined by $f(\mathbf{r}, \mathbf{v}, t-\tau)$ alone (and for instance not only if the whole earlier course in time is given), and that $f(\mathbf{r}, \mathbf{v}, t)$ proceeds in a determinate manner."

Freudenthal, H. 999

La structure logique de la mécanique statistique (Analyse d'un ouvrage récent).

Nieuw Arch. Wisk. (3) **11** (1963), 95–109.

This paper is essentially a critical review of the logical foundations of a recent book by Jancel [*Les fondements de la mécanique statistique classique et quantique*, Gauthier-Villars, Paris, 1963; MR **27** #5566].

D. ter Haar (Oxford)

Ron, Amiram 1000

Quantum kinetic equations for electrons in a periodic system.

J. Mathematical Phys. **5** (1964), 748–755.

The approach to equilibrium of a system of interacting electrons in a periodic potential is studied by deriving a quantum kinetic equation in the self-consistent field approximation. The electrons are interacting via the Coulomb potential and moving in the periodic potential of a crystal. Starting from the Hamiltonian of the system, a hierarchy of equations for the n-electron quantized density matrices ($n = 1, 2, \cdots$) is obtained. This hierarchy

is truncated in a systematic manner consistent with the self-consistent field approach, which leads to a closed set of equations. Irreversibility is then introduced by making the assumption, due to Bogoliubov, that the two-electron correlation function reaches an asymptotic value in a time much shorter than the relaxation time of the one-electron distribution function f ("adiabatic hypothesis"). The kinetic equation for f, obtained by this method, governs the approach to equilibrium of the system, taking proper account of both the individual and collective aspects of the electron interactions. It may be applied to problems of transport theory in crystals, when electron-electron scattering and band transitions are important.
K. Schram (Utrecht)

Kromminga, A. J. 1001
Some applications of many-time Green's functions to the many-boson problem.
Ann. Physics **26** (1964), 307–321.
Author's summary: "The spectral representation of many-time Green's functions is generalized to Green's functions in which several operators may appear in normal order. The Heisenberg picture matrix elements are given for the Green's function having one external interaction line and one external quasi-particle line and for the Green's function having two external interaction lines. It is shown how these Green's functions can be used to calculate the chemical potential and the correlation function."
R. Balescu (Brussels)

Baxter, R. J. 1002
Direct correlation functions and their derivatives with respect to particle density.
J. Chem. Phys. **41** (1964), 553–558.
Author's summary: "An equation relating the many-body distribution functions of a classical statistical mechanical system and their derivatives with respect to the particle density is stated and proved. It is found possible to define a hierarchy of symmetric functions of 2, 3, 4, etc., variables, of which the two-variable function is the Ornstein-Zernike correlation function. These functions, termed the direct correlation functions, and their derivatives with respect to density satisfy a strikingly simple relation. The possibility of a closure of the equations is discussed and a superposition approximation suggested. The resulting equation for the two-particle direct correlation function has the advantage that a knowledge of the function at one density predicts its values at another, thereby differing from most approximations, which require the solution of an integral equation in each case."

Balian, R.; De Dominicis, C. 1003
A generating function method for eliminating anomalous diagrams in statistical mechanics.
Physica **30** (1964), 293–302.
Authors' summary: "A new derivation is given for the expansion of the Gibbs potential and thermal Green's function in terms of the true occupation numbers. In this derivation, the so-called anomalous diagrams are eliminated directly with the aid of a generating function. This procedure casts into evidence the reason for the formal resemblance of the results with the zero-temperature expansions. The analogy between the present method and the proof of the Hugenholtz-Van Hove theorem is stressed."

Zalewski, K. 1004
Remark on the exact solution of the master equation for a model proposed by Janner, Van Hove and Verboven.
Physica **30** (1964), 475–478.
Author's summary: "The presence of slowly damped oscillations in the Janner-Van Hove-Verboven strong coupling solution is shown to be a consequence of the Paley-Wiener theorem. The occurrence of similar terms is predicted for arbitrary coupling strength."

Baer, S.; Lebowitz, J. L. 1005
Convergence of fugacity expansion and bounds on molecular distributions for mixtures.
J. Chem. Phys. **40** (1964), 3474–3478.
Authors' summary: "We have generalized the methods developed recently by Groeneveld and Penrose for a one-component system to obtain a lower bound on the domain of convergence of the Mayer fugacity expansion of the pressure,

$$p = \sum b_{l_1,\cdots,l_\omega} \prod z_\alpha^{l_\alpha},$$

where z_α is the fugacity of the αth component, $\alpha = 1, \cdots, \omega$. This series is convergent for

$$\sum_{\alpha=1}^{\omega} |z_\alpha| \leq [\exp(1+2\Phi/kT)B]^{-1},$$

where

$$B = \{\max(\alpha,\beta)\} \int |\exp[-(kT)^{-1}\phi_{\alpha\beta}(r)] - 1| \, dr$$

and where the interaction potential $\phi_{\alpha\beta}(r)$ of a pair of particles of species α and β satisfies

$$\sum_{1 \leq i < j \leq s} \phi_{\alpha_i\alpha_j}(|x_i - x_j|) \geqq -s\Phi$$

for all α, x, and s. (For a positive interparticle potential, $\phi_{\alpha\beta}(r) \geqq 0$, $\Phi = 0$.) Consequently the system remains in a single phase in this region. We have also generalized the inequalities of Lieb, Penrose, Lebowitz, and Percus to this case. For positive potentials upper [lower] bounds are gotten for the pressure and the distribution functions by expanding in a Taylor series up to terms of even [odd] total order in the fugacities."

Soda, Toshio 1006
Some dynamical properties of an impurity in the hard-sphere Bose gas.
J. Mathematical Phys. **5** (1964), 769–774.
By starting from the Hamiltonian taken from his previous work [same J. **5** (1964), 142–146; errata, ibid. **5** (1964), 1000; MR **28** #917] the author discusses the dynamical properties of the impurities in the hard-sphere Bose gas. First, the effective potential between two impurities via exchange of a quasi-particle is calculated with the result that it is of the Yukawa type in addition to the hard-core. The necessary conditions for the existence of a bound state between two impurities are then studied. It is shown that the impurity mass must be at least nine times as large as that of the Boson, and the Bose gas density must

fall into a certain range. Then the elastic and inelastic cross-sections for the scattering of a quasi-particle on an impurity are calculated. Finally the Cerenkov radiation, which is the emission of a quasi-particle from an impurity travelling in the Bose gas faster than a certain critical speed, is discussed. *K. Nishijima* (Champaign, Ill.)

Götze, Wolfgang 1007

On transport theory of Bose systems. I, II.

Z. Physik **177** (1964), 300–319; *ibid.* **177** (1964), 348–359.

Author's summary: "(I) The concept of thermodynamical correlation functions for non-equilibrium states used by Kadanoff and Baym [*Quantum statistical mechanics*, Benjamin, New York, 1962] in transport theory is generalized by means of Bogoliubov's quasi-averages in order to describe transport phenomena in systems with Bose-Einstein condensation. The one-particle Green's function is decomposed into one part describing the condensed particles and a second one describing the elementary excitations. The coupled equations of motion for these functions are derived and some consequences for the case of slowly varying disturbances are discussed. (II) The self-energies defined in the [first] paper are calculated explicitly for a dilute Bose gas with weak repulsive interaction. The transport equations are written down and the coupled system of kinetic equations are solved in order to calculate the propagation of normal and second sound." *R. Balescu* (Brussels)

Bocchieri, P.; Loinger, A. 1008

Remarks on a theorem in classical ensemble theory.

Phys. Lett. **4** (1963), 188–190.

The authors clarify some of the implications of their ergodic theorem [S. Albertoni and the authors, J. Mathematical Phys. **1** (1960), 244–248; MR **22** #12200] to discuss the objections raised by others [J. S. Lomont, ibid. **2** (1961), 858–860; MR **26** #4735; A. Galindo, ibid. **3** (1962), 1185–1190; MR **26** #5919].

N. L. Balazs (Stony Brook, N.Y.)

Leff, Harvey S. 1009

Systematic characterization of mth-order energy-level spacing distributions.

J. Mathematical Phys. **5** (1964), 756–762.

Energy-level spacing distributions for complex spectra are defined by a general joint probability distribution for N consecutive eigenvalues. The limiting processes involved are explained and used to obtain formal representations of the energy-level spacing distribution for which also general asymptotic results are found.

D. ter Haar (Oxford)

Leff, Harvey S. 1010

Class of ensembles in the statistical theory of energy-level spectra.

J. Mathematical Phys. **5** (1964), 763–768.

Nearest-neighbour spacing distributions and level density distributions are obtained for particular cases of systems characterized by the joint probability density for N consecutive eigenvalues. The results are given in terms of asymptotic properties of orthogonal polynomials.

D. ter Haar (Oxford)

Lebowitz, J. L.; Penrose, O. 1011

Convergence of virial expansions.

J. Mathematical Phys. **5** (1964), 841–847.

By eliminating the fugacity z from the Mayer cluster expansions of the pressure p and the density ρ of a real gas confined to a volume V, one obtains the finite-volume virial expansion of p in powers of ρ, which involves volume-dependent coefficients $\beta_k(V)$. (The temperature enters these discussions only parametrically.) By replacing each $\beta_k(V)$ by its limit as V goes to infinity, one obtains the conventional virial expansion.

The authors investigate the radius of convergence $R(V)$ of the finite-volume virial expansion, as well as the radius of convergence R of the conventional expansion. Lower bounds are obtained for both these, and upper bounds are obtained for the special case that the intermolecular potential has a "hard core". The value of the density ρ at a phase transition must exceed $\liminf_{V\to\infty} R(V)$, but need not exceed (or be related in any way to) R.

To this reviewer, the most interesting conclusion (supported by a specific example) is: "the actual values of R and of $\liminf_{V\to\infty} R(V)$ have, in general, no physical significance since they may be determined by singularities off the real positive ρ-axis". Thus, divergence of the virial expansion at a certain density cannot be taken as proof of a phase transition at that density.

J. M. Blatt (Kensington)

Wilson, Th. A. 1012

Minimum principle for a nonequilibrium stochastic system.

Physica **30** (1964), 493–496.

Author's summary: "An urn model of a linear stochastic system is investigated. By imposing a boundary condition, the regular addition or removal of balls from certain urns, a steady nonequilibrium distribution can be induced. The quantity which is a minimum when the state of the system is steady is the square of the difference of the contents of two urns, multiplied by the probability of transfer between the urns and summed over all urn pairs. The minimum quantity is approximately equal to the entropy production rate if the departure from equilibrium is small."

Henry, J. 1013

Interaction entre systèmes de spins par l'intermédiaire du réseau. (English summary)

Acad. Roy. Belg. Bull. Cl. Sci. (5) **49** (1963), 697–707.

Author's summary: "The evolution of n-systems of spins in interaction via the lattice is considered. Equations for the evolution of the n spin temperatures $T_s^{(i)}$ are established and solved. Generally speaking, the inverse of each spin temperature varies in time as the sum of n exponentials. The relaxation times of these exponentials can be expressed in terms of transition probabilities corresponding to spin-spin interaction via the lattice. The case of two spin systems is considered in more detail."

Katsura, Shigetoshi; Inawashiro, Sakari 1014

Linear Heisenberg model of ferro- and antiferromagnetism.

J. Mathematical Phys. **5** (1964), 1091–1106.

The partition function and other thermal functions of the Heisenberg Hamiltonian of a magnetic linear chain are evaluated by second-order thermodynamic perturbation theory, with the Z-component interactions as the perturbation. *D. Mattis* (Yorktown Heights, N.Y.)

Wilde, Kenneth A. 1015
Anharmonicity in vibrational state sums.
J. Chem. Phys. **41** (1964), 448–451.
Author's summary: "Refinements have been made in the treatment of vibrational state sums. These modifications are particularly important at energies greater than the dissociation energies. It was not possible to extend the available analytical treatments of harmonic oscillators, and numerical sums were computed for several anharmonic examples and compared with previous work. Values of the empirical correction factor a were given and the previously derived approximate equations involving a extended to the anharmonic case."

Jones, Gerald L. 1016
The influence of initial correlations on the approach to equilibrium.
J. Mathematical Phys. **5** (1964), 651–658.
In the case of a simple model (Lorentz gas) some explicit calculations are performed on the problem of the decay of initial correlations in a gas. The method used is essentially the one developed by R. Zwanzig [*Lectures in theoretical physics*, Vol. III, pp. 106–141, Interscience, New York, 1961; MR **23** #B937]. *R. Balescu* (Brussels)

Naze, J. 1017
Application de la méthode des valeurs propres à l'équation de Boltzmann des gaz faiblement ionisés.
Rarefied Gas Dynamics (Proc. 3rd Internat. Sympos., Palais de l'UNESCO, Paris, 1962), *Vol. I, pp.* 252–259. *Academic Press, New York*, 1963.
Assuming weak ionization of a gas whose molecules are themselves in thermal equilibrium, the ions and electrons obey the linear Boltzmann equation. If the charged particles interact with the neutral molecules according to an induced-dipole interaction, $1/r^5$, the eigenfunctions and eigenvalues of the collision operator are known [Wang Chang and Uhlenbeck, "On the Propagation of Sound in Monatomic Gases", Engrg. Res. Inst. Univ. Michigan, Ann Arbor, Mich., 1952]. Exact solutions of the transport equations for mean local velocity (leading to the conductivity) and temperature can then be obtained, consisting of a forced term (apparently due to an oscillating electric field and a constant magnetic field, but the paper lacks details at this point) and a transient term. The relaxation times are eigenvalues of the collision operator. *A. Siegel* (Boston, Mass.)

İmre, Kaya; Özizmir, Ercüment 1018
Correlations in plasmas. I. Ternary correlations.
Nuclear Fusion **4** (1964), 1–9.
Authors' summary: "The kinetic stage of an interacting, multi-component system is studied using Bogoliubov's formalism including terms up to the second order in interaction strength λ, but to all orders in $n\lambda$, n being the density. It is shown that the second-order s-particle distribution function can be expressed in terms of zero- and first-order binary and zero-order ternary correlation functionals. This functional form is also shown to be deducible in a prescribed manner from the so-called cluster expansion series of the equilibrium theory. The latter observation provides a means for guessing the structures of the higher order distribution functionals. The equations satisfied by the first-order binary and zero-order ternary correlation functionals are derived. They constitute a coupled set of equations determining the second-order contribution to the kinetic equation. Explicit expressions are found for the correlation functions when the one-particle distribution function is Maxwellian. The results, in this case, are in agreement with those obtained from the equilibrium theory."

Laval, Guy; Pellat, René 1019
Équations d'évolution de la fonction de distribution à une particule d'un plasma stable ou faiblement instable.
C. R. Acad. Sci. Paris **258** (1964), 3997–4000.
The BGKY hierarchy is solved approximately for a homogeneous electron plasma in absence of external fields. It is shown that in this case the solution can be reduced to the solution of the linearized Vlasov equation. The knowledge of the latter allows one to obtain the two-body correlation, and hence the kinetic equation, as well for stable as for unstable plasmas. It is shown that in the latter case the equation derived by the reviewer [J. Mathematical Phys. **4** (1963), 1009–1019] is obtained, including a contribution from the initial correlations which is neglected by the latter author. It is also shown that by dropping certain terms from this equation, the equations obtained by D. Pines and J. R. Schrieffer [Phys. Rev. (2) **125** (1962), 804–812; MR **25** #1899], by W. E. Drummond and D. Pines [(Plasma Phys. and Controlled Nuclear Fusion Res. Conf. Proc., Salzburg, 1961) Nuclear Fusion (1962), suppl., 1049–1057], and by A. A. Vedenov [Plasma Phys.-Accelerators-Thermonuclear Res. **5** (1963), no. 3, 169–186] are obtained. This relation among the various existing kinetic equations for unstable plasmas is especially interesting. *R. Balescu* (Brussels)

Sestero, Arrigo 1020
Structure of plasma sheaths.
Phys. Fluids **7** (1964), 44–51.
Author's summary: "A model is given to provide a microscopic description of plasma sheaths, using the Vlasov equation for ions and electrons coupled with Maxwell's equations for the fields. The sheaths considered connect two different constant states of a plasma in a magnetic field. In the charge-neutral approximation a solution is obtained in closed form in terms of quadratures. The solution depends essentially on three free parameters, related to the differences in density and electric potential between the two end states, and to the value of β, the ratio of the plasma pressure to the magnetic pressure, on one side of the sheath. The scaling law changes considerably from one solution to another. Solutions are obtained which scale according to some representative electron Larmor radius, or ion Larmor radius, or piecewise according to both. As a limiting case, when the density is allowed to go to zero on one side of the sheath, one obtains the

boundary layer between a plasma and a containing vacuum magnetic field. As a peculiar feature in this limiting situation, the curve for the electric potential appears to be very sensitive to the slightest changes in the distribution functions, in the asymptotic region where the plasma density becomes vanishingly small."

D. C. Montgomery (College Park, Md.)

McQuarrie, Donald A. 1021

Virial coefficients and the classical theory of fluids for the square-well potential.

J. Chem. Phys. **40** (1964), 3455–3471.

Author's summary: "The term $g_2(r)$ proportional to the square of the density in the expansion of the radial distribution function in powers of the density has been evaluated for a square-well potential for the Born-Green-Yvon integral equation, the Kirkwood integral equation, the Percus-Yevick integral equation, the hyperchain approximation, and an integral equation derived by Meeron. The square-well potential used has a hard-core diameter of σ and attractive diameter of 2σ. Each $g_2(r)$ is then used to calculate D_p, the fourth virial coefficient calculated from the pressure equation, and D_c, that calculated from the compressibility equation, for the above five liquid theories. This allows a comparison over the entire temperature range, rather than at infinite temperature, the only value given by the previously used hard sphere potential. It is shown that the Percus-Yevick equation yields virial coefficients in best agreement with the exact value over a large temperature range. All the results presented are low-order polynomials in $f=\exp(\varepsilon/kT-1)$, where ε is the depth of the well, and reduce to the hard-sphere results when f goes to zero."

Widom, B. 1022

On the radial distribution function in fluids.

J. Chem. Phys. **41** (1964), 74–77.

Author's summary: "It is shown that if $g(r)$ is the radial distribution function and $\phi(r)$ the pair potential, then the function $F(r)=g(r)\exp[\phi(r)/kT]-1$ has a number of properties analogous to those of the pair correlation function $G(r)=g(r)-1$ which are known from the Ornstein-Zernike theory. The integral of $F(r)$ over all space is related to the fluctuations in a certain well-defined physical quantity, in the same way that the integral of $G(r)$ is related to fluctuations in density. The integral of $F(r)$, furthermore, is shown to be always positive, so that on the average $g(r)$ exceeds $\exp[-\phi(r)/kT]$. Finally, it is shown that if one makes a hypothesis analogous to one which is familiar in the Ornstein-Zernike theory, then it follows that the pair correlation function $G(r)$ cannot vanish more rapidly than the pair potential $\phi(r)$ as $r\to\infty$."

Leimdörfer, Martin 1023

On the transformation of the transport equation for solving deep penetration problems by the Monte Carlo method.

Chalmers Tekn. Högsk. Handl. No. 286 (1964), 12 *pp.*

Author's summary: "The present report describes a modification of the 'exponential transformation' of the Boltzmann transport equation which removes a singularity in the transformed scattering kernel, discussed by the author in an earlier paper. The effect is obtained by limiting the tracks of the particles to the problem boundary, whereby a zero 'virtual' cross-section may be permitted. A sampling technique for the new inter-collision distance is given."

Leimdörfer, Martin 1024

On the use of Monte Carlo methods for calculating the deep penetration of neutrons in shields.

Chalmers Tekn. Högsk. Handl. No. 287 (1964), 14 *pp.*

Author's summary: "The exponential transformation, previously applied by the author to problems concerning deep penetration of photons in plane geometry, is here applied to problems of deep penetration of neutrons in spherical geometry. The representation of nuclear data is briefly described and the extension of the exponential transformation technique to spherical geometry is discussed on a quite general basis, to show the proper way of derivation in other systems of coordinates. In a sample problem, concerning the dose rate versus distance (up to 210 cm) from a point fission source in concrete, the present method is compared with the Moments Method and excellent agreement is shown."

Résibois, P.; Dagonnier, R. 1025

On the hydrodynamics of a weakly interacting Fermi gas.

Physica **29** (1963), 1057–1078.

A general transport equation for weakly interacting fermions is derived for a hydrodynamically inhomogeneous system with an arbitrary characteristic length of the inhomogeneity. This transport equation covers a wide range of phenomena including macroscopic transport phenomena and zero sound. Instead of the ordinary n-representation (the occupation number representation) of the density operator, the (ν, N)-representation $(N=(n+n')/2,\ \nu=n-n')$ is used. This representation enables us to select easily the relevant processes in the dynamical behaviour of the system on the basis of the formalism developed by Prigogine and Résibois, and it plays an essential rôle in the derivation of this transport equation. In particular, it permits us to formulate in a simple way the influence of "localization" of energy and momentum in connection with their conservation laws. As an example, it is shown that the attenuation of zero sound at high frequencies can be understood in terms of ordinary two-particle interaction, provided the finite duration of the collision and the localization effects are correctly taken into account.

S. Ono (Tokyo)

Résibois, P.; Dagonnier, R. 1026

On the theory of impurity resistance in quantum systems.

Acad. Roy. Belg. Bull. Cl. Sci. (5) **49** (1963), 954–972.

Authors' summary: "Using the special representation introduced by I. Prigogine and S. Ono [Physica **25** (1959), 171–178; MR **23** #B932], we derive an exact formula for the conductivity tensor resulting from the scattering of a free electron by randomly distributed impurities. This formula is the exact analogue of the general expression obtained by R. Balescu [ibid. **27** (1961), 693–706; MR **24** #B335] for classical systems. This formula is applied to the explicit calculation of this conductivity tensor up to the order λ^0. Special attention is paid to the λ^0 correction

coming from a fourth-order transition, which has a singular behavior in the limit of an infinite system, although no intermediate are identical with each other."

Bellman, Richard; Kalaba, Robert 1027
Invariant imbedding and the integration of Hamilton's equations.
Rend. Circ. Mat. Palermo (2) **12** (1963), 172–182.
In a series of papers [cf. the authors and Wing, J. Mathematical Phys. **1** (1960), 280–308; MR **22** #7339], the authors have successfully applied the invariant imbedding technique to various kinds of transport problems of current interest. In these problems it is easy to derive the functional equations for certain reflection and transmission functions. In this paper it is shown that, by constructing a countercurrent flow model associated with each dynamical process [cf. the authors, Quart. Appl. Math. **21** (1963), 166–168; MR **27** #6421], one can express the solution of the canonical equations of motion in terms of these reflection and transmission functions. Finally, harmonic motion and the motion of a particle in uniform and central inverse-square force fields are considered as examples of this approach. *S. Ueno* (Kyoto)

Petschek, Albert G. 1028
Fluctuations in multiple capture processes.
J. Mathematical Phys. **5** (1964), 998–999.
Author's summary: "The fluctuation in the number of many-fold captures by an element exposed to a high neutron flux is computed. It is found that the fluctuations are identical to those that would be obtained by selecting nuclei at random from an infinite supply with the appropriate average composition."

RELATIVITY

See also 572, 979, 1050.

Podlaha, M. 1029
Note on the axioms of the special theory of relativity.
Acta Phys. Austriaca **17** (1963/64), 266–270.
The author derives the Lorentz transformation from two axioms: (i) the principle of relativity, and (ii) the principle of time-dilatation. (i) means that no one uniformly moving frame of reference is singled out by experiments; (ii) asserts the validity of the usual dilatation formula, as applied to a pair of events whose spatial separation is orthogonal (in one frame of reference) to the relative velocity of the second frame. The author apparently feels that, whereas the experimental evidence for these axioms is impeccable, the evidence for the more usual assumption (that the velocity of light is independent of the motion of its source) is still open to question.
R. H. Boyer (Liverpool)

Fedorov, F. I. 1030
Composition of parameters of the Lorentz group. (Russian)
Dokl. Akad. Nauk SSSR **143** (1962), 56–59.
In two of his previous papers [Dokl. Akad. Nauk BSSR **5** (1961), 101–104; MR **26** #2287; ibid. **5** (1961), 194–198; MR **26** #3788] the author has introduced a special parametrization of the Lorentz group of motions consisting in a correspondence between complex 3-vectors $\mathbf{q}=\mathbf{a}+i\mathbf{b}$ and the Lorentz group, and has shown that to every Lorentz group matrix can be assigned a properly chosen 3-vector $\mathbf{q}$.
In this paper, this parametrization is analyzed in detail and the author derives certain conclusions, e.g., that by means of the law of composition of these 3-vectors one can obtain the velocity addition formula of special relativity and many other relations between the Lorentz group and the Lorentz matrix. *T. P. Anđelić* (Belgrade)

Romain, J. E. 1031
Rigidity criteria for one-dimensional translations. (Italian summary)
Nuovo Cimento (10) **31** (1964). 1060–1069.
Author's summary: "A special-relativistic rigidity criterion for a body undergoing a nonuniform one-dimensional translation has recently been proposed by Evett in the form of a differential equation. An integrated form of this criterion is given. Evett's definition is compared with the classical concept of rigidity, with Born's definition, with 'radar' rigidity and with a stronger form of the latter in which the velocity of light in the accelerated frame of reference attached to the body is required to be independent of time. It turns out that there exists one particular type of motion for which Evett's definition and the classical definition are simultaneously satisfied, and one type of motion for which both Evett's and Born's criteria are satisfied. Any law of motion that satisfies Evett's requirement is compatible with both the weaker and the stronger form of 'radar' rigidity."
N. L. Balazs (Stony Brook, N.Y.)

Kulhánek, J. 1032
On the theory of de Broglie waves in space-time. (Italian summary)
Nuovo Cimento (10) **31** (1964), 978–985.
Continuing the method of a previous paper [Nuovo Cimento (10) **16** (1960), 1092–1097; MR **22** #10737] the author uses a generally invariant Hamiltonian $H=(g_{\mu\nu}p^\nu p^\mu)^{1/2}$ for the motion of a point particle, and writes down the covariant Hamilton-Jacobi equation, using the proper time s as an independent variable. Then the 4-velocity of the particle is one. It is shown that the Hamilton-Jacobi equation can be regarded as the characteristic equation of a class of non-linear second-order differential equations, and in this way a variety of wave-like equations can be formally associated with the motion of a particle. *K. J. Le Couteur* (Canberra)

Salija, R. N. 1033
A property of the energy-momentum tensor of a gravitational field. (Russian. Georgian summary)
Soobšč. Akad. Nauk Gruzin. SSR **32** (1963), 555–558.
The author shows that the four-momentum vector P_i corresponding to an energy-momentum tensor invariant under space transformations

$$x^{l'} = x^{l'}(x^k), \quad x^{4'} = x^4,$$

is not a four-vector with respect to Lorentz transformations in a continuum with a Schwarzschild metric.
A. H. Klotz (Liverpool)

Kühnel, Adolf 1034
A difference between the two approximation methods in general relativity.
Acta Phys. Polon. **24** (1963), 399–405.
Two different perturbation schemes are generally used to solve the equation of general relativity: the method of Einstein, Infeld and Hoffmann (EIH) assumes slow motion and expands every quantity in powers of c^{-1}; in the "fast motion approximation" (fma) one demands only small deviations from a Minkowsky metric and expands them in powers of the gravitational constant G. The author compares the two methods and shows that, as expected, Newton's law of motion can be deduced from the linear fma by keeping only the terms in the field of order c^{-2}. He solves for the field in the second-order fma and points out that, by expanding the latter in powers of c^{-1}, one does not obtain a term of order c^{-4} (linear in the gravitational constant) which appears instead in the corresponding expression in the EIH method. {The reviewer considers this remark as trivial. It is no wonder that the second approximation of an expansion in powers of G does not include terms linear in G; the missing term, which corresponds to a retardation effect, is to be found in the linear fma when terms of order c^{-4} are considered [see also the reviewer, Nuovo Cimento (10) **4** (1956), 898–906; MR **18**, 703].}
B. Bertotti (Frascati)

Edelen, Dominic G. B. 1035
Deformations and momentum-energy complexes.
Arch. Rational Mech. Anal. **16** (1964), 316–324.
In this paper the author (1) assumes that the metric tensor h_{AB} of the space-time is unknown (the reviewer is indebted to the author for this statement, which is not in the paper), (2) assumes that a field Y^A is known, the trajectories of which constitute a congruence normal with respect to the (unknown) metric, (3) assumes that the Lie derivative $\underset{Y}{\pounds}\, h_{AB}$ of the (unknown) tensor h_{AB} is known. The main aim of the paper is to get as much information as possible on the momentum-energy tensor from these few given data. To this effect he derives first the (unknown) covariant derivative $U_{A;B}$ of the (unknown) unit vector U_A of Y^A and then finds the integrability conditions connected with $U_{A;B}$. In this way he obtains the vector $\lambda_C = {}_{\mathrm{def}} B_{CA} U^A$, where B_{CA} is the (unknown) contracted curvature tensor and finally the momentum-energy tensor (for $n > 3$)

$$T_{AB} = \frac{2}{\kappa}\lambda_{(A}U_{B)} + \frac{1}{n-3}\left[S - \frac{n-4}{\kappa}\lambda_E U^E\right] U_A U_B + S_{AB}.$$

Here κ is the general relativistic gravitational constant, S_{AB} is an arbitrary symmetric tensor satisfying $S_{AB}Y^B = 0$, and $S = S_E{}^E$. This formula is applied among other things to the investigation of the time-like eigenvector of T_{AB}.
V. Hlavatý (Bloomington, Ind.)

Roche, Claude 1036
Les équations de Hamilton en relativité générale.
C. R. Acad. Sci. Paris **258** (1964), 5359–5362.
The Hamiltonian formalism for the gravitational field [P. A. M. Dirac, Proc. Roy. Soc. London Ser. A **246** (1958), 333–343; MR **20** #725] is applied to the case of small deviations from a given Einstein metric. *A. Peres* (Haifa)

Upadhyay, M. D.; Mishra, R. S. 1037
Maxwell fields.
J. Math. Pures Appl. (9) **43** (1964), 67–85.
The following problem was proposed by Hlavatý [Ann. Mat. Pura Appl. (4) **53** (1961), 21–39; MR **24** #B804]: Given a four-dimensional Riemannian metric $g_{\alpha\beta}$, the Ricci tensor of which is $R_{\alpha\beta}$, find an antisymmetric tensor $F_{\alpha\beta}$ (the electromagnetic field), a unit vector v^α (the matter velocity) and three scalars μ (the matter density), ε (the charge density) and p (the pressure) such that the Einstein-Maxwell equations

$$R^{\alpha\beta} - \tfrac{1}{2}Rg^{\alpha\beta} = \mu u^\alpha u^\beta + pg^{\alpha\beta} + \tfrac{1}{4}g^{\alpha\beta}F_{\gamma\delta}F^{\delta\gamma} - F^\alpha{}_\gamma F^{\gamma\beta},$$

$$V_\alpha F^{\alpha\beta} = \varepsilon u^\beta,$$

are satisfied.
Hlavatý solved this problem in the case where $F_{\alpha\beta}$ is not a null field (i.e., at least one of its invariants does not vanish). In the present paper, the authors solve this problem for null electromagnetic fields. *A. Peres* (Haifa)

Mishra, R. S. 1038
Geometry of electromagnetic null field.
Rend. Circ. Mat. Palermo (2) **12** (1963), 155–171.
Canonical forms are derived for the field of the title (i.e., for an antisymmetric second rank tensor, in Minkowski or Riemann space-time, with both invariants zero). Hlavatý found similar results by other methods [*Geometry of Einstein's unified field theory*, Noordhoff, Groningen, 1957; MR **20** #5067]. *F. A. E. Pirani* (Waltham, Mass.)

Bouche, Liane 1039
Étude d'une généralisation de la théorie du champ unifié relativiste d'Einstein-Schrödinger.
Ann. Inst. H. Poincaré **18**, 1–97 (1963).
The author assumes that the connection of the theory has the torsion vector a priori equal to zero and tries to find the equations of motion by means of a variational principle. The corresponding condition of the conservation (i.e., the divergence-free momentum-stress energy tensor) can be checked only approximately because of the complexity of the calculation. The author hoped to be able to find in the approximation the Lorentz force vector using this new approach, but, according to her, this hope "est donc réduit à néant dès que les équations antisymétriques du champ sont vérifiées". This short description of the highlights of this long paper (almost 100 pages) is by no means an adequate image of the broad frame and of the richness of topics characterizing this work.
{The following reviewer's remarks refer to the reviewer's book, *Geometry of Einstein's unified field theory* [Noordhoff, Groningen, 1958; MR **20** #5067]: (1) The conformally invariant connection has the torsion vector equal to zero (Theorem 11.1, p. 158). (2) $g_{(\lambda\mu)}$ and the initial unified connection are "physically admissible" in the Schrödinger sense (p. 177) and give rise to the unified

law of inertia which leads in successive approximations to the Newtonian law of gravitation, to the Newtonian relativistic second law for the Lorentz force vector (Section 15, p. 190–195). (3) The approximation method usually applied by different authors to $g_{\lambda\mu}$ distorts the structure of this tensor (but does not distort the structure of $g_{(\lambda\mu)}$), and therefore can never lead to physically satisfactory results. However, there are finite equations of motion satisfying the law of conservation (Section 24, p. 212–213).} *V. Hlavatý* (Bloomington, Ind.)

Udeschini, Paolo 1040

Sopra una costante universale nella teoria unitaria di Einstein.

Ist. Lombardo Accad. Sci. Lett. Rend. A **97** (1963), 68–80.

In general relativity the contracted curvature tensor is a geometric object and is therefore dimensionless. The momentum-energy tensor (being a physical object) is not dimensionless. Therefore, in the field equations of this theory the momentum-energy tensor must be multiplied by a universal constant of physical dimensions which makes the product dimensionless.

In the Einstein unified field theory the gravitational field, as well as the electromagnetic field, is built up from the dimensionless tensor $g_{\lambda\mu} \neq g_{\mu\lambda}$. In particular, the electromagnetic field $f^{\lambda\nu}$ is identified with the dual tensor $*g^{[\lambda\nu]}$ to $g_{\lambda\nu}$. This tensor is obviously dimensionless, so that in the identification one must use a constant Ψ of physical dimensions such that the product on the right-hand side of $*g^{[\lambda\nu]} = \Psi f^{\lambda\nu}$ is dimensionless. In order to find the dimensions of Ψ the author represents $f^{\lambda\nu}$ by the electric vector **E** and magnetic vector **H** and uses the dimensionalities of **E** and **H**. In this way he obtains

$$[\Psi] = [L^{1/2}TM^{-1/2}\mu_{(0)}{}^{-1/2}],$$

where length, time, and matter are symbolized by L, T, M, and $\mu_{(0)}$ represents the magnetic permeability. In the second half of his paper the author uses his own and the reviewer's previous results in the unified field theory to compute Ψ. Because the description of this method requires rather lengthy explanations of the above-mentioned results, the interested reader is referred to the literature [e.g., the reviewer, J. Rational Mech. Anal. **3** (1954), 103–146; MR **15**, 654; ibid. **3** (1954), 147–179; MR **15**, 654; ibid. **3** (1954), 645–689; MR **16**, 408; ibid. **4** (1955), 247–277; MR **16**, 870; the author, Rend. Sem. Mat. Fis. Milano **27** (1957), 50–74; MR **21** #616]. *V. Hlavatý* (Bloomington, Ind.)

ASTRONOMY

See also 328, 329.

Blackman, R. B. 1041

Methods of orbit refinement.

Bell System Tech. J. **43** (1964), 885–909.

Author's summary: "During the past six or seven years, several methods of orbit refinement were developed specifically for use with artificial satellites and spacecraft. This article describes these methods and the classical method in a uniform mathematical formalism in order to facilitate comparisons of their relative advantages and disadvantages for practical systems applications. However, such comparisons are made in this article only to the extent that motivated the development of the new methods." *I. G. Izsak* (Cambridge, Mass.)

Geyling, F. T. 1042

Perturbation methods for satellite orbits.

Bell System Tech. J. **43** (1964), 847–884.

The author reviews a number of perturbation techniques proposed recently in connection with the orbits of artificial satellites. Unfortunately, he pays undue attention to sloppy engineering methods that are all too often claimed to be "simpler" than the sound classical methods of celestial mechanics. It must be said that such simplifications, if any, were achieved by oversimplifying the problem itself. *I. G. Izsak* (Cambridge, Mass.)

Idlis, G. M. 1043

The force between heavy radiating bodies. (Russian. English summary)

Akad. Nauk Kazah. SSR Izv. Astr. Inst. **10** (1960), 3–14.

The analysis of the general interaction of radiating heavenly bodies reveals that, in addition to the force of gravity and the force of a radiational pressure, there exists the force of the reaction which is inversely proportional to a distance attraction. The author gives some conclusions about a criticism of the hypothesis of an accretion and about the kinematics of the Metagalaxy. *M. Kiveliovitch* (Paris)

Danby, J. M. A. 1044

Stability of the triangular points in the elliptic restricted problem of three bodies.

Astronom. J. **69** (1964), 165–172.

The author is concerned with the elliptic restricted problem of three bodies in which the relative orbit of two finite bodies with mass ratio $\mu/(1-\mu)$ is an ellipse of eccentricity e, and a third body, of negligible mass, moves in their field of gravity. Possible equilibrium configurations of the system are equilateral triangles, the stability of which is investigated for general μ and e using the variational equations of the general equations of motion. These equations being homogeneous, with coefficients that are periodic functions of the time, two possible methods for solving them suggest themselves. The first is to use Floquet's theory and calculate the characteristic exponents; the second is to apply a theorem of Lyapunov according to which a transformation, with periodic coefficients and with the same period as before, can be found in such a way that the transformed differential equations have constant coefficients. In either method the solution is expanded in powers of μ or e. The author finds that for small μ the use of Floquet's theory is preferable while for small e that of Lyapunov's transformation. By combining analytical and numerical approaches he succeeds in determining in the (μ, e)-plane the region in which the triangular points are stable.

The reviewer would like to remark that it is still an

unsolved problem whether these equilateral triangle solutions are stable for all times in the sense of Lyapunov.

E. Leimanis (Vancouver, B.C.)

Szebehely, Victor; Giacaglia, Giorgio E. O. 1045

On the elliptic restricted problem of three bodies.

Astronom. J. **69** (1964), 230–235.

From the summary of the authors: "Allowing nonzero eccentricity for Jupiter's orbit, the differential equations of motion of the restricted problem are presented. The simple form of the equations is obtained by using the true anomaly of the primaries as the independent variable and by introducing a special set of dimensionless variables describing the position of the third body. The short- and long-time effects of the eccentricity of the orbits of the primaries are discussed in connection with the generalized Jacobi integral. The similarity between the equations for the circular and for the elliptic case permits the introduction of regularizing transformations following methods applicable for the circular case. The major effect of the eccentricity is that the regularized equations are in the form of integro-differential equations."

E. Leimanis (Vancouver, B.C.)

Simon, R. 1046

Sur l'instabilité gravitationnelle de Jeans. (English summary)

Acad. Roy. Belg. Bull. Cl. Sci. (5) **49** (1963), 927–934.

Author's summary: "This paper discusses some numerical results concerning the evolution of an initial disturbance of finite extent in Jeans's medium, when the linear dimension a of the initial perturbation is equal to or smaller than Jeans's critical length. The characteristic time of gravitational instability at the centre of the initial disturbance increases when a decreases, but it is found that this increase is logarithmic and therefore extremely slow."

Tassoul, J. L. 1047

Sur la stabilité gravitationnelle d'un plasma homogène, dissipatif et illimité. (English summary)

Acad. Roy. Belg. Bull. Cl. Sci. (5) **49** (1963), 935–946.

Author's summary: "This paper deals with the gravitational stability of an extended, homogeneous, viscous and thermally conducting plasma in nonuniform rotation. When the uniform magnetic field and the angular velocity are both at right angles to the direction of a wave perturbation, it is shown that Jeans's criterion is slightly modified by the thermal conductivity while viscosities and remaining forces have no effect on the appearance of instability. Furthermore, under the same assumptions, it is found that magnetic field and rotation have a stabilizing effect provided that the system is a perfect magnetic gas. In the last section, a general survey of gravitational stability for such a dissipative plasma is given, regardless of the directions of the magnetic field and angular velocity."

Liboff, R. L. 1048

A note on gravitational instability.

Phys. Lett. **3** (1962/63), 322–323.

En 1902, Jeans a étudié le cas de stabilité d'un nombre infini des points matériels qui s'attirent suivant la loi de Newton. L'analyse macroscopique a montré que le système n'est pas stable et le degré d'instabilité croit avec la longueur d'onde de la perturbation. L'étude microscopique du même problème entrepris récemment par l'auteur a montré qu'en dehors du résultat de Jeans, la perturbation disparaît pour une longueur d'onde plus petite que d (analogue à la distance de Debye), $d = C^2/\alpha m n_G$, C étant la vitesse thermique de l'équilibre, n_G est la densité numérique de l'équilibre, m la masse des éléments et α la constante de la force suivant laquelle la loi de Newton s'écrit: $\Delta G = -\alpha m n$, où G est le champ des forces. Dans cette note, l'auteur établit par une analyse microscopique le résultat de Jeans que le maximum de la croissance est $1/w_0$, où $w_0{}^2 = \alpha m n_G$, ce qui est réalisé à la limite pour des longueurs d'onde de la perturbation. La démonstration est obtenue en utilisant l'équation de Vlasov-Boltzmann linéarisée appliquée à une distribution de Maxwell liée à un champ de forces Newtonien.

M. Kiveliovitch (Paris)

Kopal, Zdeněk 1049

Radiative transport of heat in lunar and planetary interiors.

Icarus **3** (1964), 8–44.

Author's summary: "The aim of the present investigation has been to formulate and solve the problem of the radiative transfer of energy in solid globes of planetary size—both in their interiors as well as surface layers—with the radiative loss into space taken into account. Following a general statement of this problem in the introductory section, the time-dependent partial differential equation of radiative transfer has been formulated in Section II in terms of spherical polar coordinates, and a method developed which permits us to construct a solution of this equation, to an arbitrary order of accuracy, both for the emissivity or temperature as the dependent variable. In the latter case, the well-known equation of heat conduction is shown to obtain as a limiting case if the mean free path of the energy-carrying photons tends to zero.

"In the third section, the problem of the radiative transfer of energy in planetary interiors, with radiative loss at the boundary, is solved analytically for the case of a homogeneous globe characterized by constant absorption coefficient. It is pointed out that, in such a case, a suitable substitution can render the problem linear if the emissivity —rather than temperature—is adopted as the dependent variable; the problem is then solvable in terms of Fourier series arising by the inversion of a certain finite Fourier sine transform of the emissivity.

"In Section IV, the same technique is applied to the solution of a plane-parallel problem of radiative energy transfer in surface layers subject to a periodic insulation from above; and Section V contains the derivation of an explicit analytical solution of this problem in a closed form. The Appendix has been devoted to a formulation of the integral equation of the time-dependent and spherically symmetrical transfer problem, and the errors arising from the neglect of the time-dependence of the intensity (as distinct from that of the emissivity) of the radiation, committed by all earlier investigators of the problem, are explicitly evaluated."

Lifshitz, E. M. [Lifšic, E. M.]; Khalatnikov, I. M. [Halatnikov, I. M.] 1050
Problems of relativistic cosmology.
Uspehi Fiz. Nauk **80** (1963), 391–438 (*Russian*); *translated as Soviet Physics Uspekhi* **6** (1964), 495–522.

This paper gives a detailed account of a programme of fundamental work in cosmology carried out in recent papers by the authors [cf. e.g., Ž. Èksper. Teoret. Fiz. **39** (1960), 800–808; MR **23** #B414].

In Part I the authors set themselves the problem: how general to the solutions of the field equations of general relativity is that particular property possessed by the isotropic world models, viz., a time singularity in the finite past? The authors point out that the criterion for the generality of a solution is the number of physically arbitrary functions of the spatial coordinates that it contains, as distinct from those arbitrary functions that permit an arbitrary choice of the reference frame. These physically arbitrary functions in the general solution are shown to be eight in number.

To solve the problem the authors adopt, as can always be done, a "synchronous" reference frame in which

$$g_{0\alpha} = 0, \quad g_{00} = -1,$$

the space-time metric having then the form

$$-ds^2 = -dt^2 + dl^2, \quad dl^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta,$$

in which $\alpha, \beta = 1, 2, 3$. It is then demonstrated that in this reference frame the field equations of general relativity (without the cosmological constant) possess in their general solution a "fictitious" singularity occurring in the finite past at each spatial point, when the determinant of the metric vanishes. In general, however, the density and pressure of matter at each point remain finite, so that the singularity is not physical. The geometrical nature of the origin of this fictitious singularity, viz., the crossing of the geodesic lines $x^\alpha = \text{const}$, is pointed out. By a suitable choice of the spatial coordinates this crossing can be made simultaneous in coordinate time t throughout the reference frame. By transforming the reference frame 4-dimensionally, foregoing its synchronous property, the singularity may be removed.

In the special case of the isotropic world models the reference frame can be chosen both synchronous and co-moving with the constituent matter, unlike, for example, a homogeneous model with local spin. For this reason the singularity in the isotropic models is physically real. The authors succeed in generating a wide class of "special" solutions which are in general anisotropic (but without spin) and possess a real singularity. Such solutions, however, belong to a narrower class than the general solution, notwithstanding, as the authors observe, that such special solutions with a real singularity in the past may in fact be relevant to the real universe, where in the nature of things the initial conditions cannot be regarded as arbitrary. On the other hand, those special solutions with a real singularity in the future, such as an isotropic oscillating model in a stage of contraction, are unlikely to be realised in the real world since any perturbation from its special character would be sufficient to render the singularity fictitious.

This brings the authors to Part II in which the stability of the isotropic world models is examined in detail. The field equations are varied in a linear approximation under small perturbations. Three cases are considered, viz., perturbations of mass density, rotational perturbations and gravitational waves. The conclusions are that, generally speaking, the expansion of the world exerts a stabilising influence on the development of perturbations. The disturbance attenuates with time or remains constant in amplitude in all cases save one. This is the case of long wave perturbations of mass density when the change in density increases slowly with time. If the pressure is small, as in the later stages of the expansion of the universe, the perturbation in density can become large.

On the other hand, a contracting world would be essentially unstable, and perturbations in it must ultimately become large.

There are 8 appendices and 17 references.

W. Davidson (London)

GEOPHYSICS

See also 370, 859, 900.

Nemčinov, S. V. 1051
Investigation of the solution of the equation for the prediction of the pressure field of the atmosphere. (Russian)
Izv. Akad. Nauk SSSR Ser. Geofiz. **1960**, 1635–1641.

L'auteur expose une méthode de solutions d'un système des équations aux dérivées partielles du second ordre à coefficients constants qu'il a établit dans son mémoire sur "la solution des équations d'un champ de pression en vue d'une prévision" [mêmes Izv. **1959**, 1821–1830; MR **22** #5469].

M. Kiveliovitch (Paris)

Djubjuk, A. F. 1052
On the problem of prediction of the pressure field by a complete system of hydromechanical equations. (Russian)
Izv. Akad. Nauk SSSR Ser. Geofiz. **1960**, 1642–1648.

En utilisant les résultats précédents [Dokl. Akad. Nauk SSSR **123** (1958), 266–268] (sur la prévision hydrodynamique des champs basique et cinématique), l'auteur généralise le problème en introduisant les forces de frottement internes. Il construit en outre un système d'équations en coordonnées sphériques pour le cas local.

M. Kiveliovitch (Paris)

Chambers, Ll. G. 1053
Long waves on a rotating earth in the presence of a semi-infinite barrier.
Proc. Edinburgh Math. Soc. (2) **14** (1964/65), 25–31.

Author's summary: "The problem considered is that of long gravity waves approaching, from an arbitrary direction, a semi-infinite barrier, the whole system being in rotation. It is shown that the rotation gives rise to a wave in the shadow region whose amplitude depends upon the angle of incidence, but whose form is independent of it and which travels along the barrier without attenuation in that direction. The work is an extension and simplification of previous work by Crease, involving the use of methods previously developed by the author."

ECONOMICS, OPERATIONS RESEARCH, GAMES

See also 36, 673, 677, 678, 693, 709, 717, 1087, 1095.

Kornai, János; Lipták, Tamás 1054

Two-level planning: A game-theoretical model and iterative computing procedure for solving long-term planning problems of the national economy. (Hungarian. Russian and English summaries)

Magyar Tud. Akad. Mat. Kutató Int. Közl. 7 (1962), 577–621 (1963). (1 *insert*)

From the authors' summary: "The familiar simplex algorithm and the decomposition procedures so far worked out (Dantzig and Wolfe [Operations Res. 8 (1960), 101–111; Econometrica 29 (1961), 767–778; MR 25 #1953]), did not make it possible—at least under Hungarian circumstances—to compute linear programming problems suited to solving long-term planning problems of the national economy. Setting out from this fact, a new algorithm has been proposed which provides a solution to within any prescribed degree of accuracy of such problems, in a finite number of steps."

Krivenkov, Ju. P. 1055

A production model in dynamic programming. (Russian)

Dokl. Akad. Nauk SSSR 156 (1964), 752–755.

Properties of a continuous-type bottleneck model are investigated in order to eliminate the use of δ-functions and also to give a new classification of these problems, including cases previously not examined.

P. L. Ivănescu (Bucharest)

Flehinger, Betty J.; Miller, James 1056

Incentive contracts and price differential acceptance tests.

J. Amer. Statist. Assoc. 59 (1964), 149–159.

The authors develop the theory of the incentive contract, i.e., a contract in which the price paid for a product depends on the outcome of an acceptance test. In this way the acceptance test is viewed not merely as a tool for obtaining information but as a motivation for product improvement.

G. Weiss (Queens, N.Y.)

Pistoia, Angelo 1057

Sull'operatore non lineare $M(h)$. (French, English, Spanish, and German summaries)

Giorn. Ist. Ital. Attuari 26 (1963), 21–29.

This article is related to the work of R. E. Bellman and S. E. Dreyfus [*Applied dynamic programming*, Appendix IV, Princeton Univ. Press, Princeton, N.J., 1962; MR 25 #3791]. Assume that $h(x)>0$ $(x>0)$ and possesses continuous first and second derivatives. Also $d[h'(x)/h(x)]/dx<0$; $h'(x)/h(x)$ tends to infinity as x tends to zero; it tends to zero as x tends to infinity. The non-linear operator $H(z)=\mathrm{Max}_{x>0}\,[e^{-zx}h(x)]$ exists and is unique. For the class of positive and continuous functions $f(x)$ and for $z>0$, $M(f)=\mathrm{Max}_{x>0}\,[e^{-zx}f(x)]$ exists and is unique. For the two positive and continuous functions $f(x)$ and $g(x)$ the "composition"

$$f(x)*g(x) = \operatorname*{Max}_{0\leq y\leq x}\,[f(y)g(x-y)]$$

gives $M(f*g) = M(f)\cdot M(g)$. Applications to capital investment and probability theory are indicated, as well as a generalisation for functions of several variables.

G. Tintner (Los Angeles, Calif.)

Nikaidô, Hukukane 1058

Generalized gross substitutability and extremization.

Advances in game theory, pp. 55–68. *Princeton Univ. Press, Princeton, N.J.*, 1964.

The author studies the existence of equilibrium and related questions without using fixed-point theorems and related topological methods. It is necessary to make stronger assumptions than usual on the excess demand functions, specifically, that they satisfy the condition that $(\partial E_i/p_j)+(\partial E_j/p_i)\geqq 0$ for $i\neq j$, where E_i is the excess demand for, and p_i the price of, the ith commodity. This assumption is weaker than the much-used assumption of gross substitutability, $\partial E_i/\partial p_j\geqq 0$ for $i\neq j$, and is referred to by the author as generalized gross substitutability. The domain over which prices vary is more general than usual; P is a nonempty subcone of the non-negative orthant with origin deleted, and is open in that orthant. The excess demand functions are homogeneous of any degree between 0 and 1 inclusive (instead of only degree 0, as is usual). It is then demonstrated that an equilibrium always exists provided that a certain boundary condition is satisfied, specifically, that for any nonzero boundary point $\bar{p}\in\bar{P}$, $\notin P$, and for any sequence $\{p^\nu\}$ converging to $\bar{p}$, there exists some $q\in P$, such that

$$\sum_{i=1}^{n}\bar{p}_i = \sum_{i=1}^{n} q_i, \qquad \Phi(q) < \limsup_{\nu\to\infty}\Phi(p^\nu),$$

where

$$\Phi(p) = \sum_{i=1}^{n}\{\max[E_i(p), 0]\}^2.$$

The extent to which the functions $E_i(p)$ are determined by their values on the boundary of P is studied; in particular, if $E_i(p)$ is homogeneous of degree less than 1 and if P includes all nonzero, non-negative vectors, then under the above assumptions, they must be identically zero. Finally, the stability of a generalized form of the Brown-von Neumann equations for arriving at an equilibrium is demonstrated.

K. J. Arrow (Athens)

Briggs, F. E. A. 1059

Solution of the Hitchcock problem with one single row capacity constraint per row by the Ford-Fulkerson method.

Naval Res. Logist. Quart. 9 (1962), 107–120.

This paper is concerned with a computation procedure for the Hitchcock transportation problem with capacity constraints of the type that the sum of certain elements in a row has an upper bound.

Formally, the problem is the following. Maximize

$$z = \sum_{i\in M}\sum_{j\in N} d_{ij}x_{ij}$$

given $\sum_{j\in N} x_{ij}=a_i$ $(i\in M)$, $\sum_{i\in M} x_{ij}=b_j$ $(j\in N)$, $\sum_{j\in S_i}x_{ij}\leqq c_i$ $(i\in M)$, $x_{ij}\geqq 0$ $(i\in M, j\in N)$.

The algorithm is along the lines of those given by Ford and Fulkerson for other variants of the problem.

N. S. Mendelsohn (Winnipeg, Man.)

Young, Henry A. 1060

On the optimum location of checking stations.

Operations Res. **11** (1963), 721–731.

From the author's summary: "Items move uniformly from the initial to the terminal point of a route. Checking stations that detect and repair imperfect items can be located along the route. If the condition of an item changes from 'perfect' to 'imperfect', an operating cost is incurred which increases with the distance traveled in the imperfect condition to the nearest checking station. Given an arbitrary increasing operating cost function, data on the likelihood of failure of an item at any point enroute, the volume of traffic per unit time and the cost per unit time of maintaining a checking station, the dynamic programming technique is used to find the number and placement of checking stations which minimize the sum of operating and checking costs."

With the added assumption that the cumulative probability of breakdown is a constant multiple of distance traveled, it is shown that equidistant location of checking stations is optimal, and an algorithm for calculation of the optimal number of stations as a function of route length is constructed. *A. Wouk* (Madison, Wis.)

Zehna, Peter W. 1061

Inventory depletion policies.

Studies in applied probability and management science, *pp.* 263–287. *Stanford Univ. Press, Stanford, Calif.*, 1962.

A stockpile has n items of given ages. The field life of an item is a known function $L(S)$ of the age S of the item when issued. A new item is issued when the life of the previously issued item is ended. The problem is to determine an issue policy that maximizes the total field life, regardless of the specific initial ages of the items. Such a policy, if it exists, is called optimal. C. Derman and M. Klein [Management Sci. **4** (1958), 450–456] and G. J. Lieberman [ibid. **5** (1958), 102–105; MR **20** #5084] have investigated conditions on $L(S)$ for which either the policy LIFO (last in, first out) or FIFO (first in, first out) is optimal. The author extends these results and also investigates related stochastic models, in which the field life is a random variable with known distribution. Only special results are obtained when the other assumptions of the model are not relaxed. More general results are obtained when the modification is introduced that an item is issued regularly at a given interval t, regardless of whether the previously issued item is exhausted. An issue policy is again optimal if it maximizes total utility.

A further stochastic process model of a continuing operation is suggested in which the stockpile consists of two items. When the previously issued item is exhausted, one of the two items is issued and is immediately replaced in the stockpile by an item of zero age. The amount of field life obtained from an item is a random variable, and it is assumed that this has a gamma distribution with a mean inversely proportional to the age at issue. For the two policies of LIFO and FIFO, the stationary distributions of field life at issue are obtained by examination of imbedded Markov processes. For the exponential case it is demonstrated numerically that LIFO gives the greater expected utility. The general problem is not solved. *F. G. Foster* (London)

Beale, E. M. L. 1062

The simplex method using pseudo-basic variables for structured linear programming problems.

Recent advances in mathematical programming, *pp.* 133–148. *McGraw-Hill, New York*, 1963.

Author's summary: "A procedure is described for solving linear programming problems that consist of separate subproblems with a few linking variables that occur in all (or several) subproblems. This is the simplex method, organized so that the advantages of the special structure of the problem are preserved."

Rosen, J. B. 1063

Convex partition programming.

Recent advances in mathematical programming, *pp.* 159–176. *McGraw-Hill, New York*, 1963.

The author is concerned with solving a set of independent linear programming problems except for some linking variables, which may enter nonlinearly in the constraints. If y denotes the vector of linking variables, and x_i the vector of unlinked variables in the ith subproblems, the whole problem may be written in the form: Minimize $\sum_{i=1}^{l}\psi_i(y)$, where $\psi_i(y)=\min_{x_i}\{\sum_{i=1}^{l}c_i^Tx_i|A_i^Tx_i\geq b_i(y)\}$. The vector x_i is not required to be non-negative, though particular components may be restricted by explicit inequalities.

The proposed computational procedure is as follows. Start with an arbitrary value of y, say y^0, and solve the linear programming problems defining the optimum values of the x_i. Some slack variables, the nonbasic unlinked variables, in each problem will then take the value zero, and there will be a vector r_i representing the reduced costs on these variables, or the shadow prices on the corresponding inequalities. Solving the subproblems for any fixed y is known as Problem 1.

Having solved Problem 1 for all subproblems, we now consider the problem of selecting a better value for y keeping all the nonbasic unlinked variables in each subproblem equal to zero. This is called Problem 2. In principle it has nonlinear constraints, but we approximate it by one with a nonlinear objective function and linear constraints. This is done by using the shadow prices on the nonlinear constraints to throw the nonlinearities into the objective function. Specifically, the objective function for Problem 2 is the expression $\sum_{i=1}^{l}r_i^Tb_i(y)$. The constraints are obtained by taking local linear approximations to the b_i.

The solution to Problem 2 will be a new vector y^1. In view of the approximations made, it may not be feasible. So the value of y^0 used in the next major iteration is taken as $y^0+\theta(y^1-y^0)$, where $0\leq\theta\leq 1$, and θ is chosen as large as possible consistent with keeping the exact problem feasible and all nonbasic unlinked variables equal to zero.

Theorem 2 is important. It shows that one can recognize whether a set of values for the unlinked and linking variables that optimize Problems 1 and 2 separately also optimize the whole problem. This will not necessarily be so, because the values of the linking variables will generally be such as to cause degeneracy in one or more subproblems. The difficulty can perhaps be seen more easily in the reviewer's version of the method [#1062 above]. Problems 1 and 2 are both solved, but the problem as a whole is not, whenever a nonbasic variable has to be made pseudo-basic in order to make it basic subsequently. In

cases of degeneracy, some constraints whose slack variables are basic may still have nonzero shadow prices in Problem 2. And these shadow prices, multiplied by the entries in the current Problem 1 tableaux, must be added to the vector of reduced costs for the nonbasic variables in Problem 1 to determine the true reduced costs on these variables for the problem as a whole. If any of these variables has a negative true reduced cost, the overall solution can be improved by making one or more of them basic in Problem 1, in place of others that are basic but take the value zero. When we revert to Problem 2 we will no longer insist on such variables remaining equal to zero, and further progress can be made.

This process converges to a local minimum, which must be a global minimum if the $b_i(y)$ are convex, though if the problem is nonlinear, the number of major iterations may not be finite, contrary to the author's Theorem 3.

This procedure has been programmed for the IBM 7090 computer, using the gradient projection method for both Problems 1 and 2. *E. M. L. Beale* (London)

Moroz, A. I. 1064

A model of discrete linear programming.

Avtomat. i Telemeh. **23** (1962), 553–564 (*Russian. English summary*); *translated as Automat. Remote Control* **23** (1962), 501–512.

An integer programming problem with bounded variables $0 \leq x_j \leq m_j$ can be reduced by setting $x_j = x_{1j} + x_{2j} + \cdots + x_{m_j j}$ to a canonical form with $x_{ij} = 0$ or 1. In this canonical form the solution of the discrete problem and the continuous problem $0 \leq x_{ij} \leq 1$ coincide, and the simplex method is applicable. The author exhibits a procedure which in part overcomes the difficulties encountered from the increase in size of the problem. Given the problem: minimize $\sum_{j=1}^n c_j x_j$ subject to $\sum_{j=1}^n a_{ij} x_j = b_i$, $i = 1, 2, \cdots, m$, and $x_j = 0$ or 1. If the columns of (a_{ij}) are denoted by P_j, and there is a fundamental admissible trial solution constructed from $P_1, \cdots, P_m$, then any $P_j = \sum_{i=1}^m q_{ij} P_i$. In a simultaneous replacement of r basis vectors, $1 \leq r \leq m$, say $P_1, \cdots, P_r$, necessary and sufficient for the new trial solution to be admissible is

$$\sum_{j=1}^{r} q_{ij} = 1, \quad i = 1, \cdots, r,$$
$$= 0, \quad i = r+1, \cdots, m.$$

Necessary and sufficient for it to be fundamental is $\det(q_{ij}) \neq 0$, $i, j = 1, \cdots, r$. If the first trial solution is non-degenerate, the new one will also be non-degenerate. The usefulness of the procedure for transportation problems is indicated by a theorem of Dantzig that the q_{ij} in a transportation problem are always either 0, 1 or -1 and thus that the verification of the conditions and the computation are considerably simplified. *A. Wouk* (Madison, Wis.)

Healy, W. C., Jr. 1065

Multiple choice programming. (A procedure for linear programming with zero-one variables).

Operations Res. **12** (1964), 122–138.

The author presents a practical algorithm for solving the important class of mixed integer programming problems where the integer variables must be either zero or one, and where they are divided into sets (called integer sets) such that the variables in each set sum to unity. Each integer variable must appear in at least one such set.

The procedure consists of solving a series of linear programming problems where at each stage except the first the linear programming problem is formed by adding to the original problem one constraint on the objective function for each integer set, based on limits derived from the preceding solutions. These objective function constraints, which are progressively tightened, act as the driving force towards the multiple choice programming solution. It is stated that the process can be expected to break down if it encounters a non-integer solution having the same profitability as the optimum integer solution.

A justification for the algorithm is given, but a proof of convergence is not known to the author. It is claimed that all problems presented so far have been solved in a reasonable number of iterations. Since the paper was written the reviewer knows of one problem which proved insoluble using this method due to the large amount of dual degeneracy encountered.

A worked example and a list of applications are given. An experimental computer code exists and computational experience is described.

R. P. Harvey (San Francisco, Calif.)

Grabowski, W. 1066

Problem of transportation in minimum time.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 107–108.

A theorem is stated without proof which transforms the problem of transportation in minimum time into an ordinary transportation problem.

G. Berman (Waterloo, Ont.)

Verkhovskii, B. S. [Verhovskiĭ, B. S.] 1067

Multidimensional transportation type problems in linear programming.

Dokl. Akad. Nauk SSSR **151** (1963), 515–518 (*Russian*); *translated as Soviet Physics Dokl.* **8** (1964), 655–657.

The paper deals with the problem of finding all the elements of the s-dimensional matrix $\{x_{i_1 \cdots i_s}\}$ that satisfy the conditions

$$(1) \qquad \sum_{k \in M_j} x_{i_1 \cdots i_s} = a_{j(i_1 \cdots i_s)}, \qquad x_{i_1 \cdots i_s} \geq 0,$$

and minimize the functional

$$(2) \qquad L = \sum_{k \in M} p_{i_1 \cdots i_s} x_{i_1 \cdots i_s};$$

here M_j are given subsets of the set M of all coordinates $(i_1 \cdots i_s)$ (with $i_k = 1, \cdots, n_k$; $n_k \geq 2$ for all $k = 1, \cdots, s$); $j(i_1 \cdots i_s)$ are the coordinates of the cell whose indices are not the elements of M_j; $\sum_{k \in M_j}$ denotes the summing over all the values of the coordinates whose numbers belong to M_j, and where the numbers $p_{i_1 \cdots i_s}$ and $a_{j(i_1 \cdots i_s)} \geq 0$ are given.

After proving a theorem on the solvability of the above problem, the author points out cases in which the above problem can be reduced to one or several simpler problems. Some rules for computation are also given.

P. L. Ivănescu (Bucharest)

Heller, I. 1068

On linear programs equivalent to the transportation program.

J. Soc. Indust. Appl. Math. **12** (1964), 31–42.

Let the program for the transportation problem be represented in the form min $q'y$ subject to $Dy=s$, $y\geqq 0$, where D is a matrix of $m+n$ rows, mn columns, and rank $m+n-1$. Consider also the second program min $p'x$ subject to $Ax=b$, $x\geqq 0$, where A has mn columns and rank $m+n-1$. It is no restriction to suppose that A has $m+n$ rows. The author gives necessary and (constructively) sufficient conditions that there exist a nonsingular matrix T and a permutation matrix P such that $TAP=D$. If these conditions are met, then a solution to the second program is given by $x=Py$, where y is a solution to the first program with $q=P'p$, $s=Tb$. This transformation is valuable because transportation problems are more easily solved than general linear programs.

R. L. Graves (Chicago, Ill.)

Balinski, Michel L.; Gomory, Ralph E. 1069

A mutual primal-dual simplex method.

Recent advances in mathematical programming, pp. 17–26. *McGraw-Hill, New York*, 1963.

The authors develop a simplex method for the mutual primal-dual problems: $x\geqq 0$; $Ax=b$, $yA\geqq c$; $cx=\min$, $yb=\max$. Their method differs from the usual simplex method in that they do not construct an extended tableau with artificial variables in order to find a basic feasible solution.

Using the initial combined primal-dual tableau, and assuming non-optimality, a primal simplex pivot choice rule is used until degeneracies occur; at this point a dual simplex pivot choice rule is used on a sub-tableau corresponding to primal basic variables at zero value until the degeneracies are resolved. If sub-degeneracies occur, a sub-subtableau is used with a primal pivot choice rule, etc. At each step there is an improvement of some goal in the hierarchy of tableaux. The process is finite and is continued until one of the four following conditions obtain: (1) Primal and dual optimal; (2) Primal and dual infeasible; (3) Primal feasible, dual infeasible; (4) Dual feasible, primal infeasible.

The greatest interest in such a method is in application to highly degenerate problems such as network flow problems. The authors hope that an application of the mutual primal-dual simplex method to the transportation problem will lead to a more efficient computational algorithm than the Hungarian method. The "out of kilter" algorithm of Fulkerson [RAND Corp. P-1825 (1960)] has some elements of these ideas; one can show the simplex counterpart of each step of the algorithm. Thus, the mutual primal-dual method is most valuable in the insight gained about special algorithms used to solve network flow problems. *R. J. Swersey* (Richmond, Calif.)

Ivănescu, Petru L. 1070

Programmation polynomiale en nombres entiers.

C. R. Acad. Sci. Paris **258** (1964), 424–427.

From the author's summary: "A method is proposed for solving integer programming problems in which the constraints and the objective function are polynomials with integer coefficients and bounded variables. The method has three stages: (1) the constraints are transformed into congruences over the finite field GF(2); (2) the congruences are resolved parametrically; (3) the new form of the objective function (which has its variables in GF(2) and integral values) is minimized."

J. J. Stone (Waltham, Mass.)

Mangasarian, O. L. 1071

Nonlinear programming problems with stochastic objective functions.

Management Sci. **10** (1963/64), 353–359.

The author considers several nonlinear programming problems related to maximization with respect to $x\varphi(c,x)$ subject to $g(x)\geqq 0$, where c and x are vectors and g is a vector function. Let c be a random vector whose distribution is known and let E denote expectation with respect to c. Denote the solution to the fourth problem below by $\bar{x}(Ec)$. The main result is:

$$E\max_x\{\varphi(c,x)|g(x)\geqq 0\}\geqq \max_x\{E\varphi(c,x)|g(x)\geqq 0\}\geqq$$
$$E\varphi(c,\bar{x}(Ec))\geqq \max_x\{\varphi(Ec,x)|g(x)\geqq 0\}.$$

The paper is very closely related to and contains other results like those in some work of Madansky [Management Sci. **6** (1959/60), 197–204; MR **22** #1457].

R. L. Graves (Chicago, Ill.)

Restrepo, Rodrigo A. 1072

Games with a random move.

Advances in game theory, pp. 29–38. *Princeton Univ. Press, Princeton, N.J.*, 1964.

The author considers a class of games that includes some models of poker investigated by von Neumann and Morgenstern [*Theory of games and economic behavior*, Princeton Univ. Press, Princeton, N.J., 1944; MR **6**, 235] and Karlin and the author [*Contributions to the theory of games*, Vol. 3, pp. 337–363, Princeton Univ. Press, Princeton, N.J., 1957; MR **19**, 1244]. These games yield three types of strategy spaces. A solution is exhibited for the first type. Some qualitative properties of the solution are given for the second type of strategy space. The solution for the third type is not known.

M. Dresher (Pacific Palisades, Calif.)

Johnson, Selmer M. 1073

A search game.

Advances in game theory, pp. 39–48. *Princeton Univ. Press, Princeton, N.J.*, 1964.

The following "hide and seek" game is studied: Blue chooses an integer $\leqq n$ (hiding regions). Red then guesses which of these n integers was chosen (seeking regions), is told whether the guess is too high or too low, and repeats until his guess is correct. The payoff of Blue is the total number of guesses made by Red.

The author introduces a method for reducing the size of the game matrix and then derives some necessary conditions for optimality. The optimal strategies and the game value for $n\leqq 11$ are exhibited. These game values are as follows: 1, 3/2, 9/5, 2, 20/9, 12/5, 23/9, 27/10, 31/11, 35/12, 187/62. *M. Dresher* (Pacific Palisades, Calif.)

Mycielski, Jan 1074
Continuous games with perfect information.
Advances in game theory, pp. 103–112. *Princeton Univ. Press, Princeton, N.J.*, 1964.

Let M be a metric space, let S be a class of convergent sequences in M, and let φ be a real function on M. φ is said to belong to $K_n(M)$ if $\{s:\varphi(s)<r\}$ or $\{s:\varphi(s)\leq r\}$ is of Borel class B_n for all r in a dense set of real numbers. The 2-person game Γ_φ is played by determining a sequence $s=\{s_n\}\in S$, the player who chooses s_n being determined by $s_0,\cdots,s_{n-1}$ according to some rule. The pay-off function is $\varphi(\lim s_n)$. Applying some known theorems concerning infinite games it is shown that Γ_φ is determined if $\varphi\in K_1(M)$, and that in case the mapping $l:S\to M$, where $l(s)=\lim s_n$, is continuous, Γ_φ is determined if $\varphi\in K_2(M)$. For example, let the players choose alternate terms of a sequence $\{x_n\}$ of real numbers, player I winning in case $\lim x_n$ (or $\lim x_{2n}$) belongs to a set X. If $x_{n+1}-x_n$ is required to be positive and decreasing, this game is determined when $X\in B_1$. If $|x_{n+1}-x_n|$ is required to be $\leq 2^{-n}$, or if the sequence $x_{n+1}-x_n$ is required to alternate in sign and decrease numerically, the game is determined when $X\in B_2$. Finally, it is indicated how certain continuous games of pursuit and evasion can be approximated by infinite positional games of this sort.

J. C. Oxtoby (Bryn Mawr, Pa.)

Berkovitz, Leonard D. 1075
A variational approach to differential games.
Advances in game theory, pp. 127–174. *Princeton Univ. Press, Princeton, N.J.*, 1964.

The author discusses a two-person zero-sum game with perfect information for which the payoff to the maximizing player is of the form $g(T,x_T)+\int_{t_0}^T f(t,x(t),y(t),z(t))\,dt$. There are side equations $x'=G(t,x,y,z)$ and other constraints. The vector x with n components is called the state, y with $\bar{s}$ or σ components and z with s components are respectively chosen by the maximizing and minimizing players from given sets of "permissible" functions called pure strategies. The formulation of the game, including the definition of saddle point, runs to twelve pages and involves a multiplicity of concepts, hypotheses and notation. The author obtains necessary conditions by way of known results for the ordinary Bolza problem of the calculus of variations. Under hypotheses including the existence of a field, the definition of which is in terms of seven conditions, a sufficiency theorem is obtained. Among questions that appear to be open are: existence theory for solutions in pure strategies, the possibility of a notion of mixed strategies, and theorems on existence of fields in the large in support of sufficiency. *G. M. Ewing* (Norman, Okla.)

Berkovitz, Leonard D. 1076
A differential game with no pure strategy solution.
Advances in game theory, pp. 175–194. *Princeton Univ. Press, Princeton, N.J.*, 1964.

This sequel to the article reviewed above [#1075] gives two examples not having solutions in pure strategies, the first of which, as pointed out by the author, does not fit his general formulation precisely.

G. M. Ewing (Norman, Okla.)

Isbell, J. R. 1077
Homogeneous games. III.
Advances in game theory, pp. 255–265. *Princeton Univ. Press, Princeton, N.J.*, 1964.

Part II appeared in Proc. Amer. Math. Soc. **11** (1960), 159–161 [MR **22** #9314]. A strong simple game on a set N of players is called homogeneous if it admits a transitive group of permutations of N. A natural question is: For what n do there exist homogeneous (strong simple) n-player games? That these games exist has been shown for many even numbers n and for some numbers $4p$, p a prime. The author proves the following: Let $n=2^Km$, with m odd, then for the existence of a homogeneous game of 2^Km players, it is necessary that $K<2m-4$; for infinitely many m, it is sufficient that $2^K<m$.

M. Dresher (Pacific Palisades, Calif.)

Galmarino, Alberto Raul 1078
On the cardinality of solutions of four-person constant-sum games.
Advances in game theory, pp. 327–344. *Princeton Univ. Press, Princeton, N.J.*, 1964.

It is shown that the number of imputations in a finite solution of a four-person constant-sum game has an upper bound. Moreover, such a game cannot have a countably infinite solution. For n-person constant-sum games, $n\geq 5$, these questions remain open.

W. H. Mills (Princeton, N.J.)

Hebert, Michael H. 1079
The doubly discriminatory solutions of four person constant-sum games.
Advances in game theory, pp. 345–375. *Princeton Univ. Press, Princeton, N.J.*, 1964.

A solution of an n-person game is called discriminatory if there is a player who receives the same amount in all its imputations and doubly discriminatory if there are two such players. In the present paper all doubly discriminatory solutions of four-person constant-sum games are determined. Every such solution lies entirely in one of the faces of the tetrahedron of imputations, i.e., there is at least one player who is discriminated against completely. If the essential four-person constant-sum games are put in a one-to-one correspondence with the points of a cube Q in the usual way, then those games that do not possess doubly discriminatory solutions correspond to the interior and some of the boundary points of a convex polyhedron whose volume is half that of the cube. Thus, in this sense, half of these games possess doubly discriminatory solutions and half do not. *W. H. Mills* (Princeton, N.J.)

Peleg, Bezalel 1080
On the bargaining set $\mathscr{M}_0$ of m-quota games.
Advances in game theory, pp. 501–512. *Princeton Univ. Press, Princeton, N.J.*, 1964.

An n-person cooperative game is called an m-quota game if there exist n real numbers $w_1, w_2, \cdots, w_m$, called the quotas of the players, such that the characteristic function of the game satisfies $v(S)=\sum_S w_i$ for each m-person coalition S, $2\leq m<n$, and $v(S)=0$ otherwise. (This definition with some differences originated from L. S. Shapley and G. K. Kalisch [Shapley, *Contributions to the theory of games*, Vol. 2, pp. 343–359, Princeton Univ. Press, Princeton,

N.J., 1953; MR **14**, 779; Kalisch, ibid., Vol. IV, pp. 163–177, Princeton Univ. Press, Princeton, N.J., 1959; MR **21** #7121].) A player is called weak if his quota is negative. The bargaining set $\mathcal{M}_0$ is a variant of a bargaining set in which objections and counter-objections are permitted for any two non-empty disjoint subsets of the players which intersect the same coalitions in the coalition structure of the payoff configuration under consideration. (For the definition of the various bargaining sets see R. J. Aumann and the reviewer [*Advances in game theory*, pp. 443–476, Princeton Univ. Press, Princeton, N.J., 1964].)

Generalizing the reviewer's study of the case $m=2$ [ibid., pp. 477–499], the author proves that for coalition structures which contain the maximal number of m-person coalitions, if no player is weak, a payoff configuration belongs to $\mathcal{M}_0$ if and only if each member of a formed m-person coalition receives his quota. If weak players exist and if $m(m+1)\leq n$, the condition is that the weak players must receive a zero payment and none of the other players obtains more than his quota. The case $m(m+1)>n$ remains open, except for the case $m=n-1$; see the reviewer's paper [J. Math. Anal Appl. **6** (1963), 230–256; MR **28** #4953]. *M. Maschler* (Jerusalem)

Selten, Reinhard 1081
Valuation of n-person games.
Advances in game theory, pp. 577–626. Princeton Univ. Press, Princeton, N.J., 1964.

The author states eleven postulates which define a unique value to each player for any n-person cooperative game with transferable utility. He shows that in the case of constant-sum games his postulates lead to the Shapley value, whereas in the general case they lead to the reviewer's "modified Shapley value". The author's treatment, unlike Shapley's and the reviewer's, is based on the extensive form, rather than the normal form. Besides postulates closely related to Shapley's (symmetry, maximum joint payoff, additivity), his most important assumptions are monotonicity postulates, such that the value of the game to a given player cannot be increased by forbidding a move to him, or by increasing the other players' payoffs at his expense, or by informing the other players about his previous moves. (The first of these is similar to Nash's axiom VI for two-person cooperative games.) The author's approach is an interesting alternative to those based on bargaining or arbitration models. It would be worthwhile to extend it to games without transferable utility. *J. C. Harsanyi* (Berkeley, Calif.)

BIOLOGY AND BEHAVIORAL SCIENCES

See 665.

INFORMATION, COMMUNICATION, CONTROL

See also 15, 16, 28, 29, 323, 639, 721.

Barndorff-Nielsen, O. 1082
Subfields and loss of information.
Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2**, *369–379 (1964).*

Suppose a random experiment has the possible outcomes $A_1, A_2, \cdots, A_n$ but that the observer can only determine which of the outcomes $B_1, \cdots, B_m$ occurs, each B_j being a union of certain A_i's. The author introduces a minimax-type measure of the loss of information due to observing the B's instead of the A's. He then derives several properties of this measure. *L. L. Campbell* (Kingston, Ont.)

Hu, Kuo-Ting 1083
Three kinds of converses to Shannon's theorem in information theory.
Acta Math. Sinica **11** (1961), 260–294 (*Chinese*); *translated as Chinese Math.* **2** (1963), 293–332.

Considered are random sequences $\{X_n\}$, $n=1, 2, \cdots$, each X_n having a finite alphabet of cardinality $a^{(n)}$, with $a^{(n)}\to\infty$. Asymptotic information properties of such sequences are proved (see also R. L. Dobrušin's work [Uspehi Mat. Nauk **14** (1959), no. 6 (90), 3–104; MR **21** #6299]) in a self-contained treatment, most of which is expository. Using a form of Feinstein's lemma [*Foundations of information theory*, sec. 6.2, McGraw-Hill, New York, 1958; MR **20** #1594], the author proves several theorems wherein he infers uniformly small error probability and/or satisfaction of the information transmission criterion (both versions of the Shannon coding theorem [A. Hinčin, Uspehi Mat. Nauk **11** (1956), no. 1 (67), 17–75, MR **17**, 1098], here modified to apply to the sequences considered) from various hypotheses on entropy stability, information stability [M. Pinsker, *Information and information stability of random variables and processes*, Holden-Day, San Francisco, Calif., 1964], and probability compressibility. The precise definition of these terms is too involved to be repeated here. Examples are given to indicate that the alternative forms of the Shannon coding theorem are not equivalent, neither implying the other without auxiliary assumptions.

Finally, converses of the Shannon theorems are given in terms of the information stability and/or compressibility of the source and/or channel. The latter work, part of which is new, is related to that discussed elsewhere by the same author [Teor. Verojatnost. i Primenen **7** (1962), 271–282; MR **26** #1223]. *F. J. Beutler* (Ann Arbor, Mich.)

Shen, Shi-yi [Shen, Shih-yi] 1084
A necessary and sufficient condition for satisfaction of the information criterion in the Shannon theorem.
Acta Math. Sinica **12** (1962), 389–407 (*Chinese*); *translated as Chinese Math.* **3** (1963), 419–438.

A recent paper by Hu [#1083] establishes the notation, definitions, and analytical background for the paper under review; the latter is essentially a continuation of the former. The asymptotic properties of a random sequence $\{X_n\}$, $n=1, 2, \cdots$, are again considered, with $H^{(n)}\to\infty$, where $H^{(n)}$ is the entropy of X_n (which has a finite alphabet). Relations among various types of information stability and compressibility [M. Pinsker, *Information and information stability of random variables and processes*, Holden-Day, San Francisco, Calif., 1964] are demonstrated. These concepts are applied to provide a necessary and sufficient condition for the satisfaction of the information transmission criterion (the second form of the Shannon coding theorem [A. Hinčin, Uspehi Mat. Nauk **11** (1956), no. 1 (67), 17–75; MR **17**, 1098]). The last part of the

paper presents several theorems stating assumptions under which uniformly small error probability is implied by satisfaction of the information transmission criterion, and vice versa. *F. J. Beutler* (Ann Arbor, Mich.)

Zaitsev, A. G. [Zaĭcev, A. G.] 1085

The analytical design of systems that reproduce a useful signal in the presence of noise.

Avtomat. i Telemeh. **24** (1963), 143–150 (*Russian. English summary*); *translated as Automat. Remote Control* **24** (1963), 139–145.

Author's summary: "The paper determines the differential equation or the transfer function of an optimal system that reproduces a regular time function in the presence of random stationary noise. The optimality criterion is proposed in the form of the minimum sum of the integral quadratic error with the weight a and the dispersion of the random error with the weight c."

W. H. Fleming (Providence, R.I.)

Senin, A. G. 1086

Some questions in the analysis and synthesis of measuring systems for extrapolating a random field. (Russian)

Izv. Sibirsk. Otd. Akad. Nauk SSSR Ser. Tehn. Nauk **1964**, *no.* 1, 54–62.

The author is concerned with the extrapolation of a random field $\varphi(x, t)$, a function of time and one space coordinate. The particular problem he formulates is the determination of an admissible dynamic operator, the synthesis of a measuring system, ensuring an optimal estimate of the extrapolated signal. The proposed solution is based on a least squares criterion. In particular, he considers the extrapolation of both a nonhomogeneous and a homogeneous field. For the nonhomogeneous field $\varphi(x, t)$ is a stationary process in each point of the coordinate space, and for two arbitrary sections the processes $\varphi(x, t)$ and $\varphi(y, t)$ are stationarily related. For the homogeneous field the correlation function depends only on the shift of the space and time variables. We omit the details of the proposed solutions which involve the solution of a two-dimensional Fredholm integral equation.

S. Kullback (Washington, D.C.)

de Leve, G. 1087

Decision rules for adjusting Markovian processes. (French summary)

Bull. Inst. Internat. Statist. **39** (1962), *livraison* 2, 423–433.

Suppose that a system moves randomly in time through a state space ψ and that the motion of the system is governed by a stationary Markov process. Suppose further that in each time interval the motion of the system produces a loss. These losses are additive and depend only on the states of the system during the given time interval.

Assume now that a decision-maker can, at various chosen time points, intervene in the process by replacing the Markov transition function by another one chosen from a given set. A strategy is a choice of intervening time points and for each chosen time point a choice of a transition function. Clearly a decision-maker should try to choose a strategy so as to minimize in some sense the expected loss path.

The author discusses some strategies which are too involved to describe here. Neither the exact assumptions nor proofs are given in the paper, and the author promises a further paper giving precise results.

J. R. Blum (Albuquerque, N.M.)

Wyner, A. D.; Ash, R. B. 1088

Analysis of recurrent codes.

IEEE Trans. Information Theory **IT-9** (1963), 143–156.

This paper analyses codes of the type found by Hagelbarger [Bell System Tech. J. **38** (1959), 969–984; MR **21** #7812]. For burst-correcting codes, a lower bound is derived for the length of the guard space for a given information rate and burst length. Some codes which meet the bound and some which are close are given. The paper also considers codes which correct errors provided no more than e errors occur in n successive digits. Upper and lower bounds on n for fixed e are derived, and for $e = 1$ it is shown that the lower bound can always be met.

W. W. Peterson (Gainesville, Fla.)

Johnson, Selmer M. 1089

Improved asymptotic bounds for error-correcting codes.

IEEE Trans. Information Theory **IT-9** (1963), 198–205.

Author's summary: "The author has previously developed a new upper bound on nonsystematic binary error-correcting codes, using a sphere-packing approach and combinatorial analysis. A significant refinement is now added; together with a detailed study of the asymptotic behavior of the upper bound, this enables one to show that any large code must correct almost all sequences with a larger number of errors than the code was designed for. This excess is expressed numerically as a fraction of the designed error-correcting capability of the code. The fraction is a function of the ratio of the sequence length and the designed error-correcting capability. A possible application might be in the use of a larger code giving almost certain error correction rather than a smaller one with certain correction capability."

W. W. Peterson (Gainesville, Fla.)

Kislicyn, S. S. 1090

Average length of a binary code with minimal redundancy where the probabilities of the coded symbols are mutually close. (Russian. English summary)

Teor. Verojatnost. i Primenen. **7** (1962), 342–343.

For a zero-memory source with n symbols and symbol probabilities $p_1, p_2, \cdots, p_n$, where $p_1 \geqq p_2 \geqq \cdots \geqq p_n$ with the restriction $p_1 \leqq p_{n-1} + p_n$, the exact value for the average length of Huffman's minimum-redundancy binary code is derived. *S. Kotz* (Toronto, Ont.)

Imaz, Carlos 1091

★**Teoria de control [Control theory].**

Centro de Investigación y Estudios Avanzados del IPN, Mexico City, 1962 [1963]. iii + 86 *pp*.

This is a set of seminar notes on some mathematical aspects of control theory. After some introductory material the author discusses first closed-loop control systems and their stability in the setting of Lyapunov's second method. He then takes up in Chapter IV linear

control systems and the existence, uniqueness and synthesis of time-optimal controls. Chapter V is concerned with the problem of complete controllability and energy optimization, Chapter VI with Pontryagin's maximum principle and some of its consequences for linear systems. In the final Chapter VII the author discusses his notion of threshold control and strong ε-stability.

These notes will provide Spanish-speaking mathematicians and engineers with a good and mathematically sound introduction to this important subject.

H. A. Antosiewicz (Los Angeles, Calif.)

Litvin-Sedoĭ, M. Z. 1092

The synthesis of correcting loops in non-linear oscillatory and control systems. (Russian. English summary)

Qualitative methods in the theory of non-linear vibrations (Proc. Internat. Sympos. Non-linear Vibrations, Vol. II, 1961), pp. 233–240. *Izdat. Akad. Nauk Ukrain. SSR, Kiev,* 1963.

In the first part the author proves that all solutions of the system $x'=y$, $y'=-\phi(|y|)y-h(x)+p(t)$ which start near the origin are bounded provided $\phi(|y|)\geq 0$, $H(x)=\int_0^x h(s)\,ds$ is increasing in $0\leq |x|\leq a$ and $|p(t)|\leq \rho$. In the second part he exhibits a very special class of systems of n autonomous equations containing a single control parameter to which this result may be extended under suitable assumptions. Such systems arise in various applications. *H. A. Antosiewicz* (Los Angeles, Calif.)

Jury, E. I. 1093

A simplified stability criterion for linear discrete systems.

Proc. IRE **50** (1962), 1493–1500.

Author's summary: "In this study a simplified analytic test of stability of linear discrete systems is obtained. This test also yields the necessary and sufficient conditions for a real polynomial in the variable z to have all its roots inside the unit circle in the z-plane. The new stability constraints require the evaluation of only half the number of Schur-Cohn determinants. It is shown that for the test of a fourth-order system only a third-order determinant is required and for the fifth-order, one second-order and one fourth-order determinant are required. The test is applied directly in the z-plane and yields the minimum number of constraint terms. Stability constraints up to the sixth-order case are obtained and for the nth-order case are formulated. The simplicity of this criterion is similar to that of the Lienard-Chipard criterion for the continuous case which has a decisive advantage over the Routh-Hurwitz criterion. Finally, general conditions on the number of roots inside the unit circle for n even and odd are also presented in this paper."

M. Aoki (Los Angeles, Calif.)

Meyer, K. R. 1094

On a system of equations in automatic control theory.

Contributions to Differential Equations **3** (1964), 163–173.

The author studies a control system described by the differential equation $\dot v=Av-b\mu$, where the control μ is governed by the equations

$$\dot\mu = \psi(w)\phi(\sigma),$$
$$\sigma = c'v-\rho\mu,$$
$$w = 1-\theta\mu \operatorname{sgn}\sigma.$$

These equations were introduced by Letov to describe the automatic roll stabilization system of the *Queen Mary*. A lemma attributed to Kalman and Yacubovich is used in conjunction with the direct method of Liapunov to establish conditions under which the system is asymptotically stable in the large.

S. E. Dreyfus (Santa Monica, Calif.)

Mangeron, D. 1095

The Bellman equations of dynamic programming concerning a new class of boundary value problems with "total derivatives".

J. Math. Anal. Appl. **9** (1964), 141–146.

The author derives a functional equation for a class of boundary-value problems by applying the principle of optimality of dynamic programming to a certain type of partial differential equation in two variables.

M. Aoki (Los Angeles, Calif.)

Borisenok, I. T. 1096

A control system with doubling activating devices. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, *no.* 2, 83–96.

In order to increase the reliability of automatic control systems one sometimes uses parallel actuating elements. In such systems there may exist conditions for sustaining stable oscillations with considerable amplitude. In this paper the cases of actuating elements with linear and non-linear characteristics are investigated. The discussion covers the cases when either the actuating elements are in order or they are defective. *I. Cederbaum* (Tel-Aviv)

Falb, Peter L. 1097

Infinite dimensional control problems. I. On the closure of the set of attainable states for linear systems.

J. Math. Anal. Appl. **9** (1964), 12–22.

For control systems described by finite-dimensional state vectors, attainable sets and admissible controls [E. Roxin, Michigan Math. J. **9** (1962), 109–119; MR **25** #305] have been related to the existence of time-optimal controls [L. W. Neustadt, J. Math. Anal. Appl. **7** (1963), 110–117; MR **27** #4714] and to the "bang-bang" principle [J. P. LaSalle, Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 573–577; MR **21** #4065].

The author extends some of the above to Banach spaces. Let E be the interval $[t_0, t_1]$, and take X and Ω to be Banach spaces. $L_p(E, X)$ and $L_p(E, \Omega)$ are defined in the obvious manner; it is also assumed that $L_2(E, \Omega)$ is reflexive. If $\mathscr{W}$ is a subset of $L_2(E, \Omega)$, the set of attainable states $\mathscr{A}(E, \mathscr{W})$ is defined by the Bochner integral $\mathscr{A}(E, \mathscr{W})=\{\int_E K(\tau, w(\tau))\,d\tau : w\in\mathscr{W}\}$, where $K(\cdot,\cdot)$ is a mapping from $E\times\Omega$ to X possessing certain linearity, boundedness, and measurability properties. Define $\mathscr{U}=\{u : u(\cdot)$ is Bochner measurable and $u(\tau)\in U$ for each $\tau\in E\}$, where $U\subset\Omega$ is closed, bounded, and convex. Then $\mathscr{A}(E, \mathscr{U})$ is shown to be closed. Under auxiliary conditions on $\mathscr{A}$, $\mathscr{A}(E, \mathscr{U})=\mathscr{A}(E, \mathscr{U}_1)$, where $\mathscr{U}_1\subset\mathscr{U}$, and $\mathscr{U}_1$ contains the extremal points of $\mathscr{U}$. If, in particular, X and Ω are finite-dimensional and if U is either a sphere or cube, $\mathscr{A}(E, \mathscr{U})=\mathscr{A}(E, \hat{\mathscr{U}})$, where $\hat{\mathscr{U}}=\{u : u\in U$ and $u(\tau)$ is in the set of extremal points of U for all $\tau\in E\}$. This

result is essentially equivalent to LaSalle's "bang-bang" principle. An example shows that this principle need not hold for infinite-dimensional spaces, e.g., when X and Ω are separable Hilbert spaces.

The time-optimal problem of attaining a moving point in X is considered. It is shown that a minimum-time solution exists provided that the moving point falls in the set of attainable states at some time, and that sufficient hypotheses are placed on the set of admissible controls and their relation to the control system. This generalizes the work of LaSalle [*Contribution to the theory of nonlinear oscillations*, Vol. V, pp. 1–24, Princeton Univ. Press, Princeton, N.J., 1960; MR **26** #2704]. In particular, the time-optimal problem involving time-invariant differential equations and a fixed target point has a time-optimal solution under conditions generalizing those of the finite-dimensional case [R. Bellman, I. Glicksberg, and O. Gross, Quart. Appl. Math. **14** (1956), 11–18; MR 17, 1206]. *F. J. Beutler* (Ann Arbor, Mich.)

Odanaka, Toshio 1098

A functional equation arising in control process with certain probability criterion.

J. Operations Res. Soc. Japan **5** (1962/63), 102–109.

Author's introduction: "An equation we have used is the following. Consider a system specified at any time t by a finite-dimensional vector $x(t)$, satisfying a vector differential equation $dx/dt = g(x, y, z)$, $x(0) = c$, where c is the initial state, $y(t)$ is a random forcing term possessing a known distribution, and $z(t)$ is a forcing chosen, via a feedback process, so as to minimize the functional defined by the probability that $|\max_{0 \le t \le T} x(t)|$ or $|\min_{0 \le t \le T} x(t)|$ exceeds a specified bound. The functional equation is introduced by the technique of dynamic programming which may be used to obtain an analytic approach to problems of this genre. The purpose of the present note is to discuss the character of the solution of the functional equation introduced. The method of successive approximations that we employ is a very powerful analytic tool for the treatment of functional equations of this general class."

Kosmodem'ianskii, V. A. 1099
[**Kosmodem'janskiĭ, V. A.**]

On a type of variational problems.

Prikl. Mat. Meh. **27** (1963), 1111–1116 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1705–1714.

It is desired to maximize a functional $J = J(x_s(T), T)$ under conditions

$$\dot{x}_s - f_s(x_1, \cdots, x_n, u_1, \cdots, u_m, t) = 0, \quad s = 1, \cdots, n,$$

(plus initial and terminal conditions), where $m-r$ of the control variables u_j are given except for the timing of certain discontinuities. The size of the discontinuities is restricted by relationships

$$\psi_k(x_1, \cdots, u_m, t) = 0, \qquad k = 1, \cdots, r, \quad 0 \leqq t \leqq T.$$

The solution is given in terms of Lagrangian multipliers. As an example, the optimal time for separating the first stage from a two-stage rocket is considered with the object of maximizing the terminal velocity after a fixed total flight time T in vertical flight through the gravitational field of the earth when aerodynamic forces are absent. *M. J. Beckmann* (Providence, R.I.)

Kosmodem'janskiĭ, V. A. 1100

On the theory of multi-stage rockets. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 286–289.

The author considers the problem of choosing the staging times for a multi-stage rocket so as to maximize either the range, altitude, horizontal velocity, or some other related variable at final stage burnout. It is assumed that the thrust and burning rate of each stage, as well as the payload, launch weight, and complete steering program are fixed, and that the dry weight of each stage is a known function of the staging times. Since the maximization is with respect to a finite number of parameters (the staging times), this is a problem in the differential calculus. The author appears to be unaware of this fact, and instead uses machinery from the calculus of variations to formally derive the Euler-Lagrange equations, transversality conditions, and a complex expression for the staging times (which contains two misprints). The results are applied to an example of a two-stage rocket whose motion takes place in a vertical plane under the action of a uniform gravitational field with no air resistance. Since rocket trajectory optimizations generally involve a proper choice of parameters additional to the staging times (e.g., the steering program), the results of this paper are of questionable value. *L. W. Neustadt* (Ann Arbor, Mich.)

Pšeničnyĭ, B. N. 1101

A numerical method for computing the time-optimal control in linear systems. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 52–60.

This article deals with a time-optimal control problem which may be described as follows: Given the differential equation (1) $\dot{x} = A(t)x + G(t, u)$, where x and G are n-vectors, u is an r-vector, and both G and the matrix A are continuous in a compact set U in r-space, a time t_0 and n-vectors x_0 and x_1; find a measurable function $u(t)$ whose range is contained in U such that the solution of (1) with $u = u(t)$ and $x(t_0) = x_0$ satisfies the relation $x(t') = x_1$ for some $t' \geqq t_0$, and such that t' is minimal. It is well known that the time-optimal "control" function $u(t)$ can be obtained directly in terms of a properly chosen solution of the adjoint equation (2) $\dot{\psi} = -A^*\psi$. The author presents an iterative computational procedure for obtaining one such proper solution of (2). The mathematical derivations are rigorous, and a convergence proof is presented. However, the author appears unaware of prior results for this problem. Namely, an almost identical iterative procedure has been described by the reviewer [J. Math. Anal. Appl. **1** (1960), 484–493; MR **23** #B1612; ibid. **7** (1963), 110–117; MR **27** #4714] and by Eaton [ibid. **5** (1962), 329–344; MR **25** #4214], whose convergence proof is similar to, but less elegant than, the one in the paper under review. For another similar treatment of this problem, see Babunašvili [Dokl. Akad. Nauk SSSR **155** (1964), 295–298; MR **28** #3896]. *L. W. Neustadt* (Ann Arbor, Mich.)

Eaton, J. H. 1102

Errata and addenda to "An iterative solution to time-optimal control".

J. Math. Anal. Appl. **9** (1964), 147–152.

As the title implies, this paper is concerned with the correction of some erroneous results presented in an earlier paper [same J. **5** (1962), 329–344; MR **25** #4214]. The

major change is in the iterative procedure for finding a time-optimal control which causes interception of a target on a subspace of the state space.

P. E. Sarachik (New York)

Kolosov, G. E.; Stratonovič, R. L. 1103

A problem on optimal control synthesis which can be solved by dynamic programming methods. (Russian. English summary)

Avtomat. i Telemeh. **24** (1963), 1165–1173.

Authors' summary: "A statistical problem of an optimum controller design is solved by the asymptotic method described by Stratonovič [Avtomat. i Telemeh. **23** (1962), 1439–1447; MR **28** #1052]. An approximate equation of the switching line for the first order system is obtained which depends on the noise intensity at the input and on the control plant random actions."

W. H. Fleming (Providence, R.I.)

Gnoenskiĭ, L. S. 1104

On a synthesis problem for control systems. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 1022–1024.

The author considers an automatic control system whose behavior under perturbing effects is described, on $0 \leqq t \leqq T$, by the ordinary differential equation (1) $L_1(y)+c(t)y=f(t)$, plus homogeneous initial conditions. $L_1(y)$ represents a prescribed linear differential operator of order n with constant or variable coefficients. The "disturbance" $f(t)$ belongs to a class F of uniformly bounded piecewise-continuous functions, while the "variable gain" $c(t)$ belongs to another class H of uniformly bounded piecewise-continuous functions.

The problem is to choose $c(t)$ so as to minimize the "maximum cumulative error"; that is, if the solution of (1) is denoted by $y(t, f(\tau), c(\tau))$, an element $c_0(t)$ in H must be found for which

$$I = \min_c \max_f |y(T, f(t), c(t))|,$$
$$c \in H, \quad f \in F, \quad 0 \leqq t \leqq T,$$

is realized. An iterative scheme is given which leads to the determination of I, under certain sufficient conditions. Also, estimates are given for the number of steps required for this and for the number of switchings of $c_0(t)$ in the interval $0 \leqq t \leqq T$. *B. A. Fleishman* (Troy, N.Y.)

Tumanjan, G. B. 1105

Minimization of indecomposable Boolean functions by an approximate functional decomposition. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 525–528.

From the author's summary: "In this paper some properties and relations among nondecomposable Boolean functions characterizing switching circuits are investigated. A well-defined method for minimization of such functions by applying an approximate nondivisible decomposition is given. Application of this method, together with the complete minimization algorithm, considerably extends the class of Boolean functions, both decomposable and nondecomposable, which can be minimized."

I. Cederbaum (Tel-Aviv)

Nečiporuk, È. I. 1106

On the synthesis of logical networks in incomplete and degenerate bases. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 299–301.

The author studies the complexity of realizations of Boolean functions by logical nets of two types: networks of functional elements and superpositions. In either case there is a basis F of Boolean functions, and a net constructed of elements corresponding to functions of F may be called an F-net. Each element of F is assigned a non-negative weight, and the weight of an F-net is the sum of the weights of the basis elements in it. Let X_F denote the set of Boolean functions realizable by F-nets and let $Z(F)$ denote the set of elements of F having weight 0; then the basis F is said to be of type $(X_{Z(F)}, X_F)$. A basis F is called complete if X_F contains all Boolean functions, non-degenerate if $Z(F)$ contains no function depending on more than one variable. The Shannon function $L_F(n)$ is the smallest number w such that every function of n variables belonging to X_F is realizable by an F-net of weight not exceeding w.

For complete bases F, asymptotic expressions for $L_F(n)$ have been obtained by R. E. Kričevskiĭ [same Dokl. **126** (1959), 1195–1198; MR **22** #3630], O. B. Lupanov [Problemy Kibernet. **3** (1960), 61–80; MR **23** #A3681], and the author [Dokl. Akad. Nauk SSSR **136** (1961), 560–563; MR **29** #1155]. The present article gives, without proof, asymptotic expressions for $L_F(n)$ for certain incomplete and degenerate bases. Among the bases considered are those of types (L_i, C_j) and (A_i, C_j), where L_i $(1 \leqq i \leqq 5)$, C_j $(1 \leqq j \leqq 4)$ and A_i $(1 \leqq i \leqq 4)$ are certain classes of Boolean functions defined by E. Post [*The two-valued iterative systems of mathematical logic*, Princeton Univ. Press, Princeton, N.J., 1941; MR **2**, 337].

G. N. Raney (Storrs, Conn.)

Ostrovskiĭ, Ju. I. 1107

A method of coding the rows of the transition matrix. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 382–393.

Author's summary: "The first stage of the synthesis of a finite automaton according to a given transition matrix which consists in coding lines of the matrix by binary numbers is considered. An algorithm for coding is proposed in which the number of transition matrix lines with competitions between relays is minimum."

Ginsburg, Seymour; Spanier, Edwin H. 1108

Quotients of context-free languages.

J. Assoc. Comput. Mach. **10** (1963), 487–492.

Es werden context-freie Sprachen (CFL) betrachtet [cf. Chomsky, IRE Trans. Information Theory **IT-2** (1956), 113–124], d.h. solche Wortmengen über einem Alphabet Σ, welche aus einem Anfangssymbol S mittels Produktionsregeln (in denen noch Hilfsvariable vorkommen können) erzeugt werden. Jede reguläre Menge [Kleene, *Automata studies*, pp. 3–41, Princeton Univ. Press, Princeton, N.J., 1956; MR **17**, 1040] ist nach Chomsky [Information and Control **2** (1959), 137–167; MR **21** #4107] ebenfalls eine CFL. Der (Rechts-) Quotient X/Y zweier Wortmengen X, Y über Σ ist die Menge derjenigen Worte w über Σ, zu denen es ein Wort y aus Y gibt, so daß wy aus X ist. Ergebnisse: (1) Ist X regulär, so ist X/Y

regulär. (2) Ist X eine CFL, Y regulär, so ist X/Y eine CFL. (3) Das Problem, für beliebige CFL X und Y zu entscheiden, ob X/Y eine CFL ist, ist rekursiv unlösbar. Ergebnis (3) wird zurückgeführt aus das als unlösbar nachgewiesene Postsche Korrespondenz-Problem [Post, Bull. Amer. Math. Soc. **52** (1946), 264–268; MR **7**, 405].

Walter Oberschelp (Herford)

Schützenberger, M. P. 1109

On context-free languages and push-down automata.

Information and Control **6** (1963), 246–264.

This note describes a special type of one-way, one-tape automata in the sense of Rabin and Scott that idealizes some of the elementary formal features used in the so-called "push-down store" programming techniques. It is verified that the sets of words accepted by these automata form a proper subset of the family of the unambiguous context-free languages of Chomsky's, and that this property admits a weak converse.

J. C. Shepherdson (Bristol)

Varshavskii, V. I. [Varšavskiĭ, V. I.]; 1110
Vorontsova, I. P. [Voroncova, I. P.]

On the behavior of stochastic automata with a variable structure.

Avtomat. i Telemeh. **24** (1963), 353–360 (*Russian. English summary*); *translated as Automat. Remote Control* **24** (1963), 327–333.

This paper extends to stochastic finite automata the investigations of M. L. Cetlin [Avtomat. i Telemeh. **22** (1961), 1345–1354; MR **25** #4973; Dokl. Akad. Nauk SSSR **139** (1961), 830–833; MR **24** #B480] on the behavior of finite automata in random media. A stationary stochastic automaton A is specified by finite sets $\{x_l\}$, $\{a_i\}$, and $\{y_v\}$ of inputs, states, and outputs, respectively, a function $F(a_i)$ giving the output y_{v_i} associated with any state a_i, and probabilities $\Pi_{ij}{}^l$ of a transition from state a_i to state a_j under input x_l.

A stationary stochastic automaton with two inputs, $x=0$ (a nonpenalty) and $x=1$ (a penalty), is said to function in the stationary random medium $C=C(p_1, p_2, \cdots, p_r)$ if, whenever the output is y_v, the probability that the next input will be 1 is p_v. The system consisting of the medium and the automaton is described by a Markov chain, and, assuming ergodicity, the stationary probabilities of the system are determined, and $M(A, C)$, the limiting expectation of a penalty, may be found. If $M(A, C)=\min(p_1, p_2, \cdots, p_r)$, the automaton is said to have optimal behavior in C, and it is said to have expedient behavior if $M(A, C)<(p_1+p_2+\cdots+p_r)/r$.

This analysis is extended to adaptive stochastic automata in which $\Pi_{ij}{}^l$ increases or decreases according to whether transitions from a_i to a_j under input x_l lead to nonpenalties or penalties. Under certain conditions limiting state probabilities have meaning for such an automaton functioning in a stationary random medium, and $M(A, C)$ may be defined. An example is given of a two-state automaton having expedient behavior in any medium $C(p_1, p_2)$, and of a two-state automaton whose behavior is sometimes optimal, sometimes expedient, and sometimes neither optimal nor expedient, depending on p_1 and p_2 as well as the initial values of the probabilities $\Pi_{ij}{}^l$. Results of computer simulations of adaptive stochastic automata in nonstationary random media are also reported.

R. M. Karp (Yorktown Heights, N.Y.)

AUTHOR INDEX

Mathematical Reviews

Vol. 29, No. 2 February 1965 Reviews 1111–2139

GENERAL

Rothe, R. 1111

★**Höhere Mathematik für Mathematiker, Physiker, Ingenieure. Teil I: Differentialrechnung und Grundformeln der Integralrechnung nebst Andwendung.**

Siebzehnte Auflage.

B. G. Teubner Verlagsgesellschaft, Stuttgart, 1962. 211 *pp.* *DM* 10.40.

This is an unaltered reprinting of an earlier edition of the first volume of a course of higher analysis [see #1112–#1116 following]. This volume is devoted to the concept of number and function, the differential and integral calculus, functions of one and several variables, differential geometry of plane curves, and complex functions.

Rothe, R. 1112

★**Höhere Mathematik für Mathematiker, Physiker, Ingenieure. Teil II: Integralrechnung. Unendliche Reihen. Vektorrechnung nebst Anwendungen.**

Dreizehnte Auflage.

B. G. Teubner Verlagsgesellschaft, Stuttgart, 1962. 210 *pp.* *DM* 11.60.

This second volume [see #1111 above] is devoted to a continuation of the integral calculus, series and power series, integrals involving a parameter, curvilinear integrals and integrals in complex domain, determinants and vectors with applications.

Rothe, R. 1113

★**Höhere Mathematik für Mathematiker, Physiker, Ingenieure. Teil III: Flächen in Raume. Linienintegrale und mehrfache Integrale. Gewöhnliche Differentialgleichungen reeller Veränderlicher nebst Anwendungen.**

Neunte Auflage.

B. G. Teubner Verlagsgesellschaft, Stuttgart, 1962. 244 *pp.* *DM* 13.60.

This third volume [see #1111, #1112 above] is devoted to the topics mentioned in the heading of the review.

Rothe, R. 1114a

★**Höhere Mathematik für Mathematiker, Physiker, Ingenieure. Teil IV: Hefte 1/2, 3/4, 5/6. Übungsaufgaben mit Lösungen zu Teile I, II, III.**

B. G. Teubner Verlagsgesellschaft, Stuttgart, 1962. *Heft* 1/2: 109 *pp.* *DM* 5.80; *Heft* 3/4: 108 *pp.* *DM* 5.80; *Heft* 5/6: 106 *pp.* *DM* 5.80.

1—M.R.

Rothe, R. 1114b

★**Höhere Mathematik für Mathematiker, Physiker, Ingenieure. Teil V: Formelsammlung.**

Sechste Auflage.

B. G. Teubner Verlagsgesellschaft, Stuttgart, 1962. 124 *pp.* *DM* 9.40.

These three "Hefte" contain exercises with solutions for Parts I, II and III of the series mentioned above [#1111–#1113], while Part IV consists of a set of formulae referenced for Parts I, II and III.

Rothe, R.; Szabó, István 1115

★**Höhere Mathematik für Mathematiker, Physiker, Ingenieure. Teil VI: Integration und Reihenentwicklung im Komplexen. Gewöhnliche und partielle Differentialgleichungen.**

Zweite, verbesserte Auflage.

B. G. Teubner Verlagsgesellschaft, Stuttgart, 1958. 251 *pp.* *DM* 20.80.

This volume, which was written by Szabó, is a continuation of the first three parts [#1111–#1113 above] and is devoted to selected topics in function theory and applications to special functions (in particular, to the gamma and beta functions and the Gaussian Ψ-function), differential equations (in particular, to the hypergeometric equations and the Bessel equation), and to partial differential equations. Exercises are included at the end of the various sections.

Rothe, R.; Schmeidler, W. 1116

★**Höhere Mathematik für Mathematiker, Physiker, Ingenieure. Teil VII: Räumliche und ebene Potentialfunktionen. Konforme Abbildung. Integralgleichungen. Variationsrechnung.**

Zweite, verbesserte Auflage.

B. G. Teubner Verlagsgesellschaft, Stuttgart, 1960. 218 *pp.* *DM* 22.40.

This volume, written by Schmeidler, is divided into three parts. The first part deals with potential theory in the plane and in space, together with classical topics in conformal mapping. The second part treats linear integral equations, while the third part is devoted to the classical calculus of variations. Each section contains exercises.

Yoshino, Y. 1117

★**The Japanese abacus explained.**

With an introduction by Martin Gardner.

Dover Publications, Inc., New York, 1963. xiii + 240 *pp.* $1.25.

An unaltered reprinting of the book published earlier [Kyo Bun Kwan, Tokyo, 1937].

HISTORY AND BIOGRAPHY

Mercier, André 1118

★Antikes und modernes Denken in Physik und Mathematik. Eine philosophische Betrachtung.

Erfahrung und Denken, Bd. 13.

Duncker & Humblot, Berlin, 1964. 154 *pp.* *DM* 19.80.

Vor dem Hintergrund der Einheitlichkeit der drei "Bestrebungen des menschlichen Geistes" (Wissenschaft, Kunst und Moral) und der Mystik (Religion) im antiken Denken entwickelt der Verfasser ein Bild der Zerrissenheit dieser vier Erkenntnisformen und ihrer Wertverschiebung in der modernen westlichen Welt. "Ein Großteil zeitgenössischer Denker im Westen meint, daß Philosophie auf objektives Wissen allein abgestellt hat . . . Infolge ihres alles überragenden Erfolges dient dabei die Physik als Modell. . . . Da aber einerseits die Physik seit Jahrhunderten ohne ihre mathematischen Grundlagen nicht mehr denkbar ist, andererseits die Mathematik sich als die reinste Form menschlicher Denkkraft erwiesen hat . . ., (muß) der Abstand und die Gegenwart der Antike im neuzeitlichen Denken auch in Beziehung auf die Mathematik beleuchtet" werden. Entgegen der Ankündigung im Titel wird aber mehr die Stellung der Physik und Mathematik im antiken und modernen Denken behandelt. Der Verfasser verfolgt so mathematische (S. 40 ff.) und physikalische (68 ff.) Denkweise von den frühesten Anfängen (Ägypten, Mesopotamien) über die Griechen und die beginnende Neuzeit bis in unsere Tage. Dabei wird stets "unter einem modernen Gesichtswinkel die Situation der Antike betrachtet". Bezeichnend dafür ist die S. 124 geäusserte Ansicht: "Für vieles wäre es besser, die Historiker . . . der Wissenschaft würden es nicht mehr aus ihren Archiven ausgraben und uns damit bekannt machen". Diesem mangelnden historischen Bewusstsein, das zu teilweise falscher Einschätzung und Beurteilung der Eigenheiten antiken mathematischen und physikalischen Denkens führt, steht das echte pädagogische Verantwortungsgefühl gegenüber, mit dem (nach der Behandlung des "Einflusses der exakten Wissenschaften auf die neuere Philosophie"; 98 ff.) der Standpunkt von Mathematik und Physik in der heutigen Welt des menschlichen Geistes skizziert wird, wo wenigstens innerhalb des Bereichs der Wissenschaft antike Einheit und Komplexität wieder erreicht zu werden scheint. Allerdings "bestehen Mathematik und Physik heute nicht mehr (wie in der Antike) als sakrale, sondern als rein weltliche Betätigungsfelder des Geistes" (128). Diesem Verlust stehe ein Gewinn an anderer Stelle gegenüber. Eine Bewertung dieser beiden Situationen wird jedoch bewußt vermieden.

F. Krafft (Hamburg)

Struik, Dirk J. 1119

★Abriss der Geschichte der Mathematik.

Zweite, bearbeitete Auflage.

VEB Deutscher Verlag der Wissenschaften, Berlin, 1963. xv+237 *pp.* *DM* 9.00.

Translation into German of the author's *A concise history of mathematics* [Dover, New York, 1948; MR **10**, 173].

Szabó, Árpád 1120

The transformation of mathematics into deductive science and the beginnings of its foundation on definitions and axioms.

Scripta Math. **27**, 27–48*A* (1964).

Erster Teil eines ursprünglich in Russisch [in den Istor.-Mat. Issled. **12** (1959), 321–392; MR **23** #A3622] veröffentlichten Berichtes über die Ergebnisse zahlreicher vorausgegangener Studien des Verfassers. Die wissenschaftliche Mathematik der alten Griechen unterscheidet sich von den zumeist nur rezeptartigen Regeln vorzugsweise der Babylonier grundsätzlich durch methodische Beweise. Am Anfang (5. Jh. v. Chr.) überwiegen sowohl in der Arithmetik wie in der Geometrie anschauliche Elemente die nur langsam zurückgedrängt werden und noch bei Euklid durchgefühlt werden können. Das schon bei Hippokrates v. Chios wirksame Streben nach systematischem und rein begrifflichem Vorgehen weist auf ältere derzeit noch unbekannte Vorbilder zurück. Die bei Parmenides zunächst in dialektischer Form auftretende indirekte Schlußweise wird von den Mathematikern übernommen und zu Unmöglichkeitsbeweisen umgebildet.

J. E. Hofmann (Ichenhausen)

A paper of Omar Khayyam. 1121

Translated by Ali R. Amir-Moéz.

Scripta Math. **26**, 323–337 (1963).

This is a literal translation of a hitherto unknown treatise of Khayyam. He seeks to solve the following problem Find a point on a quadrant of a circle in such manner that when a normal is dropped from the point to one of the bounding radii, the ratio of the normal's length to that of the radius equals the ratio of the segments determined by the foot of the normal. Khayyam shows that this is equivalent to a second problem: Find a right triangle having the property that the hypotenuse equals the sum of one leg plus the altitude on the hypotenuse. This in turn is shown to be equivalent to solving the equation

$$x^3+200x = 20x^2+2000.$$

He then digresses to classify such equations, to review previous work done in solving them, and to announce his intention of making a systematic study of the subject. This was evidently carried out, resulting in Khayyam's famous work on the cubic.

The author returns to the attack, finding a positive root of this particular cubic at the intersection of a rectangular hyperbola and a circle. He finds an approximate numerical solution by interpolation in trigonometric tables with the equivalent expression $1/\sin\theta = \cos\theta/\text{vers}\,\theta$.

The language of the original was evidently Arabic, although this is not stated by the translator. No indication is given as to where the manuscript or manuscripts of the source may be found. The translator's unorthodox transliterations from Arabic into Latin characters make it difficult to guess at the original word unless the reader is already familiar with it. A text publication of this treatise together with a commentary would be of great interest as a supplement to this translation.

E. S. Kennedy (Beirut)

Aaboe, Asger 1122

On a Greek qualitative planetary model of the epicyclic variety.

Centaurus **9** (1963), 1–10.

In a Greek papyrus of the second century A.D. [P. Mich. 149, edited, translated and annotated by F. E. Robbins in:

Michigan Papyri, Vol. III, pp. 62–117, Ann Arbor, Mich., 1936] there is evidence of a planetary model of epicyclic type in which the planet travels on the epicycle in the opposite direction to that correctly adopted by Ptolemy. In the paper under review it is shown that assuming the angular velocities of the center of the epicycle and the angular velocities of the planet on the epicycle to be the same as those in Almagest (and this is a very reasonable assumption), a model of the above mentioned type can make neither Venus nor Mars retrograde.

O. Schmidt (Copenhagen)

Boyer, Carl B. 1123

Clairaut le Cadet and a theorem of Thâbit ibn Qurra.

Isis **55** (1964), 68–70.

This note gives the history of the following generalization of the Pythagorean theorem: From vertex C of any triangle ABC draw lines cutting the opposite side AB in A' and B' so that angles $AA'C = BB'C = A$. Then $\overline{AC}^2 + \overline{BC}^2 = AB(AA' + BB')$.

Enunciated by Thābīt ibn Qurra (fl. 850 A.D.), the theorem was attributed by Montucla to the younger brother of A. C. Clairaut. It does not appear, however, in the single publication by Clairaut le Cadet (d. 1731). It was rediscovered and published by W. Lietzmann in 1937.

E. S. Kennedy (Beirut)

Toomer, G. J. 1124

A further note on Tamil astronomical tables.

Centaurus **9** (1963/64), 254–256.

The principal object of this note is to acknowledge an error pointed out by van der Waerden in the author's earlier paper [Centaurus **9** (1963), 11–15; MR **27** #5671], thus confirming a calculation and a contention of van der Waerden. Reference to additional literature is made.

Boerma, E. 1125

Galileis "Discorsi", ihre Entstehung und ihre Bedeutung für unsere Demonstrationsversuche.

Math. Naturwiss. Unterricht **17** (1964/65), 49–53.

The author reports on a number of elementary experiments in physics which are described in Galileo's famous *Discorsi e dimostrazioni matematiche* [Elfevirii, Leiden, 1638; reprinted, Boringhieri, Torino, 1958]. Many of them have found their way into the modern classroom of high school or college—testimony to a genius who (born exactly 400 years ago) invented them for the first time.

C. J. Scriba (Hamburg)

Ivanov, I.; **Dobrescu, Al.** 1126

Discussion of a problem of Leonardo da Vinci. (Romanian. **Russian and French summaries)**

Gaz. Mat. Fiz. Ser. A **15** (**68**) (1963), 686–693.

Authors' summary: "Cet article s'occupe d'une affirmation de Leonardo da Vinci concernant le temps nécessaire à un mobile pour parcourir un arc de cercle et la corde correspondante, affirmation qui a suscité à son temps une discussion ample. On reproduit le texte original et on l'analyse d'une façon détaillée."

★**Leonard Euler. Letters to colleagues** 1127

[Леонард Эйлер. Письма к ученым].

Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1963. 397 *pp.* 1.60 *r.*

Of Euler's correspondence, more than 2500 letters are still in existence, among them more than 500 written by Euler. While some have been published earlier, it was only in recent years that greater parts of this important correspondence for the history of mathematics and the sciences in the 18th century has been made available to a larger public. Collaboration between the German Academy of Sciences in Berlin and the Academy of Sciences of the USSR produced two volumes in 1959 and 1961 [*Die Berliner und die Petersburger Akademie der Wissenschaften im Briefwechsel Leonhard Eulers*, Teil I, *Der Briefwechsel L. Eulers mit G. F. Müller, 1735–1767*, Akademie-Verlag, Berlin, 1959; MR **22** #12029; Teil II, *Der Briefwechsel L. Eulers mit Nartov, Razumovskij, Schumacher, Teplov und der Petersburger Akademie, 1730–1763*, Akademie-Verlag, Berlin, 1961; MR **23** #A1496; cf. also T. A. Krasotkina, Istor.-Mat. Issled. **10** (1957), 117–158, Ju. H. Kopelevič, ibid. **10** (1957), 95–116; Istor.-Astronom. Issled. **5** (1959), 271–444].

As a further contribution the Academy of Sciences of the USSR has issued the collection of letters in hand, taken from originals that are preserved in the archives of the Academy at Petersburg and the University Library at Tartu (Dorpat). The volume contains in all 49 letters from Euler to J. S. Bailly (1), G. B. Bülfinger (2), C. Bonnet (5), F. Weggersløff (2), C. Wolff (1), G. Heinsius (7), J. N. de l'Isle (2), W. J. G. Karsten (1), M. Knutzen (2), H. Kühn (1), N. L. de la Caille (1), G. M. Lowitz (1), G. J. Marinoni (2), P. L. Maupertuis (3), P. Naudé (1), G. Poleni (8), B. le Bovier de Fontenelle (1), Gabrielle Emilie du Châtelet (2) and C. L. G. Ehler (6) (arranged according to the names of the recipients in the order of the Russian alphabet). The letters to Marinoni, Naudé, Poleni and Ehler are largely mathematical in content.

The team of editors under the direction of V. I. Smirnov has supplemented the original letters (which are written in Latin, French and German) by Russian translations and notes on the recipients of the letters, as well as on their contents, making use of the reply letters where possible. The notes and the index of names are given in Russian, only the names have been transliterated. Western readers must be warned that figures and tables are often included solely in the Russian translation, but not in the original Euler text. The volume is decorated with a plate and several portraits and facsimiles.

A number of minor errors have been noted, especially in the reproduction of Euler's old-fashioned German. A bad example is on p. 100 where, among other mistakes, "nicht" has been rendered as "mich" in two places. But such slips will not seriously detract from the value of this publication.

C. J. Scriba (Hamburg)

★**Giovanni Poleni (1683–1761) nel** 1128

bicentenario della morte, Padova, 17 dicembre 1961.

Supplemento al Vol. LXXIV degli "Atti e Memorie dell'Accademia Patavina di Scienze Lettere ed Arti".

Accademia Patavina di Scienze Lettere ed Arti, Padova, 1963. 147 *pp.*

Collected in this volume are commemoration lectures, together with some additional material relating to Poleni,

astronomer, meteorologist, mathematician, physicist, engineer and architect at Padua, whose name is remembered in the history of mathematics chiefly for his construction of a calculating machine. The lectures deal with Poleni the mathematician and astronomer, the theoretical and practical hydraulics engineer, and the architect, mentioning in passing some of his many other activities. There are extensive bibliographical articles on books, papers and unpublished material by Poleni and on publications about him, and a note on his arithmetical machine. Reprinted in the appendix are recent letters from the Royal Society and the Academies in Paris, Berlin and Moscow, containing information on his membership, correspondence and related subjects. (Eight letters from Euler to Poleni concerning mathematical and astronomical questions have recently been published in Leonard Euler, *Letters to colleagues* (Russian) [#1127 above].) Many interesting illustrations make this publication a worthy commemoration issue. *C. J. Scriba* (Hamburg)

Biermann, Kurt-R. 1129
Lagrange im Urteil und in der Erinnerung A. v. Humboldts.
Monatsb. Deutsch. Akad. Wiss. Berlin **5** (1963), 445–450.
J. L. Lagrange was considered by many of his contemporaries not only as a leading mathematician but even as the prototype of a mathematician. For Humboldt, this is testified by his speaking of Gauss as of another Lagrange, as well as by many similar utterances. The posthumous publication, by Humboldt, of a letter from Lagrange to Laplace concerning the latter's intention to follow Lagrange in moving to Berlin is another example of Humboldt's interest in Lagrange the mathematician and the man. *C. J. Scriba* (Hamburg)

Taliaferro, R. Catesby 1130
★The concept of matter in Descartes and Leibniz.
Notre Dame Mathematical Lectures, No. 9.
University of Notre Dame Press, Notre Dame, Ind., 1964. iii + 33 *pp.* $2.00.
In dieser mit sorgfältigen Literaturhinweisen ausgestatteten und auch die Stellungnahme der führenden Zeitgenossen berücksichtigenden Studie behandelt der Verfasser vor allem die Probleme der Lichtbrechung und des unelastischen Stoßes. Unter der mit angeführten Zweitliteratur fehlt erstaunlicher Weise das ausgezeichnete Werk von Y. Belaval [*Leibniz critique de Descartes*, Gallimard, Paris, 1960; MR **22** #12028].
J. E. Hofmann (Ichenhausen)

★Briefwechsel zwischen Leibniz und Christian Wolff. 1131
Aus den Handschriften der Koeniglichen Bibliothek zu Hannover. Herausgegeben von C. I. Gerhardt.
Georg Olms Verlagsbuchhandlung Hildesheim, 1963. iv + 188 *pp.* (1 *insert*) *DM* 24.00.
As a companion volume to Leibniz's *Mathematische Schriften* [H. W. Schmidt, Halle, 1849–1863; reprinted, Olms, Hildesheim, 1962; MR **25** #4979a–g] C. I. Gerhardt had published the volume under review in 1860 [H. W. Schmidt, Halle, 1860]. For this reason, the correspondence between Leibniz and Wolff, mathematician and philosopher, was omitted from *Der Briefwechsel von Gottfried Wilhelm Leibniz mit Mathematikern*, edited by C. I. Gerhardt [Mayer & Müller, Berlin, 1899; reprinted, Olms, Hildesheim, 1962; MR **26** #6030]. Wolff came into contact with Leibniz through Mencke, the editor of the Leipzig *Acta Eruditorum*, in 1704, and the correspondence (in Latin) lasted until Leibniz's death in 1716. Wolff, being 35 years younger than Leibniz, by whose recommendation he was made professor of mathematics at the university of Halle in 1707 (and of philosophy in 1709), had acquired a comprehensive knowledge of the mathematical and technical literature of his days but was not a creative mathematician. This is reflected in the correspondence, which is mostly on mathematics and natural philosophy. From 1712 onward we observe the rise of the quarrel for priority between Newton and Leibniz and their respective followers which was stirred up by the appearance of the famous *Commercium epistolicum* [London, 1712/13]. Except for the brief introduction Gerhardt gives no annotations, nor is there an index. *C. J. Scriba* (Hamburg)

Couturat, Louis 1132
★Opuscules et fragments inédits de Leibniz.
Extraits des manuscrits de la Bibliothèque Royale de Hanovre.
Georg Olms Verlagsbuchhandlung, Hildesheim, 1961. xvi + 683 *pp.* *DM* 82.00.
In 1901 L. Couturat, author of several excellent studies in the history of science, mathematics and logic, had published his book *La logique de Leibniz* [F. Alcan, Paris, 1901; reprinted, Olms, Hildesheim, 1961]. In order to enable his readers to form a competent judgement based on the original sources, Couturat edited the volume (now offered as a reprint) in 1903 [F. Alcan, Paris, 1903]. In contrast to Gerhardt's Leibniz editions [cf. the reviews quoted in #1131 above], his work met high standards, reproducing the manuscripts with painstaking accuracy, with notes and explanations. An elaborate index of names and subjects and systematic, as well as chronological, tables complete the edition, which is opened by an introduction well worth reading by any editor of manuscripts or other collections of scientific documents.

Most of the pieces printed in this volume are very short —from a sentence to a few pages in length. The common denominator is that they are all related to Leibniz's thoughts on logic, be it the study of logic as such or its applications in various fields. There are a great many notes and observations on mathematics; it is impossible to describe their contents in detail with a few words. Almost all are written in Latin. *C. J. Scriba* (Hamburg)

LOGIC AND FOUNDATIONS

See also 1248, 1756, 1928, 2134, 2136, 2138, 2139.

Lightstone, A. H. 1133
★The axiomatic method. An introduction to mathematical logic.
Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964. x + 246 *pp.* $7.95.
This introductory text is intended for students of mathematics familiar with a little modern algebra. It is divided

into two parts. The first contains informal accounts of two-valued logic, quantifiers and intuitive set theory. The second part opens with a review of algebraic concepts and proceeds to set up formalisations of sentence and predicate logic (the Hilbert-Bernays system); consistency is proved by Henkin's method, and Vaught's and Robinson's completeness criteria are established. There are several sets of exercises to which answers are provided.

R. L. Goodstein (Leicester)

Harbeck, Gerd 1134

★Einführung in die formale Logik.

Friedr. Vieweg & Sohn, Braunschweig, 1963. vi + 114 *pp.* *DM* 6.80.

Table of contents: (I) Grundlegende Begriffe; (II) Aussagenlogik; (III) Prädikatenlogik (Mengenlehre, Quantorenlogik, Syllogistik).

Schmidt, H. Arnold 1135

★Mathematische Gesetze der Logik. I. Vorlesungen über Aussagenlogik.

Die Grundlehren der mathematischen Wissenschaften. Bd. 69.

Springer-Verlag, Berlin, 1960. xxiv + 555 *pp.* *DM* 79.00.

Angesichts des Umfangs und der Reichhaltigkeit dieses Bandes muß Ref. sich leider auf eine zusammenhängende Erörterung beschränken; die dem Verf. eigentümliche Terminologie ist dabei aus dem Kontext zu erschließen. Es werden behandelt die "alternäre" (d. h. klassische oder 2 wertige), die "derivative" (sowohl die positive Logik wie den Minimalkalkül umfassende), die intuitionistische und die strikte Aussagenlogik; letztere auch in erweiterter Form mit Einbeziehung der Modalitäten. Von diesen Systemen werden meist mehrere Varianten dargestellt; so erscheint die alternäre Logik sowohl (nach Whitehead und Russell) mit Disjunktion und Negation, wie auch (nach Frege und Łukasiewicz) mit Implikation und Negation als Grundverknüpfungen, und außerdem (wie nachher die derivative Logik) als reine Implikationslogik. Wichtiger ist noch, daß mehrere Methoden der "Kodifikation" studiert werden, und zwar insbesondere folgende "Kodifikate": (i) "normaldeduktive", (ii) "aufschichtende" und (iii) "entwickelnde"; es kommen noch hinzu die Algebra der Logik und die wertende Logik. Zu (i): Die normaldeduktiven Kodifikate verwenden Einsetzung und Abtrennung. Zu (ii): Die aufschichtenden Kodifikate entsprechen den Sequenzenkalkülen Gentzens, wobei jedoch die Sequenz $U_1, U_2, \cdots, U_m \rightarrow V_1, V_2, \cdots, V_n$ durch eine Implikation $U_1 \wedge U_2 \wedge \cdots \wedge U_m \rightarrow V_1 \vee V_2 \vee \cdots \vee V_n$ ersetzt wird, und die Ableitungsregeln die jeweils entsprechende Umformung innerhalb einer solchen Implikation vorzunehmen gestatten (vgl. W. Craig [J. Symbolic Logic **22** (1957), 250–268; MR **21** #3317], wo jedoch nur die Quantoren in dieser Weise behandelt werden). Zu (iii): Ein entwickelndes Kodifikat liegt nur für die derivative Implikationslogik vor: es ist gekennzeichnet durch eine verwickelte, obzwar den Verhältnissen angemessene, Schlußregel. Die groß angelegte Darstellung der Aussagenlogik (die sich durch Sorgfalt und Klarheit auszeichnet) ist durch die (jetzt wohl allgemeine) Einsicht gerechtfertigt, daß schon auf diesem, anscheinend trivialen Gebiet Zusammenhänge aufgewiesen werden können, die auch für die höheren Gebiete der Logik von grundlegender Bedeutung sind. Es wird im allgemeinen jedes Kodifikat zunächst an und für sich beschrieben und studiert und erst nachher in seinem Verhältnis zu anderen Kodifikaten untersucht. Auf Fragen der Unabhängigkeit, Vollständigkeit (für Kodifikate der alternativen Logik sowie für die Algebra der Logik) und Entscheidbarkeit wird Nachdruck gelegt. Entscheidungsverfahren (teilweise neu) werden entwickelt für die Algebra der Logik, die alternäre Logik, die derivative Implikationslogik, die natürliche derivative Logik (d. h. für den Minimalkalkül), die intuitionistische Logik und die verschärfte strikte Logik (d. h. für das System S5 von Lewis).—Die Ausstattung ist mustergültig.

E. W. Beth (Zbl **88**, 246)

Monk, Donald 1136

Singulary cylindric and polyadic equality algebras.

Trans. Amer. Math. Soc. **112** (1964), 185–205.

This paper deals with the representation problem for finite-dimensional cylindric, polyadic, and polyadic equality algebras. An algebra is said to be "singulary" if it is generated by elements supported by singletons. Let α be a natural number. The main result is that every singulary algebra of dimension α is representable. At the end of the paper the author reformulates the result as a completeness theorem for singulary predicate calculus with only α individual variables. The method used in the proof of the main result is, as the author points out, essentially an algebraic version of the decision procedure for singulary predicate calculus due to Behmann [Math. Ann. **86** (1922), 163–229]. However, some new difficulties arise because only finitely many individual variables are available. The proof also provides a recursive algorithm for deciding whether an identity holds in all singulary algebras of dimension α.

H. J. Keisler (Madison, Wis.)

Scott, Dana 1137

More on the axiom of extensionality.

Essays on the foundations of mathematics, pp. 115–131. *Magnes Press, Hebrew Univ., Jerusalem,* 1961.

The axiom of extensionality asserts that any two sets having the same extension are identical. Let Z^* [ZF^*] be the systems obtained by omitting this axiom from the usual formulations Z [ZF] of Zermelo [Zermelo-Fraenkel] set theory. In this beautiful paper the author constructs: (i) a model for Z in Z^*; (ii) a model for ZF^* in Z. Call a set x extensional if whenever $y \varepsilon x$ and z has the same extension as y, then $z \varepsilon x$. The model (i) consists of all those sets which have an extensional transitive closure (with respect to ε); identity in the model is interpreted as equality of extension. This is the same kind of model (but discovered independently) as was used by the reviewer [J. Symbolic Logic **24** (1959), 287–300; MR **25** #18] (hereinafter "AE") to prove the consistency of Gödel-Bernays set theory relative to a non-extensional form (GB^-) of it. The author's definitions and proofs are much neater than those of AE.

The model (ii) is constructed as follows. For any ordinal α define $Q_\alpha = \bigcup_{\beta<\alpha} (SQ_\beta \times 2)$, where SA is the power set of A and $2 = \{0, 1\}$. The universe M of the model is $Q_{\omega+\omega}$; each $x \varepsilon M$ is an ordered pair $\langle a, j \rangle$, $a \subseteq M$, $j \varepsilon 2$, and may be described as the set a labelled by j. Membership is interpreted by η; $y \eta x \leftrightarrow y \varepsilon a$; identity is interpreted as identity. M can be described in Z; the proof that all axioms of

$ZF^{\neq}$ other than the axiom of replacement (Axiom R) hold in the model is trivial. To see that Axiom R holds, observe that the structure of $x \varepsilon M$ may be described by a downward branching tree T_x each link of which represents η and each node of which is labelled with 0 or 1 and corresponds to some $y \varepsilon M$. Define the rank ρx of x as the least n for which $x \varepsilon Q_{\omega+n}$; $\rho x = 0$ if and only if T_x is finite. Define $\psi_n(x)$ as the $x' \varepsilon M$ whose tree is obtained by changing the labels at all those nodes of T_x which correspond to elements of rank $> n$. ψ_n is an η-automorphism of M which leaves invariant just those x for which $\rho x \leqq n$. Let $\Phi(x, y)$ be a formula, Φ^M an interpretation of it in M, and suppose that for all $x \eta z \varepsilon M$ there is at most one $y \varepsilon M$ for which $\Phi^M(x, y)$. Choose $n \geqq \rho z$ and $\geqq \rho u_i$ where the u_i were substituted for the free variables of Φ (other than x and y) to obtain Φ^M. Let $x \eta z$ and suppose $\Phi^M(x, y)$; then x and Φ^M are invariant under ψ_n, and hence so is y; therefore $\rho y \leqq n$. Thus the image of z by Φ^M is in M, i.e., every instance of Axiom R holds in M.

Since M is a set in ZF, the consistency of $ZF^{\neq}$ can be proved in ZF; it follows that ZF is not relatively interpretable in $ZF^{\neq}$. The essential difference between $ZF^{\neq}$ and GB^- is that GB^- permits abstracts, which may select a unique object with a given extension (see, in particular, 2.12 of AE). {Reviewer's comment: The argument of AE could be carried out in a class-free system ZF^- which permitted set-abstracts.} The author points out that the model (i) could be extended to give a model for ZF in a system ZF^E; which is like $ZF^{\neq}$, but Axiom R is modified: if $\Phi(x, y)$ assigns a unique extension for y to each x, then the image of a set under Φ is a set. ZF^E lies between $ZF^{\neq}$ and ZF^-. In the reviewer's opinion, it is the natural non-extensional system; for, although $ZF^{\neq}$ arises naturally, it is, as (ii) shows, something of a curiosity.

{Correction: in definition (xxii), replace

'$x = \langle \phi``(n, m, a), j \rangle$' by '$x \varepsilon \langle \phi``(n, m, a), j \rangle$'.}

R. O. Gandy (Manchester)

Grzegorczyk, A.; Mostowski, A.; Ryll-Nardzewski, C. 1138

Definability of sets in models of axiomatic theories.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **9** (1961), 163–167.

T is a consistent first-order theory with finitely or denumerably many formulas and denumerably many variables x_i, $i \geqq 0$. $\mathfrak{M}$ is a set of relations on the integers indexed by the formulas of T, with each relation having the same number of arguments as its index. If the theorems of T are true in $\mathfrak{M}$, then $\mathfrak{M}$ is a model of T. The set of such models is denoted by M. Write $\models_{\mathfrak{M}} \varphi$ if formula φ is satisfied in $\mathfrak{M}$ by an assignment in which for all n, n is assigned to x_n, and write $[\varphi]$ for $\{\mathfrak{M} \mid \models_{\mathfrak{M}} \varphi\}$. Let $\pi_0, \pi_1, \cdots$ be a sequence of formulas, each with exactly one free variable, and assume for all $n > 0$, $\vdash_T \pi_n \to \pi_0$. T is ω-closed if $(\varphi)\{[(n)_{n>0} \vdash_T \pi_n \to \varphi] \to [\vdash_T \pi_0 \to \varphi]$. With respect to the sequence π_n, $\mathfrak{M}$ is an ω-model if $(\varphi)\{\varphi$ is 1-place and $\models_{\mathfrak{M}} (Ex_1)[\pi_0$ and $\varphi] \to (En)_{n>0} \models_{\mathfrak{M}} (Ex_1)[\pi_n$ and $\varphi]\}$, and is an ω-standard model if $(i)_{i>0}(En)_{n>0}[\models_{\mathfrak{M}} \pi_0(x_i) \to \pi_n(x_i)]$. M_ω denotes the set of ω-standard models. A topology is placed on M by taking $\{[\varphi] \mid \varphi$ is a formula of $T\}$ as a basis for open sets. Then M is absolute G_δ, and if T is ω-closed, then M_ω is a residual G_δ set in M. Further, a model elementarily equivalent to an ω-standard model is an ω-model. Hence the following results (some of which were previously known): If T is ω-closed, then $M_\omega \neq \varnothing$ and T has a complete consistent ω-closed extension. To every ω-model of T there is an elementarily equivalent ω-standard model. The principal notion of the paper is that of a set Z of integers definable in $\mathfrak{M}$ by the formula $\varphi: n \in Z$ if and only if $\models_{\mathfrak{M}} (Ex_1)[\pi_n$ and $\varphi]$. M_Z is the set of models of T in which Z is definable. Z is said to be representable in T with respect to the sequence π_n if there is a formula φ with one free variable x_1 such that $n \in Z \to \vdash_T (Ex_1)[\pi_n$ and $\varphi]$ and $n \notin Z \to \vdash_T \sim (Ex_1)[\pi_n$ and $\varphi]$. The representability of Z in T implies $M_Z = M$. A principal lemma is proved: For any Z, M_Z is an F_σ set in M, and if M_Z is not of first category in M, then there is a finite extension of T in which Z is representable. Now dropping the requirement that $\mathfrak{M} \in M$, denoting the class of sets Z definable in all models (all ω-models) of T by $\mathfrak{F}$ (by $\mathfrak{F}_\omega$), writing $\mathfrak{F}'$ and $\mathfrak{F}_\omega'$ similarly for models in M, and denoting by $\mathfrak{F}_0$ the class of sets representable in T, the principal result of the paper is obtained by use of the principal lemma. If T is axiomatizable and all recursive sets (R) are representable in T, then $\mathfrak{F}_0$, $\mathfrak{F}$ and R coincide. Further, if T is ω-closed and all hyperarithmetic (HA) sets are represented in T and the set of theorems of T is Π_1^1, then $\mathfrak{F}_0$, $\mathfrak{F}_\omega$ and HA coincide. If every set representable in a finite extension of T is representable in T, then $\mathfrak{F}_0 = \mathfrak{F}$. If T is ω-closed, $\mathfrak{F}_0 = \mathfrak{F}_\omega$. Applying these results to Z-F set theory T_0, with π_0 expressing the property of being a natural number, and π_n, $n > 0$, that of being the nth integer, one has that sets representable in all models of T_0 are recursive, while sets representable in all ω-standard models of T_0 are HA.

P. Axt (E. Lansing, Mich.)

Montague, Richard 1139

Fraenkel's addition to the axioms of Zermelo.

Essays on the foundations of mathematics, pp. 91–114. *Magnes Press, Hebrew Univ., Jerusalem*, 1961.

Fränkel's addition is the replacement (or Ersetzungs-) schema (RS). Assuming the axiom of regularity, it is a formal theorem that every set is an element of the cumulative type structure, formulated in the natural way in set theory. The author defines a (countable) subset of the cumulative type structure which can be proved to satisfy all instances of Zermelo's comprehension schema (SSF: schema of set formation); the proof needs only a few instances of the replacement axiom. And if an additional finite set A of sentences in the notation of set theory is added, a model satisfying both A and SSF can be established. In particular, the consistency of A_USSF can be established in $A_U\mathrm{RS}_F$ (where RS_F is a finite subset of RS). A consequence is that RS is not finitely axiomatisable over SSF, and the same holds for any consistent extension of RS (of course, without new constants; von Neumann-Bernays type formulations provide equivalent finite systems). A standard analysis of the argument shows that RS cannot be axiomatised by adding to SSF a set of formulae of bounded logical complexity. The author points out that in his proof (i.e., his particular construction of models) essential use is made of the axiom of regularity, though there is no evidence that it is essential to the result. It should be pointed out that also essential use is made of the power set axiom. This is perhaps essential to the result because the consistency of RS^- relative to SSF^- (in fact to classical analysis) can be proved in (primitive recursive)

arithmetic, where $^-$ means that the power set axiom is left out. The system **RS$^-$** is interesting because it is satisfied, e.g., by the hereditarily countable sets, at least as natural objects as the "extraordinary" sets which violate regularity. The paper is very agreeable to read.

G. Kreisel (Stanford, Calif.)

Montague, Richard 1140

Syntactical treatments of modality, with corollaries on reflexion principles and finite axiomatizability.

Acta Philos. Fenn. Fasc. 16 (1963), 153–167.

Der Verfasser untersucht die Möglichkeiten, eine Modalitätenlogik syntaktisch zu begründen, indem die Modalitäten nicht auf Aussagen, sondern auf Namen von Aussagen bezogen werden. Im Rahmen der gewohnlichen Prädikatenlogik mit Identität sei T eine Theorie, in der die Robinson-Arithmetik enthalten ist. Für eine Formel α mit der einzigen freien Variablen u und eine geschlossene Formel φ bezeichne $\alpha[\varphi]$ diejenige Formel, die sich aus α ergibt, wenn das Zahlzeichen der Gödelnummer von φ für u eingesetzt wird. Definiert man in T die Notwendigkeit durch eine Formel N mit der Eigenschaft, daß $N[\varphi]\to\varphi$, $N[\varphi\to\psi]\to(N[\varphi]\to N[\psi])$ und $N[\chi]$ für alle geschlossenen Formeln φ, ψ und alle geschlossenen logischen Axiome χ gilt, so kann man das Modalitätengesetz $N[N[\varphi]\to\varphi]$ nicht widerspruchsfrei hinzufügen. Hiermit und mit weiteren Sätzen ähnlicher Art wird gezeigt, daß eine Modalitätenlogik mit den üblichen Gesetzen der Notwendigkeit nicht in der angegebenen syntaktischen Weise zu gewinnen ist. Mit gleichen Methoden werden weitere Sätze bewiesen, nach denen das Formelschema $\alpha[\varphi]\to\varphi$ nicht allgemein in T gilt, wenn $\alpha[\varphi]$ formal ausdrückt, daß φ in T herleitbar ist. Verlangt man von α nur, daß $\alpha[\varphi]$ für jede in T herleitbare geschlossene Formel φ gilt, wobei $\alpha[\varphi]$ auch sonst gelten darf, so ergibt sich unter entsprechenden Voraussetzungen, daß T nicht endlich axiomatisierbar ist, falls $\alpha[\varphi]\to\varphi$ in T gilt.

K. Schütte (Kiel)

Mendelson, Elliott 1141

On non-standard models for number theory.

Essays on the foundations of mathematics, pp. 259–268. *Magnes Press, Hebrew Univ., Jerusalem*, 1961.

A Peano ring R is a ring which is elementarily equivalent (e.e.) to the ring Z of integers; with R is associated a field K of quotients which is e.e. to the field Q of rationals. This paper contains a miscellany of results, some new, some well known, about such R and K. First it is pointed out that no purely algebraic characterisation of Peano rings is known, although the work of Szmielew [Fund. Math. **41** (1955), 203–271; MR **17**, 233] does give such a characterisation for Abelian groups which are e.e. to Z. Subsequent results described include the following. (i) The ideals in R do not satisfy either the ascending or the descending chain condition, nor are they all principal ideals. (ii) There is an ideal (consisting of all r in R which are divisible by all $z\neq 0$ in Z) which is not the intersection or product of any finite number of prime, irreducible or primary ideals. (iii) R is not expressible as $Z\oplus X$. (iv) Every element of $K-Q$ is transcendental over Q; the transcendence degree of K is infinite, but K is not a pure transcendental extension of Q. The proofs of all these results are straightforward.

R. O. Gandy (Manchester)

Meredith, C. A.; Prior, A. N. 1142

Investigations into implicational S5.

Z. Math. Logik Grundlagen Math. **10** (1964), 203–220.

This paper is concerned mainly with the propositional calculus *C5* which is obtained from the Lewis calculus *S5* by omitting all formulae containing primitive functors other than strict implication. It is shown that a complete formalisation of *C5* is given by the axioms *CCpqCCqrCpr*, *CCCCpqrqCpq*, *Cpp* and the rules of substitution and modus ponens (strict implication being denoted by *C*). It is stated, without proof, that an alternative formalisation is provided by the above rules of procedure and the axioms *CCpqCCqrCpr*, *CCCPqPP*, *CqCpp*, where *P* is any strict implication, and it is proved that these may be replaced by the single axiom *CCCCCttpqCrsCCspCuCrp*. The paper contains a considerable amount of discussion of other axiom systems for *C5*.

Systems obtained from *C5* by replacing the third axiom (*Cpp*) of the original set by *CpCCpqq*, *CpCCCqpqq*, *CpCqp* are discussed. In these systems the contingent constant n is taken as an additional primitive and the systems have three additional axioms and one additional primitive rule of procedure. The paper concludes with a discussion of formalisations of full *S5* with strict implication, material implication and the logical constant 0 as primitives. Some of the discussion relates to the system obtained by taking the constant n as an additional primitive.

A. Rose (Nottingham)

Mo, Shao-kui [Mo, Shao-kuei] 1143

Notes on mathematical logic.

Acta Math. Sinica **13** (1963), 485–507 (*Chinese*); *translated as Chinese Math.* 4 (1964), 527–552.

The five loosely connected parts of this paper all pertain to algorithm theory or recursive-function theory.

(I) While recognizing normal algorithms (NA's) in the sense of A. A. Markov [Trudy Mat. Inst. Steklov. **42** (1954); MR **17**, 1038] as the hitherto simplest way of conceptually marshalling algorithmicity, the author is in favor of further standardized simplicity. He therefore adopts the (entirely "local") head-tail rules of E. L. Post [Amer. J. Math. **65** (1943), 197–215; MR **4**, 209] as the only ones to be employed, whereby words of form MP are changed into words of form PN (with P arbitrary, M and N nonempty). A head-tail algorithm (HTA) in A then consists of an alphabet A and a list of head-tail rules (with a rigid order in which they apply, given an initial word). In the main it is proved that (i) for any NA $\mathfrak{A}$ in A there is an HTA $\mathfrak{B}$ in $A\cup\{\alpha,\beta\}$ such that, for any words P, Q in A, $\mathfrak{A}$ transforms P into Q if and only if $\mathfrak{B}$ transforms αP into βQ; (ii) for any HTA $\mathfrak{C}$ in A there is an NA $\mathfrak{D}$ in A such that, for any words P, Q in A, $\mathfrak{C}$ transforms P into Q if and only if $\mathfrak{D}$ does. {Whereas some further reductions known from or implied by previous results in the field are mentioned, the author seems to have overlooked the pertinence of V. S. Černjavskiĭ's class of *čelnočnye* ("shuttle") algorithms [*Logičeskie issledovanija*, pp. 263–299, Izdat. Akad. Nauk SSSR, Moscow, 1959; MR **23** #A3072].}

(II) Consider as altering-the-middle computation (ATMC) any NA whose rules are all of the one type, whereby words of the form PAQ are changed into words of form PBQ, and where the NA conventions concerning stops and order of application are disregarded. If an ATMC is reversible, then it is an associative computation (AC) as previously

considered by Markov [vide supra]. Analogously defined are head-tail computations (HTC's), as opposed to HTA's. It is proved (i) if $\mathfrak{A}$ is an ATMC in A, then there is an HTC $\mathfrak{B}$ such that $\mathfrak{A}$ transforms P into Q if and only if $\mathfrak{B}$ transforms $P\alpha$ into $Q\alpha$; (ii) if $\mathfrak{C}$ is an HTC in A, then there is an ATMC $\mathfrak{D}$ in $A \cup \{\alpha, \beta\}$ such that $\mathfrak{C}$ transforms P into Q if and only if $\mathfrak{D}$ transforms $\alpha\beta\beta P\alpha$ into $\alpha\beta\beta Q\alpha$. It is also pointed out that most of Markov's conclusions concerning AC's can be generalized to hold for ATMC's and that any ATMC has the property of preserving adjacent equivalences in the sense that if P is transformed into Q and R into S, then PR is transformed into QS. Additional remarks pertain to the concepts of homomorphism, isomorphism, imbedding, and similarity of computations.

(III) Referring to Chapter 5 of Markov's monograph [op. cit.], it is proved that for an NA in A, where $\{a, b\} \subseteq A$, the non-self-applicability is not semi-decidable, and hence that the self-applicability of an algorithm in A cannot be completely decided.

(IV) The method of successive substitution employed is the ordinary method of functional composition. If from one function of m variables and m functions of n variables one constructs one function of n variables and characterizes this as (m, n)-substitution, then what is studied here is the reduction from (m, n)-substitution to $(1, 1)$-substitution. Likewise studied is the reduction of functionals, where a functional is understood as a computation for the construction of a new function from several old ones. {Among the more accessible references one misses V. A. Uspenskiĭ [*Lectures on computable functions* (Russian), Fizmatgiz, Moscow, 1960; MR **22** #12043].}

(V) The final section deals with the reduction of recursively generated sets. Mainly, it is proved that, under suitable hypotheses, in any recursively generated set the subset of functions of n variables is recursively generated.

E. M. Fels (Munich)

Kloss, B. M. 1144

On the definition of complexity of algorithms. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 38–40.

The author is interested in the question of how a measure of complexity may be associated with algorithms (partial recursive functions). He gives a number of results, the presentation of the proofs of which lean heavily upon Uspenskiĭ [*Lectures on computable functions* (Russian), Fizmatgiz, Moscow, 1960; MR **22** #12043]. The approach taken here consists of considering a general recursive function $K(\cdot)$, a principal (in the sense of Uspenskiĭ) enumeration η of the partial recursive functions, and the assignment, as the complexity of the function g, of the value $\min K(n)$, where n ranges over the numbers corresponding to g under the enumeration η. Two of the main results are the following. $K(\cdot)$ is a nontrivial function of complexity if and only if it takes its minimum at only a finite number of points. If $K(\cdot)$ is a nontrivial function of complexity, the function which associates with every partial recursive function its complexity is not computable.

R. M. Baer (Berkeley, Calif.)

Hmelevskiĭ, Ju. I. 1145

The solution of certain systems of equations in words. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 749–751.

Let $k > 1$, $n > 0$, $I = \{a_1, \cdots, a_k\}$, $H = \{x_1, \cdots, x_n\}$, $I \cap H = \varnothing$, where the a's and x's are distinct letters. Individual a's and x's are words; words juxtaposed are words; no other words. Consider a system of equations

$$(*) \qquad F_1 = G_1, \cdots, F_N = G_N \qquad (N \geqq 1),$$

where $F_1, \cdots, F_N, G_1, \cdots, G_N$ are n-words (reviewer's term) in $I \cup H$ (i.e., words that contain n different letters from the union of alphabets I and H). Let F be a word in $I \cup H$ and $X_1, \cdots, X_n$ words in an arbitrary set of letters. Let

$$(**) \qquad S_{X_1, \cdots, X_n}^{x_1, \cdots, x_n} F|$$

be the result of substituting, in F, words $X_1, \cdots, X_n$ into every occurrence of correspondingly indexed letters $x_1, \cdots, x_n$. If $X_1, \cdots, X_n$ are words in I, so is (**). A list of such $X_1, \cdots, X_n$ in I is a solution of the system of equations (*) above if, in I, the following equations hold:

$$S_{X_1, \cdots, X_n}^{x_1, \cdots, x_n} F_r| = S_{X_1, \cdots, X_n}^{x_1, \cdots, x_n} G_r| \qquad (r = 1, \cdots, N).$$

A. A. Markov has posed this problem (apparently unpublished): Is there an algorithm with which it is decidable whether any system like (*) has a solution? Call such an algorithm an "n-algorithm" (reviewer's term) if it applies to systems of equations of n-words. Markov succeeded in constructing a 2-algorithm (unpublished).

Call a system of equations of n-words a "W_j-system" if each individual equation of this system is an equation of j-words. Then it can be shown that W_3-systems are all that one ever needs to consider. E.g., the equation $x_1x_2 = x_3a_1x_4$ has a solution if and only if the system $x_1x_2 = x_5$, $x_3a_1 = x_6$, $x_6x_4 = x_7$, $x_5 = x_7$ does.

The author proves a theorem from which follows the existence of n-algorithms for W_2-systems. Another theorem shows that for systems like (*) one need not go beyond $N = 1$.

E. M. Fels (Munich)

Dekker, J. C. E. 1146

An infinite product of isols.

Illinois J. Math. **7** (1963), 668–680.

Let T be a regressive isol; let $a_0, a_1, \cdots$ be a sequence of non-negative integers. The author introduced the notion of the infinite sum isol $\sum_T a_n$ of the sequence $\{a_n\}$ with respect to T [Proc. Sympos. Pure Math., Vol. V, pp. 77–96, Amer. Math. Soc., Providence, R.I., 1962; MR **26** #16]. He suggested an analogous definition of $\prod_T (1 + a_n)$ to his student Ferguson, who developed the theory of such products in his dissertation [Ph.D. Dissertation, Rutgers Univ., New Brunswick, N.J., 1963]. E. Ellentuck independently investigated these same products [see #1147 below]. The corresponding problem of defining $\sum_T A_n$ and $\prod_T A_n$ for more general sequences $\{A_n\}$ of isols is much more difficult. The problem of when such operations are sensible is not yet resolved. Such operations incorporate a new and interesting aspect of Dedekind-finiteness into isols over and above the aspects presented by the now fairly well-understood theory of combinatorial functions. In the paper under review such sums and products are introduced in some special cases, and naturality properties are developed. These are undoubtedly to be regarded as an exploratory sample of the type of phenomena any general treatment will have to encompass. One such example is $\prod_T (1 + X^{2^t}) = \sum_{2^T} X^t = (X^{2^T} - 1)/(X - 1)$, where T is a regressive isol $\geqq 1$ and X is an isol $\geqq 2$.

A. Nerode (Ithaca, N.Y.)

Ellentuck, Erik 1147
Infinite products of isols.
Pacific J. Math. **14** (1964), 49–52.

The author introduces and investigates the notion $\prod_T (1+a_n)$, where T is a regressive isol and $\{a_n\}$ is a sequence of non-negative integers. This notion was independently introduced by Dekker and investigated by Ferguson [see #1146 above]. The following naturality property is established. Let f be a recursive combinatorial function, let g be the recursive combinatorial function $g(m)=\prod_{0\le n\le m-1}(1+f(n))$, let G be the extension of g to isols, let T be a regressive isol. Then $\prod_T (1+f(n))=G(T)$. Thus the product of 1, 2, 3, $\cdots$ with respect to T is $T!$, and for finite $a>1$ the product of $a, a, a, \cdots$ with respect to T is a^T. *A. Nerode* (Ithaca, N.Y.)

Rabin, Michael O. 1148
Real time computation.
Israel J. Math. **1** (1963), 203–211.

A multitape Turing machine operates in real-time if and only if it performs exactly one operation per input symbol and prints a zero or one on each operation (on an output tape). The set of (input) sequences for which a real-time Turing machine, T, yields a one is said to be defined by T. The author shows constructively that there exists a set of sequences which is defined by a real-time, two-tape Turing machine, but which cannot be defined by a real-time, one-tape Turing machine. Thus, for "time-limited" computations a two-tape machine is computationally more powerful than a one-tape machine.

This result is very instructive and contributes new techniques to the emerging theory of computational complexity of recursive sequences and functions. This theory is mainly concerned with the classification of computable problems by their degree of computational difficulty, the study of the properties of these complexity classes, their relation to each other and their dependence on the (abstract) computing devices. Other contributions to this new topic of research have been made by H. Yamada [IRE Trans. **EC-11** (1962), 753–760; MR **27** #2141], J. Hartmanis and R. E. Stearns [Trans. Amer. Math. Soc. to appear], M. Blum [Ph.D. Dissertation, M.I.T., Cambridge, Mass. 1964] and F. C. Hennie (unpublished).
J. Hartmanis (Schenectady, N.Y.)

Rasiowa, Helena; Sikorski, Roman 1149
★The mathematics of metamathematics.
Monografie Matematyczne, Tom 41.
Państwowe Wydawnictwo Naukowe, Warsaw, 1963. 522 pp. $12.00.

This very agreeable book gives a systematic and leisurely account of the authors' researches in elementary metamathematics; elementary in the sense that it is almost wholly concerned with various predicate logics of first order. The general idea is that the obvious operations on equivalence classes of formulae (of such a logic) constitute an algebra: boolean in the classical case, topological boolean (=closure) in the intuitionistic case, and pseudo-boolean in the modal case. In consequence, the authors begin with a self-contained study of lattices generally and the algebras mentioned, and then they apply these results to the predicate logics. Most of the specific results in the book seem to have been published previously; the authors do not indicate what is new over their previous articles. But the coherent and smooth presentation of the whole body of material is, of course, more effective than a series of research papers.

The other reason why this book is so engaging is the lucid and frank formulation of the authors' view on the relative role of mathematics and philosophy in logical research, a view which is widely held, but rarely put so clearly. Roughly speaking, they consider it a miracle, evidently beyond rational explanation, that general and "vague" philosophical conceptions should lead to such rich formal structures as intuitionistic formal systems. They state that the mathematical aspects of such systems are best brought out by analysis in algebraic rather than, e.g., combinatorial (finitistic) terms. Less explicitly, they reject the problematic aspects of higher-order consequence, e.g., in their attitude to Gödel's incompleteness theorem; classical arithmetic is considered "completely satisfactory" (p. 320) because it codifies the inferences customarily made, so to speak, perpetuating the limitations of current mathematics. Quite consistently, they regard the paradoxes as a mere aberration (pp. 9, 144), which, indeed, they are if by sets one means objects in the cumulative type structure built up by the power set construction from some definite collection. (Curiously enough, they do not seem to say in so many words that they mean just this by "set".) This is consistent with their general view because, if one regards a notion so mildly "philosophical", i.e., intensional, as constructive as vague, one will certainly boggle at those self-applicable notions of property for which the paradoxes constitute real problems.

The book illustrates beautifully the simplicity resulting from such a general view. But it also has its defects. First, paradoxically enough, for finitist or constructive metamathematics the area covered by the present book constitutes only a tiny and simple fraction; e.g. K. Schütte's *Beweistheorie* [Springer, Berlin, 1960; MR **22** #9438] gives a fairly good picture of the relative importance for (constructive) metamathematics of predicate logic on the one hand, and arithmetic, ramified analysis on the other: the latter are not treated in the book at all. The whole, quite basic, theory of arithmetisation, which is quite essential in the metamathematics of arithmetic, does not fit into the authors' approach, as they themselves emphasize (p. 6). But even if one considers only predicate logic, there is no proof of the interpolation lemma, not to speak of a proof that is uniform for both intuitionistic and classical predicate logic as in Schütte [Math. Ann. **148** (1962), 192–200; MR **26** #24]. Nor are included what are, perhaps, the two most striking results on Heyting's predicate logic (HPC), namely (i) that the HPC is complete on the topological interpretation for open subsets in the Cantor discontinuum, which has been obtained by the analysis of E. W. Beth [Meded. Kon. Nederl. Akad. Wetensch. **18** (1955), 309–342; MR **19**, 625], and (ii) S. A. Kripke's proof of undecidability of the monadic fragment of HPC [Z. Math. Logik Grundlagen Math. **8** (1962), 113–116; MR **28** #2975]. Thus, while there can be no doubt of the expository virtues of the algebraic treatment, the omissions cast some doubt on its fruitfulness (suggestiveness). It may be remarked that the methods used in the proofs quoted, though set-theoretic, are perhaps more closely connected with the intended interpretation of HPC. The same applies to another obvious set-theoretic interpretation of HPC in the reviewer's article [Infinitistic Methods (Proc. Sympos. Foundations of

Math., Warsaw, 1959), pp. 103–140, Pergamon, Oxford, 1961; MR **26** #3599]. Finally, and conceptually more interesting than any of these, there is the reinterpretation of logical constants obtained from the definition of forcing in Cohen [Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 1143–1148; MR **28** #1118] if $(x)_\alpha$, p. 1146, line 4, and (x), line 9 from the bottom, are read as $\neg(\exists x)_\alpha\neg$, $\neg(\exists x)\neg$, respectively. When formulated abstractly, i.e., not only for the case of quantifiers ranging over the ordinals of $\mathfrak{M}$ [loc. cit.], the set of valid statements is also generated by intuitionistic rules, because the abstract formulation is formally identically with the interpretation above.

Perhaps some of these defects are inherent in the authors' "non-philosophical" approach: for, at least so far, the heuristically most powerful ideas in mathematical logic have come from either trying to formulate a philosophical conception mathematically or to decide the validity of such a conception. But, in the reviewer's opinion, there is a wide area of metamathematics which would fit beautifully into the authors' scheme, namely, the truly combinatorial (formal) ideas introduced into proof theory by Gentzen, in particular, the duality of cut-free rules used for introducing logical constants into hypothesis and conclusion, respectively, of a sequent, the idea of using infinite proof figures by Novikov [Mat. Sb. (N.S.) **12 (54)** (1943), 231–261; MR **5**, 197] and the development by Schütte and others. There are few objects in mathematics that lend themselves more readily to a lattice-theoretic analysis than trees. And if they are well-founded, they lend themselves to methods of proof by (transfinite) induction. This is precisely the characteristic feature of the area of metamathematics here recommended for exposition in the mathematical spirit of the authors.

G. Kreisel (Stanford, Calif.)

Hansel, Georges 1150

Nombre minimal de contacts de fermeture nécessaires pour réaliser une fonction booléenne symétrique de n variables.

C. R. Acad. Sci. Paris **258** (1964), 6037–6040.

This author establishes new bounds on the minimum number A_n of relay contacts required to realize a symmetric boolean function of n variables. The new bounds on A_n are

$$n \log_2 n \leqq A_n \leqq n\left(1+\frac{1-\log_2 e+\log_2\log_2 e}{\log_2 n}\right)\log_2 n.$$

This result shows that, as n increases, A_n goes to $n \log_2 n$.

E. G. Wagner (Yorktown Heights, N.Y.)

SET THEORY

See also 1138, 1278.

Klaua, Dieter 1151

Eine Formulierung des Tarskischen Unerreichbarkeitsaxioms mittels Allmengen.

Z. Math. Logik Grundlagen Math. **10** (1964), 115–117.

The author considers Tarski's unattainability axiom, which asserts the existence in the theory of sets of arbitrarily large cardinal and ordinal numbers, from the point of view of his theory of universal sets ("Allmenge") [the author, same Z. **3** (1957), 303–316; MR **20** #2276]. Tarski's axiom postulates for any set N the existence of a family M of sets containing as members: (1) the set N, (2) all subsets of any of its elements, (3) the power set of each of its elements, and (4) each of its subsets of cardinality smaller than M. The author proposes an equivalent form of this axiom which calls for the existence for each set N of a universal set A with these four properties, emphasizing thus to a greater degree the limit number character of the "exorbitant" numbers.

E. J. Cogan (Bronxville, N.Y.)

Simauti, Takakazu 1152

A note on the construction of ordinal numbers.

Comment. Math. Univ. St. Paul. **12** (1964), 37–39.

The author considers general schemata for constructing ordinals, generalizing Ackermann's construction and Takeuti's ordinal diagrams. Some sufficient conditions are given for such schemata to actually give a well-ordering. {Remarks: On p. 38 the operations φ, ψ presumably can have parameters. The equality $B_\rho{}^c=\{\alpha|\alpha<\rho\}$ on p. 38 appears to be false. It is not clear how to carry through the "transfinite induction until $\varphi_1(\delta)$" indicated at the bottom of p. 38.}

D. Monk (Boulder, Colo.)

Bressler, D. W.; Sion, M. 1153

The current theory of analytic sets.

Canad. J. Math. **16** (1964), 207–230.

Les auteurs examinent la théorie des ensembles boréliens, sousliniens et analytiques, d'une manière plus générale qu'autrefois, de façon à inclure les résultats récents dus surtout à Choquet et Sion. On considère d'abord sans topologie une famille H d'ensembles et les familles dites H-boréliennes, anneau H-borélien ou champ H-borélien, qui sont les plus petites contenant H, fermées pour la réunion (non vide) dénombrable, et respectivement pour l'intersection dénombrable, la différence, la complémentation par rapport à la réunion des ensembles de H. La famille H-souslinienne (ou souslinienne relative à H) est celle des ensembles $\bigcup_{s\in S}\bigcap h(s|\bar{n})$ où S est l'ensemble des suites d'entiers $\geqq 0$, $\bar{n}$ l'ensemble des entiers de 0 à N inclus, $s|\bar{n}$ la restriction de la suite s à ses $n+1$ premiers éléments, $h(\alpha)$ une application de l'ensemble de ces suites tronquées α dans H. On donne d'abord des propriétés d'inclusion ou identité concernant de telles familles, comme l'invariance quand, partant d'une famille H-souslinienne, on en prend la famille relative souslinienne ou borélienne.

Puis, dans un espace topologique, on définit un ensemble analytique comme une image continue dans cet espace d'un ensemble $K_{\sigma\delta}$ (intersection dénombrable d'une réunion dénombrable de fermés compacts) d'un autre espace topologique. On examine dans des conditions plus ou moins restrictives les relations d'inclusion des diverses familles précédentes. Un exemple simple est que la famille souslinienne relative aux ensembles fermés compacts est contenue dans la famille analytique, et, si l'espace est séparé, la famille analytique est contenue dans la famille souslinienne relative à celle des fermés. Puis on donne des théorèmes de séparation. Par exemple, deux ensembles disjoints de la famille K-souslinienne (où K est l'ensemble des fermés compacts) sont séparés par la famille K-borélienne. On étudie les images des ensembles boréliens ou analytiques. Ainsi, si f est une application continue de

X séparé, réunion dénombrable de compacts dans Y séparé, l'image $f^{-1}(A)$ de A analytique dans Y est analytique dans X.

L'auteur termine en insistant sur un aspect d'approximation des ensembles analytiques par des compacts contenus, qui explique leur intérêt en analyse et leur rôle dans la théorie de la mesure et de la capacité, dont divers théorèmes sont obtenus et généralisés comme application du résultat général suivant. Appelons capacitance une famille P d'ensembles de l'espace X, invariante par agrandissement d'un de ses ensembles et telle que, si la réunion d'une suite croissante d'ensembles appartient à P, l'un d'eux appartient à P. Soit alors H une famille d'ensembles de X, fermée par réunion ou intersection finie et A un ensemble H-souslinien. Si A appartient à une capacitance P, il existe une suite décroissante $\{\alpha_n\}$, telle que $\alpha_n \in H \cap P$ et $\bigcup \alpha_n \subset A$. Dans tout ce travail on examine naturellement le cas classique des espaces métriques complets séparables. *M. Brelot* (Paris)

COMBINATORIAL ANALYSIS

See also 1220, 1464, 1469, 1635, 1711.

Turyn, Richard J. 1154

The multiplier theorem for difference sets.

Canad. J. Math. **16** (1964), 386–388.

The reviewer showed [same J. **15** (1963), 121–124; MR **26** #41] that for a cyclic difference set with parameters v, k, λ such that $k-\lambda=2q$, where q is prime, $(q, v)=1$ and $(v, 7)=1$, the number q is a multiplier. The author removes the restriction that $(v, 7)=1$ and generalizes this result to abelian difference sets such that $k-\lambda=2q^a$, where q is prime, $(q, v)=1$ and a is odd, i.e., he proves that q is a multiplier for such a set.

Morris Newman (Washington, D.C.)

Nečiporuk, È. I. 1155

On the complexity of superpositions in bases that contain nontrivial linear formulas with zero weights.

Dokl. Akad. Nauk SSSR **136** (1961), 560–563 (*Russian*); *translated as Soviet Physics Dokl.* **6** (1961), 6–9.

Post [*The two-valued iterative systems of mathematical logic*, Princeton Univ. Press, Princeton, N.J., 1941; MR **2**, 337] showed there are exactly five classes L of functions on Boolean algebras, closed under composition, that are linear, that is, expressible in terms of symmetric difference and set difference. Let a weight be attached to each expression for a Boolean function in terms of members of L together with conjunction $(x \cdot y)$ and negation $(x+1)$ by counting the number of conjunctions and negations. Define $L(n)$ to be the least l such that every Boolean function of no more than n arguments is expressible by a function of weight no more than l. It is shown that for each of the five classes, $L(n) \sim 2^{n-1}/n$.

The proof is combinatorial, and would appear difficult to reconstruct from the argument presented.

R. C. Lyndon (Ann Arbor, Mich.)

Yalavigi, C. C. 1156

A tournament problem.

Math. Student **31** (1963), 51–64 (1964).

The paper designs tournaments in which v players participate. In each match of the tournament two disjoint sets of players, m in each, become opposing teams. Pairs of opposing teams are so chosen that each of the $\frac{1}{2}v(v-1)$ pairs of players belongs to the same team in $m-1$ matches and to opposing teams in m matches. F. Scheid [Amer. Math. Monthly **67** (1960), 39–41; MR **23** #A72] gave tournament designs for some small values of m and v. The present paper obtains two infinite families of tournaments by using the method of generators. In one family m is a power of 2 and v is a power of a prime p with $p\equiv 1 \pmod 4$. In the second family m is a multiple of 3 and v is a power of a prime with $v \equiv 1 \pmod 6$).

E. N. Gilbert (Murray Hill, N.J.)

Lagrange, René 1157

Sur les combinaisons d'objets numérotés.

Bull. Sci. Math. (2) **87** (1963), 1ière partie, 29–42.

The combinations in question are those of numbers 1 to n, p at a time, taken in rising order, and specified by the set of differences $\{d_1, d_2, \cdots, d_{p-1}\}$ with d_j the difference of the numbers in the $j+1$ and j positions. It is shown that the number of combinations of numbers 1 to n, p at a time, with smallest difference at least k and largest difference at most k', is given by, in a slight modification of the author's notation,

$$\Gamma(n, p; k, k') = \sum_{i=0} (-1)^i \binom{p-1}{i}\binom{q-ij}{p},$$

with $q=n-(p-1)(k-1)$, $j=k'-k+1$. Further, if the difference of consecutive numbers is taken as the smallest positive residue, modulo n, the similar number with smallest difference at least k and largest difference unrestricted, is given by

$$\Gamma^*(n, p; k) = \frac{n}{p}\binom{q-k}{p-1}.$$

Finally, if the diameter of a p-combination is taken as the difference of its largest and smallest numbers, it is shown that the number of p-combinations of n numbers with smallest difference (in the first sense) at least k, and diameter at most δ, is given by

$$\Gamma(n, p; k; \delta) = \frac{np-(p-1)(k+\delta)}{p}\binom{\delta-(p-1)(k-1)}{p-1}.$$

J. Riordan (Murray Hill, N.J.)

ORDER, LATTICES

See also 1136, 1155, 1269, 1270, 1279, 1520, 1776.

Luxemburg, W. A. J. 1158

On finitely additive measures in Boolean algebras.

J. Reine Angew. Math. **213** (1963/64), 165–173.

The author attempts to generalize Koshi's theorem in some sense. Generally, if we consider a linear lattice L on a Boolean algebra B, then from a theorem about L we conclude a corresponding theorem about B. Thus Theorem 2.1, which the author claims to be useful, is an

immediate consequence of Theorem 13.2 of the reviewer's book, *Modulared semi-ordered linear spaces* [Maruzen, Tokyo, 1950; MR **12**, 420]. Koshi's theorem is formulated about positive linear functionals on sequentially continuous linear lattices. Thus we obtain a corresponding theorem about measures on sequentially continuous Boolean algebras. The author generalizes this theorem to non-sequentially continuous Boolean algebras.

H. Nakano (Detroit, Mich.)

MacLaren, M. Donald 1159

Atomic orthocomplemented lattices.

Pacific J. Math. **14** (1964), 597–612.

Taking as a guide the lattice of all closed subspaces of a separable Hilbert space, the author investigates complete, irreducible, atomic, orthocomplemented, semi-modular lattices. Among other things, he shows that such a lattice is the lattice of closed subspaces of a suitable semi-inner product space. *J. McLaughlin* (Ann Arbor, Mich.)

Maeda, Fumitomo 1160

Modular centers of affine matroid lattices.

J. Sci. Hiroshima Univ. Ser. A-I Math. **27** (1963), 73–84.

Let L be a matroid lattice, i.e., a complete lattice with the following three properties: (i) every element of L is a join of atoms; (ii) if p is an atom, and S is a set of atoms with $\bigcup S \geqq p$, there exists a finite subset S_1 of S with $\bigcup S_1 \geqq p$; (iii) any two elements of L having a common lower neighbor have a common upper neighbor. Then, in L, the concepts of point, line, and parallel lines can be defined in the usual way. Let L satisfy the following further conditions: (iv) there exists an element of L of rank 4; (v) $(c \cup a) \cap b = c \cup (a \cap b)$ if $c \leqq b$ and $a \cap b \neq 0$; (vi) two different lines which are parallel to one and the same line never intersect. Under these conditions, the author calls L an affine matroid lattice.

Let us call a line l complete if there exists no line parallel to l, and let us call an element of L anti-complete if it contains no complete line. If p is a point then the set of the anti-complete elements of L containing p is known to have a maximum $I(p)$. The author of the present paper calls an element a of L $\|$-closed if $I(p) \leq a$ for every point p contained in a. He proves that the set M of all $\|$-closed elements of L is a sublattice of L with respect to unrestricted joins and meets, that M is a modular matroid lattice, that the points of M are the elements $I(p)$ of L, and that M is irreducible if L is non-modular. He calls M the modular center of L.

H. F. J. Lowig (Edmonton, Alta.)

Maeda, Fumitomo 1161

Parallel mappings and comparability theorem in affine matroid lattices.

J. Sci. Hiroshima Univ. Ser. A-I Math. **27** (1963), 85–96.

Let L be a semimodular lattice as defined in the paper on "Modularity in the theory of lattices" by L. R. Wilcox [Ann. of Math. (2) **40** (1939), 490–505]. For such a lattice, the author proves the following theorem: If $a\|b$ (i.e., $a \cap b = 0$, and $a \cup b$ covers both a and b), p and q are points contained in a and b, respectively, and $T = ((a_1 \cup q) \cap b$; $a_1 \in L(p, a))$, then T is an isomorphism of $L(p, a)$ onto $L(q, b)$, its inverse is $((b_1 \cup p) \cap a$; $b_1 \in L(q, b))$, and if $a_1 \in L(p, a)$ and $b_1 \in L(q, b)$, $Ta_1 = b_1$ if and only if $a_1 \cup q = b_1 \cup p$, and $a_1 \| b_1$ if this condition is satisfied. He calls T a parallel mapping of $L(p, a)$ onto $L(q, b)$.

Let, especially, L be an affine matroid lattice. If a is anti-complete (see preceding #1160) and non-zero, and r is a point, the author defines $r(a)$ as that element b of L which has the property that $a \|\| b$ (i.e., $a\|b$ or $a = b$) and $r \leqq b$. With this notation, he states and proves the following theorem: If a and b are anti-complete, p and q are points contained in a and b, respectively, and r is any point, then there exist a', a'', b', b'' such that $a = a' \cup a''$, $a' \cap a'' = p$, $b = b' \cup b''$, $b' \cap b'' = q$, $a' \|\| b'$, and $r(a'') \cap r(b'') = r$. In this case, $r(a') = r(b') = r(a) \cap r(b)$. He calls this theorem a comparability theorem.

The author proves a number of further theorems on affine matroid lattices. E.g., he proves that every affine matroid lattice is a Wilcox lattice, i.e., a lattice of the sort considered at the end of p. 497 and the beginning of p. 498 of the paper by L. R. Wilcox quoted above.

H. F. J. Lowig (Edmonton, Alta.)

Hales, A. W. 1162

On the non-existence of free complete Boolean algebras.

Fund. Math. **54** (1964), 45–66.

The main result of this paper is the answer to a question raised by Rieger in 1951 [Fund. Math. **38** (1951), 35–52; MR **14**, 347] whether there exists a free complete Boolean algebra on ω complete generators. The question is answered in the negative by the author, a result that was obtained independently and about the same time by Gaifman [ibid. **54** (1964), 229–250]. The author also investigates the existence of free complete lattices on a set of complete generators. Applying a technique developed by Crawley and Dean [Trans. Amer. Math. Soc. **92** (1959), 35–47; MR **21** #7163] it is shown that the free complete lattice on three complete generators does not exist. (The free complete lattices on one and two complete generators are the ordinary free lattices.) The first step is the definition of "polynomials" [cf. G. Birkhoff, *Lattice theory*, Amer. Math. Soc., New York, 1948; MR **10**, 673]. Let γ be a cardinal (identified with the corresponding initial ordinal), and let $a_0, a_1, \cdots, a_i, \cdots, 0 \leqq i < \gamma$, be symbols (generators). Then for each i, $0 \leqq i < \gamma$, a_i is a polynomial of rank $r(a_i) = 0$. If $\mathscr{A}$ is a set of polynomials defined previously, and if $\mathscr{A}$ has cardinality $\mathscr{B}$, then the symbols $\bigcup \mathscr{A}$ and $\bigcap \mathscr{A}$ are polynomials of rank $r(\bigcup \mathscr{A}) = r(\bigcap \mathscr{A}) = \max(\mathscr{B}, \sup_{A \in \mathscr{A}}[r(A)+1])$. The collection of all such polynomials is denoted by $L(\gamma)$. A valuation of $L(\gamma)$ is defined as a mapping f of $\{a_i : 0 \leqq i < \gamma\}$ onto a subset of a complete lattice L which completely generates L. Such an f can be extended in a natural way to a mapping f^* from $L(\gamma)$ to L. A_1 and A_2 in $L(\gamma)$ are identified if for every valuation f, one has $f^*(A_1) = f^*(A_2)$. Notice that one may not conclude that $L(\gamma)$ should be a set in general. Now the following statements are equivalent: $L(\gamma)$ (after identifications) is a set; $L(\gamma)$ is a free complete lattice on γ complete generators; there exists a free complete lattice on γ complete generators; the cardinality of complete lattices with γ complete generators is bounded. In order to show that the free complete lattice on three complete generators does not exist, the author develops an effective method of deciding when two polynomials in $L(3)$ are

equal. Alternatively, for every regular cardinal β, let $L_\beta(3)$ be the subset of $L(3)$ consisting of polynomials of length less than β, then $L_\beta(3)$ can be made in a natural way into a β-complete lattice with three complete generators. Its normal completion is a complete lattice with three complete generators. Hence the cardinality of complete lattices with three complete generators is unbounded. The author now proceeds to prove that there does not exist a free complete Boolean algebra on ω complete generators. Again use is made of polynomials $B(\omega)$ (which are a natural extension of the $L(\omega)$). However the situation is now much more complicated because the presence of the distributive law prevents giving an effective method of deciding when two polynomials are equal. Again the statement that there exists a free complete Boolean algebra on ω complete generators is equivalent to the statement that the cardinality of complete lattices on ω complete generators is bounded. The method used in this case is a more delicate one than that used in the case of the free complete lattice. The first step is to choose an ordinal-indexed collection of polynomials in $B(\omega)$. One must prove now that these polynomials are pairwise distinct. "The choice of the polynomials suggests the method of constructing complete Boolean algebras with ω complete generators of arbitrary large cardinality. These complete Boolean algebras, in turn, establish the pairwise inequality of the chosen polynomials." The last part of the paper is devoted to a generalisation of the previous result to Boolean algebras which satisfy a certain degree of distributivity. It is proved that there does not exist a free complete (γ, ∞)-distributive Boolean algebra on γ complete generators. *Ph. Dwinger* (Delft)

Wooyenaka, Yuki 1163

On postulate-sets for Newman algebra and Boolean algebra. I, II.
Proc. Japan Acad. **40** (1964), 76–81; *ibid.* **40** (1964), 82–87.

These papers are concerned with various generalizations of Boolean algebras. Taken together, they are divided into three parts. The first gives two sets of axioms, both of which are equivalent to the axiom system for a Newman algebra in Birkhoff's *Lattice theory* [Amer. Math. Soc., New York, 1940; MR **1**, 325]. The merit of these axiom systems lies in the fact that neither demands the existence of distinguished elements. The author then defines a quasi-Boolean ring to be "an algebra satisfying all postulates for the Boolean ring with unity except the associative law for multiplication". A generalized quasi-Boolean ring (GQBR) is defined analogously except that the unity is not required. Extensions of the two axiom systems mentioned above produce axiom systems for the Boolean algebra and the quasi-Boolean algebra. The second paragraph gives a few results and examples concerning the systems of Paragraph 1. Among other things the author shows that a finite GQBR contains 2^m elements where $m \geqq 2$ and exhibits a non-associative GQBR of order 4. Other generalizations of Boolean rings are discussed in which one or the other of the distributive laws do not hold. It turns out that these too, when finite, must have 2^m elements, where $m \geqq 2$. Examples of these are also given. Paragraph 3 is concerned with independence.
B. Brainerd (Toronto, Ont.)

GENERAL MATHEMATICAL SYSTEMS

See also 1163, 1239.

Ehresmann, Charles 1164

★Catégories structurées. Sous-structures et catégories ordonnées. Structures quotient.
Topologie et Géométrie Différentielle, Séminaire dirigé par Charles Ehresmann.
Faculté des Sciences de Paris, Paris, 1963. 191 *pp.* (*not consecutively numbered*)

A series of lectures given by the author in the seminar mentioned in the heading. These lectures will be (or have been) published as individual papers as follows: "Catégories structurées, I, II" [Ann. Sci. École Norm. Sup. (3)]; "Sous-structures et catégories ordonées" [Fund. Math. **54** (1964), 211–228]; "Structures quotient, I, II" [Comment. Math. Helv. **38** (1964), 220–242; ibid. **38** (1964), 243–283].

Foster, Alfred L.; Pixley, Alden 1165

Semi-categorical algebras. I. Semi-primal algebras.
Math. Z. **83** (1964), 147–169.

Let $\mathfrak{A} = (A, \mathfrak{o}_1, \mathfrak{o}_2, \cdots)$ be an algebra and let $S = (n_1, n_2, \cdots)$ be its species. (Here $\mathfrak{o}_i$ is an n_i-ary primitive operation, $n_i \geqq 0$.) A number of recent investigations were devoted to the study of primal algebras (see the first author [same Z. **58** (1953), 306–336; MR **15**, 194; ibid. **59** (1953), 191–199; MR **15**, 194; Math. Ann. **143** (1961), 29–58; MR **23** #A85; ibid. **148** (1962), 173–191; MR **27** #5714], A. F. Pixley [Proc. Amer. Math. Soc. **14** (1963), 105–109; MR **26** #3630], A. Astromoff [Ph.D. Dissertation, Univ. California, Berkeley, Calif., 1963], E. S. O'Keefe [Pacific J. Math. **11** (1961), 1505–1510; MR **25** #3881], F. M. Sioson [Math. Z. **75** (1960/61), 201–210; MR **25** #2018]). The "first primal structure theorem" runs thus: A primal algebra is categorical. The notion "categorical algebra" means a finite algebra $\mathfrak{B}$ having at least two elements and having the property that every $\mathfrak{B}$-algebra is isomorphic with a subdirect power of $\mathfrak{B}$. The term "$\mathfrak{B}$-algebra" means an algebra (of the same species as $\mathfrak{B}$) which satisfies all (global) identities of $\mathfrak{B}$. In the present paper the authors give a more general result ("semi-primal structure theorem"): A semi-primal algebra is semi-categorical. The notion "semi-primal algebra" generalizes primals and means a finite algebra $\mathfrak{W}$ with at least two elements such that every conservative map $f(\xi, \eta, \cdots, \zeta)$ in $\mathfrak{W}$ is expressible (that is, there exists a (finite) composition $\Phi(\xi, \eta, \cdots, \zeta)$ of indeterminates $\xi, \eta, \cdots, \zeta$ via the primitive operations of $\mathfrak{W}$ such that $f = \phi$ for all elements $\xi, \eta, \cdots, \zeta$ of $\mathfrak{W}$). A map $f(\xi, \eta, \cdots, \zeta)$ from $A, A, \cdots, A$ into A is "conservative" if, for each subalgebra $\mathfrak{A}_i$ ($\subseteq \mathfrak{A}$) and for each sequence $\alpha, \beta, \cdots$ of elements of $\mathfrak{A}$, $\alpha, \beta, \cdots \in \mathfrak{A}_i$, then $f(\alpha, \beta, \cdots) \in \mathfrak{A}_i$. The notion "semi-categorical algebra" generalizes categoricals and means a finite algebra $\mathfrak{B}$ with at least two elements such that every $\mathfrak{B}$-algebra is isomorphic with a subalgebra of a direct power of $\mathfrak{B}$. The proof of the generalized theorem is based in part on a slightly more general version of a criterion of Astromoff [loc. cit.]. Further, it is shown that, in the case of finite $\mathfrak{W}$-algebras, the generalized (subdirect structure) theorem becomes direct. Then several new classes of semi-primals are exhibited. *H.-J. Hoehnke* (Berlin)

THEORY OF NUMBERS

See also 1141, 1154, 1211, 1212, 1213, 1214, 1223, 1224, 1255, 1332, 1722, 1767.

Cohn, J. H. E. 1166

On square Fibonacci numbers.

J. London Math. Soc. **39** (1964), 537–540.

The author proves that among the Fibonacci numbers defined by $u_1=u_2=1$, $u_{n+2}=u_{n+1}+u_n$, only u_1, u_2 and $u_{12}=144$ are perfect squares. The argument is elementary and employs the associated numbers v_n satisfying the same recursion as the u_n but with initial terms $v_1=1$, $v_2=3$. *T. M. Apostol* (Pasadena, Calif.)

Wrathall, Claude P. 1167

New factors of Fermat numbers.

Math. Comp. **18** (1964), 324–325.

The author has discovered eleven new prime factors of Fermat numbers $F_m=2^{2^m}+1$ for $m=19, 21, 25, 26, 27, 30, 32, 38, 42$ and 52. These are all of the form $K2^n+1$ with K odd and $21\leq n\leq 54$. A summary of the present status of F_m is given. From this we see that F_m is a prime for $m=0(1)4$. F_5 and F_6 are completely factored. One or more small prime factors are known of 40 Fermat numbers. F_7, F_8, F_{13}, F_{14} are composite but no factors are known. F_{17} is the least Fermat number whose character is not known. *D. H. Lehmer* (Berkeley, Calif.)

Karst, Edgar 1168

A remarkable quartic yielding certain divisors of Mersenne numbers.

Nordisk Tidskr. Informations-Behandling **3** (1963), 122–123.

According to a criterion first announced by Reuschle in 1856, 2 is an octic residue of the prime $p=16n+9$ if and only if

$$(*)\qquad p = u^2+64v^2 \quad (v \text{ odd}).$$

For reasons not explained, the author chooses $u=43$, $v=x^2+x-31$ and obtains from (*) a quartic in x which is a prime p for $x=0, 1, 2, 3, 5, 7, 8, 20$. In each case, then, p divides $2^{(p-1)/8}-1$. *D. H. Lehmer* (Berkeley, Calif.)

Karst, Edgar 1169

Some new divisors of Mersenne numbers.

Nordisk Tidskr. Informations-Behandling **4** (1964), 28–29.

Lists are given of primes q having 2 as 8th and 16th power residues. More specifically, all 12 primes q between 240000 and 400000 for which 2 is not a cubic residue but 2 is a 16th power residue are given. The results are due to Kravitz. A similar list due to Riesel for octic residuacity for $120000<q<8024000$ is given.
D. H. Lehmer (Berkeley, Calif.)

Muskat, Joseph B. 1170

Criteria for solvability of certain congruences.

Canad. J. Math. **16** (1964), 343–352.

Let χ_p be a primitive rth power multiplicative character mod p and ζ_r be a primitive rth root of unity. Define $\tau(\chi_p)=\sum_{n=1}^{p-1}\chi_p(n)\zeta_p{}^n$; $\tau(\chi_p)$ is called a Gaussian sum, or Lagrange's resolvent. N. C. Ankeny [Pacific J. Math. **10** (1960), 1115–1124; MR **22** #9479] obtained criteria for rth power residuacity. In the present paper these criteria are studied and other criteria are developed. In the case $q\equiv -1 \pmod r$ a rational integral criterion is given involving the trace of $\tau(\chi_p)^r$. A Jacobi function is defined in terms of Gaussian sums and relations between Jacobi function criteria and those involving rth power Gaussian sums are explored. *W. H. Simons* (Vancouver, B.C.)

Carlitz, L.; Whiteman, A. L. 1171

The number of solutions of some congruences modulo a product of primes.

Trans. Amer. Math. Soc. **112** (1964), 536–552.

Let m be an odd integer with prime power decomposition $m=p_1{}^{\alpha_1}p_2{}^{\alpha_2}\cdots p_r{}^{\alpha_r}$. Suppose g_i is a primitive root $(\bmod\, p_i{}^{\alpha_i})$ and that g is an integer satisfying $g\equiv g_i \pmod{p_i{}^{\alpha_i}}$, $1\leq i\leq r$; g is determined $(\bmod\, m)$ by the g_i. Put $d=\operatorname{lcm}(\phi_1(p_1{}^{\alpha_1}), \phi_2(p_2{}^{\alpha_2}), \cdots, \phi_r(p_r{}^{\alpha_r}))$. The authors are interested in determining the number $N(g)$ of solutions to the congruence $g^s+1\equiv g^t$, where $0\leq s, t\leq d-1$. This has applications in the construction of difference sets. They consider the cases when m is the product of two primes, $m=pq$, and where $e=\gcd(p-1, q-1)$ is 2, 4, or 6. In addition, they consider the case when m is the product of three primes, $m=pqr$, and $e=\gcd(p-1, q-1)=\gcd(p-1, r-1)=\gcd(q-1, r-1)=2$. Their answers are expressed in terms of Jacobi sums. When $e>2$, the results depend upon the choice of primitive roots g_i.
D. G. Cantor (Seattle, Wash.)

Rademacher, H. 1172

Some remarks on certain generalized Dedekind sums.

Acta Arith. **9** (1964), 97–105.

The author introduces a generalized Dedekind sum

$$(1)\quad s(h, k; x, y) = \sum_{\mu \bmod k}\left(\left(h\left(\frac{\mu+y}{k}+\frac{x}{h}\right)\right)\right)\left(\left(\frac{\mu+y}{k}\right)\right),$$

where h, k are coprime integers, $k>0$, and x, y are real. Special cases of this sum, for rational x, y, have been considered by previous authors. When x and y are both integers, (1) reduces to the classical Dedekind sum.

The main object of the paper is to prove the reciprocity formula:

$$s(h, k; x, y)+s(k, h; y, x) = ((x))((y))+\frac{1}{2}\left\{\frac{h}{k}\psi_2(y)+\frac{1}{hk}\psi_2(hy+kx)+\frac{k}{h}\psi_2(x)\right\},$$

where $\psi_2(y)=B_2(y-[y])$, $B_2(u)=u^2-u+\frac{1}{6}$. The relation

$$s(h, k; x, y) = s(h-mk, k; x+my, y)$$

is also proved; this permits calculation of s by a euclidean algorithm. *J. Lehner* (College Park, Md.)

Hooley, C. 1173

On the distribution of the roots of polynomial congruences.

Mathematika **11** (1964), 39–49.

Let $f(x)$ be an irreducible polynomial of degree $n>1$

having integer coefficients. Denote by $v_1, \cdots, v_{l_k}$ the roots of the congruence $f(v) \equiv 0 \pmod k$ and arrange the numbers v_i/k, $1 \leqq i \leqq l_k$, $1 \leqq k < \infty$, as a sequence $s_1, s_2, \cdots$ so that the corresponding denominators are in ascending order (the arrangement in a group corresponding to a fixed value of k is immaterial).

The author proves that the sequence $s_1, s_2, \cdots$ is uniformly distributed in (0, 1). He uses in his proof several lemmas of independent interest, and also applies his method to the estimation of Kloosterman sums.

P. Erdős (Budapest)

Miech, R. J. 1174

Almost primes generated by a polynomial.

Acta Arith. **10** (1964/65), 9–30.

An "almost prime" in a given sequence of integers is an integer with a bounded number of prime factors. The author considers the sequence $G(1), G(2), \cdots$, where $G(n)$ is an integer-valued polynomial and proves that infinitely many almost primes exist in this sequence. The actual theorems proved are somewhat too complicated to state in a short review, but roughly speaking, he shows that there are infinitely many integers m such that $G(m)$ has at most $9h/5 + k \log k$ prime factors, where $G(n)$ is of degree h and is the product of k polynomials irreducible over the rationals. A corollary of his work, for example, is that there are infinitely many integers of the form n^2+1 with at most 4 prime factors. The methods of proof are sieve methods and involve generalizations of Selberg's methods for the twin prime case.

Morris Newman (Washington, D.C.)

Gandhi, J. M. 1175

On logarithmic numbers.

Math. Student **31** (1963), 73–83 (1964).

Put

$$e^{-zt} \log(1-x^n) = -\sum_{r=1}^{\infty} G_r^{(n)}(t)x^r/r!,$$

$$A_r^{(n)}(t) = [tG_r^{(n)}(t) + G_{r+1}^{(n)}(t)]/n.$$

Numerous relations involving the $G_r^{(n)}(t)$ and $A_n^{(r)}(t)$ are obtained. For example, we cite

$$\sum_{n=1}^{r+1} \mu(n)A_r^{(n)}(t) = (-t)^r \quad (r \geqq 0),$$

$$\sigma(r+1) = \frac{1}{r!}\sum_{n=1}^{r+1}\sum_{n=1}^{r+1}\sum_{j=0}^{r}\binom{r}{j} nA_j^{(n)}(t)t^j,$$

where $\sigma(r+1)$ is the sum of the divisors for $r+1$.

The case $t = \pm 1$ is of special interest. Thus, for example, the coefficients of $\cosh x \log(1-x^n)$ and $\sinh x \log(1-x^n)$ are expressible in terms of these numbers. Tables of $G_r^{(n)}(\pm 1)$ and $A_r^{(n)}(\pm 1)$ for $1 \leqq n \leqq r \leqq 10$ are given, as well as tables of such sums as $\sum_{n=1}^{r} G_r^{(n)}(1)$ for $1 \leqq r \leqq 10$.

Finally the author derives some congruence properties of the numbers. For example,

$$G_r^{(n)}(\pm 1) \equiv 0 \pmod{r(r-1)\cdots(r-n+1)}$$

$$G_{r+p}^{(n)}(1) \equiv -G_r^{(n)}(1) \pmod p,$$

provided $n > 1$, p is a prime $> r$ and $r+1 > (t+1)n$, where $t = [r/n]$.

L. Carlitz (Durham, N.C.)

Daykin, D. E.; Baines, M. J. 1176

Coprime mappings between sets of consecutive integers.

Mathematika **10** (1963), 132–136.

Put $\Gamma = \{s+1, s+2, \cdots, s+n\}$, $\Pi = \{t+1, t+2, \cdots, t+n\}$, where $s \leqq t$. A mapping of Γ onto Π, $i \to f(i)$, such that $(i, f(i)) = 1$ for all $i \in \Gamma$ is called a coprime mapping. The authors prove that if

$$\Gamma = \{1, 2, \cdots, n\}, \quad \Pi = \{n+1, n+2, \cdots, 2n\},$$

a coprime mapping can be constructed. This settles the case $t = n$ of the following conjecture of D. Newman. If $\Gamma = \{1, 2, \cdots, n\}$ and $\Pi = \{t+1, t+2, \cdots, t+n\}$, where t is any integer, then there exists a coprime mapping of Γ onto Π.

The authors also prove another of Newman's conjectures. If $1 \leqq a_1 < a_2 < \cdots < a_{n+1} \leqq 2n$, where the a_i are integers, then there exists a pair a_λ, a_μ satisfying $a_\lambda \leqq n < a_\mu$ and $(a_\lambda, a_\mu) = 1$.

L. Carlitz (Durham, N.C.)

Graham, R. L. 1177

On a conjecture of Erdős in additive number theory.

Acta Arith. **10** (1964/65), 63–70.

A sequence $\{s_k\}$ of non-negative integers is said to be complete if every sufficiently large integer n can be written as $n = \sum_1^\infty \varepsilon_u s_u$, where $\varepsilon_u = 0$ or 1. For real t and α set $s_k = [t\alpha^k]$ and $S_t(\alpha) = \{s_k\}$. It is not known for what values of $t > 0$, $1 < \alpha < 2$, the set $S_t(\alpha)$ is complete. The author determines the set T of value (t, α), $0 < t < 1$, $1 < \alpha < 2$, for which $S_t(\alpha)$ is complete. Its area is stated to be approximately 0.85.

H. B. Mann (Columbus, Ohio)

Titsworth, Robert C. 1178

Equivalence classes of periodic sequences.

Illinois J. Math. **8** (1964), 266–270.

Let A be the set of all sequences with period p, taking values in a set with b elements. Each $a \in A$ may be regarded as a function on Z_p, and the group of transformations (*) $x' = kx + t$, with $(k, p) = 1$, acts on Z_p. Regarding $a, a' \in A$ as equivalent if $a'(x) = a(kx+t)$ for some t and k, the author counts the number of equivalence classes in A. By Pólya's lemma, this number is

$$N(p) = \frac{1}{p\varphi(p)} \sum_{t=0}^{p-1} \sum_{\substack{k=1 \\ (k,p)=1}}^{p-1} b^{C(k,t)},$$

where $C(k, t)$ is the number of cycles of the permutation (*). A formula for $C(k, t)$ is derived. In the case where $(p, k-1)$ divides t, it simplifies to

$$C(k, t) = C(k, 0) = \sum_{d|p} \varphi(d)/E_k(d),$$

where $E_k(d)$ is the exponent of k mod d. For $b = 2$, there is a table of $N(p)$ $(p = 1, 2, \cdots, 20, 30, 31)$.

N. J. Fine (University Park, Pa.)

Davenport, H.; Lewis, D. J.; Schinzel, A. 1179

Polynomials of certain special types.

Acta Arith. **9** (1964), 107–116.

It is well known that if $f(x)$ is a polynomial with integral coefficients such that $f(x)$ is a kth power for every positive integer x, then $f(x) = g(x)^k$, where $g(x)$ is a polynomial with integral coefficients. The authors show that the conclusion holds under the weaker hypothesis that every

arithmetic progression contains an integer x such that $f(x)$ is a kth power. This result, in turn, is a corollary of the following more general theorem: Let $F(x, y)$ be a polynomial with integral coefficients. Suppose that every arithmetic progression contains an integer x such that $F(x, y) = 0$ has an integral solution in y. Then there exists a polynomial $h(x)$ with rational coefficients such that $F(x, h(x)) = 0$ identically.

The second principal result of the paper is the following. Let K be any normal algebraic number field of degree n with integral basis $w_1, w_2, \cdots, w_n$ and let $N(u_1, u_2, \cdots, u_n) = \text{norm}(u_1w_1 + u_2w_2 + \cdots + u_nw_n)$ be the norm-form corresponding to K. Let $f(x)$ be a polynomial with rational coefficients and suppose that every arithmetic progression contains an integer x such that $f(x) = N(u_1, u_2, \cdots, u_n)$ for rational $u_1, u_2, \cdots, u_n$. Suppose further that either K is cyclic or the multiplicity of every zero of $f(x)$ is relatively prime to n. Then $f(x) = N(v_1(x), v_2(x), \cdots, v_n(x))$ identically, where $v_1(x), v_2(x), \cdots, v_n(x)$ are polynomials with rational coefficients. An example shows that if K is not cyclic and the multiplicities are not prime to n, the conclusion need not follow. An interesting corollary, proved by letting $K = Q(i)$ (Q is the rational field), is that if $f(x)$ is a polynomial with integral coefficients such that every arithmetic progression contains an integer x for which $f(x)$ is a sum of two squares, then $f(x) = v_1^2(x) + v_2^2(x)$ identically, where $v_1(x)$ and $v_2(x)$ are polynomials with integral coefficients. The authors do not know what happens when $f(x)$ is a sum of two cubes for every sufficiently large x.

The highly original proofs involve classical (1930) algebraic number theory. Among the tools employed are Hilbert's irreducibility theorem and a theorem of Bauer [Math. Ann. **77** (1916), 353–356].

J. B. Kelly (Tempe, Ariz.)

Schinzel, A.; Sierpiński, W. 1180

Sur l'équation diophantienne

$$(x^2-1)(y^2-1) = \left[\left(\frac{y-x}{2}\right)^2 - 1\right]^2.$$

Elem. Math. **18** (1963), 132–133.

The authors give the complete solution of the title equation in natural numbers x, y, $x<y$, showing that this equation can be written $(x+y)^2(x^2-6xy+y^2+8)=0$, which for $x>0$, $y>0$, is equivalent to $x^2-6xy+y^2+8=0$, or $x^2-2(\frac{1}{2}(y-3x))^2=1$. A similar problem was proposed by J. A. Hunter [Amer. Math. Monthly **69** (1962), 316–317] and solved by R. Venkatachalam Iyer [ibid. **70** (1963), 574]. The complete solution of the more general equation $(x^2-1)(y^2-1)=(z^2-1)^2$ is unknown.

W. Ljunggren (Oslo)

Motzkin, T. S. 1181

Linear diophantine inequalities applied to generalized Faber polynomials.

Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 251–253.

Gegeben seien die positiven Zahlen $a_1, a_2, \cdots, a_K$ mit $\sum_1^K a_j = 1$; ist $k_1, k_2, \cdots$ eine Folge mit Werten aus der Indexmenge $\{1, \cdots, K\}$, so seien Zahlen ε_{kn} ($k = 1, 2, \cdots, K$; $n = 1, 2, \cdots$) erklärt durch $\varepsilon_{kn} = (\sum_{\nu=1, k_\nu=k}^n 1) - na_k$. Ein Lemma von J. L. Walsh [Ann. Mat. Pura Appl. (4) **39** (1955), 267–277; MR **17**, 1077] besagt in der Formulierung des Verfassers: die Folge $\{k_n\}$ kann so gewähl[t] werden, daß stets $|\varepsilon_{kn}| \leqq K'$ gilt, wobei K' die kleinst[e] ganze Zahl $\geqq \log K/\log 2$ ist. Unabhängig hiervon bewei[st] Verfasser, daß durch Wahl der Folge $\{k_n\}$ jede einzelne d[er] folgenden ganz oder teilweise schärferen Abschätzunge[n] erreicht werden kann: $|\varepsilon_{kn}| \leqq \frac{1}{2}K'$; $-(K+1/K-2)$; $\varepsilon_{kn} \leqq 1 - 1/K$; $\varepsilon_{kn} = \sum_{j=1}^K \theta_{jn}(\delta_{jk} - a_k)$, wobei $0 \leqq \theta_{jn} \leqq$ $\prod_{j=1}^K \theta_{jn} = 0$, $\delta_{jk} =$ Kroneckersymbol.

H. Tietz (Zbl **91**, 6[…]

Mahler, K. 118[2]

On the approximation of algebraic numbers by algebra[ic] integers.

J. Austral. Math. Soc. **3** (1963), 408–434.

Let K be an algebraic number field of degree n over th[e] rationals, and let $\kappa \in K$ have conjugates $\kappa^{(1)}, \cdots, \kappa^{(n)}$. Let $h(\kappa)$ be the smallest positive rational integer such tha[t] the polynomial $h(\kappa)(x-\kappa^{(1)})\cdots(x-\kappa^{(n)}) = b_0x^n + \cdots + b_n$ has rational integral coefficients. Put

$$q(\kappa) = \max(|b_0|, \cdots, |b_n|).$$

For arbitrary algebraic numbers $\xi_1, \cdots, \xi_n$, put

$$\Delta(\kappa) = \prod_{j=1}^n \min(1, |\kappa^{(j)} - \xi_j|).$$

Let $\Sigma = \{\kappa_l\}$ be an infinite sequence of distinct elements [of] K, and put $h(\kappa_l) = h_l$, etc. Suppose that there are constant[s] c', c'', σ and τ, such that $c'>0$, $c''>0$, $\sigma>0$, $0\leqq\tau\leqq 1$, f[or] which $\Delta_l \leqq c'q_l^{-\sigma}$ and $h_l \leqq c''q_l^{\tau}$ for all l. The main resul[t] of the paper is that under these circumstances, $\sigma \leqq 1+$[…] The condition $h_l \leqq c''q_l^{\tau}$ holds automatically for $c''=\tau=1$ and this case gives the reviewer's extension of Roth['s] theorem [the reviewer, *Topics in number theory*, Vol. [2], Addison-Wesley, Reading, Mass., 1956; MR **18**, 283]. Th[e] other extreme, $c''=1$, $\tau=0$, gives information about th[e] approximation of algebraic numbers by algebraic intege[rs] from K. Several other conclusions are drawn, amon[g] them the following new type of result on inhomogeneou[s] approximation: If α is a real quadratic irrationality, if [β] is a real algebraic number, and if $\varepsilon>0$, then there are onl[y] finitely many pairs of rational integers $x\neq 0$, y such tha[t] $|x\alpha - y - \beta| < |x|^{-(1+\varepsilon)}$.

W. J. LeVeque (Ann Arbor. Mich.)

Brjuno, A. D. 118[3]

The expansion of algebraic numbers into continue[d] fractions. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 211–221.

The author has carried the regular continued fractio[n] expansions of the cube roots of 2, 3 and 7 to 2500, 100[0] and 4500 terms, respectively. From these expansions, dat[a] are presented on the function

$$\phi_n(k) = n^{-1} \sum_{\substack{a_i \geqq k \\ i=1}}^{n} 1$$

for n at intervals of 250 and $k = 2(1)10(10)100$. Here a_i i[s] the ith partial quotient. These are compared with th[e] expected limits $\phi(k) = \lim_{n\to\infty} \phi_n(k) = \log((k+1)/k)/\log 2$. Further data include the exceptionally large a's for a[ll] these irrationals and the first 500 a's for the cube roo[t] of 2.

These data support the author's conjecture that th[e]

regular continued fraction for an algebraic number of degree > 2 behaves like that of almost any real number.

D. H. Lehmer (Berkeley, Calif.)

Meijer, H. G. 1184

Irrational power series.

Nederl. Akad. Wetensch. Proc. Ser. A **66** = *Indag. Math.* **25** (1963), 682–690.

The author generalizes a theorem of the reviewer [Proc. Amer. Math. Soc. **11** (1960), 699–702; MR **22** #9483] and proves that if α is a real number, k a positive integer and $g(x)$ a polynomial of degree $\geqq 1$, then the power series

$$\sum_{n=0}^{\infty} g([\alpha n^k])x^n$$

represents a rational function of x if and only if α is a rational number. Essentially the same result is proved by W. Schwarz [Arch. Math. **13** (1962), 228–240; MR **25** #4084].

Morris Newman (Washington, D.C.)

Popken, J. 1185

Irrational power series.

Nederl. Akad. Wetensch. Proc. Ser. A **66** = *Indag. Math.* **25** (1963), 691–694.

The author generalizes the work of H. G. Meijer described in the preceding review [#1184] and proves some very general results, among which the following is both elegant and beautiful: Let $g(u)$ denote a polynomial with algebraic coefficients. Let $\varphi(n)$ be an integer-valued function, and suppose that the power series $G = \sum_{n=1}^{\infty} g(\varphi(n))x^{-n}$ converges for $|x| > 1$. Then G is either a rational function of x or a function having the unit circle as a natural boundary. The proof makes use of a generalization of Liouville's theorem on approximation of algebraic numbers and on a result of Pólya relating transfinite diameter to a certain limit involving Hankel determinants.

Morris Newman (Washington, D.C.)

Gundlach, Karl-Bernhard 1186

Die Bestimmung der Funktionen zur Hilbertschen Modulgruppe des Zahlkörpers $Q(\sqrt{5})$.

Math. Ann. **152** (1963), 226–256.

La determinazione delle funzioni invarianti per il gruppo modulare hilbertiano Γ del campo $K = Q(\sqrt{5})$—ove Q è il campo razionale—si vale in modo essenziale dello studio di una particolare funzione Θ (prodotto di 10 funzioni thêta speciali) della quale interessano, tra l'altro, gli zeri appartenenti al campo fondamentale del gruppo Γ. La conoscenza della funzione Θ consente di determinare le forme modulari relative al gruppo Γ ed in conseguenza le funzioni invarianti. Nella dimostrazione vengono usate proprietà speciali del campo fondamentale del gruppo Γ; non sembra quindi che i risultati ottenuti possano estendersi direttamente a tipi di campi diversi da K.

M. Rosati (Rome)

Walfisz, Anna [Val'fiš, Anna] 1187

Über die summatorischen Funktionen einiger Dirichletscher Reihen. II.

Acta Arith. **10** (1964/65), 71–118.

The author further generalizes her results [Soobšč. Akad. Nauk Gruzin. SSR **27** (1961), 9–16; MR **25** #3017; ibid. **26** (1961), 9–16; MR **26** #83] on identities of the Hardy-Landau type for the sum function of a Dirichlet series satisfying a certain functional equation. The results are too long to be quoted here. It is shown that the main theorem includes the well-known identities of Hardy, Voronoi, Arnold Val'fiš, and Oppenheim.

H.-E. Richert (Marburg)

Gupta, H. 1188

On the coefficients of the powers of Dedekind's modular form.

J. London Math. Soc. **39** (1964), 433–440.

Let $\varphi(x)$ denote the Euler product, $\varphi(x) = \prod_{n=1}^{\infty} (1-x^n)$, and for arbitrary r define $p_r(n)$ as the coefficient of x^n in $\varphi(x)^r$, i.e., $\varphi(x)^r = \sum_{n=0}^{\infty} p_r(n)x^n$. Then $\varphi(x)$ is a polynomial in r of degree n which was studied by the reviewer [J. London Math. Soc. **30** (1955), 488–493; MR **17**, 15], who gave the polynomials explicitly for $n \leqq 10$. The coefficients of $p_r(n)$ tend to become large, and the author observes that if $p_r(n)$ be expressed in terms of the binomial coefficients $\binom{r}{k}$, $0 \leqq k \leqq n$, the coefficients involved are much smaller. Thus, setting

$$(-1)^n p_r(n) = \sum_{k=0}^{n} C_k(n)\binom{r}{k},$$

the author gives recurrence formulas for the integers $C_k(n)$ and gives their actual values for $0 < k \leqq n \leqq 50$. The tables were checked against tables of the partition function by choosing $r = -1$. It would be of fundamental interest to compute the zeros of these polynomials.

Morris Newman (Washington, D.C.)

Sudler, C., Jr. 1189

An estimate for a restricted partition function.

Quart. J. Math. Oxford Ser. (2) **15** (1964), 1–10.

According to Euler,

$$\prod_{k=1}^{\infty} (1-x^k) = \sum_{-\infty}^{+\infty} (-1)^m x^{(3m^2+m)/2} = \sum_{h=0}^{\infty} c_h x^h,$$

so $c_h = -1, 0, 1$. Let

$$P_n(x) = \prod_{k=1}^{n} (1-x^k) = \sum_h a_{nh}x^h.$$

The coefficients a_{nh} and $c_h = \lim_{n\to\infty} a_{nh}$ have interpretations in terms of certain partitions of h into distinct parts. Nicol and Vandiver have discussed a_{nh} and Nicol asked whether they are uniformly bounded. Motzkin announced a negative answer. By studying $\Pi_n(\theta) = |P_n(e^{2\pi i\theta})|$, the author proves this result, showing that $M_n =_{\text{def}} \max_h |a_{nh}|$ satisfies $\log M_n = Kn + O(\log n)$, where

$$K = \max_{1/2<w<1} g(w) = g(w_0) = 0.19861\cdots,$$

and $g(w) = w^{-1}\int_0^w \log(2\sin\pi t)\,dt$. He conjectures that $\lim M_{n+1}/M_n = e^K = 1.2197\cdots$.

N. J. Fine (University Park, Pa.)

Wright, E. M. 1190

Proof of a conjecture of Sudler's.

Quart. J. Math. Oxford Ser. (2) **15** (1964), 11–15.

This paper is based on the preceding review [#1189]. Let $N=\frac{1}{4}n(n+1)$, $L=(1/n)(N-h)w_0-n/4$, $\varphi_n=Be^{Kn}/n$, $B=2e^K(1-e^{2K}/4)^{-1/4}=2.7424\cdots$. The author proves that $M_n\sim\varphi_n$. Also, if $h=N+o(n^{3/2})$, $a_{nh}=\varphi_n\cos 2\pi L+o(\varphi_n)$. He obtains a good estimate for $\log \Pi_n(\theta)$ in the range $1/2\leq|w|\leq(1+w_0)/2$. *N. J. Fine* (University Park, Pa.)

Wright, E. M. 1191
A closer estimate for a restricted partition function.
Quart. J. Math. Oxford Ser. (2) **15** (1964), 283–287.
See the two preceding reviews [#1189, #1190]. Let $m=h-N$, $S=2\pi mn^{-2}$, $C^2=-g''(w_0)/2$. The author shows that if $m=o(n^2)$, then

$$a_{nh} = Bn^{-1}e^{K_1 n}\{\cos K_2 n+o(1)\},$$

where K_1 and K_2 are certain power series in S:

$$K_1 = K-\frac{S^2}{4C^2}+\cdots,\qquad K_2 = \frac{\pi}{2}+w_0S+\cdots.$$

N. J. Fine (University Park, Pa.)

Rogers, Kenneth 1192
The Schnirelmann density of the squarefree integers.
Proc. Amer. Math. Soc. **15** (1964), 515–516.
Let $A(x)$ be the number of square-free integers $\leq x$. Although $A(x)/x\to6/\pi^2$, the Schnirelmann density $(=\inf A(x)/x)$ is less than this, as the author shows with the help of the UCLA computer and the estimate

$$|A(x)/x-6/\pi^2| \leq 12/\pi^2x^{1/2}+8/3x^{3/4},$$

which he establishes. In fact, $A(x)/x$ has a unique minimum at $x=176$, the minimum value being 53/88.
J. Lehner (College Park, Md.)

Barban, M. B.; Linnik, Ju. V.; Čudakov, N. G. 1193
On the distribution of primes in short progressions mod p^n. (Russian)
Dokl. Akad. Nauk SSSR **154** (1964), 751–753.
Consider the remainder in

$$\pi(x, D, k) = \varphi(D)^{-1}[\mathrm{li}(x)+R(x, D, k)],$$

where x, D, k denote positive integers, $(k, D)=1$. The authors are interested in the dependence of R on D. It is stated that, for each constant $A>0$, we have

$$R(x, D, k) = O(x(\log D)^{-A})$$

provided: (I) D is restricted to the powers p^n of a fixed prime p; (II) $x\gg D^{8/3+\varepsilon}$.

The proof is outlined in some detail. Use is made of: (i) An upper bound involving D, due to M. B. Barban [Mat. Sb. (N.S.) **61 (103)** (1963), 418–425], on the total number of zeros inside $\sigma\leq\sigma'\leq1$, $|t|\leq T$, of the functions $L(s,\chi)$ having D as modulus; (ii) A zero-free region near $\sigma=1$, depending on D, of the $L(s,\chi)$ with $D=p^n$, analogous to one by S. M. Rozin [Izv. Akad. Nauk SSSR Ser. Mat. **23** (1959), 503–508; MR **22** #4680].
J. H. B. Kemperman (Rochester, N.Y.)

Barban, M. B.; Vinogradov, A. I. 1194
On the number-theoretic basis of probabilistic number theory. (Russian)
Dokl. Akad. Nauk SSSR **154** (1964), 495–496.
With m as a positive integer, let $(m)_r$ denote the largest divisor of m having all its prime factors $\leq r$. Then

$$(1)\qquad N_n\{(m)_r \geq u\} \leq n\exp(-c_1\beta),$$

as was shown by M. B. Barban [Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk **1961**, no. 5, 3–9; MR **25** #2051]. Here n, r and u are variable integers ≥2, c_1 is an absolute constant, $\beta=\log u/\log r$, while $N_n\{\cdot\}$ denotes the number of integers $1\leq m\leq n$ having the stated property. The result (1) appears as one of the two so-called fundamental lemmas (namely, Lemma 1.2) in the book of J. Kubilius [*Probabilistic methods in the theory of numbers* (Russian), Second, enlarged edition, Gosudarstv. Izdat. Politič. i Naučn. Lit. Litovsk. SSR, Vilna, 1962; MR **26** #3691; English transl., Amer. Math. Soc., Providence, R.I., 1964; MR **28** #3956]. In the present note, the authors improve (1) to

$$(2)\qquad N_n\{(m)_r \geq u\} \leq n\exp(-c_2\beta\log\beta).$$

In fact, (2) turns out to be an easy consequence of a bound on $N_n\{(m)_r=m\}$ due to A. I. Vinogradov [Dokl. Akad. Nauk SSSR **109** (1956), 683–686; MR **19**, 16]. Using (2), and further a method due to R. V. Uždavinis [Dissertation, Vilna, 1961], the authors also obtain a more precise form for the remainder term in a central limit theorem for strongly additive functions; the new remainder is smaller by a factor $[\log\log(1/\mu_n)]^{-1}$, where μ_n has the usual meaning; see Kubilius, op. cit., p. 108.
J. H. B. Kemperman (Rochester, N.Y.)

Vinogradov, A. I. 1195
The sieve method in algebraic fields. Lower bounds. (Russian)
Mat. Sb. (N.S.) **64 (106)** (1964), 52–78.
The author extends the method for "lower" bounds due to A. Selberg, and proves the following theorem. Let K be an algebraic number field over the rationals. Then every integer $\zeta\in K$ of sufficiently high norm can be represented in the form

$$(\mathrm{A})\qquad \zeta = \alpha+\beta,$$

where α, β are integers of K such that the ideal (α) is devisible by at most 2 prime ideals, and the ideal (β) by at most 3. In fact, the number of solutions of (A) exceeds

$$cS_K(\zeta)\frac{|N\zeta|}{\log^2|N\zeta|},$$

where $c>0$ is a constant depending only on K, and $S_K(\zeta)$ is the "singular" series:

$$S_K(\zeta) = \prod_{P|\zeta}\left(1-\frac{1}{NP}\right)^{-1}\prod_{P\nmid\zeta}\left(1-\frac{2}{NP}\right)\cdot\left(1-\frac{1}{NP}\right)^{-2}$$

and P stands for a prime-ideal (of K). The method of proof also requires estimates for the number of zeros of the Dedekind zeta function in portions of the critical strip (see the proof of Lemma 4, pp. 57–61). See also G. J. Rieger [J. Reine Angew. Math. **208** (1961), 79–90; MR **28** #3024], Tatuzawa [J. Math. Soc. Japan **7** (1955), 409–423; MR **18**, 113], R. Brauer [Amer. J. Math. **69** (1947), 243–250; MR **8**, 567]. *S. Chowla* (University Park, Pa.)

Uchiyama, Saburô 1196
A further note on the sieve method of A. Selberg.
J. Fac. Sci. Hokkaido Univ. Ser. I **17** (1963), 79–83.

The author continues the work originated in his paper [same J. **16** (1962), 189–192; MR **27** #1432]. Previously a universal upper bound for the remainder term in the "upper" sieve of A. Selberg was obtained under certain restrictions. Here the analogous problem is treated for the "lower" sieve, and similar results are obtained under similar restrictions. A full description is given in the review of the previous paper [MR **27** #1432].

Morris Newman (Washington, D.C.)

Dress, Andreas 1197

Zu einem Satz aus der Theorie der algebraischen Zahlen.

J. Reine Angew. Math. **216** (1964), 218–219.

The author gives a simple and elementary proof of the following theorem: Let K be a commutative field and M_K a set of valuations of K for which the product formula $\prod_{v\in M_K}|a|_v^{l_v}=1$ holds for $a\in K$, $a\neq 0$. If $f(x)$ is a polynomial in K, then there exist infinitely many valuations $v\in M_K$ such that $f(x)$ splits in linear factors in K_v.

If K is an algebraic number field and M_K is the set of all valuations of K, then the theorem follows immediately from the "Density Theorem" of Kronecker.

T. Tamagawa (New Haven, Conn.)

Siegel, Carl Ludwig 1198

Zu zwei Bemerkungen Kummers.

Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II **1964**, 51–57.

Let $\alpha(x)$ denote the number of irregular primes $\leqq p$ and $\beta(x)$ the number of regular primes $\leqq x$. Kummer conjectured that $\alpha(x)/\beta(x)\to\frac{1}{2}$. Recent computation by Selfridge, Nicol and Vandiver [Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 970–973; MR **17**, 348] suggests that the limit $\frac{1}{2}$ is incorrect. The author gives a heuristic argument that $\alpha(x)/\beta(x)\to e^{1/2}-1=0.6487\cdots$.

Let h_p denote the class number of the field K_p generated by a pth root of unity and let g_p denote the first factor of the class number. Kummer conjectured that

$$(*)\qquad g_p\sim P=2^{(3-p)/2}\pi^{(1-p)/2}p^{(p+3)/4}.$$

This implies

$$(**)\qquad \log g_p\sim\tfrac{1}{4}p\log p.$$

The author gives a proof of (**). He remarks that since $g_p\leqq h_p$, it follows that $h_p=1$ for only a finite number of fields K_p, presumably only for $p=3, 5, 7, 11, 13, 17, 19$.

(It may be noted that Ankeny and Chowla [Canad. J. Math. **3** (1951), 486–494; MR **13**, 538] proved that $\lim_{p=\infty}(\log(g_p/P)/\log p)=0$, where P has the same meaning as in (*).)

L. Carlitz (Durham, N.C.)

Kubota, Tomio; Leopoldt, Heinrich Wolfgang 1199

Eine p-adische Theorie der Zetawerte. I. Einführung der p-adischen Dirichletschen L-Funktionen.

J. Reine Angew. Math. **214/215** (1964), 328–339.

If x is a p-adic unit, p an odd prime (the case $p=2$ is similar), let $\omega(x)$ be the p-adic limit of x^{p^n}. Then $x\equiv\omega(x)\pmod p$ and $\langle x\rangle=x/\omega(x)$ belongs to the set U of units congruent to 1 modulo p. If χ is any character with "Führer" f, to each power series $A(u)=\sum_{n\geq 0}a_n(u-1)^n$ converging for $u\in U$, define

$$\mathfrak{M}_\chi{}^n(A)=\frac{1}{p^n[f,p]}\sum_{\substack{x=1\\(x,p)=1}}^{p^n[f,p]}\chi(x)A(\langle x\rangle).$$

Let $\mathfrak{M}_\chi(A)=\lim_{n\to\infty}\mathfrak{M}_\chi{}^n(A)$. Let $P_s(u)=\sum_{n\geq 0}\binom{s}{n}(u-1)^n$. The authors show that the function

$$L(s,\chi)=\frac{1}{s-1}\mathfrak{M}_\chi(P_{1-s})$$

is a p-adic analogue of the usual Dirichlet L-function. Information is gathered concerning $L(1,\chi)$, including the size of $|L(1,\chi)|_p$ and approximations of $L(1,\chi)$ by Bernoulli numbers. The authors claim the analogy will extend to a p-adic class number formula for the class number of real abelian number fields in terms of $L(1,\chi)$.

D. J. Lewis (Ann Arbor, Mich.)

Jacobinski, H. 1200

Über die Hauptordnung eines Körpers als Gruppenmodul.

J. Reine Angew. Math. **213** (1963/64), 151–164.

Let G be the Galois group of a finite Galois extension K over a finite algebraic number field k, and let $\mathfrak{o}$ and $\mathfrak{O}$ denote the maximal orders of k and K, respectively. For any subgroup H of G and for any ideal $\mathfrak{A}$ of K which is invariant under G, let $r[H,\mathfrak{A}]$ denote the subring of all x in $k[H]$, the group ring of H over k, such that $x\mathfrak{A}\subset\mathfrak{A}$, the action of $k[H]$ on K being defined in the obvious manner. The author first proves that if the inertia group T of a prime ideal of K, which is essentially (i.e., not tamely) ramified for the extension K/k, is always an invariant subgroup of G, then the ring $r[G,\mathfrak{A}]$ is generated by $\mathfrak{o}[G]$, the group ring of G over $\mathfrak{o}$, and by the rings $r[T,\mathfrak{A}]$ with T ranging over all inertia groups as stated above. Next, the author gives an alternate proof to the following theorem of Leopoldt [same J. **201** (1959), 119–149; MR **21** #7195]: Let k be the rational field, and let G be abelian. Then $r[G,\mathfrak{O}]$ is isomorphic to $\mathfrak{O}$ as an $\mathfrak{o}[G]$-module, and the ring $r[G,\mathfrak{O}]$ is generated over the subring $\mathfrak{o}[G]$ by the idempotents $e(V)=[V:1]^{-1}\sum\sigma$ $(\sigma\in V)$, where V ranges over all ramification groups of the prime ideals of K. Finally, the author proves that the second part of the above theorem of Leopoldt can be obtained also for any abelian extension K/k such that every prime ideal of k, which is essentially ramified in K, is unramified over the rational field, and such that for any intermediate field E of two successive ramification fields for the extension K/k, the different of K/E is divisible by the degree $[K:E]$.

K. Iwasawa (Cambridge, Mass.)

Billevič, K. K. 1201

A theorem on unit elements of algebraic fields of order n. (Russian)

Mat. Sb. (*N.S.*) **64 (106)** (1964), 145–152.

Let K be an algebraic number field of nth degree. In a previous paper [same Sb. (N.S.) **40 (82)** (1956), 123–136; MR **19**, 533] the author obtained a theorem on unit elements by making use of sequences of lattice-points, and thereby established a table of fundamental units of K for the case $n=3$ and $n=4$. The present article contains an extension of this theorem to the general case. The underlying idea is almost the same as that in the previous paper, but the result is much too complicated to be explained here.

E. Inaba (Tokyo)

FIELDS AND POLYNOMIALS

See also 1186, 1200, 1201, 1232.

Krull, Wolfgang 1202

★Elementare und klassische Algebra vom modernen Standpunkt. Band I.

Dritte, erweiterte Auflage.

Walter de Gruyter & Co., Berlin, 1963. 148 *pp.* *DM* 3.60.

This book is devoted to elementary ring and field theory as applied to the solution of equations and related topics. It leads up to Abel's impossibility theorem and a short sketch of Galois theory. The last chapter discusses ordered and real fields. The material is similar to the corresponding parts of van der Waerden [*Moderne Algebra*, Bd. I, J. Springer, Berlin, 1930] but given a more rapid and concise treatment. On p. 133, line 9, "in $\mathfrak{H}^*$" should be transposed to follow "oder". *K. L. Chung* (Stanford, Calif.)

Cavior, S. R. 1203

Equivalence classes of functions over a finite field.

Acta Arith. **10** (1964/65), 119–136.

Let GF(q) denote the finite field of order q, and f, g, h functions (polynomials) in a single indeterminate with coefficients in GF(q). Let ϕ denote a permutation function. The author discusses five different types of equivalence relations defined by means of (1) $g\phi=h$: right equivalence; (2) $\phi g=h$: left equivalence; (3) $\phi_1 g\phi_2=h$: weak equivalence; (4) $\phi^{-1}g\phi=h$: similarity; (5) $\phi_1 g=h$, $g\phi_2=h$: strong equivalence. (The reviewer [Trans. Amer. Math. Soc. **75** (1953), 405–427; MR **15**, 291] had previously considered the case of right equivalence from a different point of view.) In each case the author obtains criteria for equivalence, the number of equivalence classes, the number of functions in a class and the number of automorphisms. Considerable use is made of the partition Π_f of the function f; this is defined to be the decomposition of GF(q) into distinct subsets $\{S_i\}$ such that $\alpha, \beta\in S_i$ if and only if $f(\alpha)=f(\beta)$. Thus, for example, it is proved that g and h are left equivalent if and only if $\Pi_g=\Pi_h$. On the other hand, if $\Pi_g=\{S_i: i=1,\cdots,t\}$, $\Pi_h=\{Q_i: i=1,\cdots,r\}$, then g and h are weakly equivalent if and only if $\{O(S_i): i=1,\cdots,t\}$ is a permutation of $\{O(Q_i): i=1,\cdots,r\}$; here $O(S)$ denotes the number of elements in the set S. The number of right equivalence classes is $\binom{2q-1}{q-1}$; the number of left equivalence classes is $H(q)$, where $H(n)$ is the "exponential" number defined by

$$e^{e^x-1}=\sum_0^\infty H(n)x^n/n!;$$

the number of weak equivalence classes is equal to the number of unrestricted partitions of q. For similarity the number of classes is not found explicitly but is shown to depend upon an unsolved problem about permutations.

In the last section of the paper the author obtains necessary and sufficient conditions for the solvability of certain functional equations over a finite field, namely, $fg=f$, $gf=f$, $ff=f$, $fg=h$.

Finally the author points out that most of the results of the paper hold for arbitrary finite sets; in this case, of course, functions are not necessarily representable as polynomials. *L. Carlitz* (Durham, N.C.)

Runge, Carl 1204

Eine Vorzeichenregel in der Theorie der algebraischen Gleichungen.

Jber. Deutsch. Math.-Verein. **66** (1963/64), *Abt.* 1, 52–66.

In this paper A. Ostrowski presents some unpublished results of Carl Runge, which were delivered originally in a lecture in 1907 and deposited in the form of a preliminary draft in the reading room at Göttingen. The present article consists of the original Runge text, together with corrections, amplifications and background material by Ostrowski. The principal result is: Let $g(x)$ be a polynomial such that

$$g(x)=a_0(x-x_1)(x-x_2)\cdots(x-x_n)$$
$$+a_1(x-x_1)(x-x_2)\cdots(x-x_{n-1})+\cdots+a_{n-1}(x-x_1)+a_n,$$

where $a_0, a_1, \cdots, a_n, x_1, x_2, \cdots, x_n$ are real numbers, and let V be the number of changes of sign in the sequence $a_0, a_1, \cdots, a_n$. Suppose that $g(x)$ also has such a representation in terms of the numbers

$$a_0', a_1', \cdots, a_n', \quad x_1', x_2', \cdots, x_n',$$

and that V' is the number of changes of sign in $a_0', a_1', \cdots, a_n'$. Suppose also that there are numbers u, v with $u<v$ such that $x_i\leqq u$ and $x_i'\geqq v$ for $i=1, 2, \cdots, n$. Then (a) $V'\leqq V$, and (b) the number of roots w of $g(x)=0$ for which $u<w\leqq v$ is not greater than $V-V'$. *H. Levi* (New York)

Mikusiński, J.; Schinzel, A. 1205

Sur la réductibilité de certains trinômes.

Acta Arith. **9** (1964), 91–95.

Let a and b be rational integers such that $ab\neq 0$ and $|a|-|b|\neq 0, \pm 1$. One of the authors conjectured that there exist only a finite number of ratios m/n such that the polynomial x^n+ax^m+b is reducible over the rationals [Schinzel, *Neki nerešeni problemi u matematici*, pp. 63–70, Matematička Biblioteka, 25, Zavod za Izdavanje Udžbenika Socijalisticke Republike Srbije, Belgrade, 1963]. The present article treats the special case when $|a|$ is a prime number and $|b|=1$, and shows that the above-mentioned conjecture is valid in this case. *E. Inaba* (Tokyo)

Jakovlev, A. V. 1206

The imbedding problem for fields. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 645–660.

This article gives a detailed account of the author's work, the outline of which has already been published [Dokl. Akad. Nauk SSSR **150** (1963), 1009–1011]. *E. Inaba* (Tokyo)

ABSTRACT ALGEBRAIC GEOMETRY

See also 1289.

Grothendieck, A. 1207

Éléments de géometrie algébrique. I. Le langage des schémas.

Rédigés avec la collaboration de J. Dieudonné.

Inst. Hautes Études Sci. Publ. Math. No. 4 (1960), 228 *pp.*

Grothendieck, A. 1208
Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes.
Rédigés avec la collaboration de J. Dieudonné.
Inst. Hautes Études Sci. Publ. Math. No. 8 (1961), 222 *pp.*

Grothendieck, A. 1209
Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I.
Rédigés avec la collaboration de J. Dieudonné.
Inst. Hautes Études Sci. Publ. Math. No. 11 (1961), 167 *pp.*

Grothendieck, A. 1210
Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. II.
Rédigés avec la collaboration de J. Dieudonné.
Inst. Hautes Études Sci. Publ. Math. No. 17 (1963), 91 *pp.*

This is the second portion of Chapter III, which contains two sections. In the first one, § 6, one finds a Künneth formula on cohomologies with coefficients in coherent algebraic sheaves. To be precise, let Y be a prescheme and $f_i: X^{(i)} \to Y$ $(i = 1, 2)$ morphisms of preschemes, separated and quasi-compact. Let $\mathscr{P}^{(i)}$ be a quasi-coherent $\mathcal{O}_{X^{(i)}}$-modules. Then the formula asserts: If $\mathscr{P}^{(i)}$ is Y-flat for at least one i and all the higher direct images

$$\mathscr{H}^n(f_i, \mathscr{P}^{(i)}) \ (= R^n f_{i*}(\mathscr{P}^{(i)}))$$

are Y-flat for at least one i, then there exist canonical isomorphisms $(n \in Z)$:

$$\mathscr{H}^n(f_1 \times_Y f_2, \mathscr{P}^{(1)} \otimes_{\mathcal{O}_Y} \mathscr{P}^{(2)}) = \bigoplus_{n_1+n_2=n} \mathscr{H}^{n_1}(f_1, \mathscr{P}^{(1)}) \otimes_{\mathcal{O}_Y} \mathscr{H}^{n_2}(f_2, \mathscr{P}^{(2)}).$$

Note that all the assumptions are automatically satisfied if $X^{(i)}$ are algebraic schemes over a field $k(Y = \mathrm{Spec}(k))$. Quite generally (without the flatness assumptions), the "global hypertor" $\mathscr{T}or_n{}^Y (f_1, f_2; \mathscr{P}^{(1)}, \mathscr{P}^{(2)})$ $(n \in Z)$ bridges the gap between the two sides of the formula by means of the pair of "Künneth spectral sequences" which converge to it. The "local hypertors" play roles in the spectral sequences (cf. Theorem (6.7.8)).

The bi-∂-functor $\mathscr{T}or^Y.\,(f_1, f_2; \mathscr{P}^{(1)}, \mathscr{P}^{(2)})$ in $\mathscr{P}^{(1)}$ and $\mathscr{P}^{(2)}$ is "homological" in general, while the higher direct image functor $\mathscr{H}^{\cdot}(f_1 \times_Y f_2, \mathscr{P}^{(1)} \otimes_{\mathcal{O}_Y} \mathscr{P}^{(2)})$ is not "cohomological". (Observe that this is not so in $\mathscr{P}^{(2)}$, when f_2 is the identity morphism, $\mathscr{P}^{(1)} = \mathcal{O}_X{}^{(1)}$ and f_1 is a quadratic transformation of nonsingular algebraic k-schemes.) Now consider the case: $X^{(2)} = Y$ and $f_2 = \mathrm{id}_Y$. Then one finds a "complex" $\mathscr{R}.$ of quasi-coherent $\mathcal{O}_Y$-modules, which is Y-flat, such that there exists an isomorphism

$$\mathscr{T}or^Y.\,(f_1, \mathrm{id}_Y; \mathscr{P}^{(1)}, \mathscr{Q}) \xrightarrow{\approx} \mathscr{H}.\,(\mathscr{R}. \otimes_{\mathcal{O}_Y} \mathscr{Q})$$

of ∂-functors in quasi-coherent $\mathcal{O}_Y$-modules $\mathscr{Q}$ (cf. Proposition (6.10.1)). For simplicity, let Y be affine. If $\mathscr{P}^{(1)}$ is Y-flat, then $\mathscr{R}.$ can be so chosen that it is bounded below and has vanishing positive parts, and that its terms $\mathscr{R}_j$ are all of the form $\mathcal{O}_Y{}^{m(j)}$ with integers $m(j)$. Moreover, the above isomorphism becomes the following useful presentation of the higher direct image functor:

$$\mathscr{H}^{-n}(f_1, \mathscr{P}^{(1)} \otimes_{\mathcal{O}_Y} \mathscr{Q}) \xrightarrow{\approx} \mathscr{H}_n(\mathscr{R}. \otimes_{\mathcal{O}_Y} \mathscr{Q})$$

$(n \in Z)$ (cf. Theorem (6.10.5)).

In § 7, one finds a systematical study of the variation of cohomologies with coefficients in coherent algebraic sheaves which vary in an algebraic family. Let $f: X \to Y$ be a proper morphism with a locally noetherian prescheme Y, and let $\mathscr{P}$ be an $\mathcal{O}_X$-module, coherent and Y-flat. Consider the ∂-functor ("homological" by the flatness of $\mathscr{P}$) $\mathscr{T}_p(\mathscr{Q}) = \mathscr{H}^{-p}(f, \mathscr{P} \otimes_{\mathcal{O}_Y} \mathscr{Q})$ in quasi-coherent $\mathcal{O}_Y$-modules $\mathscr{Q}$. Then Theorem ((7.7.5), I) (also Theorem (7.6.9)) asserts the "upper semi-continuity" of the integer-valued function on $Y: y \rightsquigarrow \mathrm{rank}_{k(y)} \mathscr{T}_p(k(y))$, where $k(y)$ denotes the residue field (as well as the single-pointed $\mathcal{O}_Y$-module associated with it) of the local ring $\mathcal{O}_{Y,y}$. Part II of the same theorem gives various equivalent conditions for the right exactness of the functor $\mathscr{T}_p$, one of which is the left exactness of $\mathscr{T}_{p-1}$. The Euler-Poincaré characteristic $EP(f, \mathscr{P}; y) = \sum_i (-1)^i \mathrm{rank}_{k(y)} \mathscr{T}_i(k(y))$ has naturally a stronger continuity; namely, under the same assumptions, it is constant on each connected component of Y (cf. Theorem (7.9.4)).

The Künneth formula and the other theorems are proven in greater generality than stated in this review; for instance, the $\mathscr{P}^{(i)}$ are replaced by complexes $\mathscr{P}.^{(i)}$ of quasi-coherent $\mathcal{O}_{X^{(i)}}$-modules, in which case, however, the higher direct images must be accordingly replaced by "hypercohomologies for given morphisms". As is seen in the paper, this generalization has offered a better formalism of the whole theory. Notice an indication of this in the type of Theorem (6.10.5) cited above. One can also guess (viewing the K-theory in a general scheme-theoretical setup, the Riemann - Roch - Hirzebruch - Grothendieck theory, etc.) that the generalization will become essential in future chapters of this work.

H. Hironaka (Waltham, Mass.)

Cassels, J. W. S. 1211
Arithmetic on curves of genus 1. II. A general result.
J. Reine Angew. Math. **203** (1960), 174–208.

On généralise ici les résultats d'un article antérieur [même J. **202** (1959), 52–99; MR **22** #24; errata, MR **22**, p. 2545] dont on reprend les notations. On désigne par C une courbe algébrique de genre un sur un corps de nombres algébriques R, par P un point de C défini sur R et par $X \to \nu X$ une application φ de C en C qui est définie sur R et laisse le point P invariant. On précise au préalable certaines propriétés des recouvrements et des conséquences du théorème de finitude de Mordell-Weil. En particulier, les théorèmes de descente permettent de définir un groupe $M^{(1)}$ de ν-recouvrements particulier. On obtient alors le résultat suivant (théorème I): si ν^2 est un entier rationnel et si chaque point du noyau de φ est défini sur R, il existe une application bilinéaire symétrique gauche des paires d'éléments de $M^{(1)}$ dans les entiers rationnels modulo un.

J. Guérindon (Zbl **94**, 26)

Cassels, J. W. S. 1212
Arithmetic on curves of genus 1. III. The Tate-Šafarevič and Selmer groups.
Proc. London Math. Soc. (3) **12** (1962), 259–296.

L'auteur continue l'étude du groupe G des points d'une courbe C de genre 1, définie sur un corps de nombres k, algébrique et de degré fini [voir #1211 ci-dessus]. Il passe aux complétions P-adiques de k et donne un algorithme permettant de déterminer effectivement le groupe de Tate-Šafarevič, pris modulo les éléments qui sont divisibles par une puissance élevée d'un nombre premier fixé q. Enfin on étudie le problème de savoir si deux espaces homogènes principaux sur C sont dans la même classe, élément du groupe de Weil-Chatelet. *J. Guérindon* (Zbl **106**, 37)

Cassels, J. W. S. 1213
Corrigendum: "Arithmetic on curves of genus 1. III. The Tate-Šafarevič and Selmer groups".
Proc. London Math. Soc. (3) **13** (1963), 768.
The author points out an imprecision in the reasoning of § 5 of the paper mentioned in the title [#1212] and shows how to rectify it.

Cassels, J. W. S. 1214
Arithmetic on curves of genus 1. IV. Proof of the Hauptvermutung.
J. Reine Angew. Math. **211** (1962), 95–112.
Let k be an algebraic number field, $\mathscr{C}$ an abelian variety of dimension 1 and Ш the corresponding Tate-Šafarevič group of classes of principal homogeneous spaces for $\mathscr{C}$ defined over k which are everywhere locally trivial. The author constructs a pairing $l(\xi, \eta)$ over Ш with values in the group Q/Z of the rationals modulo 1 and he shows that $l(\xi, \eta)$ is linear in η and that $l(\xi, \eta)$ is skew-symmetric. Tate's lemma is then given: If A is a Γ-module which, when considered only as a Z-module, is isomorphic to $Z/qZ \oplus Z/qZ$, where q is a rational prime and Γ is the Galois group of the algebraic closure $\bar{k}$ of the algebraic number field k over k; then an element of $H^2(\Gamma, A)$ is trivial if it is everywhere locally trivial. By means of Tate's lemma, it is proved that if q is a prime, a necessary and sufficient condition for $\xi \in$ Ш to be divisible by q in Ш is that $l(\xi, \eta) = 0$ for all $\eta \in$ Ш with $q\eta = 0$. From this lemma follows the theorem: Let m be a natural number and suppose that $l(\xi, \eta) = 0$ for all $\xi \in$ Ш such that $m\xi = 0$; then $\eta = mH$ for some $H \in$ Ш. One proves also the following theorems: (i) Let $(\mathscr{D}, \mu)$ be a homogeneous space over an abelian variety $\mathscr{C}$ of dimension 1 defined over an algebraic number field. Suppose that $(\mathscr{D}, \mu)$ corresponds to an element of Ш. Then the map induced by μ of the divisors on $\mathscr{D}$ defined over k of degree 0 into the points of $\mathscr{C}$ defined over k is surjective. (ii) In the language of Lang and Tate [Amer. J. Math. **80** (1958), 659–684; MR **21** #4960] the index of an element of Ш is always equal to its order in the group Ш.

{Part V has already appeared [J. London Math. Soc. **38** (1963), 244–248; MR **26** #6171].} *P. Abellanas* (Madrid)

Segre, Beniamino 1215
Intorno ad una congettura di Lang e Weil.
Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **34** (1963), 337–339.
The author establishes in this note a contradiction between some statements in Picard and Simart [*Théorie des fonctions algébriques de deux variables indépendantes*, Tome I, Gauthier-Villars, Paris, 1897] and the well-known Weil conjectures relating the zeta-function of a variety to its Betti numbers. By counting rational points, he shows that these conjectures (or rather, a weaker version of them given by Lang and Weil [Amer. J. Math. **76** (1954), 819–827; MR **16**, 398]) predict that the surface $x^3z + y^3 + z^2 = 0$ must be irregular. Yet Picard and Simart list—without proof, it should be noted—all the irregular surfaces of degree four, and this one is not among them. The author concludes that the conjectures in question "cannot hold without restriction"; we reply, "$(xdy - 2ydx)(2z + x^3)^{-1}$".
A. Mattuck (Cambridge, Mass.)

LINEAR ALGEBRA

See also 1226, 1256, 1734.

Eberlein, P. J. 1216
On a conjecture of L. Mirsky's.
Quart. J. Math. Oxford Ser. (2) **15** (1964), 278.
The conjecture is that if A is an $n \times n$ matrix with characteristic roots $\omega_1, \cdots, \omega_n$ and if $\Omega = \operatorname{diag}(\omega_1, \cdots, \omega_n)$, then $\inf\{\|A - X\|^2 : X \text{ normal}\} = \frac{1}{2}\{\|A\|^2 - \|\Omega\|^2\}$. Here the norm signs indicate an arbitrary unitarily invariant norm.

The author asserts that the conjecture is true for the Euclidean norm when $n = 2$, and proves that it is false for the Euclidean norm when $n \geqq 3$. (The Euclidean norm is the square root of the sum of the squares.) A counterexample (for $n = 3$) is

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

{Reviewer's comment: If the operator norm is used, the conjecture is false even for $n = 2$. (The operator norm of A is $\sup\{\|Af\| : \|f\| \leqq 1\}$.) A counterexample is $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. For this particular A, the left term of the conjectured equation has been evaluated by the reviewer and J. E. McLaughlin [Pacific J. Math. **13** (1963), 585–596; MR **28** #477]; the equation becomes $\frac{1}{4} = \frac{1}{2}$.}
P. R. Halmos (Ann Arbor, Mich.)

Mirsky, L. 1217
Inequalities and existence theorems in the theory of matrices.
J. Math. Anal. Appl. **9** (1964), 99–118.
This expository article is a survey of some of the results obtained by various authors during the last ten years, on inequalities concerning matrices. Most results covered are concerned with the necessary and sufficient conditions for the existence of a matrix of a certain type with certain prescribed quantities. Such are the conditions on the existence of: (1) a real, symmetric matrix with prescribed diagonal elements and eigenvalues; (2) a real, symmetric, positive definite matrix with prescribed eigenvalues and values of the leading principal minors; (3) a real, symmetric, positive definite matrix A with prescribed diagonal elements of A and A^{-1}; (4) a complex matrix A with prescribed eigenvalues of A and $\frac{1}{2}(A + A^*)$; (5) a complex matrix with prescribed eigenvalues and singular values; (6) an orthogonal matrix with prescribed diagonal elements; (7) a doubly

stochastic matrix with prescribed diagonal elements; (8) an incidence matrix with prescribed row-sums and column-sums. The paper also discusses some known inequalities between the eigenvalues of Hermitian matrices A, B and $A+B$; or between the singular values of arbitrary matrices A, B and AB. The bibliography contains 51 references. *Ky Fan* (Evanston, Ill.)

Sanchez, Antonio Valle [Valle Sanchez, Antonio] 1218
On the Jordan canonical form corresponding to a square matrix and the non-singular matrices relating them. (Spanish)
Gac. Mat. (Madrid) (1) **16** (1964), 20–34.
Assuming known the distinct characteristic roots $\lambda_1, \cdots, \lambda_r$ of a complex matrix A, the author determines the block structure of the Jordan normal form J of A by examination of the ranks of $(A-\lambda_i I)^j$, $i=1, \cdots, r; j=1, 2, \cdots$.

The determination of P, such that $P^{-1}AP=J$, is accomplished by a standard analysis of the null spaces of the matrices $(A-\lambda_i I)^j$, $i=1, \cdots, r; j=1, 2, \cdots$. Four numerical examples are worked out.
R. F. Rinehart (Cleveland, Ohio)

Malyšev, A. V. 1219
A new variant of the proof of the Stuff-Minkowski theorem on the finiteness of the number of edges of the Hermite reduction region. (Russian)
Uspehi Mat. Nauk **16** (1961), *no.* 2 (98), 127–129.
The author gives a proof of the theorem that, in the space of real positive definite quadratic forms in n variables, the reduced forms constitute a convex polyhedral cone with a finite number of faces. His proof is a little more geometrical than the standard proof given by Siegel [Abh. Math. Sem. Hansischen Univ. **13** (1940), 209–239; MR **2**, 148]. In spite of the title of the paper the author uses the term "reduced" in the sense of Minkowski.
P. T. Bateman (Zbl **101**, 36)

Heap, B. R.; Lynn, M. S. 1220
The index of primitivity of a non-negative matrix.
Numer. Math. **6** (1964), 120–141.
If A is a non-negative irreducible matrix which is primitive, denote by $\gamma(A)$ the first exponent r for which $A^r>0$. An improved bound for $\gamma(A)$ is given, taking into account the lengths of circuits in the graph of A. If the graph of A contains $k\geqq 2$ circuits of relatively prime lengths $p(1), \cdots, p(k)$, then $\gamma(A)\leqq\sum_1^k m_i+d(A)+\bar\phi(p(1), \cdots, p(k))$, where $d(A)$ is the diameter of the graph, m_i are, roughly speaking, distances of an arbitrary vertex from the union of the first i circuits and $\bar\phi$ is the minimum over all permutations of $\phi(p(\xi_1), \cdots, p(\xi_k))$. The function ϕ is explicitly given and is such that the system $\sum a(i)p(i)=r$ has a non-negative integer solution $a(i)$ whenever $r\geqq\phi(p(1), \cdots, p(k))$. A great number of corollaries are obtained using either explicit expressions or bounds for the numbers occurring in the main theorem. In particular, the bound for $\gamma(A)$ may be considerably improved if the set $p(1), \cdots, p(k)$ is a minimal set of relatively prime numbers, i.e., if every proper subset has g.c.d. >1. Another bound for $\gamma(A)$ uses the notion of the inward demi-degree of a set of vertices. Several interesting examples are given to illustrate the power of the different bounds for $\gamma(A)$.
V. Pták (Prague)

Marcus, Marvin; Yaqub, Adil 1221
Compounds of skew-symmetric matrices.
Canad. J. Math. **16** (1964), 473–478.
The problem treated is that of finding all real skew solutions A of the matrix equation (1) $C^{n-r}(A^T)=C_r(B)$, where $C_r(X)$ denotes the rth compound of X, $C^{n-r}(X)$ the $(n-r)$th supplementary compound of X, and the superscript T denotes transpose. This is a generalization of a problem solved by H. Schwerdtfeger [Portugal. Math. **20** (1961), 39–41; MR **23** #A3150], equivalent to the case $r=1$ of the present problem, answering a question posed by W. R. Utz [ibid. **18** (1959), 225–229; MR **23** #A3749].

The cases where B is nonsingular or singular are treated separately, and the results are embodied in the following two theorems and corollary. Theorem 1: Let B be an n-square nonsingular matrix over the real field. Every solution of (1) appears in the form $A=(d(B))^{1/(n-r)}B^{-1}$. Thus there is no real solution if $n-r$ is even and $d(B)<0$. Corollary: In the nonsingular case, the solution A is skew symmetric if and only if B is skew symmetric; and since in this case $d(B)>0$, the solution always exists. Theorem 2: Let B be an n-square singular matrix over the reals. A necessary and sufficient condition for the existence of a nontrivial n-square real symmetric matrix A satisfying (1) is that both of the following conditions hold: (i) $\rho(A)=n-r$ and $\rho(B)=r$ and $n-r$ is even; (ii) there exists a real orthogonal matrix U such that the only nonvanishing r-square subdeterminant of U^TBU is the first principal one; in fact, $d(U^TBU[1, \cdots, r|1, \cdots, r])=a>0$.

The proof of Theorem 1 is brief and rests on known properties of compounds, i.e., the multiplicative property of the function $C_r(X)$ and the compound matrix formulation of the Laplace determinantal expansion. The proof of Theorem 2 is lengthy and invokes the orthogonal reduction of the skew symmetric matrix A to canonical form, again taking advantage of the multiplicative character of $C_r(X)$.
R. F. Rinehart (Cleveland, Ohio)

Minc, Henryk 1222
Permanents of (0, 1)-circulants.
Canad. Math. Bull. **7** (1964), 253–263.
A (0, 1)-matrix is a matrix in which each entry is either 0 or 1. The circulants discussed in this paper are defined as follows. Let P_n be the $n\times n$ permutation matrix having unity in the positions (1, 2), (2, 3), $\cdots$, $(n-1, n)$, $(n, 1)$, and zeros elsewhere. Put $Q(n, k)=\sum_{j=1}^{k-1}P_n{}^j$. The permanent of a matrix $A=(a_{ij})$ is denoted by $p(A)=\sum_R a_{1\rho_1}a_{2\rho_2}\cdots a_{n\rho_n}$, where $R: i\to\rho_i$ runs through all permutations of the symmetric group. The author's work is related to the conjecture that $p(Q(n, r))\geqq n!(r/n)^n$. He disproves certain statements made by N. S. Mendelsohn [same Bull. **4** (1961), 29–38; MR **23** #A2332]. Amongst his results are explicit formulae for $p(Q(n, 3))$ and $p(Q(n, 4))$, and he shows that there exists a doubly stochastic matrix S such that $p(S)<p(Q(n, r)/r)$.
W. Ledermann (Brighton)

Cigler, Johann 1223
Ein gruppentheoretisches Analogon zum Begriff der normalen Zahl.
J. Reine Angew. Math. **206** (1961), 5–8.
Es sei X eine kompakte abelsche Gruppe mit abzählbarer Basis und T ein steiger Endomorphismus von X auf sich,

der bezüglich des Haarschen Maßes μ auf X ergodisch ist. Ein Element $x \in X$ heißt normal bezüglich T, wenn die Folge $\{T^n x\}$ in X gleichverteilt ist, d.h. wenn

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x) = \int_X f(x)\, d\mu(x)$$

für alle stetigen komplexwertigen Funktionen f auf X gilt. Der Verfasser beweist folgende Verallgemeinerungen bekannter Sätze über (bezüglich einer ganzzahligen Basis $a > 1$) normale Zahlen: Fast alle $x \in X$ sind normal bezüglich T. Für jedes $k \geqq 2$ ist das Element $x \in X$ genau dann normal bezüglich T, wenn es normal bezüglich T^k ist. Ist x normal bezüglich T und S ein mit T vertauschbarer stetiger Endomorphismus von X auf sich, dann ist auch Sx normal bezüglich T; ist außerdem der Kern dieses Endomorphismus S endlich, dann ist auch jedes S-Urbild von x normal bezüglich T. Die beim Beweis dieser letzten Behauptung verwendete Existenz der Menge B, deren Rand das Maß 0 hat, scheint dem Referenten nicht voll geklärt zu sein, jedoch läßt sich nach Mitteilung des Verfassers der Beweis auch ohne Verwendung dieser Voraussetzung führen. Durch die Begriffsbildungen des Verfassers werden auch die normalen k-tupel von Maxfield [Pacific J. Math. **3** (1953), 189–196; MR **14**, 851] und Resultate von Rohlin [Izv. Akad. Nauk SSSR Ser. Mat. **13** (1949), 329–340; MR **11**, 40] erfaßt. *G. Helmberg* (Zbl **103**, 19)

Schmidt, Wolfgang M. 1224
Normalität bezüglich Matrizen.
J. Reine Angew. Math. **214/215** (1964), 227–260.

Let ξ be an n-dimensional vector and A an n-dimensional matrix, with real elements in both cases. Say that ξ is normal with respect to A if the sequence $\{A^k\xi;\, k = 1, 2, 3, \cdots\}$ is uniformly distributed modulo 1, i.e., their images modulo 1 are uniformly distributed in the unit cube. The matrix A is said to be good if almost every vector ξ is normal with respect to A. Let

$$P(x) = (x-\lambda_1)^{a_1} \cdots (x-\lambda_r)^{a_r}$$

be the characteristic polynomial of A, with distinct λ_i. Let C_n denote the n-dimensional vector space whose components are complex numbers, and $C^{(i)}$ the subspace of C_n consisting of those vectors ζ of C_n such that $(A-\lambda_i)^{a_i}\zeta = 0$. Then C is the direct sum of the $C^{(i)}$ taken over $i = 1, \cdots, r$. By $(A-\lambda_i)C^{(i)}$ is meant the set of all $(A-\lambda_i)\zeta$ with $\zeta \in C^{(i)}$. Let $C(A)$ denote the direct sum of those $C^{(i)}$ for which $|\lambda_i| > 1$ and of those $(A-\lambda_i)C^{(i)}$ for which $|\lambda_i| = 1$. Say that a subset M of C_n lies in a rational plane if there is a non-zero n-dimensional vector u with rational integer components such that the scalar product of u with every element of M is zero. It is proved that A is good if $C(A)$ lies in no rational plane. Conversely, if $C(A)$ lies in a rational plane, then almost no ξ is normal with respect to A. A corollary to this is that A is good if all its eigenvalues exceed 1 in absolute value. This corollary is a special case of a result of W. Philipp [Arch. Math. **12** (1961), 429–433; MR **27** #3614].

Say that the matrix A is almost integral if its elements are rational numbers and its eigenvalues are algebraic integers. Under the assumption that A is almost integral, it is established that A is good if and only if neither zero nor a root of unity is among its eigenvalues. This and other results of the paper are closely related to work of V. A. Rohlin [Izv. Akad. Nauk SSSR Ser. Mat. **13** (1949), 329–340; MR **11**, 40], J. Cigler [#1223 above], and to the previously cited work of W. Philipp.

Say that an almost integral A is almost ergodic if neither zero nor a root of unity is among its eigenvalues. Write $A \sim B$ in case $A^m = B^k$ for some natural numbers m and k. Write $A \to B$ in case every vector ξ which is normal with respect to A is also normal with respect to B. Under the assumption that A and B are almost ergodic, then $A \sim B$ implies $A \to B$, and the author conjectures that the converse of the implication holds. The converse does hold in the one-dimensional case by work of J. W. S. Cassels [Colloq. Math. **7** (1959), 95–101; MR **22** #4694] and the author [Pacific J. Math. **10** (1960), 661–672; MR **22** #7994]. The converse also holds for the special class of two-dimensional matrices defined in the usual way from complex numbers. This is implied by the following result: Assuming that A and B are almost ergodic, $A \not\sim B$, $AB = BA$, and that all the eigenvalues of B exceed 1 in absolute value, then $A \not\to B$. *I. Niven* (Eugene, Ore.)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 1202, 1236, 1237, 1242, 1243.

Jensen, Chr. U. 1225
On characterizations of Prüfer rings.
Math. Scand. **13** (1963), 90–98.

This paper deals with the ideal-theoretic characterization of Prüfer rings in various respects. In the first place it is proved that an integral domain R is a Prüfer ring if and only if the lattice formed by ideals of R is distributive. As an application a Dedekind domain is characterized as a domain such that any proper residue class $R/\mathfrak{a}$ ($\mathfrak{a} \neq 0$) is a principal ideal ring. Theorems 3 and 4 deal with the characterization of Prüfer rings involving the multiplicative structure of ideals. The restricted cancellation law (i.e., $\mathfrak{a}\mathfrak{b} = \mathfrak{a}\mathfrak{c}$, with $\mathfrak{a}$ ($\neq 0$) a finitely generated ideal, implies $\mathfrak{b} = \mathfrak{c}$) also characterizes a Prüfer ring. At the end the equivalent conditions for the cancellation law are discussed in an integral domain. *Y. Nakai* (Hiroshima)

Sharpe, D. W. 1226
On certain polynomial ideals defined by matrices.
Quart. J. Math. Oxford Ser. (2) **15** (1964), 155–175.

Suppose T is a commutative integral domain with identity, x_{ij} are indeterminates, and $R = T[x_{11}, \cdots, x_{rs}]$ is the polynomial ring over T in the indeterminates x_{ij}. Denote by A the matrix with r rows and s columns which has as its (i, j) entry the indeterminate x_{ij}. The author provides the following information about the ideal (A) generated by the 2×2 subdeterminants of the matrix A: (1) the ideal (A) is prime, (2) the first syzygy of (A) (i.e., he determines an exact sequence $(0)\to K\to F\to(A)\to(0)$ of R modules with F free and displays a set of generators for K), (3) if T is noetherian, then the grade of (A) [the length of a maximal R-sequence contained in (A)] is $(r-1)(s-1)$ and the grade of (A) is the homological dimension of $R/(A)$, that is, the ideal is perfect.
K. R. Mount (Evanston, Ill.)

Luh, Jiang 1227

A note on strongly regular rings.

Proc. Japan Acad. **40** (1964), 74–75.

An associative ring R is called strongly regular if to each element x of R there is an element a in R such that $x=x^2a$. Theorem: R is strongly regular if and only if each element of R is contained in a subset of R which is a group under the ring multiplication.

L. G. Kovács (Canberra)

Lihtman, A. I. 1228

On group rings of p-groups. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **27** (1963), 795–800.

Sei G eine Gruppe und R ein Ring mit Einselement und der Charakteristik p^m für eine Primzahl p. Sei weiter $O_R(G)$ das Fundamentalideal des Gruppenringes $R(G)$ von G über R. Das Hauptergebnis der Arbeit ist der folgende Satz: Ist G eine endlich erzeugte p-Gruppe und $O_K{}^a(G)=O_K{}^{a+1}(G)$ für ein geeignetes a, so ist $O_R(G)$ dann und nur dann im Jacobson-Radikal von $R(G)$ enthalten, wenn G endlich ist. Daraus folgt mit Hilfe eines Ergebnisses von A. I. Kostrikin [dieselben Izv. **23** (1959), 3–34; MR **24** #A1947] ein Korollar, das eine teilweise Lösung eines Problemes von I. Kaplansky [Report of a Conference on Linear Algebras, pp. 1–3, Problem 9, Washington, D.C., 1957; MR **20** #3179] liefert: Sei G eine Gruppe vom Exponent p. Dann ist $O_R(G)$ dann und nur dann im Jacobson-Radikal von $R(G)$ enthalten, wenn G lokal endlich ist. Im letzten Teil der Arbeit werden Darstellungen von unendlichen p-Gruppen über Körpern der Charakteristik p untersucht.

C. Hering (Frankfurt a.M.)

Taft, Earl J. 1229

Orthogonal conjugacies in associative and Lie algebras.

Trans. Amer. Math. Soc. **113** (1964), 18–29.

Let A be either: (1) an associative algebra over a field F with radical R such that A/R is separable; or (2) a Lie algebra over F of characteristic zero, with radical R (all algebras finite-dimensional). Let G be a completely reducible group of linear transformations of A consisting of automorphisms and anti-automorphisms. If F is modular, suppose further that G is finite of order prime to the characteristic. Earlier results of the author [Illinois J. Math. **1** (1957), 565–573; MR **20** #4586] and of Mostow [Amer. J. Math. **78** (1956), 200–221; MR **19**, 1181] show that A admits a Wedderburn (or Levi) decomposition $A=T+R$, where T is a G-stable subalgebra. Now assume the characteristic is not 2, and let S be any separable (semisimple) G-stable subalgebra of A. Theorem: There is an automorphism φ of A of the form $a\to a-az'-za+zaz'$, where $z+z'-zz'=0$ (of the form exp (ad z) in the Lie case), such that $S^\varphi\subseteq T$, where $z\in R$ ($z\in N\subset R$, N the maximal nilpotent ideal of A in the Lie case), and where $z^\sigma=z$ for all automorphisms $\sigma\in G$, $z+z^\sigma-zz^\sigma=0=z+z^\sigma-z^\sigma z$ (associative case) or $z^\sigma=-z$ (Lie case) whenever $\sigma\in G$ is not an automorphism. Thus any two G-stable Wedderburn or Levi factors are conjugate by such an automorphism of A. If A is a solvable Lie algebra of characteristic zero, and if G consists of automorphisms of A, the author shows that any two G-stable Cartan subalgebras of A (Mostow has shown that these exist) are conjugate under an exp (ad z) as above, where z may be taken from the intersection of the lower central series of A.

G. B. Seligman (New Haven, Conn.)

Levitzki, J. 1230

On nil subrings.

Israel J. Math. **1** (1963), 215–216.

Herstein and Small [Notices Amer. Math. Soc. **10** (1963), 662] announced the following result. Let R be a ring satisfying the ascending chain condition on right and left annihilators, then the nil subrings of R are nilpotent. The author left, amongst his papers, two results which imply this theorem. These results have been prepared for publication by S. A. Amitsur. For any ring R and subset S of R, denote by $r(S)$ [$l(S)$] the right [left] annihilator of S, by $L(R)$ the lower radical of R, and by $N(R)$ the sum of all nilpotent ideals of R. Then the two results are: (1) If R satisfies the maximum condition on sequences of annihilators of the form $r((b_1))\subseteq r((b_2))\subseteq\cdots$, where $b_{i+1}=b_ir_ib_i$ for arbitrary r_i in R, then $S=L(S)$ for every nil subring S of R. (2) If R satisfies the maximum condition on right and left annihilators of the form $t(A_1)\subset t(A_1A_2)\subset t(A_1A_2A_3)\subset\cdots$, where $t=r, l$, and where A_k is a nil ring, and $A_1A_2\cdots A_k$ lies in $A_1A_2\cdots A_{k-1}$, then for every subring S of R, $N(S)$ is nilpotent.

A. Rosenberg (Ithaca, N.Y.)

Forster, Otto 1231

Über die Anzahl der Erzeugenden eines Ideals in einem Noetherschen Ring.

Math. Z. **84** (1964), 80–87.

In a noetherian ring any ideal can be generated by a finite number of generators. The author proposes the problem of finding an upper bound for the number of generators for a certain class of ideals in a noetherian ring. More generally, let A be a noetherian ring and let M be a finite A-module. For any $\mathfrak{p}\in X=\operatorname{Spec}(A)$ he puts $L_\mathfrak{p}(M)=M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$ and $\dim L_\mathfrak{p}(M)=\dim_{A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}}L_\mathfrak{p}(M)$. The main theorem is as follows. We set

$$b_\mathfrak{p}(M) = \dim\mathfrak{p}+\dim L_\mathfrak{p}(M) \quad \text{if } L_\mathfrak{p}(M)\neq 0,$$
$$= 0 \quad \text{if } L_\mathfrak{p}(M)=0$$

and $\mathfrak{b}(M)=\sup_{\mathfrak{p}\in X}b_\mathfrak{p}(M)$, then M has a set of generators consisting of $\mathfrak{b}(M)$-elements. It is to be noted that in the assertion of the theorem it is necessary to assume that M is a finite A-module, otherwise, a counter-example exists. Some applications of this theorem are discussed in § 2 and § 3.

Y. Nakai (Hiroshima)

Yaqub, Adil 1232

On the ring-logic character of certain rings.

Pacific J. Math. **14** (1964), 741–747.

Let $(R,\times,+)$ be a ring and let $K=\{\rho_1,\rho_2,\cdots\}$ be a group of transformations of R. Then the K-logic of $(R,\times,+)$ is the system $(R,\times,\rho_1,\rho_2,\cdots)$ whose class R is identical with that of the ring and whose operations are the binary multiplication of the ring and the unary operations which are the elements of K. $(R,\times,+)$ is called a ring-logic mod K if (1) the "+" is equationally definable in terms of its K-logic $(R,\times,\rho_1,\rho_2,\cdots)$, and (2) the "+" of the ring is fixed by its K-logic. The present paper is related to the work of A. L. Foster [see, for example, Math. Z. **58** (1953), 306–336; MR **15**, 194].

The following are the author's main results: (1) Given the Galois field $(F_{p^k},\times,+)$ with p^k elements (p prime) and the (normal) group of permutations D on F_{p^k} such that $D=\{\text{identity}, \frown, \frown^2, \cdots, \frown^{p^k-1}\}$, where

$$x^\frown = \xi x+(1+\xi x+\xi^2x^2+\cdots+\xi^{p^k-2}x^{p^k-2})$$

and ξ is a multiplicative generator of F_{p^k}, then $(F_{p^k}, \times, +)$ is a ring-logic mod D. (2) If $(R, \times, +)$ is a finite commutative ring with zero radical, then it is a ring-logic mod D. In this case D is the direct product of the normal permutation groups of those Galois fields which are the direct summands of R. (3) Let R be a ring with unit such that $x^n = x$ for all x in R, where n is a fixed integer greater than 1. Then R is a ring-logic mod D. This D is presumably defined similarly to the one in (2).

B. Brainerd (Toronto, Ont.)

NON-ASSOCIATIVE ALGEBRA

See also 1229, 1290, 1571.

Lorenzen, Hans-Peter 1233

★Quadratische Darstellungen in Jordanalgebren.

Dissertation zur Erlangung des Doktorgrades der Mathematisch-Naturwissenschaftlichen Fakultät der Universität Hamburg, Hamburg, 1963. ii + 41 *pp.*

The author gives a definition of representation of a Jordan algebra that is valid for any characteristic, and reduces to the classical definition if the characteristic is not 2 or 3. The study of these representations is transferred to an appropriate enveloping algebra, somewhat in the manner of Lie algebra theory. In the second part of the paper, the notion of a quadratic mapping of modules is introduced. This leads to an interesting new way of defining a Jordan algebra, which turns out to be useful for describing the representations. The details are too technical to reproduce here. *W. E. Jenner* (Chapel Hill, N.C.)

Leger, G. 1234

A note on free Lie algebras.

Proc. Amer. Math. Soc. **15** (1964), 517–518.

For a free Lie algebra L and any L-module M, $H^2(L, M) = 0$. A partial converse is proved; viz., let L be a Lie algebra over a field K such that $H^2(L, K) = 0$, and let T be a subspace of L which generates L and such that $T \cap [L, L] = (0)$. Then L is free. *W. T. van Est* (Leiden)

Simonjan, L. A. 1235

On two radicals of Lie algebras. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 281–283.

Let L be a Lie algebra (of any dimension) over a field of characteristic zero. A subalgebra M of L is called subinvariant if there is an ascending normal series from M to L. If the series can be taken finite, M is called accessible. If M is generated by a single element x, then x is itself called subinvariant or accessible, respectively. If all the elements of L are subinvariant [accessible], L is called a B-algebra [SB-algebra]. For general L, the B-radical, $B(L)$, is defined as the sum of all B-ideals (i.e., ideals which are B-algebras), and the SB-radical, $SB(L)$, is defined correspondingly. The author outlines the proof of the following results: (i) $B(L)$ is a B-ideal, (ii) $SB(L)$ is an SB-ideal, (iii) denote by $LN(L)$ the locally nilpotent radical of L and by $RN^*(L)$ the RN^*-radical (formed from ideals having an ascending normal series with abelian factors), then $B(L) = LN(L) \cap RN^*(L)$. *P. M. Cohn* (Chicago, Ill.)

HOMOLOGICAL ALGEBRA

See also 1164, 1226, 1234, 1281.

Maranda, J.-M. 1236

Injective structures.

Trans. Amer. Math. Soc. **110** (1964), 98–135.

The object of this paper is to introduce a unified theory of quotient structures and completions of rings and modules. For example, suppose R is a ring and R' an extension. A left R-submodule B of R' is a weak left inverse [left inverse; inverse] of an ideal A of R if $BA \subset R \subset AB$ [$BA = R \subset AB$; $BA = R = AB$]. There is a canonical extension R_1 such that an ideal is weakly left invertible in some extension if and only if it is weakly left invertible in R_1. A smaller extension R_2 serves to test left invertibility. If an ideal is invertible, a left inverse is an inverse. The set of weakly left invertible [left invertible; invertible] ideals is closed under multiplication. Other results relate submodules purely with respect to a topologizing set of ideals to completions with respect to the associated topology.

The unifying notions are those of an injective structure in a category and of a radical of a category of modules. The former consists of a pair $(\mathfrak{S}, \mathfrak{Q})$, where $\mathfrak{S}$ is a set of maps, called proper morphisms, and $\mathfrak{Q}$ a set of objects, called injectives, such that $f: A \to B$ in $\mathfrak{S}$, $Q \in \mathfrak{Q}$ implies $\operatorname{Hom}(B, Q) \to \operatorname{Hom}(A, Q)$ is surjective. $\mathfrak{S}$ [$\mathfrak{Q}$] must consist of all f [Q] such that this is true for all $Q \in \mathfrak{Q}$ [$f \in \mathfrak{S}$]. Further, every object must map by a proper morphism into an injective.

Injective structures are closely related to "relative" abelian structures and like them may be transferred from one category to another by a functor having a left adjoint [cf. S. MacLane, *Categorical algebra*, Boulder, Col., 1963] or even, as the author observes, by a functor provided with a "left adjoint system" having only some of the properties of an adjoint functor. Indeed, any injective structure arises thus from an adjoint system to the inclusion of the full subcategory of injectives. If this adjoint system is an adjoint functor, the injective structure is called regular.

A radical of the category of right modules over a ring R is a subfunctor $\mathfrak{r}$ of the identity with the property $\mathfrak{r}(A/\mathfrak{r}A) = 0$, all A. $\mathfrak{r}$ is a torsion radical if it is left exact. If $(\mathfrak{S}, \mathfrak{Q})$ is an injective structure $\mathfrak{r}A = \ker \varphi$, where $\mathfrak{S} \in \varphi: A \to Q \in \mathfrak{Q}$, defines a radical associated with $(\mathfrak{S}, \mathfrak{Q})$. Torsion radicals always arise this way: let $\mathfrak{Q} = \{A \mid \mathfrak{r}A = 0\}$.

A submodule $A \subset B$ is $\mathfrak{r}$-dense if $\mathfrak{r}(B/A) = 0$. The family of $\mathfrak{r}$-dense right ideals in R determines $\mathfrak{r}$; appropriate families may be abstractly characterized. The applications to invertible ideals cited above arise in this way.

A regular injective structure in a category of modules determines an extension of the ground ring (i.e., a ring structure in the injective assigned to the ring by the adjoint functor) as well as a functor into the new category of modules. This is the construction which provides the quotients and completions discussed.

The restriction to categories of modules of the author's theory of radicals and construction of extensions is probably a strategic misfortune. The former can be generalized to abelian categories: so can the latter, using a quotient category invented by Grothendieck and heavily exploited in Gabriel's thesis [Bull. Soc. Math. France **90** (1962), 323–448], with which, as the author observes in proof, this paper has a considerable intersection. A good deal of

what is done here might then probably be exhibited as an application of Gabriel's theory of localizations.

A. Heller (Urbana, Ill.)

Govorov, V. E. 1237

Semi-injective modules. (Russian)

Algebra i Logika Sem. **2** (1963), *no.* 6, 21–50.

Bezeichne Λ einen assoziativen Ring mit Eins. Betrachtet werden nur unitäre Λ-Moduln. Ein Λ-Rechtsmodul A wird semi-injektiv (hier kurz: s.i.) genannt, wenn jeder Endomorphismus eines Teilmoduls von A induziert wird von einem Endomorphismus von A. Für $\Lambda = Z$ (Ring der ganzen Zahlen) wurde diese Eigenschaft bereits von V. Dlab [Czechoslovak Math. J. **10 (85)** (1960), 244–254; MR **22** #A8054] und A. P. Mišina [Vestnik Moskov. Univ. Ser. I Mat. Meh. **1962**, no. 4, 39–43; MR **26** #2499] untersucht. Der Definition ist jeder quasi-injektive Λ-Modul im Sinne von R. E. Johnson und E. T. Wong [J. London Math. Soc. **36** (1961), 260–268; MR **24** #A1295] sicher s.i. Dual zu s.i. wird s.p. (semi-projektiv) erklärt.

Theorem 1 [bzw. 1′]: Sei K eine Klasse von Λ-Rechtsmoduln derart, daß mit $A \in K$ auch $A \oplus \Lambda \in K$ [bzw. $A \oplus C \in K$ für jeden Λ-Modul C, dessen Erzeugendenzahl diejenige von A nicht übersteigt]. Genau dann sind alle Moduln aus K s.i. [bzw. s.p.], wenn sie injektiv [projektiv] sind. Theorem 2: Für Λ sind folgende Bedingungen äquivalent: (a) alle Λ-Rechtsmoduln sind quasi-injektiv, (b) alle endlich erzeugbaren (kurz: e.e.) Λ-Rechtsmoduln sind quasi-injectiv, (c) alle Λ-Rechtsmoduln sind s.i., (d) alle e.e. Λ-Rechtsmoduln sind s.i., (e) alle Λ-Rechtsmoduln sind s.p., (f) alle e.e. Λ-Rechtsmoduln sind s.p., (g) der Ring Λ ist halbeinfach.

Ein (nicht notwendig kommutativer) Körper Γ heißt zweiseitiger Quotientenkörper (z.Q.) von Λ, wenn $\Lambda \subseteq \Gamma$ und Γ sowohl Rechts- wie Linksquotientenkörper von Λ sind. Theorem 3: Ein nullteilerfreier Ring Λ ist genau dann s.i. als Λ-Rechtsmodul, wenn Λ einen z.Q. Γ besitzt und rechts vollständig ganz-abgeschlossen [O. Zariski und P. Samuel, *Commutative algebra*, Vol. II, Van Nostrand, Princeton, N.J., 1960; MR **22** #11006] in Γ ist. Theorem 4: Für einen Λ-Rechtsmodul A, der zugleich s.i. und s.p. ist, sind folgende Bedingungen äquivalent: (a) alle Teilmoduln von A sind s.p., (b) alle Restklassenmoduln von A sind s.i. Theorem 5: Wenn alle zyklischen Λ-Rechtsmoduln über einem nullteilerfreien Ring Λ s.i. sind, so ist Λ rechts und links semi-erblich (s.e.) im Sinne von A. Cartan und S. Eilenberg. Theorem 6: Wenn Λ links s.e. ist und einen z.Q. besitzt, so ist Λ auch rechts s.e. Theorem 7: Sei Λ ein nullteilerfreier rechts Noetherscher Ring. Genau dann sind alle zyklischen Λ-Rechtsmoduln s.i., wenn (1) Λ rechts erblich (im Sinne von Cartan-Eilenberg) ist und (2) $a\Lambda \subseteq \Lambda a$ für alle $a \in \Lambda$. Theorem 8: Ein direktes Produkt $\Lambda = \prod_\alpha \Lambda_\alpha$ von Ringen Λ_α ist genau dann ein s.i. Λ-Rechtsmodul, wenn alle Λ_α-Moduln Λ_α s.i. sind.

Der Ring Λ heißt D-Ring, wenn folgendes gilt: (1) Λ ist Primring; jedes zweiseitige Ideal von Λ ist als Λ-Bimodul (2) e.e. und (3) projektiv, (4) jeder Endomorphismus eines zweiseitigen Ideals I ($\subseteq \Lambda$) als Λ-Rechtsmodul (Λ-Linksmodul) ist fortsetzbar zu einem Endomorphismus von Λ (ebenfalls als ein solcher Modul betrachtet). Die kommutativen D-Ringe sind genau die Dederkindschen Ringe (denn 3) bedeutet, daß w.gl. dim $\Lambda \leq 1$ ist, und andererseits bedeutet: jedes Primideal ist maximal, daß die Krullsche Dimension von $\Lambda \leq 1$ ist). Theorem 9: Sei Λ ein D-Ring, $\mathfrak{A}$ die Menge der Klassen als Λ-Bimoduln isomorpher Ideale von Λ. Dann ist $\mathfrak{A}$ eine Gruppe bezüglich der Operation $\otimes_\Lambda$. (Im kommutativen Fall verwandelt sich diese Gruppe in die Gruppe der projektiven Klassen, vgl. D. S. Rim [Ann. of Math. (2) **69** (1959), 700–712; MR **21** #3474; Trans. Amer. Math. Soc. **98** (1961), 459–467; MR **23** #A1690].) Theorem 10: Der Matrizenring über einem D-Ring ist ein D-Ring. Theorem 11–13 betreffen die Übereinstimmung der projektiven e.e. mit den freien Moduln für gewisse Klassen von sogenannten rechts p-zusammenhängenden Ringen (Verallgemeinerung des Begriffs "p-connected" im Sinne von H. Bass [Illinois J. Math. **7** (1963), 24–31; MR **26** #1341]).

H.-J. Hoehnke (Berlin)

Isbell, J. R. 1238

Subobjects, adequacy, completeness and categories of algebras.

Rozprawy Mat. **36** (1964), 33 *pp.*

If $\mathscr{C}$ is a category with finite sums and $P \in \mathscr{C}$, then each map $a: P \to P \amalg \cdots \amalg P$ (n summands) becomes an n-ary operation on each of the sets Map (P, X). Thus $X \to$ Map(P, X) is a functor from $\mathscr{C}$ into a category $\mathscr{A}$ of algebras of a "type" determined by the maps a. These are algebras with infinitely many operations, though they may possibly be generated by some finite subset. Such a category of algebras is called quasi-primitive if it is full and closed under products and subalgebras.

Theorem: $X \to$ Map(P, X) is a full isomorphic functor onto a quasi-primitive category $\mathscr{A}$ if and only if (i) every family of objects of $\mathscr{C}$ has a sum, (ii) every pair of maps $f, g: X \to Y$ in $\mathscr{C}$ has an equalizer (= left root = projective limit), (iii) every pair f, g has a co-equalizer (the dual), (iv) P is projectively finite, (v) P is projective, (vi) P is a generator. This result is accompanied by a series of examples showing the independence of the several conditions on $\mathscr{C}$.

This central theorem, which is closely allied to one of Lawvere's [Ph.D. Dissertation, Columbia Univ., New York, 1963], depends on some delicate discussions of completeness and representability. For example, several sorts of completeness are studied. A "weak" left completeness is characterized by existence of products and equalizers, a "strong" one by the additional assertion that every contravariant polar ideal (= equationally defined subfunctor of some Map$(-, Y)$) is representable. It is shown that under suitable set-theoretic restrictions strong left completeness implies its dual.

Another example: If $F: \mathscr{C} \to \mathscr{S}$ (sets) then the dual (modulo logical difficulties) is $F^*X =$ Map$(F,$ Map$(X, -))$. There is a canonical $F \to F^{**}$; F is reflexive if this is an equivalence. If $\mathscr{C}$ has sums and products and every map factors into an epimorphism followed by a monomorphism, then reflexive functors are representable.

Delicate set-theoretical considerations are pervasive, since the author is unwilling to assume that his categories are small. They turn up, especially in some extensions of his earlier results [Illinois J. Math. **4** (1960), 541–552] on adequacy of subcategories, as, in particular, in the odd result that, assuming that no measurable cardinals exist, the dual of a full category of algebras is a full category of algebras.

A. Heller (Urbana, Ill.)

Higgins, Philip J. 1239
Algebras with a scheme of operators.
Math. Nachr. **27** (1963), 115–132.

The type of algebraic system which the author considers is made up of a family $\{A_i\}$ of sets, together with a family of maps $A_{i_1} \times \cdots \times A_{i_n} \to A_i$. This generalization of the more usual structure in which all A_i are the same is designed to include, inter alia, graded systems, directed systems and categories.

A scheme of operators consists of an index set, a set of "operators" and the assignment to each "operator" of a sequence $(i_1, \cdots, i_n, i)$ of indices. The notion of an algebra over the scheme is clear from the remark above. That of the free algebra of "words" may be defined and, taking some set of words as relations, that of an algebra over the scheme satisfying some family of relations. As might be expected, these form a category which may be characterized as a subcategory of the category of all algebras over the scheme by means of "variety" conditions of the type familiar from group theory. It is further proved that any such variety contains "free" algebras satisfying the usual conditions of universality.

A homomorphism of operator-schemes is defined in the evident fashion, as also the notion of a homomorphism of algebras subordinate to such a homomorphism of schemes. The corresponding universal problem has a solution, leading to a functor from the first category to the second. Applications are to tensor algebras, enveloping algebras of Lie rings and direct limits.

It would be interesting to see the connection with the more abstract theory of Lawvere [Ph.D. Dissertation, Columbia Univ., New York, 1963] and Isbell [#1238 above]. *A. Heller* (Urbana, Ill.)

Shukla, U. 1240
A relative cohomology for associative algebras.
Proc. Amer. Math. Soc. **15** (1964), 461–469.

Following the idea used in his previous work [Ann. Sci. École Norm. Sup. (3) **78** (1961), 163–209; MR **24** #A2605], a relative cohomology $H^*(\Lambda, \Gamma; M)$ associated with the K-algebra map $(f: \Gamma \to \Lambda)$ and Λ-bimodule M is constructed, where K is an arbitrary commutative ring. However, no comparison theorem with Hochschild's relative cohomology [G. Hochschild, Trans. Amer. Math. Soc. **82** (1956), 246–269; MR **18**, 278] nor with the author's absolute cohomology (when $\Gamma = K$) is given.
D. S. Rim (Waltham, Mass.)

Lambek, J. 1241
A module is flat if and only if its character module is injective.
Canad. Math. Bull. **7** (1964), 237–243.

The purpose of this expository note is to establish the fact mentioned in the title. *A. Heller* (Urbana, Ill.)

Williamson, Susan 1242
Crossed products and hereditary orders.
Nagoya Math. J. **23** (1963), 103–120.

Let R be a rank one discrete valuation ring. Let S be the integral closure of R in a finite Galois extension of the quotient field of R with Galois group G. For f any 2-cocycle of G in $U(S)$, the group of units of S, define $D(f, S, G)$ to be the free S-module on generators u_a, a in G, with multiplication given by $su_a tu_b = sa(t)f(a, b)u_{ab}$, i.e., a crossed product. It has been shown by M. Auslander and D. S. Rim [Illinois J. Math. **7** (1963), 566–581; MR **27** #5787] that if S is tamely ramified, then $D(f, S, G)$ is a hereditary order, and $D(1, S, G)$ is a maximal order if and only if S is unramified. This is generalized here by showing that S is tamely ramified if and only if $D(f, S, G)$ is a hereditary order for all f, which in turn is equivalent to $D(1, S, G)$ being a hereditary order. Next, a certain subgroup, $H(f)$, of the inertial group of a maximal ideal of S is defined for $D(f, S, G)$, and it is shown that if S is tamely ramified, then the order of $H(f)$ is equal to the number of maximal two-sided ideals of $D(f, S, G)$. In particular, the number of maximal two-sided ideals of $D(1, S, G)$ is the ramification index of R in S. Finally, it is shown that if M is any maximal order containing $D(f, S, G)$ and C is the conductor of M in $D(f, S, G)$, then the length of $S/C \cap S = g(h-1)$, where h is the order of $H(f)$ and g is the number of maximal two-sided ideals of S.
A. Rosenberg (Ithaca, N.Y.)

Jans, James P. 1243
★Rings and homology.
Holt, Rinehart and Winston, New York, 1964. vii + 88 *pp.* $3.25.

This monograph is an introduction to rings and homology theory at an elementary level. It contains a standard treatment of semi-simple rings, an introduction of the Ext functor and homological dimension, and ends with a discussion of quasi-Frobenius rings.

Though the author's choice of content cannot escape some criticism, the book itself is smoothly written and can be read by any beginning student.
D. S. Rim (Waltham, Mass.)

GROUP THEORY AND GENERALIZATIONS

See also 1223, 1228, 1289, 1332.

Murasugi, Kunio 1244
The center of a group with a single defining relation.
Math. Ann. **155** (1964), 246–251.

Let G be a group with one single defining relation R. The author determines the center of G. The following interesting theorems are established: (1) If G has more than two generators, then the center of G is trivial; (2) If G has two generators, then the center of G is either trivial or free cyclic. A third theorem gives some properties of the relation R if the center is not trivial.

The proof is based upon some work of A. Karrass, W. Magnus, and D. Solitar [Comm. Pure Appl. Math. **13** (1960), 57–66; MR **23** #A1696].

Let a be a generator occurring in R with exponent sum 0, and let H be the normal subgroup of G generated by the other generators. The above authors showed that H can be characterised as follows. Introduce certain groups H_k $(k = 0, \pm 1, \cdots)$ which have a single defining relation. By using the "Freiheitssatz" and "Hauptform" [W. Magnus, J. Reine Angew. Math. **163** (1930), 141–165], one can show that the H_k are subgroups of H. The union of H_k and H_{k+1} is a free product with amalgamation J_k, the latter

being a free group. The whole group H can be covered by unions of groups H_k.

Using this characterisation, the question on the center is reduced to the case of free products and free groups, where it can be solved directly.

J. Mennicke (Wolfenbüttel)

Piccard, Sophie 1245

Les éléments quasi libres des groupes quasi libres.

C. R. Acad. Sci. Paris **258** (1964), 1968–1970.

The author continues her study of quasi-free groups, a concept she has defined previously [same C. R. **250** (1960), 3260–3262; MR **22** #5666]. Among other things, conditions under which an element of a quasi-free group may be quasi-free are discussed, and a method of obtaining new systems of quasi-free generators from a given system is described. Other related work of the author has appeared [ibid. **251** (1960), 2271–2273; MR **22** #8051].

R. R. Struik (Pueblo, Colo.)

Lyndon, Roger C. 1246

Length functions in groups.

Math. Scand. **12** (1963), 209–234.

The aim of this paper is to isolate certain ideas underlying the cancellation arguments in Nielsen's proof of the subgroup theorem for free groups [J. Nielsen, Math. Scand. **3** (1955), 31–43; MR **17**, 455]. To this end the author considers a group G equipped with a length function, assigning to each x in G as length a non-negative integer $|x|$, which satisfies the following five axioms. Let $d(x,y)=\frac{1}{2}[|x|+|y|-|xy^{-1}|]$. (A1) $|x|=0\Leftrightarrow x=1$; (A2) $|x^{-1}|=|x|$; (A3) $d(x,y)\geqq 0$; (A4) $d(x,y)<d(x,z)\Rightarrow d(y,z)=d(x,y)$; (A5) $d(x,y)+d(x^{-1},y^{-1})>|x|\,|y|\Rightarrow x=y$ (in practice (A4) often takes the form that $d(x,y)\geqq m$, $d(x,z)\geqq m$ implies $d(y,z)\geqq m$). An example of such a group is a subgroup of a free group or a free product of a family of subgroups. If F is a free group with a set X of free generators, then the usual length function assigning to each x the length of its normal form with respect to X satisfies (A1)–(A5) and also (A0): $x\neq 1\Rightarrow |x|<|xx|$. An element x in G such that $|x|<|xx|$ is called Archimedean. If F^* is a free product of a family of subgroups with pairwise trivial intersections, then the usual length function with respect to the free decomposition satisfies (A1)–(A5) but not (A0). Thus for any subgroup G of F or F^* the restriction of the natural length function to G satisfies (A0)–(A5) or (A1)–(A5).

The main theorem is as follows: If G is a group equipped with a length function satisfying (A1)–(A5), then G is a free product of a free subgroup and a family of non-Archimedean subgroups. As corollaries the Nielsen subgroup theorem for a free group and the Kurosh subgroup theorem for a free product are obtained. In the last section the embedding problems are discussed. For instance, it is proved that a group G with a length function satisfying (A0)–(A5) and (CO): $d(x,y)$ is always an integer and can be embedded in a free group F in such a way that the natural length function of F restricted to G is the same as the given one.

H. Nagao (Osaka)

Baumslag, Gilbert; Boone, W. W.; 1247
Neumann, B. H.

Some unsolvable problems about elements and subgroups of groups.

Math. Scand. **7** (1959), 191–201.

Diese leicht lesbare und lehrreiche Note vereinigt mehrere interessante Bemerkungen zur Gruppentheorie, die im wesentlichen einfache Korollare der algorithmischen Unlösbarkeit des Wortproblems für Gruppen, genauer für eine gewisse endlich angegebene Gruppe sind. So werden mehr als 20 spezifische Eigenschaften von Worten, bzw. von durch endliche Mengen solcher Worte erzeugten Untergruppen aufgezählt, die ebenfalls in diesem starken Sinn algorithmisch unlösbar sind. Es wird auch ein allgemeiner, eigentlich metamathematischer Satz über algorithmisch unentscheidbare (= unlösbare) Eigenschaften, ähnlich Sätzen von Adjan [Dokl. Akad. Nauk SSSR **103** (1955), 533–535; MR **18**, 455; Trudy Moskov. Mat. Obšč. **6** (1957), 231–298; MR **20** #2370] und Rabin [Bull. Amer. Math. Soc. **62** (1956), 396; Ann. of Math. (2) **67** (1958), 172–194; MR **22** #1611] angegeben, aus dem viele der Einzelergebnisse folgen. Als "Kuriosum" oder zur Abwechslung wird auch eine algorithmisch entscheidbare Eigenschaft erwähnt, nämlich, ob ein vorgegebenes Wort ein zur Kommutatoruntergruppe G' von G gehöriges Element darstellt. (Als endlich angegebene abelsche Gruppe hat G/G' natürlich ein lösbares Wortproblem.) Die einfacheren Beweise sind oft von folgender Art: Man bildet das freie Produkt einer endlich angegebenen Gruppe mit algorithmisch unlösbarem Wortproblem (w ist eine Variable, die alle ihre Worte durchläuft) mit einer einfachen "ad hoc"-Gruppe. Aus dem freien Produkt greift man eine geeignete Wortform mit w als Parameter heraus, so daß dieses Wort die betreffende Eigenschaft hat, bzw. nicht hat, dann und nur dann, wenn $w=1$, bzw. $w\neq 1$. In diesem Zusammenhang wird auch gezeigt, daß es Untergruppen endlich angegebener Gruppen gibt, die zwar nur endlich viele Erzeugende, aber ein wesentlich unendliches System von definierenden Relationen haben. Schließlich wird noch bemerkt, daß es eine endlich angegebene Gruppe gibt, für die es algorithmisch unentscheidbar ist, ob ein beliebig vorgegebener Automorphismus ein innerer Automorphismus ist. {Die Verfasser scheinen über ihre sehr endliche Fülle algorithmisch unlösbarer Probleme erstaunt zu sein. Die Lage ist jedoch in mancher Beziehung wie bei den transzendenten Zahlen: nachdem einmal ihre Existenz feststand, war auch klar, daß sie den allgemeinen Fall darstellen, was dann noch durch die Kantorschen Kardinalbetrachtungen erhärtet wurde.}

D. Tamari (Zbl **104**, 7)

Mendelsohn, N. S. 1248

An algorithmic solution for a word problem in group theory.

Canad. J. Math. **16** (1964), 509–516.

Let G be a group generated by $g_1, g_2, \cdots, g_r$ and defined by relations $R_1=R_2=\cdots=R_k=I$ (where each R_i is a word in the generators and their inverses). Let H be a subgroup generated by the words $W_1, W_2, \cdots, W_t$. The well-known Todd-Coxeter enumeration process [J. A. Todd and H. S. M. Coxeter, Proc. Edinburgh Math. Soc. (2) **5** (1936), 26–36; see also H. S. M. Coxeter and W. O. J. Moser, *Generators and relations for discrete groups*, pp. 12–18, Springer, Berlin, 1957; MR **19**, 527] enumerates, in a systematic way, the cosets of H in G. One of the many uses made of this process is the following [e.g., see J. Leech, Proc. Glasgow Math. Assoc. **5** (1962), 166–175; MR **26** #2494]. If the tables close up, then $[G:H]$ is finite, say u. In this case the tables can be used to choose (in an obvious way) a set of representatives $1, h_2, \cdots, h_u$ of the

cosets of H, and also to determine, for any element x (a word in the generators g_i and their inverses) of G, the representative, denoted by $\phi(x)$, of the coset to which x belongs, i.e., $\phi(x)=h_i$.

Schreier [see M. Hall, Jr., *The theory of groups*, pp. 94–96, Macmillan, New York, 1959; MR **21** #1996] observed that if $[G:H]$ is finite, then the elements $\alpha_{rs}=h_r g_s\{\phi(h_r g_s)\}^{-1}$ generate H, for if $x=a_1a_2\cdots a_m$ (where each $a_i=g_j$ or g_j^{-1} for some j) and we write $k_0=1$, $k_i=\phi(a_1a_2\cdots a_i)$ for $i=1,2,\cdots,m$, then $x=U\cdot\phi(x)$, where $U=(k_0a_1k_1^{-1})(k_1a_2k_2^{-1})\cdots(k_{m-1}a_mk_m^{-1})$, and each $k_i^{-1}a_ik_i$ is an α_{rs}. Thus, as the author of the paper under review points out, if the enumeration tables close up, then the $\phi(x)$ obtained from them can be used to find U explicitly.

It is therefore of interest to know conditions which insure that the tables close up. J. Leech [Proc. Cambridge Philos. Soc. **59** (1963), 257–267; MR **26** #4513] gave an argument which showed that the following conjecture is reasonable: If $[G:H]$ is finite, then the tables close up. In the present paper the author shows that this conjecture is a consequence of the Lemma: After a finite number of steps (in "building" the tables) the first r rows of all the tables are "stabilized", i.e., none of the entries are further altered because of redundancy. (A "redundancy" occurs when the tables imply that two previously defined cosets are in fact the same.) His proof of the Lemma is brief and, unfortunately, incorrect. The "proof" given is: "Suppose that at a given stage, S is the sum of the distinct integers which appear in the first r rows of the tables. If a redundancy alters the first r rows, its effect is to replace all occurrences of a certain integer by a smaller integer. Hence S is decreased. But $S\geqq\frac{1}{2}r(r+1)$. Then, after a finite number of redundancies, either S is reduced to $\frac{1}{2}r(r+1)$ in which case closure occurs, or else redundancies no longer affect the first r rows."

The effect described in the second sentence of this proof may not occur. Consider what happens if the redundancy is that the previously defined cosets k and m (with $k<m$) are the same. (The tables are set up so that for every i, the ith row begins with i and ends with i.) The tables are adjusted by replacing every occurrence of m by k, then wiping out the mth row, and then replacing each $n>m$ by $n-1$. Now if $m\leqq r$, the rth row of the adjusted tables will be the $(r+1)$st row of the unadjusted tables, and the "new" rth row may well contain integers which are larger than those which appeared in the "old" rth row, in which case the sum S could conceivably be increased.

The difficult problem of programming an electronic computer for execution of the coset enumeration process has received much attention in recent years. The work of Bandler, Felsch, Leech, Haselgrove and Maddison is reviewed by Leech [ibid. **59** (1963), 257–267; MR **26** #4513; see also J. Neubüser, Numer. Math. **2** (1960), 280–292; MR **22** #8713]. Still more remarkable work by Trotter appeared in Canad. Math. Bull. **7** (1964), 357–368.

W. Moser (Montreal, Que.)

Brumberg, N. R. 1249

The connection of wreath products of groups with other operations on groups. (Russian)

Sibirsk. Mat. Ž. **4** (1963), 1221–1234.

Let G be the wreath product of A by B, i.e., $G=\langle K, B; b^{-1}a(1)b=a(b),\ a\in A,\ b\in B\rangle$, where $K=\prod_{b\in B}A(b)$, each $A(b)$ is a copy of A and $a(b)$ denotes the element of $A(b)$ corresponding to the element a of A. Theorem 1: G is a regular product (as defined by O. N. Golovin) of $A(1)$ and B (i.e., $A(1)$ and B generate G, $A(1)$ intersects the normal closure of B in the identity, and B intersects the normal closure of $A(1)$ in the identity) if and only if A is Abelian. Theorem 2: G satisfies Mal'cev's postulate, i.e., the group generated by subgroups of $A(1)$ and B is their wreath product. Theorem 3: G satisfies MacLane's postulate if and only if A is Abelian. Theorem 4: G satisfies Golovin's postulate on embedding groups of automorphisms. Theorem 5: G is freely indecomposable. Theorem 6: G is directly decomposable if and only if every element of A, or some direct factor of A, has a unique nth root, where B is a finite group of order n.

M. Greendlinger (Ivanovo)

Baer, R. 1250

Groups with minimum condition.

Acta Arith. **9** (1964), 117–132.

If the minimum condition is satisfied by the subgroups of a group G, then we have the following two consequences: (1) The minimum condition is satisfied by the normal subgroups of G; (2) If $H\neq 1$ is an epimorphic image of G, then there exists a normal subgroup $N\neq 1$ of H such that the minimum condition is satisfied by the group of automorphisms induced in N by H. The author gives solutions of the problem to complete these conditions into a criterion for the validity of the minimum condition for subgroups. He also considers those groups which have Abelian subgroups of finite index in connection with his above studies.

K. Honda (Tokyo)

Honig, Chaim 1251

Sur les groupes sans torsion.

C. R. Acad. Sci. Paris **258** (1964), 1679–1682.

The classification theory of torsion-free abelian groups of finite rank is classical, mainly due to Kuroš [Ann. of Math. (2) **38** (1937), 175–203]. It is, for example, given in § 38 (for groups of rank 1) and in §§ 40–41 (for groups of arbitrary finite rank) in the book of Kuroš [*Gruppentheorie*, Akademie-Verlag, Berlin, 1953; MR **15**, 681]. In the present paper the author gives a new method for the theory, which is, roughly speaking, a generalization of the method of § 38 for groups of rank 1 to arbitrary finite rank case. Detailed proofs are not given here.

N. Ito (Nagoya)

Abian, Alexander 1252

On the lengths of bases of a finitely generated abelian group.

Boll. Un. Mat. Ital. (3) **19** (1964), 12–15.

The obvious upper and lower bounds are given for the number of basis elements of a finitely generated abelian group.

L. Fuchs (Budapest)

Gorčakov, Ju. M. 1253

On the existence of Abelian subgroups of infinite rank in locally solvable groups. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 17–20.

The paper contributes to a series of results started by a paper of A. Mal'cev [Mat. Sb. (N.S.) **28** (**70**) (1951), 567–588; MR **13**, 203] on the rank of Abelian subgroups of

locally soluble groups of finite rank. In 1960 V. S. Čarin [ibid. (N.S.) **52 (94)** (1960), 895–914; MR **23** #A213] proved that a locally soluble torsion-free group whose Abelian subgroups have bounded finite rank has finite rank. Later M. I. Kargapolov [Algebra i Logika Sem. **1** (1962), no. 5, 37–44; MR **27** #5822] proved that a soluble group has finite rank if all its Abelian subgroups have finite ranks. The author proves that a locally soluble group has finite rank if all its Abelian subgroups have finite ranks. The proof is based on the following interesting lemma: If r is the rank of a maximal normal Abelian subgroup of a finite p-group P, then the rank of P is not greater than $3r^2$. *A. Hulanicki* (Wrocław)

Simon, Hermann 1254
Noethersche Gruppen mit nilpotenten Normalteilern von endlichem Index.
Illinois J. Math. **8** (1964), 241–247.

In questo lavoro si studia il problema analogo a quello del lavoro precedente, per E = "G ha un sottogruppo normale, nilpotente di classe finita, di indice finito in G". Si tengano presenti le definizioni relative al lavoro precedente. Un gruppo G è detto pressochè-pseudoomogeneo ("fast-pseudohomogen") quando è dato un numero naturale n e un insieme infinito Φ di numeri primi, tali che, se E è un fattoriale finito di G, e P è un p-sottogruppo normale di E ($p \in P$), l'ordine del gruppo degli automorfismi indotto in P da E è prodotto di un divisore di n per una potenza di p. Un gruppo è poi detto pressochè-quasiomogeneo quando ogni suo sottogruppo è pressochè-pseudoomogeneo. Il teorema principale asserisce che per un gruppo G le tre condizioni seguenti sono equivalenti: (I) G è noetheriano e ha un sottogruppo normale, nilpotente di classe finita, di undice finito in G; (II) G ha un sistema finito di generatori, e ogni immagine omomorfa $H \neq 1$ di G contiene un elemento $e \neq 1$ che possiede solo un numero finito di coniugati in H; (III) G ha un sistema finito di generatori è pressochè-quasiomogeneo, e ha un sottogruppo t-quasirisolubile di indice finito in G. [Svista: a pag. 241, riga 2, in luogo di "endlich erzeugte E-Gruppen endlich sind" occorre leggere "endlich erzeugte E-Gruppen noethersch sind".] *G. Zappa* (Florence)

Lazard, Michel 1255
Sur les séries de Golod-Chafarévitch des pro-p-groupes de type fini.
C. R. Acad. Sci. Paris **258** (1964), 4403–4406.

Let $G = \text{proj lim } G_\alpha$ be an inverse limit of finite p-groups, and assume G has a finite number of generators (i.e., elements whose images generate each G_α). Let F be the field with p elements; let Alp $G = \text{proj lim } F[G_\alpha]$ be the limit of the group algebras and $\text{Alp}_n\, G$ be the nth power of its radical. Golod and Shafarevich (unpublished) introduced the numbers $a_n = \dim_F \text{Alp}_n\, G/\text{Alp}_{n+1}\, G$ as important invariants of G. The author here announces without proof that G is analytic if and only if $\log a_n = O(\log n)$; related statements about the series $\sum a_n T^n$ are also given. *A. Mattuck* (Cambridge, Mass.)

Thompson, R. C. 1256
Unimodular group matrices with rational integers as elements.
Pacific J. Math. **14** (1964), 719–726.

The reviewer and Olga Taussky proved [Comm. Pure Appl. Math. **9** (1956), 85–91; MR **17**, 829] that if A is a unimodular matrix of rational integers such that AA' is a circulant, then A must be of the form A_1T, where A_1 is a circulant of rational integers and T is a generalized permutation matrix. The author generalizes this result to the case where A is a group matrix of rational integers for a finite solvable group G. The conclusion then reads that A must be of the form A_1T, where A_1 is also a group matrix of rational integers for G and T is a generalized permutation matrix. The method makes use of the left and right regular representations of G: The fundamental point is that any matrix in the group ring of the left regular representation commutes with any matrix in the group ring of the right regular representation, and any matrix commuting with all matrices in the right regular representation must belong to the group ring of the left regular representation. *Morris Newman* (Washington, D.C.)

Berkovič, Ja. G. 1257
Some generalizations and applications of a theorem of Suzuki. (Russian)
Mat. Sb. (N.S.) **64 (106)** (1964), 357–377.

The author gives various characterizations of the non-soluble groups LF(2, 5), SL(2, 5), LF(2, 7), LF(2, 11) and SL(2, 11). For example, if G is a non-soluble finite group all of whose non-nilpotent soluble subgroups are minimal non-nilpotent groups, then G is isomorphic to LF(2, 5) or SL(2, 5). *Z. Janko* (Canberra)

Berkovič, Ja. G. 1258
Some criteria for π-solvability and π-simplicity of finite groups. (Russian)
Sibirsk. Mat. Ž. **5** (1964), 493–499.

Comparison of the number of certain non-normal subgroups of the finite group G with certain elementary functions on the order of G leads to the criteria mentioned in the title. Let us mention only the simplest (in formulation), although here the statement is not relativised by a set of primes π: If the number of non-normal subgroups of the finite group G is strictly less than 2 plus twice the number of different prime divisors of the order of G, then G is the simple group of order 60.
O. H. Kegel (Frankfurt a.M.)

Merzljakov, Ju. I. 1259
Locally solvable groups of finite rank. (Russian)
Algebra i Logika Sem. **3** (1964), no. 2, 5–16.

A solvable group G is said to be of finite rank if there is an invariant chain in G such that the quotients of this chain are abelian groups of finite rank. It is known that a solvable group G is of finite rank if all abelian subgroups of G are of finite rank. A locally solvable group is of finite rank if there is a local system all groups of which are of finite rank. If there is a bound to these ranks, the locally solvable group will be called of bounded rank. The main result of this paper is that a locally solvable group is of bounded rank if there is a bound to the ranks of its abelian subgroups. In the proof, an estimate for the relation between the two bounds is derived.
O. H. Kegel (Frankfurt a.M.)

Smirnov, D. M. 1260
Generalized soluble groups and their group rings. (Russian)
Dokl. Akad. Nauk SSSR **155** (1964), 535–537.

A variety of groups $\mathfrak{M}$ is termed normal if, for every non-abelian free group F and every proper non-trivial normal subgroup A of F, the verbal subgroup $V(\mathfrak{M}, A)$ is a proper subgroup of $V(\mathfrak{M}, F)$. Theorem 1: Let $\mathfrak{M}$ be a normal variety of groups, F a nonabelian free group, A a non-trivial normal subgroup of F. If F has a normal system $\mathfrak{S}=\{F_\alpha\}$ $(1\leqq\alpha\leqq\gamma,\ F_1=F,\ F_\alpha=A)$ connecting F to A and such that all its factors belong to $\mathfrak{M}$, then F also has a normal system $\mathfrak{S}^*$ connecting F to $V(\mathfrak{M}, A)$ and such that all its factors are subgroups of free $\mathfrak{M}$-groups. Such a system $\mathfrak{S}^*$ can be obtained by completing with respect to intersections the system $\mathfrak{S}'=\{V_\alpha\}$ $(0\leqq\alpha\leqq\gamma)$, where $V_0=F$, $V_\alpha=V(\mathfrak{M}, F_\alpha)$ $(1\leqq\alpha\leqq\gamma)$. {The reviewer has been informed of two unpublished results which are relevant: According to Peter M. Neumann, all varieties are normal except for the variety of all groups; and a result of M. J. Dunwoody contains the fact that the system $\mathfrak{S}'$ above is itself already complete with respect to intersections.} Theorem 2: If a factor group F/A of a free group F is a T-group ($T=SN$, SI, SN^*, SI^*, SD, or soluble [in the terminology of Kuroš's book *The theory of groups* (Russian), 2nd ed., GITTL, Moscow, 1953; MR **15**, 501]), then $F/[A, A]$ has a normal system of type T whose factors are all free abelian groups. Corollary: If a factor group F/A of a free group F is an SN-group and if $A^{(n)}$ denotes the nth derived group of A $(n>0)$, then $F/A^{(n)}$ can be ordered, and its group ring $P[F/A^{(n)}]$ over an arbitrary field P has no zero divisors. Theorem 3: If G is a torsion-free locally radical group (in the sense of Plotkin [Mat. Sb. (N.S.) **37 (79)** (1955), 507–526; MR **17**, 709; Amer. Math. Soc. Transl. (2) **17** (1961), 9–28; MR **23** #A1712]), then the following three assertions are equivalent: (I) The integral group ring $Z[G]$ of G is isomorphic to a matrix ring over some field; (II) For any free presentation $G\cong F/A$ of G, the group $F/[A, A]$ is isomorphic to a matrix group over some field; (III) G is a finite extension of an abelian group. Theorem 4: Let G_n denote a free group {presumably of rank at least 2} in the variety of soluble groups of derived length at most n. Neither G_n $(n\geqq 3)$ nor $Z[G_2]$ admits faithful matrix representations over any field. However, G_2 is isomorphic to a matrix group over a purely transcendental extension of the rational field. *L. G. Kovács* (Canberra)

Rukolaĭne, A. V. 1261
Some arithmetic properties of modular characters of p-solvable groups. (Russian)
Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 571–582.

Let $\mathfrak{G}$ be a finite p-solvable group, that is, a group admitting a normal series whose factors are either p-groups or groups of order prime to p. Here p is a prime number, and the order g of $\mathfrak{G}$ is $g=p^a g'$, where $(p, g')=1$. The main results of the author are the following. (1) If Φ_1 is the principal indecomposable character corresponding to the 1-character of $\mathfrak{G}$, then the values of Φ_1 on p-regular elements are 0 or powers of $p\leqq p^a$. (2) For any principal indecomposable character Φ_i and any irreducible modular character φ_i, consider the class functions of $\mathfrak{G}$ defined by

$$\omega_i'(G)=\frac{g}{c(G)}\frac{\varphi_i(G)}{\varphi_i(1)},\qquad \omega_i''(G)=\frac{g}{c(G)}\frac{\Phi_i(G)}{\Phi_i(1)}.$$

Then the values of ω_i', ω_i'' are algebraic integers. (3) Any irreducible modular character of $\mathfrak{G}$ is the restriction of some complex irreducible character to the p-regular elements. (This was also discovered by R. G. Swan [Topology **2** (1963), 85–110; MR **27** #3683].) *P. Fong* (Berkeley, Calif.)

Hall, P. 1262
A note on $\overline{SI}$-groups.
J. London Math. Soc. **39** (1964), 338–344.

Let p be an odd prime and let R be the ring of p-adic integers. Let $h\geqq 0$, $k>0$, $l>0$ be rational integers such that $h+k\geqq l$. Let $M=M_{h,k,l}$ be the group of all matrices $\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ such that $a, b, c, d\in R$, $ad-bc=1$, $b\equiv 0 \bmod p^h$; $c\equiv 0 \bmod p^k$; $d\equiv 1 \bmod p^l$. Now let $K\neq 1$ be a normal subgroup of M. Then the author proves that M/K is a finite p-group. Now a group is called an $\overline{SI}$-group if and only if every chief factor is abelian. Thus M is an $\overline{SI}$-group. But $F=\left\langle\begin{pmatrix}1 & p^m\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ p^m & 1\end{pmatrix}\right\rangle$ is contained in M for all large m, which is free by a theorem of Brenner [C. R. Acad. Sci. Paris **241** (1955), 1689–1691; MR **17**, 824]. Since F is not an $\overline{SI}$-group, the property $\overline{SI}$ is not hereditary to subgroups. This settles the question X of A. G. Kuroš and S. N. Černikov [Uspehi Mat. Nauk **2** (1947), no. 3 (19), 18–59; MR **10**, 677] negatively. (A similar example is also found by Ju. I. Merzljakov [Algebra i Logika Sem. **2** (1963), no. 5, 29–36; MR **28** #4030].)

Next, the author proves many interesting properties of M; in particular, M/K is the product of three cyclic subgroups (tricyclic). Let $T(p)$ be the class of all tricyclic p-groups and $RT(p)$ the class of all groups H such that for any $x\neq 1$ there exists a normal subgroup K of H such that H/K is tricyclic and $x\notin K$. Then the author proves that every free group belongs to $RT(p)$ for $p>3$.

Finally the author proves the following theorem: Let X be any class of groups which contains the class $RT(p)$ for some prime $p>3$, and suppose that not every countable group lies in X. Then $X>QX>SQX$. Here QX denotes the class of all X-groups H such that all factor groups of H are X-groups, and SQX denotes the class of all QX-groups H such that all subgroups of H are QX-groups. *N. Ito* (Nagoya)

Christensen, Carlton 1263
Complementation in groups.
Math. Z. **84** (1964), 52–69.

The main result of this paper is a characterization of finite metabelian groups in which every normal subgroup is complemented; they are precisely those splitting extensions of one abelian group by another in which every Sylow subgroup is elementary abelian. On the way, the author obtains a number of results on related types of groups. For instance, he obtains a necessary and sufficient condition (which, in his own words, "cannot claim to be a characterization") for a finite soluble group to have all its normal subgroups complemented, and proves that a soluble group has all its characteristic subgroups as direct factors if and only if it is a direct product of characteristically simple abelian groups. {In the statement of lemma 3.7 "infinite" should read "finite".} *M. F. Newman* (Canberra)

Zheludev, I. S. [Želudev, I. S.] 1264
Cubic and ultimate groups of complete symmetry.
Proc. Indian Acad. Sci. Sect. A **59** (1964), 191–202.
Author's summary: "It is shown that the systems of definite actions described by polar and axial tensors of the second rank and their combinations during the superposition of their elements of complete symmetry with the elements of complete symmetry of the 'grey' cube, result in 11 cubic crystallographical groups of complete symmetry. There are 35 ultimate groups (i.e., the groups having the axes of symmetry of infinite order) in complete symmetry of finite figures. 14 out of these groups are ultimate groups of symmetry of polar and axial tensors of the second rank and 24 are new groups. All these 24 ultimate groups are conventional groups since they cannot be presented by certain finite figures possessing the axes of symmetry $\overline{\infty}$, $\overline{\underline{\infty}}$, $\underline{\infty}$. Geometrical interpretation for some of the groups of complete symmetry is given. The connection between complete symmetry and physical properties of the crystals (electrical, magnetic and optical) is shown."

Newman, Morris 1265
A complete description of the normal subgroups of genus one of the modular group.
Amer. J. Math. **86** (1964), 17–24.
Die Modulgruppe Γ besteht aus allen linear-gebrochenen Abbildungen der oberen Halbebene H mit ganzen Koeffizienten und der Determinante 1 und ist freies Produkt der zyklischen Gruppen $\{x\}$, $\{y\}$,

$$x:\tau\to-\frac{1}{\tau}\qquad y:\tau\to\frac{-1}{\tau+1}.$$

Das Geschlecht eines Normalteilers G von endlichem Index ist das topologische Geschlecht des kompaktifizierten Quotientenraumes H modulo G. Verfasser stellt gruppentheoretische Untersuchungen für die Normalteiler G vom Geschlecht eins an. Unter Verwendung eines Satzes von Nielsen wird zunächst $\Gamma'\supseteq G\supset\Gamma''$ gezeigt, wobei Γ' bzw. Γ'' die erste bzw. zweite Kommutatorgruppe von Γ bedeutet. Als Hauptergebnis bekommt der Verfasser eine eineindeutige Zuordnung zwischen diesen Gruppen G und drei ganzzahligen Parametern p, m, d mit $p>0$, $0\leqq m\leqq d-1$, $m^2+m+1\equiv 0 \pmod d$. Diese Zuordnung wird explizit gegeben durch die Restklassenzerlegung

$$G=\sum A^kB^l\Gamma'',\quad A=a^pb^{mp},\quad B=b^{dp},$$
$$a=xyxy^2,\quad b=xy^2xy.$$

Der Index $[\Gamma:G]$ ergibt sich zu $6dp^2$, und es wird eine explizite Formel in diesen Parametern für die Anzahl der Normalteiler von Γ von gegebenem Index und Geschlecht eins angegeben. Da Γ′ eine freie Gruppe vom Rang 2 und $[\Gamma:\Gamma']=6$ ist, folgt nach Nielsen-Schreier, dass G frei und vom Rang $1+dp^2$ ist. *H. Klingen* (Freiburg)

Andreadakis, S. 1266
On some series of groups of automorphisms of groups.
Bull. Soc. Math. Grèce (N.S.) **4** (1963), *no.* 1, 111–120.
The paper deals with some series of subgroups of the group A of automorphisms of a group G related to the upper and the lower central series of G, G_1, G_2, $\cdots$ and Z_1, Z_2, $\cdots$, respectively. Among others the following groups are investigated: $K_n=\{\alpha\in A: x\equiv x^\alpha \bmod G_{n+1}\}$, $T_n=\{\alpha\in A: x^\alpha\equiv x \bmod Z_n\}$, $P_n=\{\alpha\in A: x^\alpha=x$ for all $x\in G_{n+1}\}$, $Q_n=\{\alpha\in A: x^\alpha=x$ for all $x\in Z_n\}$. A sample theorem: Let C_n be the centralizer mod G_{n+1} of G in the holomorph of G. If G is nilpotent of class k, then K_1 is nilpotent of class at most $k-1$ and C_n is nilpotent of class k for every $n=1,2,\cdots$. *A. Hulanicki* (Wrocław)

Neumann, B. H.; Neumann, Hanna 1267
Embedding theorems for groups.
J. London Math. Soc. **34** (1959), 465–479.
Es werden iterierte Kranzprodukte untersucht und daraus Einbettungssätze für abzählbare Gruppen gewonnen. Sind U und V gruppentheoretische Eigenschaften, so hat die Gruppe G die Eigenschaft UV, wenn es in G einen U-Normalteiler N mit V-Faktorengruppe G/N gibt. Werden U und V durch identische Relationen definiert, so ist diese Multiplikation von Eigenschaften assoziativ. A sei die Eigenschaft Kommutativität. Typische Ergebnisse der Arbeit sind: Jede abzählbare (endliche) V-Gruppe kann in eine (endliche) VAA-Gruppe mit zwei Erzeugenden eingebettet werden. Jede endliche [auflösbare der Länge m] (nilpotente) Gruppe kann in eine endliche [auflösbare der Länge $m+2$] (nilpotente) Gruppe mit zwei Erzeugenden eingebettet werden. Es gibt eine von drei Elementen erzeugte auflösbare Gruppe der Länge 3, die zu einer eigentlichen Faktorgruppe isomorph ist.
W. Kappe (Zbl **102**, 264)

Neumann, B. H. 1268
Embedding theorems for semigroups.
J. London Math. Soc. **35** (1960), 184–192.
In the present paper the notion of the wreath product of groups and the constructive procedure used by the author and H. Neumann [#1267 above] are generalized for semigroups. With the aid of this construction the author gives a new proof for a theorem of T. Evans [Proc. Amer. Math. Soc. **3** (1952), 614–620; MR **14**, 347] according to which every countable semigroup can be embedded in a semigroup generated by 2 elements. From this constructive proof it follows easily that every finite semigroup can be embedded in a finite two-generator semigroup. Furthermore, it is proved that every finitely generated periodic semigroup can be embedded in a periodic semigroup with two generators. As corollaries of this last theorem, an analogue of the Burnside reduction theorem and of the restricted Burnside conjecture for semigroups can be given. Finally, a more general theorem is stated: If the semigroup S has d generators and if it belongs to the variety defined by laws $u(x_1,\cdots,x_n)=v(x_1,\cdots,x_n)$, then S can be embedded in a two-generator semigroup satisfying the laws $u(x_1^m,\cdots,x_n^m)=v(x_1^m,\cdots,x_n^m)$, where $m\geqq 4d-1$ (or $m=3d$). *J. Szendrei* (Zbl **90**, 17)

Teller, J. Roger 1269
On the extensions of lattice-ordered groups.
Pacific J. Math. **14** (1964), 709–718.
The partially ordered (p.o.) group G is called a p.o. extension of the p.o. group A by the p.o. group Δ if there is an order-preserving homomorphism π of G onto Δ with kernel A such that π induces an order isomorphism of G/A with Δ. The p.o. extensions G of A by Δ can be described

in terms of factor sets $f(\alpha,\beta) = -r(\alpha+\beta)+r(\alpha)+r(\beta)$, where $r(\alpha) \in G$ is a representative of the coset $\alpha \in \Delta$, and by sets $Q_\alpha = \{a \in A \mid r(\alpha)+a \geqq 0\}$ for $\alpha \geqq 0$ [cf. the reviewer, Acta Math. Acad. Sci. Hungar. **1** (1950), 118–124; MR **13**, 436]. In the case of Abelian groups the following are necessary and sufficient in order that G be an l-group whenever A and Δ are: (1) $\alpha \wedge \beta = 0$ implies that $Q_\alpha \cap (Q_\beta + b + f(\alpha-\beta, \beta))$ has a smallest element for all $b \in A$; (2) $Q_\alpha + Q_\beta + f(\alpha,\beta) = Q_{\alpha+\beta}$ for $\alpha, \beta \geqq 0$. The lexico-extensions are also investigated, and a method is given for lattice-ordered extensions. Finally, the extensions of l-groups with a finite basis are considered [cf. P. Conrad, Michigan Math. J. **7** (1960), 171–180; MR **22** #6854].

L. Fuchs (Budapest)

Ševrin, L. N. 1270

The lattice-subsemigroup characteristic of non-periodic commutative groups. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 671–678.

Properties of the lattice $\Sigma'(\Gamma)$ of all subsemigroups of a semigroup Γ are found which are necessary and sufficient in order that Γ be a non-periodic abelian group. These properties are formulated in terms of certain special elements of $\Sigma'(\Gamma)$ and are too lengthy to be listed here. The lattice-subsemigroup characteristic of torsion-free abelian groups is also found.

The existence of such characteristics has been ascertained by R. V. Petropavlovskaja [Mat. Sb. (N.S.) **29** **(71)** (1951), 63–78; MR **13**, 203].

B. M. Schein (*Šaĭn*) (Saratov)

Ševrin, L. N. 1271

Remark on the paper "On lattice properties of semigroups". (Russian)

Sibirsk. Mat. Ž. **5** (1964), 727.

The author amends a certain inaccuracy in the proof of Theorem 2.10 and corrects a remark on p. 461 of his earlier paper [same Ž. **3** (1962), 446–470; MR **27** #5847]. The theorem remains valid.

B. M. Schein (*Šaĭn*) (Saratov)

Ševrin, L. N. 1272

Lattice isomorphisms of commutative non-periodic semigroups with a cancellation law. (Russian)

Mat. Sb. (*N.S.*) **63** **(105)** (1964), 43–58.

If Γ is a semigroup, denote by $\Sigma'(\Gamma)$ the lattice of all subsemigroups of Γ (including the empty set). Two semigroups Γ and Γ_1 are said to be lattice isomorphic if $\Sigma'(\Gamma)$ is isomorphic to $\Sigma'(\Gamma_1)$. We shall say that Γ is defined by $\Sigma'(\Gamma)$ if any semigroup which is lattice isomorphic to Γ is isomorphic to Γ.

In the following Γ denotes a commutative semigroup satisfying the cancellation law and having at least one element of infinite order. The main result is the following: A necessary and sufficient condition in order that Γ be defined by $\Sigma'(\Gamma)$ is that either (i) all elements of Γ are of infinite order, or (ii) if there is an element of finite order, and hence a unit element e, the set $\Gamma \setminus e$ is not a semigroup.

The proof requires a rather delicate discussion of various possibilities concerning the subsemigroups generated by two elements of Γ.

Št. Schwarz (Bratislava)

Lesohin, M. M. 1273

On the semigroup of a multiplicative commutative regular semigroup. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 3 (40), 84–87.

Let $(A, +)$ be a commutative semigroup and $F(A)$ be the (commutative) semigroup of multiplications on A that are bilinear with respect to $+$. If A is regular, a necessary and sufficient condition is found for $F(A)$ to be the unit group. If A is regular and periodic, with zero, a necessary and sufficient condition is found for $F(A)$ to be finite. The first involves the concept of nilgroup, the second, periodic groups. This generalizes results concerning abelian groups [L. Fuchs, *Abelian groups*, § 67, Publ. House Hungar. Acad. Sci., Budapest, 1958; MR **21** #5672].

S. Stein (Davis, Calif.)

Yamada, Miyuki 1274

Construction of finite commutative z-semigroups.

Proc. Japan Acad. **40** (1964). 94–98.

A semigroup having a zero element and no other idempotents is called a z-semigroup. This note presents, without proofs, a method for constructing all finite commutative z-semigroups.

Lee W. Anderson (State College, Pa.)

Howie, J. M. 1275

The maximum idempotent-separating congruence on an inverse semigroup.

Proc. Edinburgh Math. Soc. (2) **14** (1964/65), 71–79.

A congruence ρ on a semigroup is called idempotent-separating if each ρ-class contains at most one idempotent. It is shown that there exists a maximal such congruence μ on every inverse semigroup S. Let E be the set of all idempotents of an inverse semigroup S. Then $x \equiv y(\mu)$ if and only if $x^{-1}ex = y^{-1}ey$ for every $e \in E$. The centraliser of E in S is the set $E\zeta = \{z \in S : ez = ze$ for every $e \in E\}$. $x \equiv y(\mu)$ if and only if $x^{-1}x = y^{-1}y$ and $xy^{-1} \in E\zeta$. $S/\mu \simeq E$ if and only if $E\zeta = S$ (in this case S is a union of groups). μ is the identical congruence on S if and only if $E\zeta = E$.

Let us define $H\omega = \{z \in S : hh^{-1} = zh^{-1}$ for some $h \in H\}$ for $H \subset S$. A congruence ρ is called a group congruence if S/ρ is a group. W. D. Munn has found a characterisation of the minimal group congruence σ on an inverse semigroup [W. D. Munn, Proc. Glasgow Math. Assoc. **5** (1961), 41–48; MR **27** #3723]. The author gives an alternative characterisation: $x \equiv y(\sigma)$ if and only if $xy^{-1} \in E\omega$; $\sigma \cap \mu$ is the identical congruence if and only if $E\omega \cap E\zeta = E$. Let $\sigma \vee \mu$ be the join of σ and μ in the lattice of all congruences on S. Then $x \equiv y(\sigma \vee \mu)$ if and only if $xy^{-1} \in (E\zeta)\omega$. $\sigma \vee \xi = \sigma \circ \xi \circ \sigma$ for an arbitrary congruence ξ ($\circ$ denotes the multiplication of binary relations).

{Remark by the reviewer: The minimum group congruence σ on an inverse semigroup has been found by V. V. Vagner [Mat. Sb. (N.S.) **32** **(74)** (1953), 545–632, Theorem 4.39; MR **15**, 501]. The congruence μ has been considered (in another aspect) in Vagner's paper [ibid. (N.S.) **52** **(94)** (1960), 597–628; MR **22** #12156]. Vagner's characterisation of μ [loc. cit., p. 626] coincides with the first characterisation of μ given by Howie. The maximum idempotent-separating property of μ is contained implicitly in Theorem 28 of Vagner's latter paper.}

B. M. Schein (*Šaĭn*) (Saratov)

Howie, J. M. 1276

Corrigendum: "The embedding of semigroup amalgams".

Quart. J. Math. Oxford Ser. (2) **15** (1964), 277.

The proof of Theorem 3.3(B) in the author's paper [same J. (2) **15** (1964), 55–68; MR **28** #3107] is invalidated by a false statement introduced by the phrase: "Then one can easily verify . . .". The error was found by G. B. Preston.

B. H. Neumann (Canberra)

Naik, K. G. 1277

Note on conditionally commutative semi-group-S_n.

Math. Student **31** (1963), 1–3 (1964).

Let S_n, $n>2$, denote a semigroup in which $x^n=x$ for all $x\in S_n$. It is easily shown that x^{n-1} is an idempotent. S_n is called conditionally commutative if $a, b\in S_n$ and $ab=ba$ imply $a^{n-1}=b^{n-1}$. This paper proves that the following five conditions are equivalent: (1) $ab=ba$ implies $a^{n-1}=b^{n-1}$; (2) $a^{n-1}=(axa)^{n-1}$ for any $a, x\in S_n$; (3) $a^{n-1}b^{n-1}=b^{n-1}a^{n-1}$ implies $a^{n-1}=b^{n-1}$; (4) $ab^{n-1}=b^{n-1}a$ implies $a^{n-1}=b^{n-1}$; (5) $a^{n-1}b=b^{n-1}a$ implies $a^{n-1}=b^{n-1}$. Besides this theorem, the author gives two theorems. One theorem is that if a conditionally commutative semigroup has a left [right] identity, then it is a right [left] group. The other theorem is that a conditionally commutative semigroup with an identity is a group. {The reviewer notes that the proof of left cancellation is not needed, and also the proof of the latter theorem is not necessary since it is an immediate consequence of the former one.}

T. Tamura (Davis, Calif.)

Šain, B. M. 1278

Involutary semigroups of complete binary relations. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 1300–1303.

An involutary semigroup (briefly, i.s.) is a system $\langle G, \cdot, -1\rangle$, where $\langle G, \cdot\rangle$ is a semigroup and -1 its anti-automorphism, the square of which is the identity mapping. Important examples of this notion are the i.s. $\mathfrak{P}(A\times A)$ of all binary relations on a set A with both operations defined in an obvious way and its involutary subsemigroups. These are of special interest if all relations they contain are complete, i.e., their first (hence also their second) projection is equal to A. Such i.s. are completely characterised up to isomorphism by giving a sequence of axioms they have to satisfy, and they form a class which cannot be defined by any finite system of elementary axioms. However, considering partially ordered i.s., a single axiom does characterise the class of all i.s. of complete binary relations ordered by inclusion. The theory yields also a new determination of inverse semigroups. Every commutative i.s. of binary relations is isomorphic to some i.s. of complete binary relations.

K. Drbohlav (Prague)

Yusuf, S. M. 1279

A Jordan-Hölder theorem for inverse semigroups with operators.

J. Natur. Sci. and Math. **3** (1963), 129–137.

The author proves the Schreier refinement theorem and the Jordan-Hölder theorem for inverse semigroups with operators, and provides examples to show that some well-known theorems about groups do not hold for the semigroups considered.

R. McFadden (Bloomington, Ind.)

Tamura, Takayuki; Graham, N. 1280

Certain embedding problems of semigroups.

Proc. Japan Acad. **40** (1964), 8–13.

A left translation of a semigroup S is a function $\lambda: S\to S$ such that $(xy)\lambda=(x\lambda)y$ for all x, y in S. A right translation is a function ρ such that $(xy)\rho=x(y\rho)$ for all x, y in S. A left translation λ and a right translation ρ are linked if $x(y\lambda)=(x\rho)y$ for all x, y in S. An inner left translation on S is one induced by left multiplication of the elements of S by a fixed element of S, e.g., $x\to ax$, all x in S. In A. H. Clifford and G. B. Preston's book [*The algebraic theory of semigroups*, Vol. I, Amer. Math. Soc., Providence, R.I., 1961; MR **24** #A2627] the following problem is raised: Find necessary and sufficient conditions that a semigroup S be embeddable as an ideal in a semigroup T so that the left translations of S are restrictions of inner left translations of T and the right translations of S are restrictions of inner right translations of T. In the case that S is weakly reductive ($xa=xb$, $ax=bx$, all x in S, imply $a=b$), Clifford and Preston show that such conditions are equivalent to the statement (1) every left translation of S is linked with some right translation of S and every right translation of S is linked with some left translation of S.

The authors show that for any semigroup S, (1) together with (2) every left translation commutes with every right translation, are necessary and sufficient to obtain the desired embedding. Next, the requirements for embedding S are weakened, and corresponding weaker equivalent conditions are obtained. The authors relate various combinations, destroy some conjectures with counterexamples, and leave open some problems, the most interesting of which is: Can any semigroup S be embedded as an ideal in a semigroup T so that either every right translation of S is induced by some inner right translation of T, or every left translation of S is induced by some inner left translation of T?

{The left translation $\begin{pmatrix} a & b & c\\ a & b & d\end{pmatrix}$ on page 13 should read $\begin{pmatrix} a & b & c\\ a & b & b\end{pmatrix}$.}

D. R. Brown (Knoxville, Tenn.)

Koskas, Maurice 1281

Plus fin groupe homomorphe à un hypergroupe.

C. R. Acad. Sci. Paris **258** (1964), 1965–1967.

Let H be a hypergroup with composition law $*$. A complex A of H is complete if it either contains or intersects trivially every complex $x_1 * x_2 * \cdots * x_n$, and the complete closure of a complex A of H is the smallest complete complex of H containing A. Using the concept of the "finest" group homomorph of H, the complete closure operation is determined in terms of the "unitive" complex of H, defined to be the smallest, complete, unitary, reflective, multiplicatively closed complex of H. A functor from the category of hypergroups into the category of groups is defined, mapping a hypergroup onto its finest group homomorph and a hypergroup homomorphism onto the corresponding group homomorphism between the finest group homomorphs. In general, injectivity is not preserved by this functor, but it is

preserved in the case of "flat" hypergroups. Finally, a characterisation is given of the equivalences on a hypergroup H associated with reversible complexes: a left reversible complex H' of H is a complex such that for all $x, y \in H$, if $x \in H' * y$, then $y \in H' * x$, and the equivalence $x \sim y$ if and only if $x \in H' * y$ is a right regular equivalence on H regarded as a semihypergroup.

J. A. H. Shepperd (Manchester)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 1223, 1234, 1300, 1500, 1596, 1630, 1660, 1929.

Ušakov, V. I. 1282

Topological groups with bicompact classes of conjugate subgroups. (Russian)

Mat. Sb. (N.S.) **63 (105)** (1964), 277–283.

In a former paper [Dokl. Akad. Nauk SSSR **144** (1962), 65–68; MR **25** #143] the author considered groups with compact conjugation classes. Stronger results are known, such as: The set of elements with compact conjugation classes is a maximal almost periodical normal subgroup [see, for instance, the reviewer, Math. Ann. **150** (1963), 136–149, esp. p. 148; MR **27** #693; Arch. Math. **15** (1964), 161–165; MR **29** #3571]. In the present paper the results are applied on groups with compact conjugation classes of subgroups.

H. Freudenthal (Utrecht)

Ušakov, V. I. 1283

Bicompactly generated groups. (Russian)

Uspehi Mat. Nauk **19** (1964), *no.* 1 (115), 179–182.

A topological group is said to be locally finite if any finite subset is contained in a compact subgroup. Theorem: A locally compact, compactly generated, locally finite group is compact.

W. T. van Est (Leiden)

Varopoulos, N. Th. 1284

Studies in harmonic analysis.

Proc. Cambridge Philos. Soc. **60** (1964), 465–516.

The paper seems to culminate in two separate main results if one does not count a somewhat isolated appendix about the so-called Wiener-Pitt phenomenon in certain measure algebras. The first is the theorem that two locally compact group topologies on an abelian group G must coincide if a character is continuous relative to one of them if and only if it is continuous relative to the other. Apart from the observation that it is sufficient to assume that one topology is not finer than the other, the proof is completely independent of the rest of the paper. Since the hypotheses clearly imply that $(G, \mathfrak{T}) \cong (G, \mathfrak{T}')$ ($\mathfrak{T}$ and $\mathfrak{T}'$ the two topologies), the result is well known if $\mathfrak{T} \subset \mathfrak{T}'$ and if $\mathfrak{T}'$ is a σ-compact topology; the proof for the general case somehow has to reduce the problem to this case. This reduction seems possible in a variety of ways; in his method the author reproves independently a theorem of Kakutani [Proc. Imp. Acad. Tokyo **19** (1943), 366–372; MR **7**, 375] saying that a compact abelian group all of whose characters are continuous is finite. The second principal result is a complete duality theory for what the author calls abelian $\mathscr{L}_\infty$-groups. If $\mathscr{L}_n$ is a decreasing sequence of locally compact (not necessarily Hausdorff) group topologies on a group G, then their intersection $\mathscr{L}$ is called an $\mathscr{L}_\infty$-topology on G and G an $\mathscr{L}_\infty$-group. Any inductive limit of a countable system of Hausdorff locally compact abelian groups over a directed index system is, in fact, an $\mathscr{L}_\infty$-group, so that the duality theory developed here is a different and, in some respects, a more exhaustive treatment of a class of abelian topological groups investigated by Kaplan in the same direction [Duke Math. J. **17** (1950), 419–435; MR **14**, 245]. The result is as follows: Let G be an abelian $\mathscr{L}_\infty$-group. Then the continuous characters of G separate points. Any continuous character of a subgroup H of G can be extended to the whole of G (a result missing in Kaplan's approach); in fact, it can be extended in such a fashion that it takes a non-identity value on a given element outside the closure of H. If the group $G^\wedge$ of continuous characters of G is given the compact open topology, then G is exactly the group of continuous characters of $G^\wedge$ (up to natural identification) (and has, as Kaplan shows, the compact open topology relative to the compact open topology on $G^\wedge$). If H is a closed subgroup of G and H^0 its annihilator in G, then $H \cong G^\wedge/H^0$ and $H^{0\wedge} \cong G/H$ as topological groups. This very satisfactory state of affairs for the category of groups considered is achieved by a careful study of certain measure algebras of the group G for whose construction a method of forming certain inductive limits of Banach spaces is developed. This part is considerably deeper than the other parts of the paper, which are not immediately concerned with the proof of the two main results. Their content is most simply described as the effort to carry the theory of topological vector spaces over to the theory of topological abelian groups; the concepts of projective and injective limits (the former is only defined and never used) are carefully introduced and their elementary properties are studied. The theory of duality for vector spaces is developed in a parallel way for abelian groups with bimorphisms into the circle group instead of bilinear forms into the ground field. Most of the principal results of the corresponding theory of vector spaces are recovered and the paper may actually serve as a good reference source for miscellaneous features in the theory of topological abelian groups that have analogues in the theory of topological vector spaces.

K. H. Hofmann (New Orleans, La.)

Wang, Hsien-Chung 1285

Compact transformation groups of S^n with an $(n-1)$-dimensional orbit.

Amer. J. Math. **82** (1960), 698–748.

Let G be a connected, compact, and effective differentiable transformation group of the n-sphere S^n with an $(n-1)$-dimensional orbit. The fundamental result of the paper is the following. Assuming that n is even and different from 4 or odd and greater than 31, the author shows that G acts orthogonally on the sphere and belongs to one of five classes of groups. If a subgroup of the orthogonal group $O(n+1)$ is considered as a transformation group of the unit sphere in Euclidean space R^{n+1}, then these classes may be described in the following way: (1) G is a subgroup of $O(n+1)$ and is invariant and transitive on the unit sphere S^{n-1} in a hyperplane R^n; (2) let G', G'' be subgroups of $O(p+1)$, $O(q+1)$ transitive on the unit spheres S^p, S^q

in R^{p+1}, R^{q+1}, respectively, where $p>0$, $q>0$, and $p+q=n-1$; G is a subgroup of $O(n+1)$ containing the direct product $G'\times G''$ as a normal subgroup; (3) G, considered as a subgroup of $O(8r)$, $n=8r-1$, is the tensor product $\mathrm{Sp}(r)\otimes\mathrm{Sp}(2)$ over the quaternions, where $\mathrm{Sp}(m)$ is the group of all quaternion symplectic matrices of order m; (4) G, considered as a subgroup of $O(4r)$, $n=4r-1$, is either the tensor product $\mathrm{U}(r)\otimes\mathrm{U}(2)$ or $\mathrm{SU}(r)\otimes\mathrm{SU}(2)$, where $\mathrm{U}(m)$ is the unitary group and $\mathrm{SU}(m)$ is the unimodular unitary group; (5) G, considered as a subgroup of $O(2p+4)$, $n=2p+3$, is the tensor product $O(p+2)\otimes O(2)$. Cases (1) and (2) of the above classification reduce to the known classification of compact transitive transformation groups.

G. I. Kručkovič (RŽMat **1961** #12 A523)

Conner, P. E. 1286

Retraction properties of the orbit space of a compact topological transformation group.

Duke Math. J. **27** (1960), 341–357.

One of the important results of this paper is that Conjecture II implies Conjecture I where these conjectures are as follows. Conjecture I. If (G, X) denotes a compact Lie group acting on a compact connected finite-dimensional ANR, and if (G, X) has a finite number of conjugacy classes of isotropy subgroups, then the orbit space X/G is also an ANR. Furthermore, if X is an absolute retract (AR), then so is the orbit space X/G. Conjecture II. For every compact connected simple Lie group G there is a transformation group (G, M) with a finite number of conjugacy classes of isotropy groups on a compact space M such that (G, M) has no stationary points and such that $H^t(M;Z)\approx H^t(M/G;Z)=0$ for $0\leq t<\infty$.

Floyd [Amer. J. Math. **73** (1951), 363–367; MR **12**, 846] and Conner and Floyd [Ann. of Math. (2) **67** (1958), 90–98; MR **19**, 1180; Proc. Amer. Math. Soc. **10** (1959), 616–620; MR **29** #188] have obtained results about Conjecture I for finite groups and for SO(2) and SO(3). Floyd has verified Conjecture II for $\mathrm{SO}(2n+1)$. It follows that Conjecture I is true if the identity component of G is abelian or if it is isomorphic to SO(3), SO(4) or SO(5). Conjecture II also implies that if G acts on a Euclidean space, then the orbit space is contractible, so this result is true also if the identity component of G is abelian, SO(3), SO(4), or SO(5). Results are obtained concerning the local rational Betti groups of the orbit space.

If G is a finite cyclic group of prime power order or if G is a toral group, and if G acts on finite-dimensional closed or open cells, there is a stationary point from the work of Smith and Conner and Floyd. Partly from this it had often been guessed that Conjecture II was false. The fact that the opposite is true for a large class of groups and the fact that this implies Conjecture I is a significant and interesting fact.

Deane Montgomery (Princeton, N.J.)

Kunze, R. A.; Stein, E. M. 1287

Uniformly bounded representations and harmonic analysis of the 2 × 2 real unimodular group.

Amer. J. Math. **82** (1960), 1–62.

This paper contains a remarkably detailed study of representations of and harmonic analysis on the group G of 2×2 real unimodular matrices. The continuous irreducible unitary representations of G were found long ago by Gel'fand and Naĭmark [Acad. Sci. USSR J. Phys. **10** (1946), 93–94; MR **8**, 132; Izv. Akad. Nauk SSSR Ser. Mat. **11** (1947), 411–504; MR **9**, 495] and independently by V. Bargmann [Ann. of Math. (2) **48** (1947), 568–640; MR **9**, 133]. Plancherel's formula for this group was established by Harish-Chandra [Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 337–342; MR **13**, 820]. To explain the results of this paper, some notation is needed. A generic element g of G is a matrix $\begin{pmatrix} a & b\\ c & d\end{pmatrix}$; $ad-bc=1$; $a, b, c, d\in\mathbf{R}$. For $g\in G$, $x\in\mathbf{R}$, and $s\in\mathbf{C}$, let $\phi^+(g,x,s)=|bx+d|^{2s-2}$, $\phi^-(g,x,s)=\mathrm{sgn}(bx+d)\phi^+(g,x,s)$. For $g\in G$ and $x\in\mathbf{R}$, let $g(x)=(ax+c)/(bx+d)$. For complex-valued functions f on R, let

$$(v^\pm(g,s)f)(x) = \phi^\pm(g,x,s)f\circ g(x).$$

The linear operators $v^\pm(g,s)$ obviously provide representations of G. The representations $g\to v^\pm(g,\frac12+it)$ $(t\in\mathbf{R})$ acting on $L_2(\mathbf{R})$ are continuous, irreducible and unitary, and are called the two principal continuous series. The representations $g\to v^+(g,\sigma)$ $(0<\sigma<\frac12)$, acting on the Hilbert space H_σ of functions on $\mathbf{R}$ with inner product

$$a_\sigma\int_{\mathbf{R}}\int_{\mathbf{R}} f(x)\overline{f(y)}|x-y|^{-2\sigma}\,dxdy,$$

are also continuous, irreducible and unitary, and are called the complementary series. Finally, there are two discrete series of continuous irreducible unitary representations of G; these play only a minor rôle in the present paper.

A main result of the paper is that the representations of the principal and complementary series can be analytically continued into the set $\{s\in\mathbf{C}: 0<\mathrm{Re}(s)<1\}=\mathbf{S}$ in the following sense. There are a separable Hilbert space $\mathscr{H}$ and representations $g\to U^\pm(g,s)$ of G on $\mathscr{H}$ such that: (1) $g\to U^\pm(g,s)$ is a continuous representation of G for each $s\in\mathbf{S}$; (2) $g\to U^\pm(g,\frac12+it)$ $(t\in\mathbf{R})$ is unitarily equivalent to the representation $g\to v^\pm(g,\frac12+it)$ of the principal continuous series; (3) $g\to U^+(g,\sigma)$ $(0<\sigma<\frac12)$ is unitarily equivalent to the representation $g\to v^+(g,\sigma)$ of the complementary series; (4) for g fixed in G, and ξ,η fixed in $\mathscr{H}$, the function $s\to\langle U^\pm(g,s)\xi,\eta\rangle$ is analytic in S;

(5) $\sup\{\|U^\pm(g,\sigma+it)\|: g\in G\}\leq A_{\sigma,\varepsilon}(1+|t|)^{|\sigma-1/2|(1+\varepsilon)}$

for $0<\sigma<1$ and $\varepsilon>0$, where $A_{\sigma,\varepsilon}>0$ and is bounded on every interval $0<\alpha\leq\sigma\leq\beta<1$.

The authors use the representations $U^\pm(g,s)$ $(s\in\mathbf{S})$ to obtain a complete analogue of the Hausdorff-Young theorem for G. A bounded linear operator A on the separable Hilbert space $\mathscr{H}$ is in $\mathscr{B}_p$ $(1\leq p<\infty)$ if the nonnegative operator $(A^*A)^{1/2}=|A|$ has discrete spectrum $(\lambda_1,\lambda_2,\cdots,\lambda_n,\cdots)$ and $\|A\|_p=[\sum_{n=1}^\infty\lambda_n{}^p]^{1/p}$ is finite. For $f\in L_1(G)$ and $s\in\mathbf{S}$, let $\mathscr{F}^\pm(s)=\int_G U^\pm(g,s)f(g)\,dg$. This is the [operator-valued] Fourier transform of f. Let $1<p<2$, and $q=p/(p-1)$. There is a measure $\mu_{q,\sigma}$ on $\mathbf{R}$ such that for all $f\in L_1(G)\cap L_p(G)$,

$$(1)\qquad \left[\int_{-\infty}^{\infty}\|\mathscr{F}^\pm(\sigma+it)\|_q{}^q\,d\mu_{q,\sigma}(t)\right]^{1/q}\leq\|f\|_p.$$

The Fourier transform admits a [unique] linear bounded extension over all of $L_p(G)$ for which (1) holds. Furthermore, $\mathscr{F}^\pm(s)$ is analytic in s. Analogous results using only the discrete series hold for $p\in[1,2]$.

The Hausdorff-Young inequality (1), involving the

whole strip S instead of only the line $\operatorname{Re} s = \frac{1}{2}$, has some curious consequences. If $f \in L_2(G)$ and $g \in L_p(G)$, $1 \leq p < 2$, then $f * g \in L_2(G)$ and $\|f * g\|_2 \leq A_p \|f\|_2 \cdot \|g\|_p$. If $g \in L_2(G)$, then $f * g \in L_q(G)$ for all $q \in [2, \infty]$, and $\|f * g\|_q \leq A_q \|f\|_2 \cdot \|g\|_2$. Both of these assertions hold trivially for compact groups and fail for the simplest non-compact Abelian groups, e.g., for the group **R**.

This paper is nearly self-contained, and it is very carefully written. It is an important step in realizing the possibilities for detailed harmonic analysis that are implicit in the Gel'fand-Raĭkov theorem.

Edwin Hewitt (Seattle, Wash.)

Kunze, R. A.; Stein, E. M. 1288

Uniformly bounded representations. II. Analytic continuation of the principal series of representations of the $n \times n$ complex unimodular group.

Amer. J. Math. **83** (1961), 723–786.

In the paper reviewed above [#1287], the authors produced an analytic continuation of the principal series of representations of SL(2, **R**), and applied this to Fourier transforms and convolutions on SL(2, **R**). For SL(n, **C**) a similar continuation exists. The result has been announced previously [Bull. Amer. Math. Soc. **67** (1961), 593–596; MR **26** #6808]. Applications to harmonic analysis on SL(n, **C**) and analytic continuation for general semisimple Lie groups are promised for later papers.

Edwin Hewitt (Seattle, Wash.)

Ree, Rimhak 1289

Construction of certain semi-simple groups.

Canad. J. Math. **16** (1964), 490–508.

C. Chevalley [Tôhoku Math. J. (2) **7** (1955), 14–66; MR **17**, 457] showed that a complex semi-simple Lie algebra L has a basis $\{h_i, x_\alpha\}$, where the h_i are in a Cartan subalgebra H, and where the x_α are root-vectors relative to H, such that $\exp(\operatorname{ad} \lambda x_\alpha)$ has a matrix whose entries are fixed polynomials with integral coefficients in the complex parameter λ. This fact enabled him to construct a linear group acting as automorphisms of an analogue of L over an arbitrary field, a construction which yielded some new simple finite groups as well as "split" simple algebraic groups. The author shows that if $x \to X$ is a representation of L, then a basis for the representation space can be chosen so that each $\exp(\lambda X_\alpha)$ has a matrix as above; a group analogous to Chevalley's is then constructed and studied, with similar methods and results. Each such group constructed from a faithful representation has Chevalley's group as a homomorphic image. There is also a list of results obtained independently by the author but published earlier in a report by Chevalley [Séminaire Bourbaki, 1960/61, Exp. 219, Secrétariat mathématique, Paris, 1961; MR **27** #1339], relating to properties of these groups as algebraic linear groups.

The method for establishing the existence of a suitable basis for the representation space is to reduce the problem quite neatly to the fundamental irreducible representations of the simple Lie algebras, and to deal with these case by case. The author acknowledges the desirability of a unified solution of the problem; such a solution was given by D. A. Smith [Ph.D. Dissertation, Yale University, New Haven, Conn., 1963] (to appear in Trans. Amer. Math. Soc.). Smith also studied the groups in question, although not with respect to the properties treated by the author. At least some of the groups here treated are related to work of C. W. Curtis [J. Math. Mech. **9** (1960), 307–326; MR **22** #1634] and of R. Steinberg [Colloq. Théorie des Groupes Algébriques (Bruxelles, 1962), pp. 113–127, Librairie Universitaire, Louvain, 1962; MR **27** #3638].

G. B. Seligman (New Haven, Conn.)

Brown, Gordon 1290

A remark on semi-simple Lie algebras.

Proc. Amer. Math. Soc. **15** (1964), 518.

Observation: The sum of the squares of the lengths of the roots of a semi-simple Lie algebra is the dimension of a Cartan subalgebra.

W. T. van Est (Leiden)

Oliva, Waldyr 1291

Deux théorèmes sur les pseudogroupes de Lie transitifs.

C. R. Acad. Sci. Paris **258** (1964), 789–791.

Let Γ be a Lie pseudogroup operating on a differentiable manifold V_n. The infinitesimal pseudogroup $\theta(\Gamma)$ associated to Γ is, by definition, the sheaf of germs of vector fields which generate a local 1-parameter group of local transformations belonging to Γ. The infinitesimal pseudogroup $\theta(\Gamma)$ is called transitive at x if, for every tangent vector L at x, there exists a local vector field X around x such that the value of X at x equals L and such that the germ of vector fields at x determined by X belongs to $\theta(\Gamma)$.

In this short note the author announces the following two results. (1) Let Γ be a transitive Lie pseudogroup operating on V_n such that $\theta(\Gamma)$ is transitive at a point $x \in V_n$. Suppose that Γ is of order r. Then Γ is a Lie pseudogroup of order k for all $k \geq r$. (2) Let Γ be a transitive analytic Lie pseudogroup operating on an analytic manifold. Then Γ is continuous if and only if $\theta(\Gamma)$ is transitive at a point x.

These two results answer the questions proposed by the reviewer (cf. Y. Matsushima, Séminaire Bourbaki, 1954/55, deuxième édition, corrigée, Exp. 118, Secrétariat mathématique, Paris, 1959).

Y. Matsushima (Osaka)

FUNCTIONS OF REAL VARIABLES

See also 1111, 1112, 1302, 1491, 1507, 1527, 1580.

1292

Guter, R. S. [Гутер, Р. С.]; Kudrjavcev, L. D. [Кудрявцев, Л. Д.]; Levitan, B. M. [Левитан, Б. М.]

★Elements of the theory of functions. Functions of a real variable. Approximation of functions. Almost periodic functions [Элементы теории функций. Функции действительного переменного. Приближение функций. Почти-периодические функции].

Edited by P. L. Ul'janov.

Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1963. 244 *pp.* 0.68 *r.*

This textbook is written mainly for people working in applications of mathematics in those branches of sciences which use the theory of functions of a real variable, the theory of approximation in a real domain, and the theory

of almost periodic functions. It contains an outline of each of these three domains of the theory of functions; proofs are not given.

Chapter I is written by the first of the authors and contains an introduction to set theory, the theory of Lebesgue measure and the integral of real functions of a real variable, orthogonal systems, remarks on the case of n variables, and the definitions and some examples of a metric space and a normed space. Theory of approximation is given in Chapter II, written by the second author. This chapter contains interpolation of continuous functions by polynomials, uniform approximation of continuous functions by algebraic and trigonometric polynomials, approximation in mean, and approximation in a Banach space. Chapter II also contains a number of recent results and may be read with profit by mathematicians who are not specialists in the theory of approximation. Chapter III, by the third author, is devoted to the theory of almost periodic functions. First of all, it contains a study of uniformly almost periodic functions. The fundamental definitions are given and Fourier series are investigated. Next, generalizations of almost periodic functions by Stepanov, Weyl and Besicovitch are outlined, and the definition of an almost periodic function on a group is given. The last section contains analytic almost periodic functions. *J. Musielak* (Poznań)

Garnir, H. G. 1293

★Fonctions de variables réelles. Tome I.

Librairie Universitaire, Louvain; Gauthier-Villars, Paris, 1963. xi + 518 *pp.*

This book is designed for a first university course in mathematical analysis. It begins with a thorough discussion of elementary topology of n-dimensional space, limits, and continuity. Then comes a long chapter about differential calculus for functions of several variables, followed by one on uniform convergence. The implicit function theorem is omitted. This is unfortunate, since the groundwork for proving it has already been laid. The last three chapters are about the elementary functions (of one real or complex variable), techniques of formal integration, and differential equations. A second volume, on integrals, is to follow. The book is enriched by many exercises, varying in difficulty from routine calculations to proofs of some nontrivial facts about real functions (e.g., properties of convex functions and construction of a Peano curve). *W. H. Fleming* (Providence, R.I.)

Glaeser, Georges 1294

Racine carrée d'une fonction différentiable.

Ann. Inst. Fourier (*Grenoble*) **13** (1963), *fasc.* 2, 203–210.

The author is concerned with the problem of determining when a non-negative function f of class C^{2m} has a square root of class C^m. He treats the case $m=1$ in detail. Counter-examples are given to justify his hypotheses. *M. R. Hestenes* (Los Angeles, Calif.)

Beckenbach, E. F. 1295

Superadditivity inequalities.

Pacific J. Math. **14** (1964), 421–438.

If $b>0$, denote by $K(b)$ the class of continuous functions f on $[0,b]$ such that $f(0)=0$. Let $K_1(b)$ denote those f in $K(b)$ which are convex and $K_2(b)$ those f whose average function F is in $K_1(b)$. (For notation and definitions see the review of A. M. Bruckner and E. Ostrow [same J. **12** (1962), 1203–1215; MR **26** #6326].) Similarly, $K_3(b)$ [respectively, $K_4(b)$] denotes those f in $K(b)$ which are starshaped [superadditive] and $K_{i+2}(b)$ those f in $K(b)$ such that F is in $K_i(b)$, $i=3,4$. It is known [loc. cit] that $K_i(b)\subset K_{i+1}(b)$ and that equality does not always hold. The author proves this latter fact in another way, using the single function $f(x)=(1+x^{-1})\exp(-1/x)$ $(x>0)$, $f(0)=0$. For each i he determines the maximum $b=b_i$ such that f is in $K_i(b)$; the result then follows from the fact that $b_i<b_{i+1}$, $i=1,\cdots,5$. Some related results are obtained, including conditions under which a C^1 function is superadditive. *R. R. Phelps* (Seattle, Wash.)

MEASURE AND INTEGRATION

See also 1158, 1223, 1335, 1500, 1528, 1529, 1530, 1543, 1564, 1664, 2110.

Métivier, Michel 1296

Sur les mesures à valeurs vectorielles et les limites projectives de telles mesures.

C. R. Acad. Sci. Paris **256** (1963), 2993–2995.

Let V be a real linear topological (Hausdorff) space, with topology $\mathscr{J}$. Let X be a set, and let $\mathscr{R}$ be a δ-clan (in the terminology of Bourbaki) of subsets of X. A "V-valued measure on $\mathscr{R}$" is a function $\mu:\mathscr{R}\to V$ such that $\lim_n \mu(E_n)=\mu(E)$ for any increasing sequence E_n in $\mathscr{R}$ such that $\bigcup_n E_n$ is in $\mathscr{R}$, where the limit is with respect to the topology $\mathscr{J}$.

Theorem: If V and V' form a dual pair of spaces, and if μ is a V-valued measure on $\mathscr{R}$ relative to the weak topology for the pair V, V', then μ is a V-valued measure relative to any locally convex topology on V which is compatible with the duality of the pair V, V'. This generalizes a well-known result of B. J. Pettis [Proc. Amer. Math. Soc. **8** (1957), 611–616; MR **19**, 429]. The author also gives an extension theorem for set functions with values in V, and establishes (in the case that V is semireflexive) the connection between measures in this sense and "vector measures" in the sense of Bourbaki. As the title indicates, applications are made to the theory of projective limits of measures. Full details of proofs will presumably appear elsewhere. *F. B. Wright* (New Orleans, La.)

Găină, Stelian 1297

Extension of vector measures.

Rev. Math. Pures Appl. **8** (1963), 151–154.

Let $\mathscr{R}$ be a ring of sets in a set X, let V be a Banach space, and let μ be a countably additive V-valued function on $\mathscr{R}$, with bounded variation. Then μ has a unique extension to the δ-ring [δ-clan, semitribe] generated by $\mathscr{R}$. (For a more general result of this form, see M. Métevier [#1296].) *F. B. Wright* (New Orleans, La.)

Ryan, Robert 1298

Representative sets and direct sums.

Proc. Amer. Math. Soc. **15** (1964), 387–390.

Let (X, Λ, μ) be a complete measure space and let N be the class of all locally μ-null subsets of X. The measure space is said to have property (M) if there exists a mapping $\rho: \Lambda \to \Lambda$ such that: (i) $\rho(E) \doteq E \in N$ ($\doteq$ denotes the symmetric difference); (ii) if $E \doteq F \in N$, then $\rho(E) = \rho(F)$; (iii) $\rho(0) = 0$ and $\rho(X) = X$; (iv) $\rho(E \cap F) = \rho(E) \cap \rho(F)$; (v) $\rho(E \cup F) = \rho(E) \cup \rho(F)$. It is proved that the measure space has property (M) if and only if X is the direct sum of μ-summable subsets plus possibly a locally μ-null set.

R. Sikorski (Warsaw)

Lewkowicz, R. E. 1299

The measurability of an intrinsic length.

Michigan Math. J. **11** (1964), 167–175.

The Cavalieri inequality for a map $f: X \to E_n$ states that the area of $f \geq$ the integral of the length of contour, $f^{-1}(E_n \times t)$, taken in any direction. This is the case $k=2$ of an obvious generalization involving k- and $(k-1)$-dimensional area. In his book [*Surface area*, Princeton Univ. Press, Princeton, N.J., 1956; MR **17**, 596] L. Cesari completely treats the case $k=2$ with X a square. But because of the arbitrary nature of contours, a special definition of length of a contour is needed.

This inequality can be more symmetrically stated in terms of a more general formulation of Lebesgue k-dimensional area given by the reviewer [Acta Math. **102** (1959), 33–46; MR **22** #1653].

The present author pursues this formulation and proves that the integral required does indeed exist for the case $k=2$ and X compact, connected, locally connected, and metric.

R. F. Williams (Chicago, Ill.)

Hahn, F. J. 1300

Errata: "Affine transformations of compact Abelian groups".

Amer. J. Math. **86** (1964), 463–464.

The author announces that Lemma 4 of the paper being corrected [same J. **85** (1963), 428–446; MR **27** #5889] is false but becomes true if one restricts attention to compact groups X such that X has no elements of finite order. He presents a proof of the corrected lemma which he attributes to S. Kakutani. This change affects the rest of the paper at only one point. The sentence "Moreover, suppose that X has no elements of finite order" must be added to the hypothesis of Theorem 4. To facilitate comparison with the review of the paper we remark that Theorem 4 is the last theorem of Section 1.

G. W. Mackey (Cambridge, Mass.)

FUNCTIONS OF A COMPLEX VARIABLE

See also 1115, 1116, 1181, 1184, 1185, 1186, 1460, 1468, 1484, 1513, 1729, 1738.

Marstrand, J. M. 1301

The distribution of the logarithmic derivative of a polynomial.

J. London Math. Soc. **38** (1963), 495–500.

The conjecture of the reviewer and W. H. J. Fuchs [same J. **15** (1940), 162–168; MR **2**, 117; see also Amer. Math. Soc. Notices **8** (1961), 483–485] is disproved. The existence is established of a polynomial $f(z)$ of degree n (arbitrarily large) such that $|f'(z)/f(z)| > \sigma$ for a set of $z = x + iy$ in which the variation of x exceeds

$$K\sigma^{-1}n(\log n)^{1/2}(\log\log n)^{-1/2}.$$

A. J. Macintyre (Cincinnati, Ohio)

Obreschkoff, Nikola [Obreškov, N.] 1302

★Verteilung und Berechnung der Nullstellen reeller Polynome.

VEB Deutscher Verlag der Wissenschaften, Berlin, 1963. viii + 298 *pp.* *DM* 43.50.

For many years it has been known that the author had been preparing a manuscript on the real zeros of a real polynomial. The manuscript has at last been published. It has the form of a textbook which aims to acquaint a university student with both the classical and modern results in this field.

The book consists of five chapters. Part of the first chapter and most of the third and fifth chapters deal with topics connected with real zeros of real polynomials. The first begins with Rolle's Theorem and its applications to the composition theorems of Laguerre and Hermite-Poulain.

The third chapter develops the familiar upper bounds for real zeros, due to Lagrange, Maclaurin, Newton, Laguerre and Cauchy. It then takes up the rules of sign of Descartes and of Fourier-Budan, together with various proofs and generalizations, including those of Laguerre and Hurwitz. It follows this with a discussion of Sturm sequences and their generalizations and applications. It next introduces the Hermite method of a quadratic form whose rank equals the number of distinct zeros and whose signature equals the number of distinct real zeros. The third chapter also gives a detailed treatment of variation-diminishing linear transformations as developed by Motzkin and Schoenberg, and includes a proof of Schoenberg's theorem that, in a certain sense, Descartes rule is sharper than any other rule of sign.

The fifth chapter, dealing with numerical methods, discusses Newton's method of approximation and its extensions and the methods of "regula falsi", iteration and continued fractions. It includes a lengthy exposition of the Lobatschewski-Graeffe root-power method in all cases, as well as the modification of the method due to Brodetsky and Sineal. The problem of systems of equations is also touched on briefly.

The other parts of the book delve into topics which do not strictly fall under the book's title. The location of zeros in the complex plane is treated in Chapter One with the introduction of the theorems of Lucas, Laguerre and Jensen; in Chapter Two by the development of Grace's Theorem and its application to composite polynomials; and in Chapter Three by a discussion of Hurwitz polynomials (all zeros in the left half-plane) and of the Obreschkoff and Schoenberg rules of sign for the number of zeros in a sector. These topics are of course also covered in the reviewer's book [*The geometry of the zeros of a polynomial in a complex variable*, Amer. Math. Soc., New York, 1949; MR **11**, 101]. Another general topic is that of orthogonal polynomials; as treated in Chapter Four, it covers their definition and fundamental properties with special attention to the classical orthogonal polynomials. Most of

this material can be found in Szegö's book [*Orthogonal polynomials*, revised edition, Amer. Math. Soc., Providence, R.I., 1959; MR **21** #5029] except for the results due to the author, the reviewer and Turán on the zeros of linear combinations of orthogonal polynomials.

The book has a wealth of information, making it a welcome addition to the literature on the zeros of polynomials. *M. Marden* (Milwaukee, Wis.)

Obreškov, Nikola [Обрешков, Никола] 1303

★**Zeros of polynomials [Нули на полиномите].**

Izdat. Bŭlgar. Akad. Nauk., Sofia, 1963. 289 *pp.* 2.71 *lv.*

This Bulgarian monograph, which was being completed even as the above translation [#1302] of another version was in its final stages, contains essentially the same material as the German version reviewed above, although the material has been subdivided into 14 chapters.

Korevaar, J.; McCoy, T. L. 1304

Power series whose partial sums have few zeros in an angle.

J. Math. Anal. Appl. **8** (1964), 461–473.

Let $\sum a_n z^n$ be a power series different from a polynomial, and $s_n(z)$ its partial sum of order n. Let $\nu_n(\delta)$ denote the minimum of the number of zeros in any angle of opening δ and vertex 0. It was shown by Carlson and Rosenbloom (independently) that if $\nu_n(\delta)=o(n)$ for some $\delta>0$, then the power series must represent an entire function of order zero. Ganelius [Duke Math. J. **30** (1963), 533–540; MR **28** #222] sharpened this result by showing that if for some $\delta\,(>0)$ and $\alpha\;(0\leqq\alpha<1)$, it is assumed that $\nu_n(\delta)\leqq An^\alpha$, then $a_n=O(\exp\{-\varepsilon n^{2-\alpha}\})$ with $\varepsilon>0$. The authors give a new proof of this fact and obtain an estimate for ε in terms of A and δ. *A. Edrei* (London)

Ahmedov, T. K. 1305

On series of Taylor-Dirichlet type. (Russian. Azerbaijani summary)

Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Mat. Tehn. Nauk **1963**, *no.* 5, 15–23.

In this paper the author obtains results on the domains of absolute convergence of the integral $\int_0^\infty z^{\varphi(t)}e^{-zt}\,dA(t)$ and the corresponding series $\sum_0^\infty a_n z^{\mu_n} e^{-\lambda_n z}$. Here $\varphi(t)=\varphi_1(t)+i\varphi_2(t)$ with $\varphi_1(t)\geqq 0$, and $A(t)$ is a complex function of bounded variation in every finite interval on the positive real semi-axis. The results generalise in part those of G. L. Lunc [Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk **14** (1961), no. 2, 7–16; MR **24** #A212; ibid. **15** (1962), no. 5, 11–26; MR **26** #2589] and are too complicated to be quoted in detail. The essential generalisation is that now φ_2 is possibly nonzero, and the inequalities for the domains of absolute convergence or divergence involve a factor $e^{-\varphi_2(t)\operatorname{Arg} z}$; if φ_2 is zero, Lunc's results are recovered. In another direction the author generalises a special result for the series, treated by Lunc, from the case $\log n=o(\lambda_n)$ to the case $\log n=O(\lambda_n)$. There are several misprints, including one in the definition of φ_2, which might cause confusion.

J. H. Williamson (Cambridge, England)

Gupta, J. S. 1306

On the mean values of integral functions and their derivatives defined by Dirichlet series.

Amer. Math. Monthly **71** (1964), 520–523.

Simple properties of the mean value

$$\lim_{T\to\infty}(1/2T)\int_{-T}^{T}|f(\sigma+it)|^2\,dt$$

of a Dirichlet series $f(s)=\sum a_n e^{-\lambda_n s}$, having $0\leqq\lambda_n\uparrow\infty$, $\limsup_{n\to\infty}(\log n/\lambda_n)=0$, and convergent for all values of s. *H.-E. Richert* (Marburg)

Noble, M. E. 1307

Some gap theorems for Dirichlet series.

Proc. London Math. Soc. (3) **14** (1964), 471–494.

In earlier papers the author generalized a number of classical gap theorems for power series [J. London Math. Soc. **30** (1955), 220–228; MR **16**, 683; Proc. Cambridge Philos. Soc. **53** (1957), 592–598; MR **19**, 736; J. London Math. Soc. **35** (1960), 117–127; MR **22** #11111]. Now these results are extended to a certain class of Dirichlet series. Very special cases lead to Ostrowski's gap theorem and its converse and to Pólya's refinement of Fabry's gap theorem, respectively. Also the Carlson-Landau results are included. Another results of this type has been given earlier by S. Agmon [Ann. Sci. École Norm. Sup (3) **66** (1949), 263–310; MR **11**, 427]. *H.-E. Richert* (Marburg)

Sinanjan, S. O. 1308

On extending the uniqueness property of analytic functions to closed, nowhere dense sets. (Russian)

Dokl. Akad. Nauk SSSR **154** (1964), 779–782.

For a compact set E in the z-plane denote by A_E^p the closure in $L_p(E)$ $(1\leqq p\leqq\infty)$ of the set of functions $f(z)$ analytic on E. The author presents the theorem that E can be chosen as a nowhere dense set with the property that each function f of A_E^2 which is equivalent to zero on the intersection of E with some disc is equivalent to zero on all of E. A fairly complicated proof is also presented. A second theorem, presented without proof, asserts the existence of a similar set E with the hypothesis that f is equivalent to zero on the intersection of E with a disc replaced by a hypothesis concerning the order of magnitude of the integral of $|f|^2$ on a sequence of discs intersected with E. *W. Kaplan* (Ann Arbor, Mich.)

Bowen, N. A. 1309

On the limit of the modulus of a bounded regular function.

Proc. Edinburgh Math. Soc. (2) **14** (1964/65), 21–24.

The author gives a new proof, under weaker hypotheses, of a theorem of M. L. Cartwright [J. London Math. Soc. **37** (1962), 179–184, pp. 180–181; MR **25** #4106]. He first proves the following result of Vitali type with a condition on the moduli of the functions instead of on the functions themselves. Let $f_n(z)$ be regular in $|z|<1$ and uniformly bounded by 1; let $|z_n|<1$, $z_n\to 0$, and $|f_n(z_n)|\to 1$. Then $|f_n(z)|\to 1$ uniformly in each $|z|\leqq 1-\delta$. This implies that if $|f(z)|<1$ in the half-strip $\alpha<x<\beta$, $y>1$, and $|f(x_n+iy_n)|\to 1$ as $n\to\infty$, where $\{x_n+iy_n\}$ contains a sequence such that $\alpha+\delta'\leqq x_n\leqq\beta-\delta'$ $(\delta'\geqq\delta)$, the x_n have a unique limit point, $2<y_n\uparrow\infty$, and $\limsup(y_{n+1}-y_n)<\infty$

(in Cartwright's theorem this hypothesis was $|f(a+iy)|\to 1$ as $y\to\infty$), then $|f(x+iy)|\to 1$ uniformly in each $\alpha+\delta\leq x\leq\beta-\delta$. *R. P. Boas, Jr.* (Evanston, Ill.)

Bowen, N. A. 1310

On the zeros of a class of canonical products of integral order.

Proc. Edinburgh Math. Soc. (2) **13** (1962/63), 247–253.

For canonical products $\Pi(z)$ of integer order with negative zeros the asymptotic distribution of zeros is deduced from the asymptotic growth of arg $\Pi(z)$ along another radius. If the zeros are real (positive or negative), assumptions about the asymptotic growth of arg $\Pi(z)$ along two radii are used in a similar manner. For a similar analysis depending on $\log|\Pi(z)|$ in place of arg $\Pi(z)$ see the author [Proc. London Math. Soc. (3) **12** (1962), 297–314; MR **25** #199].
A. J. Macintyre (Cincinnati, Ohio)

Bowen, N. A. 1311

Some Tauberian theorems for canonical products.

J. London Math. Soc. **39** (1964), 231–234.

Let (a_n), (c_n) be non-decreasing sequences of positive numbers, and let $j(r)$, $l(r)$ denote the numbers of a_n, c_n, respectively, in $|z|\leq r$. Suppose that the products $P_1(z)=\prod(1+z/a_n)$, $P_2(z)=\prod(1-z/c_n)$ are of genus 0, that $P(z)=P_1(z)P_2(z)$, and that

$$\log P(re^{i\alpha}) \sim \pi r^\rho e^{i\rho\alpha}(J+Le^{-i\rho\pi}) \operatorname{cosec} \pi\rho \qquad (0<\alpha<\pi),$$

$$\log P(re^{i\alpha}) \sim \pi r^\rho e^{i\rho\alpha}(J+Le^{i\rho\pi}) \operatorname{cosec} \pi\rho \qquad (-\pi<\alpha<0),$$

where α, ρ are constants, $0<\rho<1$, and $\log P(z)$ denotes the branch of the function for which $\log P_1(z)$, $\log P_2(z)$ are real and positive on the positive and negative real axes, respectively. By modifying a method which he used in the case in which the sequence (c_n) was empty [Quart. J. Math. Oxford Ser. **19** (1948), 90–100; MR **9**, 577], the author shows that $j(r)\sim Jr^\rho$ and $l(r)\sim Lr^\rho$. He also mentions, without proofs, some generalisations of this result. *P. Heywood* (Edinburgh)

Betz, Albert 1312

★Konforme Abbildung.

Zweite neubearbeitete Auflage.

Springer-Verlag, Berlin, 1964. xi+407 *pp.* *DM* 49.60.

A review of the first edition (1948) of this book will be found in MR **11**, 92.

The second edition appears to be about ten percent longer than the first. In his introduction, the author states that the changes consist largely of correction of errors and the reworking of certain sections for increased clarity. The author further states that "the field of conformal mapping has not changed much since the first edition". This ignores fifteen years of work on the numerical analysis of conformal mapping which was stimulated by electronic computers. Despite this gap, the book is now, as it was when it first appeared, a unique and useful contribution to the literature of applied mathematics.

Table of Contents: Introduction and Simple Examples, Electric Fields, Further Examples and Consequences, General Results, Conformal Mapping in Other Areas of Physics, Conformal Mapping and Analytic Functions, Mapping by Means of Simple Functions, Some Composite Functions, Treatment of Certain Mapping Problems, Doubly Periodic Fields, Free Streams, Dictionary of Special Maps. *P. J. Davis* (Providence, R.I.)

Walker, Miles 1313

★The Schwarz-Christoffel transformation and its applications. A simple exposition.

Dover Publications, Inc., New York, 1964. v+116 *pp.* $1.25.

This is an unaltered reprinting of a book entitled *Conjugate functions for engineers* [Oxford Univ. Press, London, 1933].

Alenicyn, Ju. E. 1314

Conformal mappings of multiply connected domains onto multivalent canonical surfaces. (Russian)

Dokl. Akad. Nauk SSSR **150** (1963), 711–714.

Proofs of the results announced in this note are to be found in the paper reviewed below [#1315].

Alenicyn, Ju. E. 1315

Conformal mappings of a multiply connected domain onto many-sheeted canonical surfaces. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 607–644.

This paper is the detailed exposition relating the articles previously written by the present author [Dokl. Akad. Nauk SSSR **146** (1962), 267–269; MR **26** #3889; see also #1314 above]. Let G be a bounded region of finite connectivity in the z-plane, with boundary C consisting of simple closed analytic curves; let $\zeta_1,\cdots,\zeta_s$ $(s\geq 1)$ be arbitrary given points all distinct in G; and let the constants $\alpha_{0,j},\alpha_{1,j},\cdots,\alpha_{p_j,j}$ $(j=1,\cdots,s)$ be not all zero and given arbitrarily except with the restriction $\sum_{j=1}^s\alpha_{0,j}=0$. The author proves, by setting

$$S(z,\zeta,\alpha)=\sum_{j=1}^s\left[\sum_{k=1}^{p_j}\frac{\alpha_{k,j}}{(z-\zeta_j)^k}+\alpha_{0,j}\log(z-\zeta_j)\right]$$

and taking an arbitrary angle θ $(-\pi/2<\theta\leq\pi/2)$, that there exists uniquely a function $\Phi_\theta(z)$ which is analytic in the region G except for the singularities $z=\zeta_j$, takes the form $\Phi_\theta(z)=S(z,\zeta,\alpha)+F_\theta(z)$, where the function $F_\theta(z)$ is regular in the closed region $G+C$, and maps each boundary component of G onto a segment on a straight line making an angle θ with the real axis. He also proves the existence theorems of mapping functions of G onto some other types of many-sheeted canonical surfaces. By the results obtained above and by using a complete orthonormal system in the Hilbert space consisting of square-integrable analytic functions, he solves the two extremal problems about the area of the image region by an analytic function belonging to a certain class and then applies the results to derive a distortion theorem on the class of areally mean p-valent functions in a finitely connected region.

Most of the results obtained throughout this paper generalize corresponding theorems by several authors, for instance, Grötzsch [Ber. Verh. Sächs. Akad. Leipzig Math.-Phys. Kl. **83** (1931), 283–297], Grunsky [Math. Z. **45** (1939), 29–61], and Garabedian and Schiffer [Trans. Amer. Math. Soc. **65** (1949), 187–238; MR **10**, 522].
T. Kubo (Kobe)

Yong, Er-qian [Yung, Erh-chien] 1316

On the sewing theorem.

Acta Math. Sinica **14** (1964), 58–61 (*Chinese*); *translated as Chinese Math.* **5** (1964), 63–66.

The author proves the following stronger version of the title theorem [see G. M. Golusin, *Geometrical theory of functions of a complex variable* (Russian), GITTL, Moscow, 1952; MR **15**, 112]. "Suppose that $x'=g(x)$ maps the line segment l: $-1\leqq x\leqq 1$ onto itself in such a way that both end points are preserved ($g(\pm 1)=\pm 1$) and that the correspondence is continuous and one-one. Suppose further that $g'(x)$ satisfies the Hölder condition, $g'(x)\neq 0$. Then there exist two single-valued functions $\omega=f_1(z)$ and $\omega=f_2(z)$ which transform the semicircles B_1: $|z|<1$, $\mathrm{Im}\{z\}>0$ and B_2: $|z|<1$, $\mathrm{Im}\{z\}<0$ onto G_1 and G_2, respectively, where G_1, G_2 are two disjoint regions obtained from the disk $|\omega|<1$ by separating it with a Ljapunov arc λ. Moreover, $f_1(x)=f_2(g(x))$, $x\in l$."

J. F. Heyda (King of Prussia, Pa.)

Gel'fer, S. A. 1317

Typically real functions. (Russian)

Mat. Sb. (*N.S.*) **64 (106)** (1964), 171–184.

Let T denote the class of functions $f(z)=z+\sum_{n=2}^{\infty} c_n z^n$ regular and satisfying the condition $\mathrm{Im}(f(z))\,\mathrm{Im}(z)>0$ for $|z|<1$. These are the typically real functions first introduced by Rogosinski [Math. Z. **35** (1932), 93–121]. Let $T^{(2)}$ be the subclass of functions $f_2(z)$ for which $f_2(iz)/i$ also belongs to T.

For the class T the author determines a radius of starlikeness and sharp bounds for $|zf'(z)/f(z)|$. For the class $T^{(2)}$ the author determines sharp bounds for $|f_2(z)|$, $|f_2'(z)|$, and $|zf_2'(z)/f_2(z)|$. He further obtains sharp bounds for x_3, sharp upper bounds for x_{2k}, and sharp lower bounds for x_4 and x_6, where $f_2(z)=z+\sum_{n=1}^{\infty} x_n z^{2n+1}\in T^{(2)}$.

A. W. Goodman (Tampa, Fla.)

Górski, J. 1318

On the coefficients of univalent functions in the unit circle.

Colloq. Math. **11** (1963/64), 181–185.

Let $w=f(z)=z+a_2z^2+\cdots$ be analytic in the unit disk K $|z|<1$, and let Δ denote the image of K under the transformation $w=f(z)$. Let D denote the image of Δ under the transformation $\zeta=1/w=1/f(z)$, and let E denote the set complementary to D. Let $\eta^{(n)}=\{\eta_1,\eta_2,\cdots,\eta_n\}$ be an nth extremal set of points on E (in the sense used in the study of transfinite diameter). Then it is known that the limits

$$s_k=\lim_{n\to\infty}\frac{\eta_1{}^k+\eta_2{}^k+\cdots+\eta_n{}^k}{n},\qquad k=1,2,\cdots,$$

exist [Leja, Ann. Soc. Polon. Math. **23** (1950), 69–78; MR **12**, 691]. If the Taylor coefficients of the inverse function $f^{-1}(w)$ are denoted by b_k, then the author's results include a number of sharp inequalities for some expressions containing a_k, b_k and s_k. For example, it is shown that $|s_2|\leqq 6$, with equality only for the Koebe function. Again, $|a_4+b_4|\leqq 10$, with equality only for the Koebe function. Finally, $|s_3|\leqq 20$.

M. Reade (Ann Arbor, Mich.)

Krzyż, J. 1319

On some recent results in the theory of analytic functions.

Colloq. Math. **11** (1963/64), 147–150.

In this note the author summarizes some of the many results obtained by researchers in the theory of functions located in Lublin; the results are those obtained during the period 1959–1962. Some of the authors whose results are quoted are Bielecki, Krzyż, Lewandowski, Piłat and Złotkiewicz.

M. Reade (Ann Arbor, Mich.)

Janowski, W. 1320

Sur une évaluation des coefficients des fonctions holomorphes univalentes et bornées inférieurement dans le cercle $K(\infty, 1)$.

Colloq. Math. **11** (1963/64), 187–190.

With the notation of the following review [#1321] let $c_n{}^*$ be the smallest number such that $|c_n|\leqq c_n{}^*$ for every $\Phi(z)\in\Sigma(m)$. The author obtains a lower bound for $c_n{}^*$ by considering particular functions in the class $\Sigma(m)$.

A. W. Goodman (Tampa, Fla.)

Janowski, W. 1321

Évaluation de la fonctionnelle $|\log(\Phi(z_1)-\Phi(z_2))/(z_1-z_2)|$ dans la famille des fonctions univalentes et bornées inférieurement dans le cercle $K(\infty, 1)$.

Colloq. Math. **11** (1963/64), 191–194.

Let $\Sigma(m)$ denote the family of functions $\Phi(z)=z+\sum_{n=0}^{\infty} c_n/z^n$ which are regular and univalent for $1<|z|<\infty$ and for which $|\Phi(z)|\geqq m$. The author proves that under these conditions

$$\left|\log\frac{\Phi(z_1)-\Phi(z_2)}{z_1-z_2}\right|^2\leqq\prod_{j=1}^{2}\log\frac{1-m^2/|\Phi^2(z_j)|}{1-1/\rho^2},$$

where $|z_1|=|z_2|=\rho$, and the inequality is sharp.

A. W. Goodman (Tampa, Fla.)

Bielecki, A. 1322

Quelques résultats récents sur les majorantes dans la théorie des fonctions holomorphes.

Colloq. Math. **11** (1963/64), 141–145.

Let $f(z)=\sum_1^\infty a_nz^n$, $F(z)=\sum_1^\infty A_nz^n$ be two functions holomorphic in $|z|<1$. $f(z)$ is said to be (1) majorized in modulus by $F(z)$ in $|z|<\rho$ if $|f(z)|\leqq|F(z)|$ for $|z|<\rho$; (2) subordinate to $F(z)$ in $|z|<\rho$ if the set of values $f(z)$ is contained in the set of values $F(z)$ for $|z|<\rho$. The author discusses briefly the known relations between the two types of majorization and announces without proofs three new results obtained recently by the author and Z. Lewandowski. We mention only one typical result.

If $f(z)$ and $F(z)$ are starlike of order α and $|f(z)|\leqq|F(z)|$ in $|z|<1$, then $f(z)$ is subordinate to $F(z)$ in $|z|<R(\alpha)$, where $R(\alpha)$ is a root x, $0<x<1$, of the equation

$$\arcsin\frac{2x}{1+x^2+(\alpha/(1-\alpha))(1-x^2)}+2\,\mathrm{arc\,tg}\,x=\frac{\pi}{2}.$$

No larger number may be substituted for this determination of $R(\alpha)$.

M. S. Robertson (New Brunswick, N.J.)

Reade, Maxwell O. 1323

On the partial sums of certain Laurent expansions.

Colloq. Math. **11** (1963/64), 173–179.

Let $f(z)=z+\sum_1^\infty a_k z^{-k}$ be analytic and univalent for $|z|>1$. Let $S_n(z)=z+\sum_1^n a_k z^{-k}$. In an earlier paper the author [J. Math. Soc. Japan **15** (1963), 66–68; MR **26** #2595] obtained estimates on the radius of univalency for the partial sums $S_n(z)$. Improved but non-sharp estimates are now obtained for this class and for the subclass for which $\Re f'(z)>0$ in $|z|>1$.

M. S. Robertson (New Brunswick, N.J.)

Red'kov, M. I. 1324

The range of values of the functional $I(f(w), \overline{f(w)}, f'(w), f'(0))$ in the class $S_1[|f(w)|]$. (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. **169** (1963), 59–68.

Let S_1 be the class of functions $f(\omega)$ which are regular and univalent in $|\omega|<1$, with $f(0)=0$, $f'(0)>0$, and with $|f(\omega)|<1$ for $|\omega|<1$. Let $S_1[|f(w)|]$ be the subclass of S_1 for which $|f(w)|$ is a given constant at a given fixed point w. The author determines the domain of values of the functional $I=I(f(w), \overline{f(w)}, f'(w), \overline{f'(w)}, f'(0))$ for $f(z)\in S_1[|f(w)|]$, where the only assumption on I is that it have continuous first derivatives for each of its five arguments.

The solution is implicitly given by a system of five equations. The author considers in detail a number of special cases in which the computations can be carried through and the domain can be given in recognizable form.

A. W. Goodman (Tampa, Fla.)

Černikov, V. V. 1325

Extremal properties of univalent functions with real coefficients. I, II. (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. **169** (1963), 69–85; *ibid.* **169** (1963), 86–95.

The author considers six different subclasses of the class of functions that are univalent, with real coefficients in E, the unit circle; for three of the classes f is regular, and for three of the classes f has a simple pole at the origin. The author obtains six variational formulas, one for each class. As a typical example, let $S_r^\star(\rho)$ be the class of functions $z=f(w)=\sum_{n=1}^\infty C_n w^n$ that are regular, univalent, and starlike in the unit circle, and have real coefficients, and which for fixed ρ satisfy the condition $f(\rho)=\rho$. Let I be a functional of a very general form. The author obtains the precise form for a function in the class $S_r^\star(\rho)$ which maximizes I and, as may be expected, it maps E onto a domain with radial slits. For this same class he obtains the domain of values of $\ln(f(w)/w)$ and of $wf'(w)/f(w)$. From these he deduces sharp bounds for $|f(w)|$, $|\arg f(w)|$, $|\arg wf'(w)/f(w)|$ and $|wf'(w)/f(w)|$.

A. W. Goodman (Tampa, Fla.)

Èzrohi, Tat'jana G. 1326

On certain classes of p-valent functions. (Russian. Polish and French summaries)

Ann. Univ. Mariae Curie-Sklodowska Sect. A **16** (1962), 137–149 (1964).

Let $f(z)=z^p+\sum_{n=1}^\infty a_{p+n}z^{p+n}$. The author introduces the classes of functions (1) L_p, and (2) $R_p{}^0$ for which (1) $\sum_{n=1}^\infty (p+n)|a_{p+n}|\leqq p$, and (2) $\sum_{n=1}^\infty (p+n)^2|a_{p+n}|\leqq p^2$. Further, she introduces the class Q_p of functions $f(z)=z^p+\sum_{n=1}^\infty ((p+n+\gamma)b_{p+n}z^{p+n}/(p+\gamma))$ for which $\Re e^\gamma\geqq 0$ and $\sum_{n=1}^\infty (p+n)^2|b_{p+n}|\leqq p^2$. She proves that if $f\in Q_p$, then f is p-valent starlike. Further, she obtains sharp bounds for $\arg(f'(z)/z^{p-1})$ and $\arg(f(z)/z^p)$ for each of the classes L_p, $R_p{}^0$, and Q_p. Finally, similar classes are introduced, and similar results are obtained for the domain $1<|z|<\infty$.

A. W. Goodman (Tampa, Fla.)

Èzrohi, Tat'jana G. 1327

Analytic functions of the generalized Schild classes. (Russian. Polish and French summaries)

Ann. Univ. Mariae Curie-Sklodowska Sect. A **16** (1962), 151–161 (1964).

Let Sh_p denote the class of functions $f(z)=z^p-\sum_{n=1}^\infty a_{p+n}z^{p+n}$ ($p\geqq 1$ an integer) for which $a_{p+n}\geqq 0$ and $\sum_{n=1}^\infty (p+n)a_{p+n}=p$. The case $p=1$ gives the class introduced by Schild. Let $f(z)\in \mathrm{Sh}_p$, let r_0 denote the radius of convexity of f, d_0 the radius of the largest circle with center at the origin covered by the image of $|z|\leqq r_0$ under f, and let D_0 be similarly defined for the image of $|z|\leqq 1$. The author determines the minimum of the ratio d_0/D_0, which in the special case $p=1$ gives $\frac{3}{4}$, conjectured by Schild and proved by Lewandowski.

For a similar class defined on $1\leqq|z|\leqq\infty$, it is proved that d_0/D_0 can take on every value in the interval $(1,\infty)$.

The author introduces a new class of functions S_t: $f(z)=z^p-\int_b^\infty z^{p+x-1}\,d\alpha(x)$ ($b\geqq 2$, $p\geqq 1$), $\alpha(x)\in\uparrow$ in $[2,\infty)$ with $\int_b^\infty (p+x-1)\,d\alpha(x)=p$, where each such f is regular in the circle $|z|\leqq 1$ cut from -1 to 0. For these functions the author obtains the radius of convexity, some covering theorems, and sharp bounds for $|f'(z)|$.

A. W. Goodman (Tampa, Fla.)

Kir'jackiĭ, È. [Kirjackis, E.] 1328

Functions with non-zero divided difference. (Russian. Lithuanian and German summaries)

Litovsk. Mat. Sb. **3** (1963), *no.* 1, 157–168.

Let $K_n(D)$ be a class of functions $f(z)$ that are regular and single-valued in a domain D in the finite plane and satisfy the condition that the nth divided difference of f is never zero for arbitrary pairs of distinct points $z_0, z_1, \cdots, z_n$ in D. The author has shown earlier that if f belongs to $K_n(D)$, then $f(z)+P_{n-1}(z)$ has no more than n zeros in D, where $P_{n-1}(z)$ is a polynomial of degree not exceeding $n-1$. Let D be the unit disk and let $f(z)=z^n+a_2z^{n+1}+\cdots$ belong to $K_n(D)$. The author obtains bounds for $|f^{(n)}(z)|$ and $\arg|f^{(n)}(z)|$. Let $F(z)$ ($F(0)=0$, $F'(0)=1$) be univalent in the unit disk and $z^{n-1}F(z)\in K_n(E)$. It is shown that F has many of the extremal properties that are possessed by convex normalized univalent functions.

W. C. Royster (Lexington, Ky.)

Earle, Clifford J. 1329

On the homotopy classes of self-mappings of bordered Riemann surfaces.

Proc. Amer. Math. Soc. **15** (1964), 622–624.

Let W be a Riemann surface which can be represented as the interior of a bordered surface. The author proves the following theorems: (1) Let $F: W\to W$ be a quasi-conformal homeomorphism which is homotopic to the

identity. F has a continuous symmetric extension F_* to the double of W. F_* is homotopic to the identity under a homotopy leaving each border contour of W fixed. (2) Let $W_0 = W \cup B$ be a bordered Riemann surface with interior W and border B consisting of compact contours. Let $F: W \to W$ have a continuous extension mapping each contour of B into itself. Let F_* be the symmetric extension of F to the double. Let F be homotopic to the identity. Then F_* is homotopic to the identity under a homotopy leaving each contour of B fixed.

These results are corollaries of a slightly more general theorem whose proof follows by raising considerations to the universal covering surface of W and using the hyperbolic metric. *R. Accola* (Providence, R.I.)

Rešetnjak, Ju. G. 1330

Stability in Liouville's theorem on conformal mappings of space. (Russian)

Dokl. Akad. Nauk SSSR **152** (1963), 286–287.

Let $\mathscr{F}$ denote the family of homeomorphisms $f = (f_1, \cdots, f_n)$ which map the n-dimensional unit sphere $Q = \{|x| < 1\}$, $x = (x_1, \cdots, x_n)$, into itself, and which possess L^n-integrable generalized first partial derivatives. Next, for each such homeomorphism f, set

$$\theta(f) = n^{-n/2} \frac{D(f)}{|V(f)|},$$

where

$$D(f) = \int_Q \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j} \right)^2 \right)^{n/2} dx, \quad V(f) = \int_Q J(x, f)\, dx,$$

and where $J(x, f)$ is the Jacobian of f. The author asserts that there exists a universal function $\alpha(\varepsilon, r) \geqq 0$, defined for $\varepsilon > 0$ and $0 \leqq r < 1$, such that $\alpha(\varepsilon, r) \to 0$ as $\varepsilon \to 0$, and such that given any f in $\mathscr{F}$, with $\theta(f) \leqq 1 + \varepsilon$, there exists a Möbius transformation g for which $|f(x) - g(x)| \leqq \alpha(\varepsilon, r)$ in $|x| \leqq r$. This result implies the general form of Liouville's theorem established by the present author [*Some problems of mathematics and mechanics* (Russian), pp. 219–223, Izdat. Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk, 1961] and, in the case $n = 3$, by the reviewer [Trans. Amer. Math. Soc. **103** (1962), 353–393; MR **25** #3166]. No proof is given.

F. W. Gehring (Ann Arbor, Mich.)

Strebel, Kurt 1331

Zur Frage der Eindeutigkeit extremaler quasikonformer Abbildungen des Einheitskreises.

Comment. Math. Helv. **36** (1961/62), 306–323.

Let T be a topological mapping of $|Z| = 1$ onto $|W| = 1$. It is well known that among all quasi-conformal mappings of $|Z| < 1$ onto $|W| < 1$ which agree with T on the boundary and whose maximal dilatation K is bounded above there exists a quasiconformal mapping F_0 with minimum maximal dilatation K_0. The question of uniqueness of such mappings has been investigated. The following is proved: Let R be a simply connected hyperbolic Riemann surface with finite area over the z-plane and let $w = f_0(z) = K_0 x + iy$, $K_0 \geqq 1$, be a mapping of R onto a surface S over the w-plane. Further let Φ and Ψ be conformal mappings of R and S onto $|Z| < 1$ and $|W| < 1$, respectively. Then among all quasiconformal mappings of $|Z| < 1$ onto $|W| < 1$ which agree with $F_0 = \Psi f_0 \Phi^{-1}$ on $|Z| = 1$ the mapping F_0 is the only one with minimum maximal dilatation. It is shown that the condition that R should have finite area cannot be relaxed. *V. Singh* (Patiala)

Lehner, Joseph 1332

★Discontinuous groups and automorphic functions.

Mathematical Surveys, No. VIII.

American Mathematical Society, Providence, R.I., 1964. xi+425 *pp.* $12.60.

In view of the importance of the subject and the amount of work published in the last thirty years on automorphic functions and their groups, it is surprising that no comprehensive treatise in any language has appeared since the books by L. R. Ford [*Automorphic functions*, McGraw-Hill, New York, 1929] and P. Fatou [*Fonctions automorphes* (in *Théorie des fonctions algébriques d'une variable*, Tome II, deuxième édition, par P. Appell et É. Goursat), Gauthier-Villars, Paris, 1930] in 1929 and 1930, respectively. Despite the excellence of these works and of their eminent precursor, namely, the treatise of R. Fricke and F. Klein [*Vorlesungen über die Theorie der automorphen Funktionen*, Band 1, Teubner, Leipzig, 1897; Band 2, 1912], there has been for many years a great need for a book dealing with the subject in a modern way, using modern algebraic and topological concepts, particularly where Riemann surfaces and vector spaces are involved. In addition, the inclusion in book form of a concise and fairly comprehensive account of the important new ideas and techniques introduced in the last three decades by H. Petersson and others has for long been overdue. This book fills these gaps in an excellent manner and will be indispensable to the student of the subject and provide the basis from which he can begin his study of the extensive literature. It is also valuable since it provides proofs of many "known" facts which cannot, or only with difficulty, be found in the published literature. Although the scope of the book is wide, it naturally reflects the author's own special interests and there are several areas that are scarcely touched; this is particularly the case with regard to some parts of elliptic modular function theory.

In his first chapter the author gives a general historical account of the whole subject from its origin as an offshoot of elliptic function theory to the present day. The motivational advantage of such an introduction is considerable, and it is a feature that might well be copied more generally. Chapter II contains a survey of those parts of topology, Hilbert space theory, group theory, analytic function theory and hyperbolic geometry that are used in the sequel. The various types of bilinear transformations are discussed, and Ford's isometric circle is introduced. In Chapter III, discontinuous, properly discontinuous, and discrete groups of bilinear transformations are discussed and their properties are compared. In Chapter IV fundamental regions are introduced and a close study of their vertices and sides is made. {The sentence beginning at the bottom of p. 111 should be omitted as it adds nothing to the definition previously given, nor does the preceding sentence appear to amplify the one before it.} Principal circle groups are treated by the method of isometric circles and by hyperbolic geometry. Automorphic forms are defined in Chapter V, and the existence of such forms is proved by constructing Poincaré series and investigating their properties. At this stage the dimension $-k$ is assumed to be an even integer

and the multipliers take the value 1. The valence of an automorphic function is defined to be the number of times it assumes any given value in a fundamental region, and does not depend upon the value chosen. The existence of an algebraic relation connecting any two automorphic functions is proved and various consequences are deduced. After a brief summary of the properties of Riemann surfaces, including the Riemann-Roch theorem, Chapter VI continues with a detailed study of the quotient space $\Gamma\backslash\mathscr{D}^+$, where Γ is a discontinuous group with domain of existence $\mathscr{D}$, and $\mathscr{D}^+$ is the union of $\mathscr{D}$ and its parabolic vertices. A full proof of the fact that $\Gamma\backslash\mathscr{D}^+$ can be given an analytic structure making it a Riemann surface is given, and the circumstances in which this surface is compact are studied. As he defines a topology on $\mathscr{D}^+$, instead of confining his attention to $\Gamma\backslash\mathscr{D}$, the author departs from the usual practice, with considerable advantage. The Riemann-Roch theorem is applied to find the dimension of certain subspaces of regular automorphic forms of even integral dimension when the Riemann surface is compact. Chapter VII begins with Poincaré's theorem that every polygon satisfying certain conditions determines a fundamental region of a horocyclic group, and an application to triangle groups is made. Relations holding between the generating transformations are discussed, the signature of a group is defined and a canonical polygon is constructed. Automorphic forms with general multipliers are introduced in Chapter VIII, the first example being the Dedekind-Hermite η-function. Poincaré series of integral dimension $-r$ $(r>2)$ and cusp forms are defined. Petersson's scalar product of two cusp forms is introduced and the properties of the Hilbert space of cusp forms of a given dimension are obtained and the Fourier coefficients of such forms are studied and explicitly evaluated in certain cases. Chapter IX is devoted to the work of Rademacher and his school on forms of positive and zero dimension and their Fourier coefficients. Chapter X is concerned with transitivity. Roughly speaking, a circle orthogonal to the principal circle of a group having a fundamental region F with a finite number of sides is said to be transitive if the images of the circle are dense in F. In Chapter XI the modular group Γ and modular forms of arbitrary real dimension are studied. The multipliers of the η-function are obtained and Klein's absolute invariant $j(\tau)$ is introduced. Applications to number-theoretical results connected with partition theory and representations by sums of squares are briefly given. Formulae for the index and genus of the principal congruence group $\Gamma(n)$ and other subgroups of Γ are given. The final chapter contains a sketch of the theory of automorphic functions of several variables, their associated groups and fundamental regions. There are ten pages of notes amplifying the material of the text and a certain number of exercises for the student scattered throughout the text. The quality of reproduction of some of the figures does not come up to the high standard of the printing. In the text the chapter number occurs only on the first page of each chapter; since the index lists entries by chapter and section and not by page, this makes cross-reference unnecessarily difficult.

R. A. Rankin (Glasgow)

Castonguay, C. E.; Helfenstein, H. G. 1333
On analytic maps of the pseudosphere.
Monatsh. Math. **68** (1964), 97–101.

The following problem is treated. Let P be the punctured disk $0<|z|<e^{-1}$ endowed with the conformal metric $ds=-|dz|/|z|\log|z|$. P is then isometric to the upper half-pseudosphere. Let $w=f(z)$ be any analytic function of z defined in P with $f'(z)\neq 0$. Let $\delta(z)=|dw|/ds$ be the local distortion of arc length, and let $D[f]=\iint_P \log^2\delta(z)\,dxdy$. The authors show that there is a unique map f_0 which minimizes this quantity. They give an explicit expression for f_0 in terms of which they are able to estimate the minimum value $D[f_0]$ as $.33\pm.005$.

R. Osserman (Stanford, Calif.)

Kamthan, Pawan Kumar 1334
On the maximum term of an integral function and its derivatives. II.
Monatsh. Math. **68** (1964), 125–129.

Part I (by Lakshminarasimhan) appeared in Publ. Math. Debrecen **8** (1961), 308–312 [MR **25** #1284]. The author improves results of R. S. L. Srivastava [Acta Math. Acad. Sci. Hungar. **13** (1962), 275–280; MR **26** #1461]. Let $f(z)$ be an entire function of order ρ, lower order λ, with $\mu_{(s)}(r)$ and $\nu_s(r)$ the maximum term and its rank for $f^{(s)}(z)$. Put $A(r,s)=\nu_s(r)-\nu(r)+s$. Then, for positive λ and finite ρ,

$$\liminf A(r,s)\leqq\lambda s\leqq\rho s\leqq\limsup A(r,s).$$

A second result connects $\limsup A(r,s)$ and $\liminf A(r,s)$ more directly with the coefficients. Now consider two entire functions f_1,f_2 and define $B(r)=\nu(r,f_2)-\nu(r,f_1)$, and let a, b denote lim sup and lim inf of

$$(\log r)^{-1}\log\{\mu(r,f_2)/\mu(r,f_1)\}.$$

Then $\liminf B(r)\leqq b\leqq a\leqq\limsup B(r)$.

R. P. Boas, Jr. (Evanston, Ill.)

Gol'dberg, A. A. 1335
The integral over a semi-additive measure and its application to the theory of entire functions. I, II. (Russian)
Mat. Sb. (*N.S.*) **58 (100)** (1962), 289–334; *ibid.* (*N.S.*) **61 (103)** (1963), 334–349.

Measure, integral, sequence convergence, inversion of order in multiple integrals are discussed in the first twenty pages. The one-dimensional integrals satisfy, for example, the inequalities

$$\int\{f(x)+g(x)\}\,d\mu(x)\leqq\int f(x)\,d\mu(x)+\int g(x)\,d\mu(x),$$

$$\int f(x)\,d\{\mu_1(x)+\mu_2(x)\}\geqq\int f(x)\,d\mu_1(x)+\int f(x)\,d\mu_2(x),$$

where all integrals are extended over the same set.

The applications to entire functions are intended to clarify the relations between the growth index

$$h(\varphi,f)=\limsup r^{-\rho(r)}\log|f(re^{i\varphi})|$$

of an entire function $f(z)$ with respect to a proximate order $\rho(r)$ and the distribution of the zeros. Suppose $n(r,\Theta,f)$ is the number of zeros of $f(z)$ within $|z|<r$ whose argument belongs to the set Θ. Define

$$\delta(\Theta,f)=\limsup r^{-\rho(r)}n(r,\Theta,f).$$

Questions to be studied include the determination of

properties of $h(\varphi, f)$ which follow from the inequality $\delta(\Theta, f) \leq \Delta(\Theta)$ when $\Delta(\Theta)$ is a measure of the type studied. (Various special cases, e.g., Θ restricted to be a sum of a finite number of non-overlapping intervals, arise.) A fundamental result is

$$(*)\quad h(f, \varphi) \leq \int_0^\infty t^{p-\rho-1}\, dt \int_{[0,2\pi)} \bar{K}(t, \varphi-\theta, p)\, d\Delta(\theta).$$

This result is available for proximate orders $\rho(r)$ which converge to the order ρ which is not an integer. The function $\bar{K}$ depends on p, the genus of $f(z)$, and is connected with the properties of the appropriate Weierstrass factor.

There is a converse to the effect that an entire function exists whose $h(\varphi, f)$ satisfies a similar inequality. The formal expression of this inequality is

$$h(\varphi, f) \geq \int_{[0,2\pi)} d\Delta^*(\theta) \int_0^\infty t^{\rho-p-1}\bar{K}(t, \varphi-\theta, p)\, dt.$$

Some special considerations relating to the maximum modulus are also developed.

In Part II the question of functions of integer order is considered. Proximate orders associated with zeros and growth are no longer the same. Corresponding to given $\rho(r)$ we define

$$r^{\rho_1(r)} = r^\rho \int_1^r t^{\rho(t)-\rho-1}\, dt$$

or

$$r^{\rho_2(r)} = r^\rho \int_r^\infty t^{\rho(t)-\rho-1}\, dt$$

as appropriate. In the first case the analogue of (*) (and its converse)

$$h_1(\varphi, f) \leq \int_{[0,2\pi)} \cos^+ \rho(\varphi-\theta)\, d\Delta(\theta)$$

holds. ($h_1(\varphi, f)$ uses $\rho_1(r)$.)

In the second case the results are affected by the exponential factor of the canonical product for $f(z)$.

A. J. Macintyre (Cincinnati, Ohio)

Ibragimov, I. I.; Mamedhanov, D. I. 1336

The relation between weighted norms of an entire function of finite degree on lines parallel to the real axis. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 258–261.

Let $f(z)$ be an entire function of exponential type σ, $\varphi(x) \geq 1$ a function continuous on the real line, subject to the condition that $\sup \varphi(x+y)/\varphi(x)$ $(-\infty < x < \infty;$ $|y| \leq t)$ is dominated by a polynomial in t with positive coefficients. The authors consider the class $W_\sigma^{(p,\varphi)}$ for which $\|f(x)/\varphi(x)\|_p = \|f\|_{p,\varphi}$ is finite, and its subclass $Б_\sigma^{(p,\varphi)}$ for which $|f(x+iy)| \leq |f(x-iy)|$. They first obtain an inequality for $\|f(x+iy)e^{-i\omega}+f(x-iy)e^{i\omega}\|_{p,\varphi}$ when $f \in W_\sigma^{(p,\varphi)}$. Then for $f \in Б_\sigma^{(p,\varphi)}$ they estimate $\|f(x+iy)\|_{p,\varphi}$ in terms of $\|f(x)\|_{p,\varphi}$ and $\|f(x+iy)\|_{p',\varphi}$ in terms of $\|f(x)\|_{p,\varphi}$ $(p' > p)$. Finally they state analogous theorems for functions of several variables.

R. P. Boas, Jr. (Evanston, Ill.)

Hayman, W. K. 1337

★**Meromorphic functions.**

Oxford Mathematical Monographs.

Clarendon Press, Oxford, 1964. xiv+191 *pp.* $10.10.

This attractively written monograph gives a rather complete account of the Nevanlinna theory of meromorphic functions. It also treats the Ahlfors theory of covering surfaces and the theory of invariant normal families. The book can be understood by a reader familiar with basic function theory [e.g., K. Knopp, *Theory of functions*, I, Dover, New York, 1945; MR **7**, 53] and with the elements of Lebesgue theory. A very nice feature of the book is the way in which routine discussions of standard results are interspersed with non-standard applications.

Contents: Chapter I: The Poisson-Jensen formula. The First Fundamental Theorem, Convexity of $T(r)$, the Ahlfors-Shimizu interpretation of $T(r)$, Canonical products.

Chapter II: The Second Fundamental Theorem deduced from the Poisson-Jensen formula along the lines of R. Nevanlinna's treatment in his Borel tract [*Le théorème de Picard-Borel et la théorie des fonctions méromorphes*, Gauthier-Villars, Paris, 1929], functions taking the same values at the same points, fixed points of entire functions, Pólya's theorem on the order of $f \circ h$ (f, h entire).

Chapter III: The distribution of values of the derivatives of meromorphic functions: Milloux's generalization of the Second Fundamental Theorem. Pólya's Theorem on the location of the zeros of the derivatives of meromorphic functions with at least two poles. A meromorphic function of finite order such that $f(z)f^{(e)}(z)$ (for some $e \geq 2$) has only a finite number of zeros is of the form $(P(z)/Q(z))e^{R(z)}$, P, Q, R polynomials. If $f(z)f^{(e)}(z)$ $(e \geq 2)$ has no zeros, then $f(z) = e^{Az+B}$ or $f(z) = (Az+B)^{-n}$ [J. Clunie, J. London Math. Soc. **37** (1962), 17–27; MR **26** #1456].

Chapter IV: Results about deficiencies: Example of an entire function with assigned deficiencies at assigned points. Theorems restricting the possible values of the deficiencies of meromorphic functions of certain orders (non-integral order, order ≤ 1).

Some of these results have been generalized recently by replacing the hypotheses about the order by hypotheses about the lower order [A. Edrei, Duke Math. J. **31** (1964), 1–21; MR **28** #1306; I. V. Ostrovskiĭ, Dokl. Akad. Nauk SSSR **150** (1963), 32–35; MR **26** #6417; ibid. **151** (1963), 34–37; MR **27** #2631]. A result first published here is that for a meromorphic function $f(z)$ of finite order $\sum_a (\delta(a, f))^\gamma < \infty$, if $\gamma > \frac{1}{3}$ (the lower bound $\frac{1}{3}$ is best possible). (The reviewer had claimed erroneously that $\frac{1}{2}$ was the true lower bound for γ [Ann. of Math. (2) **68** (1958), 203–209; MR **20** #7106].)

Chapter V: The Ahlfors theory of covering surfaces.

Chapter VI: A normal family F of functions meromorphic in the unit circle is called invariant if $f \in F$ implies $f \circ S \in F$, where S is any linear transformation leaving $|z| < 1$ invariant. The author shows that to an invariant normal family F there is a constant C such that $f = a_0 + a_1 z + \cdots \in F$ implies $|a_1| \leq 2\mu(\log \mu + C)$; $\log M(r, f) \leq ((1+r)\log \mu + 2Cr)/(1-r)$, where $\mu = \max(1, |a_0|)$. By means of the Ahlfors theory and of a theorem of J. Dufresnoy [Ann. Sci. École Norm. Sup. (3) **58** (1941), 179–259; MR **7**, 56], it is proved that Schottky's Theorem, Landau's Theorem and Bloch's Theorem are consequences of the theory of invariant normal functions.

The book closes with a discussion of functions of bounded characteristic. *W. H. J. Fuchs* (Ithaca, N.Y.)

af Hällström, Gunnar 1338

On the main theorems of the distribution of values. (Swedish. English summary)

Nordisk Mat. Tidskr. **12** (1964), 53–67, 88.

A concise account of the Nevanlinna theory of the distribution of values of meromorphic functions, including the two main theorems of Nevanlinna, the defect relation, and the Ahlfors-Shimizu generalization. Other related topics are discussed.

Rubel, L. A. 1339

A Picardesque theorem for formal power series.

Nieuw Arch. Wisk. (3) **12** (1964), 8–11.

Let $f(z)=a_1z+a_2z^2+\cdots$ be a formal power series and define $r_n(a)=\sup\{|z|: f_n(z: a)=0\}$, where $f_n(z: a)=-a+a_1z+a_2z^2+\cdots+a_nz^n$. With the convention that $r_n(a)=\infty$ if $a_n=0$ and $\infty/\infty=1$ the author proves that if $ab\neq 0$, then $\limsup_{n\to\infty}(r_n(b)/r_n(a))\leqq 2$. The case $ab=0$ is also dealt with. Simple direct proofs of the results are given. It is noted that the case $ab\neq 0$ can also be dealt with by using a result of Szegö on the convolution of two polynomials. No indication is given as to how near the results are to being best possible.

If $T_n(f: a)=\log r_n(a)$, the characteristic function of $f(z)$ is defined to be $T_n(f)=T_n(f: 1)$. The paper ends with a number of theorems which are rephrasings of known results in terms of the characteristic function. For example, a result of Okada can be stated as follows: In order that $f(z)$ be an entire function, it is necessary and sufficient that $T_n(f)\to\infty$ as $n\to\infty$. *J. Clunie* (London)

Goldberg, J. L. 1340

Functions with positive real part in a half-plane.

Duke Math. J. **29** (1962), 333–339.

Let P be the class of functions $f=u+iv$ analytic in the open right half-plane with $u>0$. The author extends certain results of Brune [J. Math. and Phys. **10** (1931), 191–236], Richards [Duke Math. J. **14** (1947), 777–786; MR **9**, 181], and Seshu and Balabanian [IRE Trans. Circuit Theory **CT-4** (1957), 306–312] concerning the subclass of P satisfying $f(\bar z)=\overline{f(z)}$ to the full class P. With these results he obtains the useful inequality

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| \leqq \frac{u(z_0)u(z)}{x_0x}$$

for $f\in P$, $x_0=\Re(z_0)>0$, and $x=\Re(z)>0$. Using it, he is able to prove easily a number of known theorems such as $\lim_{r\to\infty}f(z)/z=\lim_{r\to\infty}u(z)/x=\lim_{r\to\infty}f'(z)=A$, $0\leqq A<\infty$, uniformly in $|\arg z|\leqq\alpha<\pi/2$, due in parts to Landau, Valiron, Carathéodory, Tsuji, Wolff, and de Kok. *G. Johnson* (Houston, Tex.)

Kiselev, P. Ja. 1341

On the approximation of analytic functions by Faber-Walsh polynomials. (Russian)

Ukrain. Mat. Ž. **15** (1963), 193–199.

Let there be given in the z-plane distinct points $a_1, a_2, \cdots, a_\nu$ and positive numbers $m_1, m_2, \cdots, m_\nu$, with $\sum_1^\nu m_j=1$. J. L. Walsh has proved [Math. Ann. **136** (1958), 23–33; MR **21** #725] that when there exists a sequence of points $\alpha_1, \alpha_2, \cdots$, of which each α_j is some a_k, such that on every closed bounded point set E containing no a_k, one has (*) $0<A_1<|V_n(z)[U(z)]^{-n}|<A_2$, where $V_n(z)=\prod_1^n(z-\alpha_k)$ and $U(z)=\prod_1^\nu(z-a_k)^{m_k}$.

Let now E be a set consisting of mutually disjoint continua K_j $(j=1,\cdots,\nu)$ and let K be the complement of E. Denote by Γ_R the level line $G(z)=\log R$ $(R>1)$, where $G(z)$ is Green's function for K with pole at infinity, by E_R^∞ and E_R respectively the sets $G(z)>\log R$ and $G(z)<\log R$. Let D_R^∞ and D_R denote the sets $|U(w)|>\mu R$ and $|U(w)|<\mu R$, respectively. In Walsh's paper it has been proved that for z on Γ_R and τ in D_R^∞ there exists an expansion

$$\frac{\psi'(\tau)}{\psi(\tau)-z}=\frac{b_0(z)}{\tau-\alpha_1}+\frac{b_1(z)}{(\tau-\alpha_1)(\tau-\alpha_2)}+\cdots,$$

where the numbers α_j satisfy the condition (*) and $b_n(z)=z^n+\cdots$ are polynomials of respective degrees n. He gave explicit expressions for these polynomials (which for $\nu=1$ reduce to those of Faber). The same author has shown that an arbitrary function $f(z)$ analytic on E can be expanded in a series (**) $f(z)=\sum_0^\infty c_kb_k(z)$, valid throughout E_ρ, the largest E_R throughout which $f(z)$ is single-valued and analytic.

The author of the present paper shows some properties of the Faber-Walsh polynomials $b_n(z)$; he gives two theorems. Theorem 1: In order that the series (**) represent a function $f(z)$ which is analytic in the interior of the level line Γ_{R_0} $(R_0>1)$ it is necessary and sufficient that $\limsup|c_n|^{1/n}=(\mu R_0)^{-1}$ as $n\to\infty$. He is also concerned with the evaluation of the degree of approximation of a class of analytic functions by Faber-Walsh polynomials. First introduce the definition: The set E is called of type S if the following conditions are fulfilled: (a) E is bounded by a finite number of closed smooth Jordan curves which are mutually disjoint; (b) the functions $w=\varphi(z)$, $z=\psi(w)$ map the complement of E into the exterior line $\Lambda: |U(w)|=\mu$, with m_j rational numbers and μ determined by $\psi(\infty)=\infty$, $\psi'(\infty)=1$; (c) for the inverse function $z=\psi(w)$ one has $\psi'(w)$ $(\neq 0)$ continuous and $\psi''(w)$ bounded on Λ. Theorem 2: If on the set E of type S we define the function $f(z)$ which is analytic on the interior points of E, continuous and satisfying a Lipschitz condition of order α $(0<\alpha\leqq 1)$ on $\bar E$, then the following inequality is valid:

$$\left|f(z)-\sum c_k\cdot\rho_k^{(n)}b_k(z)\right|<\frac{A}{n^\alpha}\qquad(z\in\bar E),$$

where A is a constant independent of n and the $\rho_k^{(n)}$ are the summation coefficients of Jackson. *D. D. Stancu* (Cluj)

Shen, Xie-chang [Shen, Hsieh-chang] 1342

Completeness of the sequence of functions $\{f(\lambda_nz)\}$.

Acta Math. Sinica **14** (1964), 103–118 (*Chinese*); *translated as Chinese Math.* **5** (1964), 112–128.

The author studies the problem of the title when $f(z)$ is represented by a series $\sum a_nz^{r_n}$ (r_n real, not necessarily integers), where completeness refers to a real mean-square approximation on an unbounded set of special type. The r_n have finite density and satisfy some further conditions. The problem is reduced to a uniqueness theorem for a class of functions analytic on the Riemann surface of the logarithm (unaccountably denoted, for the most part, by L_nz). *R. P. Boas, Jr.* (Evanston, Ill.)

Zverovič, È. I. 1343

A boundary-value problem of Carleman type for multiply connected domains. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 1270–1272.

Let D be a finite $(m+1)$-connected region with Ljapunov contour L consisting of contours $L_1, \cdots, L_m$ enclosed by contour L_0. Let $\alpha_k(t)$ be an orientation-preserving homeomorphism of L_k onto itself, where $\alpha_k'(t)$ is Hölder-continuous on L_k and $\alpha_k[\alpha_k(t)] \equiv t$. Denote by $\alpha(t)$ the function $\sum_{k=0}^{m} \omega(t, L_k)\alpha_k(t)$, where $\omega(t, L_k)$ is the harmonic measure of L_k with respect to D. The author considers the problem of finding a function $\phi(z)$, single-valued and analytic in D, Hölder-continuous in $\bar{D}$, and such that (*) $\phi^+[\alpha(t)] = G(t)\overline{\phi^+(t)}$ on L, where $G(t) \neq 0$ is a given Hölder-continuous function on L. He shows that if $\chi = \sum_{k=0}^{m} \chi_k < 0$, where $\chi_k = \operatorname{ind} G(t)|_{L_k}$, then (*) has no solution: if $\chi > 2(m-1)$, then (*) has $\chi - m + 1$ solutions.

J. F. Heyda (King of Prussia, Pa.)

Szilárd, Karl 1344

Über die topologische Natur einiger allgemeiner Sätze der Theorie der elliptischen Funktionen. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 417–423 (1964).

The author remarks that some fundamental theorems on elliptic functions (the so-called Liouville theorems) are extended to a class of pseudo-analytic functions satisfying generalized Cauchy-Riemann differential equations. Precisely, let $w = f(z)$ be a doubly periodic continuous mapping from the complex z-plane to the complex w-sphere, such that it is orientation-preserving and with discrete inverse $f^{-1}(w_0)$ unless $f(z)$ is a constant. Such a function $f(z)$ is called a doubly periodic function of class P by the author. It is known [the author, Math. Z. **26** (1927), 653–671] that such a function $f(z)$ gives an open mapping unless it is a constant. Further, if $f(z) = u(x, y) + iv(x, y)$ satisfies the differential equation

$$v_y = au_x + bu_y, \quad -v_x = du_x + cu_y\,; \; 4ac - (b+d)^2 > 0, \quad a > 0,$$

$f(z)$ satisfies the above property, in general. Now, Liouville's first three theorems on the elliptic functions are generalized to the doubly periodic functions of class P, i.e., unless $f(z)$ is a constant, the image covers the whole w-sphere, and the number of roots of $w = f(z)$ in a fundamental parallelogram is the same $n \geqq 2$ for all w. Thus, these properties are of purely topological nature.

S. Hitotumatu (Tokyo)

Balk, M. B. 1345

Degenerate bi-analytic mappings. (Russian. Armenian summary)

Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk **17** (1964), *no.* 2, 9–15.

A function $B(z)$ is bi-analytic in a domain D if it has the form $\omega = B(z) = \phi(z) + \bar{z}\psi(z)$, where ϕ, ψ are regular in D. A transformation $\omega = f(z)$ is degenerate in D if the image of D in the ω-plane has no interior points. The author shows that a bi-analytic transformation degenerate in any domain has one of the forms (a) $\omega = A(e^{i\alpha}z + \bar{z}) + B$, or (b) $\omega = A(z+c)^\gamma(\bar{z}+\bar{c}) + B$, where A, B, c, α, γ are constants, $\operatorname{Im} \alpha = 0$, $\gamma = 1$.

I. N. Baker (London)

Beck, Anatole 1346

A theorem on maximum modulus.

Proc. Amer. Math. Soc. **15** (1964), 345–349.

Let D be a domain in the complex plane. If f is analytic in D and f is not constant, and if $x \in \bar{D}$ satisfies the relation $\limsup_x |f| = \sup_D |f|$, then x is a boundary point of D (maximum modulus principle). x is called the peak point for f in D if it is the only point for which the previous relation holds. If x is the peak point for a bounded analytic function in D, then x is a peak point of D. Via W. Rudin [Trans. Amer. Math. Soc. **78** (1955), 333–342; MR **16**, 685] a boundary point x of a domain D is a removable boundary point if every function bounded and analytic in D can be continued at x. All boundary points which are not removable are essential. Now, the following theorem is proved: x is a peak point of D if and only if x is an essential boundary point. The question when x is the peak point of a bounded analytic function f which has a limit at x or whose modulus has a limit at x is left open.

A. Mallios (Athens)

Beck, Anatole 1347

On rings on rings.

Proc. Amer. Math. Soc. **15** (1964), 350–353.

Let R be a ring isomorphic to the ring of bounded analytic functions on an annulus $A = \{z : \rho_1 < |z| < \rho_2\}$, where ρ_1 and ρ_2 are not known. The paper gives a solution to the problem of finding the ratio ρ_2/ρ_1, given the ring R. As a by-product, the following result is also obtained: Given a ring R isomorphic to the ring $A(D)$ of all analytic functions on an unknown domain D, then a conformal or anti-conformal image of D is found.

A. Mallios (Athens)

Arnol'd, V. 1348

Correction to V. Arnol'd's paper: "Small denominators. I." (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 479–480.

The original paper appeared in same Izv. **25** (1961), 21–86 [MR **25** #4113].

POTENTIAL THEORY

See also 1116, 1153, 1356, 1415, 1428, 1435.

Barry, P. D. 1349

On a theorem of Kjellberg.

Quart. J. Math. Oxford Ser. (2) **15** (1964), 179–191.

Let $u(z)$ be non-constant and subharmonic in the finite plane,

$$A(r, u) = \inf_{|z|=r} u(z), \qquad B(r, u) = \max_{|z|=r} u(z),$$

$$\lambda = \liminf_{r\to\infty} \frac{\log B(r, u)}{\log r}.$$

The following is proved: The upper logarithmic density $\bar{\Lambda}$ of the set of points r, $r \geqq 1$, for which $A(r, u) > \cos \pi\alpha B(r, u)$ satisfies $\bar{\Lambda} \geqq 1 - \lambda/\alpha$ $(\lambda < \alpha < 1)$. In a special case the above result is related to a result of Kjellberg [Thesis, Univ. Uppsala, Uppsala, 1948; MR **10**, 243].

E. Reich (Stanford, Calif.)

Gowrisankaran, Kohur 1350

Extreme harmonic functions and boundary value problems.

Ann. Inst. Fourier (Grenoble) **13** (1963), *fasc.* 2, 307–356.

The main object of the paper is the detailed proof of Fatou-type theorems like those of L. Naïm [same Ann. **7** (1957), 183–281; MR **20** #6608] and J. L. Doob [ibid. **9** (1959), 293–300; MR **22** #8233] (for Green spaces) in more general axiomatic situations. Announcements of most of the results are contained in two earlier publications [the author, C. R. Acad. Sci. Paris **255** (1962), 450–451; MR **27** #335; ibid. **256** (1963), 357–358; MR **26** #5174]. By using two suitably chosen sets U and P of numerical functions $\geqq 0$ on a set Ω (corresponding to harmonic functions $\geqq 0$ and to potentials on a domain) the author discusses first the notions of thinness and fine limits. A set $E \subset \Omega$ is said to be thin at a minimal function h of U if $R_h^E \neq h$ holds for the reduced function. For each minimal $h \neq 0$ in U the system $\mathfrak{T}_h$ of all sets E thin at h is a filter. Limits with respect to $\mathfrak{T}_h$ are studied. In the case of the axiomatic theories of M. Brelot [*Lectures on potential theory*, Tata Inst. Fundamental Res., Bombay, 1960; MR **22** #9749] and of the reviewer [Math. Ann. **146** (1962), 1–59; MR **26** #1612; Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **1** (1962/63), 197–229; MR **27** #5926] a topology on the union of Ω with the set Δ_1 of all extremal rays of U is introduced in such a way that this topology induces in Ω the fine topology and that $\mathfrak{T}_h$ is the trace in Ω of the neighborhoods of the ray $\bar{h} \in \Delta_1$ represented by h. By restricting himself to Brelot's theory with the more powerful axioms the author studies then the Dirichlet problem for the fine boundary Δ_1 and generalizes first results of Naïm and Doob. For deeper results an axiom stating the resolutivity of finite, uniformly continuous functions on Δ_1 is introduced. The analogue of the Fatou-Naïm-Doob theorem is then proved by the same kind of arguments as in the classical case. The theory is completed by showing that the axiom of resolutivity is valid in the case where two potentials with the same point-support are proportional (case of unicity). In this case the exact analogue of the Martin boundary and the corresponding Dirichlet problem are studied.

H. Bauer (Hamburg)

Morse, Marston 1351

Boundary values of partial derivatives of Poisson integral.

An. Acad. Brasil. Ci. **33** (1961), 131–139.

Let S be a unit $(n-1)$-sphere in euclidean n-space E, $\mathring{J}S$ the open ball bounded by S, and JS the closed ball. Suppose Γ is a Hausdorff space and F is the restriction to $S \times \Gamma$ of a function continuous in a neighborhood W of $S \times \Gamma$, relative to $E \times \Gamma$, with partial derivatives, relative to the coordinates of E, of orders up to m, which are continuous in W. For each p let U^p be the harmonic function in $\mathring{J}S$ having boundary values F^p on S. The function U^p has partial derivatives, defined in $\mathring{J}S$, of orders up to $m-1$, which may be extended continuously to JS. For each derivative these extensions yield a function continuous on $JS \times \Gamma$. The result is used in a paper by Huebsch and the author [Ann. Mat. Pura Appl. (4) **54** (1961), 359–378; MR **26** #1898] to obtain an analytic Schoenflies extension.

Guy Johnson (Houston, Tex.)

Bauer, Heinz 1352

Kennzeichnung kompakter Simplexe mit abgeschlossener Extremalpunktmenge.

Arch. Math. **14** (1963), 415–421.

Dans un article antérieur [Ann. Inst. Fourier (Grenoble) **11** (1961), 89–136; MR **25** #443], l'auteur avait caractérisé de plusieurs manières, dans un espace vectoriel localement convexe séparé, les compacts convexes X dont l'ensemble X_e des points extrémaux est fermé et qui sont des simplexes de Choquet. L'un des critères est que l'ensemble $\mathscr{A}$ des fonctions réelles continues linéaires affines sur X est réticulé pour l'ordre naturel. L'auteur donne ici, en s'inspirant de travaux de Choquet et Meyer, une démonstration plus simple et deux autres critères: (α) toute fonction réelle continue sur $\bar{X}_e$ est prolongeable sur X selon une fonction de $\mathscr{A}$; (β) toute fonction appliquant $\bar{X}_e$ dans $[-\infty, +\infty[$ et semi-continue supérieurement (s.-c.s.) est prolongeable en une des applications s.-c.s. de X dans $[-\infty, +\infty[$, de plus linéaires affines (formant la famille $\mathscr{A}_0$). On se base sur les propriétés: (a) extension à X de l'inégalité $k_1 \leqq k_2$ donnée sur X_e pour des fonctions semi-continues de sens quelconque, k_1 convexe, k_2 concave; (b) $l \in \mathscr{A}_0$ est enveloppe inférieure des majorantes de $\mathscr{A}$.

M. Brelot (Paris)

SEVERAL COMPLEX VARIABLES

See also 1219.

Nishino, Toshio 1353

Sur les valeurs exceptionnelles au sens de Picard d'une fonction entière de deux variables.

J. Math. Kyoto Univ. **2** (1962/63), 365–372.

If $F(x, y)$ is an entire function of two complex variables, then for each fixed y the function $F(x, y)$ is an entire function of the single variable x. Hence, by Picard's theorem, it takes on all values with at most one possible exception. Letting $\pi(y)$ denote this exceptional value, if it exists, the author investigates the behaviour of $\pi(y)$ as a function of the variable y. In particular, if $F(x, y)$ is a function of finite order in x, and if $\pi(y)$ exists for a set of points $\{y\}$ with a limit point, then there is a meromorphic function $m(y)$ such that: (1) $\pi(y)$ is defined and coincides with $m(y)$ at every regular point of m, except for those points y for which $F(x, y)$ reduces to a constant; and (2) $F(x, y)$ reduces to a polynomial in x at the poles of $m(y)$.

R. C. Gunning (Princeton, N.J.)

Bergman, Stefan 1354

The distinguished boundary sets and value distribution of functions of two complex variables.

Ann. Acad. Sci. Fenn. Ser. A I No. 336/12 (1963), 28 pp.

This paper contains the contents of a lecture given at the Colloquium on Mathematical Analysis, Helsinki, September 1962. The following problems on holomorphic functions defined on domains in the space C^2 of two complex variables which are bounded by finitely many analytic hypersurfaces and possess distinguished boundary sets are discussed. (1) Generalization of Nevanlinna's first theorem on meromorphic functions. Versions of this theorem on various domains in C^2 are found in the author's earlier papers [Bull. Soc. Sci. Lettres Łódź **13** (1962), no. 14;

MR 28 #4143; *Studies in mathematical analysis and related topics*, pp. 25–37, Stanford Univ. Press, Stanford, Calif., 1962; MR 27 #3829]. (2) A generalization of Nevanlinna's second theorem. (3) Bounds for holomorphic functions f. Schottky's theorem is used to give bounds for f on a domain bounded by two analytic hypersurfaces, one of which is such that every lamina defining it goes through the origin and f omits two values on each lamina. Certain theorems on schlicht functions in C^1 lead to bounds for functions in C^2. (4) Hadamard's theory in C^1 is used to give a connection between the coefficients of the series development of f and its regularity domain [see also the author, J. Analyse Math. **11** (1963), 249–274; MR **28** #2249]. The theory of orthogonal functions is used to study the same problem.

J. Mitchell (University Park, Pa.)

Rusev, P. 1355

Analytic mappings and Bergman functions. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 279–280.

Let $G\subset C^n$ be a bounded domain, $z^0\in G$. Let K be the Bergman kernel of G, and let M be the set of all n-tuples $\varphi=(\varphi_1,\cdots,\varphi_n)$ of (not necessarily single-valued) analytic functions such that their Jacobian J_φ is single-valued in G and $|J_\varphi(z^0)|=1$. Then $K(z^0,z^0)=\inf_{\varphi\in M}\|J_\varphi\|^2$ (the norm being the Bergman norm). An application of this result is the following: If G is a polycylinder with center z^0, and $\varphi\in M$ is a one-to-one map of G onto a domain H which is a product of domains in C^1 and has volume equal to that of G, then H is a polycylinder and φ is linear.

A. Korányi (Berkeley, Calif.)

Tsurumi, Kazuyuki 1356

Potential theory in Bergman spaces.

Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **8**, 122–132 (1964).

Let $D\subset C^n$ be a bounded domain. Then D has a Bergman metric; the Beltrami first-order differential parameter and the Laplacian with respect to this metric can be easily expressed in terms of the Bergman kernel function. For subdomains B of D with a sufficiently regular boundary the author establishes Green's formula and the Dirichlet principle for functions on $\bar{B}$ harmonic with respect to the Bergman metric of D. These are generalizations of previous results of J. Mitchell [Trans. Amer. Math. Soc. **79** (1955), 401–422; MR **17**, 253] and K. Morita [Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **5** (1956), 190–212; MR **19**, 541], who considered only the classical symmetric domains. The last section of the paper gives rather explicit formulas for the solution of the Dirichlet and Neumann problems in the special cases where D is the unit ball or the unit polycylinder in C^n.

A. Korányi (Berkeley, Calif.)

Aizenberg, L. A. 1357

Integral representations of holomorphic functions of several complex variables. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 9–12.

Let $D\subset C^n$ be a bounded domain with smooth boundary. The author establishes a very general Cauchy-type integral formula involving integration on the entire boundary of D. This is a generalization of a previous result of the author [same Dokl. **151** (1963), 1247–1249; MR **28** #1321] and contains as special cases the integral formulas of Cauchy-Fantappiè, Bochner-Martinelli, and Temljakov, as well as certain representations in terms of the Szegö kernel.

A. Korányi (Berkeley, Calif.)

Bungart, L.; Rossi, H. 1358

On the closure of certain spaces of holomorphic functions.

Math. Ann. **155** (1964), 173–183.

From the authors' introduction: "In same Ann. **136** (1958), 245–318 [MR **21** #2063], Grauert and Remmert proved the following theorem. Let X be an analytic space with a separable topology. The ring of holomorphic functions on X, $H(X,\mathcal{O})$, is a Fréchet space in the topology of uniform convergence on compact subsets. Their proof is not difficult, but depends on several very deep results in the theory of analytic spaces. We offer here a (perhaps) more unpleasant, but elementary proof of this fact, based only (at least for pure dimensional analytic spaces) on the local study of varieties.

"Our method permits also the extension of the theory of the Bergman kernel function to analytic spaces. We consider, on a pure k-dimensional analytic space, the space $H^2(X)$ of square integrable holomorphic reduced k-forms on X. $H^2(X)$ is a Hilbert space, and point evaluations at the regular points are continuous linear functions on $H^2(X)$, so there exists a kernel form on the set of regular points of X. This form can be shown to extend holomorphically over the singular set.

"If X admits a holomorphic nowhere degenerate map $\pi: X\to C^k$, then we may consider the space $\mathscr{H}^2(X)$ of holomorphic functions f on X such that the form $f\pi^*(dz^1\wedge\cdots\wedge dz^k)$ ($z^1,\cdots,z^k$ are coordinates for C^k) is square integrable. $\mathscr{H}^2(X)$ is a Hilbert space of holomorphic functions (dependent of the choice of nowhere degenerate map π). Thus, for such spaces, there is also defined a kernel function."

A. Browder (Berkeley, Calif.)

Vetter, Udo 1359

Über nicht kontinuierlich operierende komplexe Transformationsgruppen auf komplexen Räumen.

Schr. Math. Inst. Univ. Münster No. 28 (1964), ii + 50 *pp.*

Let X be a complex space with a countable base of open sets. Let L be a complex Lie-group consisting of biholomorphic maps of X onto X which operates holomorphically on X. Let $\pi: X\to X/L$ be the residual map. L operates continuously on the open set U of X if and only if $\pi(U)$ is a Hausdorff subspace of X/L. Holmann [Math. Ann. **150** (1963), 327–360; MR **27** #776] showed that X/L has a complex structure if and only if L operates continuously on X. This structure can be chosen such that π is meromorphic and such that for every open subset V of X and every L-invariant holomorphic function f on $\pi^{-1}(V)$ a holomorphic function f^* on V exists such that $f^*\circ\pi=f$. The author considers the discontinuous case. He modifies the quotient space X/L to a complex space Y and the residual map π to a meromorphic map $m: X\to Y$ by requiring: (1) The meromorphic map $m: X\to Y$ is open and surjective; (2) An open, dense, L-saturated subset X' of X and a surjective map $\tau: X'/L\to m(X')$ exist such that L operates continuously on X', such that $m|X'$ is continuous, and such that τ is biholomorphic with $\tau\circ(\pi|X')=m|X'$; (3) If f is an L-invariant holomorphic

function on an open subset U of X, then a holomorphic function f^* on $m(U)$ exists such that $f^* \circ m = f$; (4) Let Y' be a complex space. Suppose that every function f which is holomorphic on $U - N$, where N is a thin analytic subset of U, is a holomorphic map satisfying (1) and (2). Then there is a proper, holomorphic modification map $\tau: Y' \to Y$ such that $\tau \circ m' = m$.

If (1) and (2) are satisfied, and if (Y', τ) is a proper modification of Y, then the requirements (1) and (2) are satisfied for Y' and $m' = \tau^{-1} \circ m$. Define $R_L = \{(x, y) | \pi(x) = \pi(y)\}$. If (1) and (2) are satisfied, then R_L is analytic in $X \times X$. If R_L is analytic in $X \times X$ and if two additional conditions are satisfied, then the requirements (1)–(4) can be fulfilled (Theorem 9).

The following theorem is proved in the second chapter. Let $(X, \mathcal{O})$ be a complex space. Let A be an analytic subset of X. Then $\mathcal{O}$ induces a structure sheaf $\mathcal{O}_A$ on A. Let $\mathcal{R}'$ be an equivalence relation on A such that the residual map $\pi': A \to A/\mathcal{R}'$ is proper. Suppose that $(A/\mathcal{R}', \mathcal{O}_{A/\mathcal{R}'})$ is a complex space. Extend $\mathcal{R}'$ to an equivalence relation $\mathcal{R}$ on X such that the equivalence class of each point $x \in X - A$ consists of x alone. Then $(X/\mathcal{R}, \mathcal{O}/\mathcal{R})$ is a complex space. *W. Stoll* (Notre Dame, Ind.)

Forster, Otto 1360

Funktionswerte als Randintegrale in komplexen Räumen.
Math. Ann. **150** (1963), 317–324.

The author shows how to represent values of holomorphic functions as boundary integrals in "complex spaces" as defined by J.-P. Serre [Ann. Inst. Fourier (Grenoble) **6** (1955/56), 1–42; MR **18**, 511]. More precisely, he proves that if X is a complex space and G is a relatively compact open set with Shilov boundary S in X, then there exists a family M of mappings whose domains are open subsets of G and cover G, where for $\mu \in M$ the values $\mu(x)$ are in the conjugate space of $\mathscr{C}(S, \mathbf{C})$ and depend w^*-holomorphically on x; and for each f holomorphic on G and continuous on the closure one has the (integral) representation $f(x) = \mu(x)(f|S)$ for each x in the domain of μ. Assuming further that X is holomorphically pre-complete, he shows that the domains of the various μ in M include all open relatively compact subsets of G. These facts yield a generalization of a theorem of Hartogs as follows: Let X, Y be complex spaces, Y irreducible. Let Y_0 be open in Y. Let G be a relatively compact open subset of X with Shilov boundary S. Now suppose H and T are open sets such that $H \supset \bar{G}$ and $T \supset S$. Then a function holomorphic on $(H \times Y_0) \cup (T \times Y)$ can be holomorphically extended (and in only one way) to $(G \cup T) \times Y$.

R. Arens (Los Angeles, Calif.)

Kajiwara, Joji 1361

Note on holomorphically convex complex spaces.
J. Math. Soc. Japan **15** (1963), 437–442.

This paper is devoted to proving the theorem that, for a 2-dimensional connected holomorphically convex complex space X, the following three conditions are equivalent: (1) X can be obtained by a proper point modification of a Stein space; (2) there exist two holomorphic functions on X which are analytically independent at each point of X; (3) X can be exhausted by strongly pseudoconvex domains. *R. C. Gunning* (Princeton, N.J.)

Siegel, Carl Ludwig 1362

★Symplectic geometry.
Academic Press, New York-London, **1964**. viii+86 pp. $4.00.

A reprinting, together with a page of errata, of the author's classical paper [Amer. J. Math. **65** (1943), 1–86; MR **4**, 242].

SPECIAL FUNCTIONS

See also 1115, 1175, 1332, 1344, 1489, 1491, 1497, 1514.

★Bessel functions. Part IV: Kelvin functions. 1363

Prepared on behalf of The Mathematical Tables Committee by Andrew Young and Alan Kirk. Royal Society Mathematical Tables, Vol. 10.
Published for the Royal Society at the Cambridge University Press, New York, 1964. xxiii+97 pp. $11.50.

The Kelvin functions are defined as follows

$$\mathrm{ber}_n\, x + i\, \mathrm{bei}_n\, x = J_n(xe^{3i\pi/4}) = M_n e^{i\theta_n},$$

$$\mathrm{ker}_n\, x + i\, \mathrm{kei}_n\, x = \tfrac{1}{2} i\pi H_n^{(1)}(xe^{3i\pi/4}) = N_n e^{i\varphi_n}$$

and occur in solutions of the diffusion or heat conduction equation in cylindrical coordinates, in particular, in problems of eddy currents in cylindrical conductors.

The main tables give the four Kelvin functions (1) to $15D$ for $n = 0, 1$ and $x = 0(.1)10$; (2) to $8S$ for $n = 0, 1, 2$ and for $x = 0(.01)2.5$ and (3) to $7S$ for $n = 0(1)10$ and $x = 0(.1)10$. In the second and third tables the polar form is also tabulated, i.e., M_n, θ_n, N_n, φ_n, with θ_n and φ_n in radians. In addition, there are various auxiliary tables to facilitate full interpolation in the tables. Everett interpolation is recommended, and δ^2 (modified where necessary) and γ^4 (a modified fourth difference) are tabulated.

The main third table was computed using recurrence relations of the form

$$\mathrm{ber}_{n+1}\, x = -n\sqrt{2}\, x^{-1}[\mathrm{ber}_n\, x - \mathrm{bei}_n\, x] - \mathrm{ber}_{n-1}\, x$$

from the basic values in the first table. These values had been obtained by summing by hand certain subseries of the power series for the functions which were evaluated on the Manchester computer to $21D$. [Other sums of these subseries produce $J_n(x)$ and this was used as a check.] In order to facilitate the evaluation of the polar form, $10D$ tables of $(1+\alpha^2)^{1/2}$ and arctan α for $\alpha = 0(.0001)1$ were prepared.

The second table was based on the values in the National Bureau of Standards tables [*Tables of the Bessel functions $J_0(z)$ and $J_1(z)$ for complex arguments*, Columbia Univ. Press, New York, 1943; MR **5**, 159; 2nd ed., 1947; MR **9**, 103]; the values for $n = 2$ were obtained from recurrence relations.

Detailed descriptions of the checks employed are given; in particular, all function values were recomputed on the Liverpool DEUCE using 62 (and even 92) binary place arithmetical routines. It is stated that each tabular value is subject to an error of at most one unit in the last place.

There is an extensive introduction which gives various properties and representations of the functions, especially asymptotic expansions. There is an up-to-date bibliography. The volume is excellently produced.

John Todd (Pasadena, Calif.)

rber, Henry 1364

irst one hundred zeros of $J_0(x)$ accurate to 19 significant igures.

Math. Comp. **18** (1964), 319–322.

e zeros were calculated by use of the McMahon series F. W. J. Olver (Editor), *Bessel functions*, Part III, mbridge Univ. Press, New York, 1960; MR **22** #10202], l early ones checked by use of the Harvard tables *ables of the Bessel functions of the first kind of orders zero l one*, Harvard Univ. Press, Cambridge, Mass., 1947; R 8, 406]. *John Todd* (Pasadena, Calif.)

rgoyne, F. D. 1365

Generalized trigonometric functions.

Math. Comp. **18** (1964), 314–316.

table of functions resulting from the investigation of ometrical properties of the curves $x^n/a^n+y^n/b^n=1$ is esented for various values of the argument. The neralized function is related, in particular cases, to gonometric and Jacobian elliptic functions.

M. D. Friedman (San Jose, Calif.)

rison, B. C. 1366

Normal elliptic integrals of the first and second kinds.

Duke Math. J. **31** (1964), 405–419.

order to give a unified treatment of elliptic integrals of e first and the second kind, the author introduces the llowing generalized hypergeometric function:

$$(a;b_1,\cdots,b_n;z_1,\cdots,z_n) = \sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{(a,m_1+\cdots+m_n)(b_1,m_1)\cdots(b_n,m_n)}{(b_1+\cdots+b_n,m_1+\cdots+m_n)m_1!\cdots m_n!}\times(1-z_1)^{m_1}\cdots(1-z_n)^{m_n},$$

here $(a,m)=\Gamma(a+m)/\Gamma(a)$. Next, the following functions e defined:

$$R_K(z_1,z_2) = R(\tfrac12;\tfrac12,\tfrac12;z_1,z_2),$$
$$R_E(z_1,z_2) = R(-\tfrac12;\tfrac12,\tfrac12;z_1,z_2),$$
$$R_F(z_1,z_2,z_3) = R(\tfrac12;\tfrac12,\tfrac12,\tfrac12;z_1,z_2,z_3),$$
$$R_G(z_1,z_2,z_3) = R(-\tfrac12;\tfrac12,\tfrac12,\tfrac12;z_1,z_2,z_3).$$

hen, in the usual notations, the complete and incomplete lliptic integrals are expressed as follows:

$$K(k) = \tfrac12\pi R_K(1-k^2,1),$$
$$E(k) = \tfrac12\pi R_E(1-k^2,1),$$
$$F(\phi,k) = \sin\phi R_F(\cos^2\phi,\Delta^2,1),$$
$$E(\phi,k) = \sin\phi R(\tfrac12;\tfrac12,-\tfrac12,\tfrac32;\cos^2\phi,\Delta^2,1),$$

here $\Delta^2=1-k^2\sin^2\phi$. In order to have a second incomplete integral that is symmetric, the author introduces he function

$$G(\phi,k) = 2\sin\phi R_G(\cos^2\phi,\Delta^2,1) = \cos^2\phi F(\phi,k)+\sin^2\phi E(\phi,k)+\Delta\sin\phi\cos\phi,$$

which satisfies $G(\frac12\pi,k)=E(k)$.

Various quantities in the theory of elliptic functions are epresented by these functions, and their properties are derived from general formulas of the R-function. For example, the quadratic Landen and Gauss transformations take a unified form, derived as a special case of a new quadratic transformation of the R-function. Finally, as an example of the applications, the surface area of an ellipsoid $x^2/\alpha^2+y^2/\beta^2+z^2/\gamma^2=1$ is given by $4\pi\alpha\beta\gamma R_G(\alpha^{-2},\beta^{-2},\gamma^{-2})$. The author emphasizes that the functions R_K, R_E, R_F, R_G are very well-suited for use in the tables of normal elliptic integrals as well as for numerical work. *S. Hitotumatu* (Tokyo)

Carlitz, L. 1367

Some generating functions of Weisner.

Duke Math. J. **28** (1961), 523–529.

The author gives an alternative direct proof of the following result of Weisner [Pacific J. Math. **5** (1955), 1033–1039; MR **19**, 264]:

$$(1-w)^{\alpha+\beta-\gamma}(1+(x-1)w)^{-\alpha}(1+(y-1)w)^{-\beta}F(\alpha,\beta;\gamma;\zeta) = \sum_{n=0}^{\infty}\frac{(r)_n w^n}{n!}F(-n,\alpha;\gamma;x)F(-n,\beta;\gamma;y),$$

where $\zeta=xyw/(1+(x-1)w)(1+(y-1)w)$. Several interesting cases associated with this generating function are discussed. *R. P. Agarwal* (Zbl **102**, 58)

Cohen, Donald S. 1368

Zeroes of Bessel functions and eigenvalues of non-self-adjoint boundary value problems.

J. Math. and Phys. **43** (1964), 133–139.

The problem considered is the equation $AJ_\nu(z)+BJ_\nu'(z)=0$, in which A, B and z have given real or complex values, and ν is sought. Particular attention is paid to the cases $A=1$, $B=0$, and $A=0$, $B=1$.

Series expansions for ν are obtained when $|\nu|\gg|z|$, $\nu\doteqdot z$ and $|\nu|\ll|z|$ by reversion of well-known expansions for $J_\nu(z)$ and $J_\nu'(z)$ applicable in these situations. {Reviewer's note. A somewhat more unified approach to the solution of this problem would be provided by the use of the uniform expansions for Bessel functions of large orders given in the reviewer's paper [Philos. Trans. Roy. Soc. London Ser. A **247** (1954), 328–368; MR **16**, 696].}

F. W. J. Olver (Washington, D.C.)

Chako, Nicholas 1369

On the evaluation of certain integrals and their application to diffraction theory.

Acta Phys. Polon. **24** (1963), 611–620.

The integral

$$I_{\alpha,\beta,\gamma,\lambda}(\rho,z,a) = \int_0^\infty e^{-z(t^2-1)^{1/2}}J_\alpha(\rho t)J_\beta(at)(t^2-1)^\gamma t^{2\lambda}\,dt,$$

where $\mathrm{Im}\{\rho\}=\mathrm{Im}\{z\}=\mathrm{Im}\{a\}=0$, and $\mathrm{Re}\{\lambda+\frac12\}>0$, is evaluated in terms of infinite series.

T. Erber (Chicago, Ill.)

Kishore, Nand 1370

A structure of the Rayleigh polynomial.

Duke Math. J. **31** (1964), 513–518.

Let $j_{\nu,m}$ ($m = 1, 2, \cdots$) denote the zeros of $z^{-\nu}J_\nu(z)$ with $|\Re(j_{\nu,m})| \leq |\Re(j_{\nu,m+1})|$, and put

$$\sigma_{2n}(\nu) = \sum_{m=1}^{\infty} (j_{\nu,m})^{-2n} \quad (n = 1, 2, \cdots),$$

$$\phi_{2n}(\nu) = 4^n \prod_{k=1}^{n} (\nu+k)^{[n/k]} \cdot \sigma_{2n}(\nu).$$

Then $\phi_{2n}(\nu)$ is the Rayleigh polynomial; it has positive integral coefficients, and its degree is $1-2n+\sum_{k=1}^{n}[n/k]$. The following theorem is proved: $\sigma_{2n}(\nu) = \sum_{i=1}^{c(n)} \{2^{N_i}P_i\}^{-1}$, where (1) $N_i = 2n - n_i > 0$; (2) P_i is a polynomial of degree $2n-1$; (3) P_i has positive integral coefficients; (4) P_i has only negative integral zeros in $[-n, -1]$. The first seven Rayleigh polynomials are exhibited.

L. Carlitz (Durham, N.C.)

Khandekar, P. R. 1371

On the product of two ultraspherical polynomials.

Math. Japon. **8** (1963), 33–36.

Put

$$(*) \qquad C_m^\mu(x)C_n^\mu(x) = \sum_{r=0}^{\min(m,n)} A_r C_{m+n-2r}(x).$$

The author shows that in general the coefficient A_r is expressible as a multiple of a ${}_9F_8$. Similarly for

$$C_m^{1-\nu}(x)C_n^\nu(x) = \sum_{r=0}^{\min(m,n)} \bar{A}_r C_{m+n-2r}^\mu(x),$$

he shows that $\bar{A}_r$ is a multiple of a ${}_4F_3$. In particular, when $m=n$, the A_r in (*) is a multiple of a ${}_4F_3$; moreover, if $\nu = 2\mu - \frac{1}{2}$, the ${}_4F_3$ can be summed. Finally, the integrals

$$\int_{-1}^{1} (1-x^2)^{2\mu-1}(C_m^\mu(x))^2(C_n^\mu(x))^2\,dx \quad (\Re\mu > 0),$$

$$\int_{-1}^{1} (1-x^2)^{\nu-1/2}C_m^{1-\nu}(x)C_n^\nu(x)C_p^\nu(x)C_q^\nu(x)\,dx \quad (\Re\nu > 0)$$

are exhibited as multiples of ${}_9F_8$'s.

L. Carlitz (Durham, N.C.)

Al-Salam, W. A. 1372

Operational derivation of some formulas for the Hermite and Laguerre polynomials.

Boll. Un. Mat. Ital. (3) **18** (1963), 358–363.

The author re-derives a number of standard formulas involving Hermite and Laguerre polynomials by "operational" methods. The formula he believes to be new is actually well known, being a special case of a more general formula given, for example, in Rainville's *Special functions*, p. 211, no. (9) [Macmillan, New York, 1960; MR **21** #6447].

P. G. Rooney (Toronto, Ont.)

Carlitz, L. 1373

A bilinear generating function for the Hermite polynomials.

Duke Math. J. **28** (1961), 531–535.

The author derives a bilinear generating function for the Hermite polynomials in the form

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{t^n}{(n!)^2} =$$

$$I_0(2t)I_0(4\sqrt{(xyt)}) + \sum_{k=1}^{\infty}(-1)^k\frac{x^{2k}+y^{2k}}{x^ky^k}I_k(2t)I_{2k}(4\sqrt{(xyt)}),$$

where $I_k(t)$ is the Bessel function of purely imaginary argument. Several interesting particular cases of the above generating function are discussed.

R. P. Agarwal (Zbl **102**, 56)

ORDINARY DIFFERENTIAL EQUATIONS

See also 1113, 1472, 1554, 1557, 1563, 1633, 1634, 1652, 1663, 1742, 1755, 1773, 2103, 2125, 2127.

Bandić, Ivan 1374

Sur les invariants de quelques équations différentielles non-linéaires du deuxième ordre qui apparaissent dans la physique théorique.

C. R. Acad. Sci. Paris **258** (1964), 4417–4420.

L'auteur démontre que les fonctions $I_1 = f^{n+3}q^{-2}$, $I_2 = f^{-2}[f^{1/2}(f^{-1/2})'' - \frac{1}{2}f' + \varphi]$ représentent les invariants absolus de l'équation

$$(1) \qquad y'' + f(x)y' + \varphi(x)y = q(x)y^n$$

pour chaque transformation $x = \alpha(t)$, $y = \beta(t)z$. En appliquant ces résultats à certains cas (1) qui apparaissent dans divers problèmes de la physique théoretique on obtient les nouvelles conditions d'intégrabilité de ces équations. Il s'agit de cas suivants: (1) $f=1$, $\varphi=\frac{1}{4}$, $q=0$; (2) $f=0$, $n=-1$, $\varphi=a$, $q=b$; (3) $\varphi=0$, $q=a$ ($a=$const, $b=$const).

M. Ráb (Brno)

Tatarkiewicz, K. 1375

Sur les équations différentielles ordinaires résolubles par des méthodes élémentaires.

Colloq. Math. **11** (1963), 113–121.

The author treats the equations

$$(*) \qquad F(y, dy/dx, \cdots, d^ny/dx^n) = 0$$

in which the independent variable does not appear. If $y = y(x)$ is a solution of (*) admitting an inverse function $x = y_{-1}(y)$, then the substitution $q(y) = y^{(k)}(y_{-1}(y))$ reduces (*) to an equation of order $n-k$ provided $G(y, y', \cdots, y^{(k-1)}, q, q', \cdots, q^{(n-k)}) \equiv F(y, y', \cdots, y^{(k-1)}, q, y'q, q''y'^2 + q'y'', \cdots)$ is independent of $y', \cdots, y^{(k-1)}$. Using this method several examples are reduced to solution by quadrature. Other substitutions which reduce (*) to solution by quadrature are investigated, as is a generalization to systems. Many special examples are given.

T. L. Sherman (Tempe, Ariz.)

Skorobogat'ko, V. Ja. 1376

The factorization of linear and non-linear differential operators into real factors. I. (Russian)

Ukrain. Mat. Ž. **15** (1963), 217–223.

Some obvious consequences of G. Mammana's factorization theorem.

P. Ungar (New York)

Bureau, Florent 1377

Sur des équations différentielles du quatrième ordre dont l'intégrale générale est à points critiques fixes.

C. R. Acad. Sci. Paris **258** (1964), 38–40.

On étudie des équations stables du quatrième ordre de la forme

$$y^{(iv)} = ay\ddot{y}+b\dot{y}\ddot{y}+cy^2\ddot{y}+dy\dot{y}^2+ey^3\dot{y}+fy^5+F(x, y),$$

$$(1)\quad F(x, y) = a_0\ddot{y}+(c_1y+c_0)\ddot{y}+d_0\dot{y}^2 + (e_2y^2+e_1y+e_0)\dot{y}+f_4y^4+f_3y^3+f_2y^2+f_1y+f_0,$$

où les a, b, c, d, e, f, avec ou sans indice, sont des fonctions analytiques de x dans un certain domaine.

Sous l'hypothèse qu'au voisinage d'un point $x=x_0$, on ait $y(x)=s(x)/(x-x_0)^r$, $r>0$, $s(x_0)\neq 0$ et $s(x)$, fonction analytique de x au voisinage de $x=x_0$, l'auteur considère les cas $r=2$ et $r=1$. Dans le cas $r=2$, il y a une seule série de pôles doubles mobiles. On donne la forme canonique des équations de cette classe. Le cas $r=1$ donne une seule série de pôles simples mobiles, ou deux séries. Les nombreuses équations de cette classe sont ramenées aux formes canoniques. *M. Bertolino* (Belgrade)

Levin, A. Ju. 1378

On the distribution of zeros of solutions of a linear differential equation. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 1281–1284.

Eine Reihe von Sätzen ohne Beweise, welche die Verteilung der Nullstellen der Lösungen von

$$x^{(n)}+p_1(t)x^{(n-1)}+\cdots+p_n(t)x = 0 \quad (a \leqq t \leqq b)$$

betreffen. *M. Ráb* (Brno)

Dnestrovskiĭ, Ju. N.; Kostomarov, D. P. 1379

Asymptotic behaviour of eigenvalues for non-selfadjoint boundary-value problems. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 267–277.

Continuing their earlier work [Dokl. Akad. Nauk SSSR 152 (1963), 28–30; MR **27** #3859], the authors consider the eigenvalue problem of the second-order differential equation

$$\frac{d^2w}{dx^2}+k^2Q(x, \lambda)w = 0,$$

$$w(+\infty, \lambda, k) = w(-\infty, \lambda, k) = 0,$$

where the parameter λ occurs in the equation non-linearly. After classifying the zeros of $Q(z, \lambda)$, $z=x+iy$, an approximate equation determining the behaviour of the eigenvalues in the neighbourhood of a point $\lambda_0 \in G$, where G is a certain domain in the plane of the complex variable, is obtained. *V. Lakshmikantham* (Aurangabad)

Moszyński, Krzysztof 1380

Notes on asymptotic distribution of eigenvalues in the Sturm-Liouville problem and related results.

Algorytmy **1**, no. 2, 7–19 (1963).

The classical regular Strum-Liouville problem $L[u]=u''+(\lambda-q(x))u=0$ with boundary conditions

$$u(a)\cos\alpha+u'(a)\sin\alpha = 0,$$

$$u(b)\cos\beta+u'(b)\sin\beta = 0,$$

where a, b, α, β are real and q is a real-valued continuous function on $[a, b]$, is considered. A proof that the eigenvalues λ_k of this system satisfy the relation

$$\sqrt{\lambda_k} = \pi k/(b-a)+O(1/k)$$

is given.

The author then shows that the series $\sum c_n\psi_n'(x)$ converges to $f'(x)$, where c_n is the nth Fourier coefficient of f and ψ_n are the normalized eigenfunctions of this system provided that f satisfies the above boundary conditions and $(L[f])'$ is bounded and integrable. Similar results hold for $\sum c_n\psi_n''(x)$. He then applies his results to verify the convergence of a simple recurrence algorithm which solves the above system. *D. O. Banks* (Davis, Calif.)

Škil', M. I. 1381

Construction of an asymptotic solution of a system of linear differential equations with a small parameter. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 572–575.

Author's summary: "Algorithms are given for the construction of an asymptotic solution of the system

$$\frac{dx}{dt} = A(\tau, \varepsilon)x+\varepsilon B(\tau, \varepsilon)e^{i\theta(\tau)}.$$

It is assumed that the roots of the characteristic equation possess constant multiplicity on a finite segment, while the external frequency becomes equal to one of the multiple roots in isolated points of the segment."

Škil', N. I. [Škil', M. I.] 1382

Asymptotic solution of a system of linear differential equations in the case of multiple roots of the characteristic equation. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, no. 2 (39), 176–185.

Der Verfasser leitet eine asymptotische Entwicklung für die Lösungen eines Systems mit kleinem Parameter

$$(1)\qquad \frac{dx}{dt} = A(\tau, \varepsilon)x+\varepsilon b(\tau, \varepsilon)e^{i\Theta}, \qquad \tau = \varepsilon t,$$

ab, wobei A eine Matrix vom Typus $n\times n$ bedeutet, $A(\tau, \varepsilon)=\sum_0^\infty \varepsilon^s A^{(s)}(\tau)$ und b ein n-dimensionaler Vektor ist, $b(\tau, \varepsilon)=\sum_0^\infty \varepsilon^s b^{(s)}(\tau)$ im Falle, dass die charakteristische Gleichung (1) $\det\|A^{(0)}(\tau)-\lambda E\|=0$ mehrfache Wurzeln mit konstanter Vielfachheit im Intervall $\langle 0, L\rangle$ hat und die Funktion $i\,d\Theta/dt$ in isolierten Punkten dieses Intervalles einer Wurzel von (1) gleich ist.

M. Ráb (Brno)

Stonic'kiĭ, A. A. 1383

Asymptotic representation of the solution of a differential equation with oscillating free term. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 299–303.

Author's summary: "An algorithm is given expressing in the 'resonance' case the formal solution of the equation

$$\frac{d^2u}{dt^2}+C(\tau, \varepsilon)u = f(\tau, \varepsilon)e^{i\theta(t,\varepsilon)}$$

through the eigenvalues and eigenfunctions of the principal part C_0 of the operator coefficient $C(\tau, \varepsilon)$ and through the solution of equations of the first order."

Waltman, Paul 1384

A bifurcation theorem.

Proc. Amer. Math. Soc. **15** (1964), 627–631.

Suppose that in the equations $dx_1/dt=f_1(x_1, x_2, \mu)$, $dx_2/dt=f_2(x_1, x_2, \mu)$ (i) f_1 and f_2 are continuous in E^3 and for each fixed μ are C^1 in (x_1, x_2), (ii) there is an interval I containing a point μ_0 such that p is a stable spiral point for $\mu<\mu_0$ and is an unstable spiral point for $\mu \geqq \mu_0$ and $R_\mu(p)$, the canonical μ-region of p, is simply connected, (iii) μ appears in the equations so as not to affect the location of critical points in the plane, and (iv) all critical points are isolated. Then there is an interval (μ_1, μ_0) such that for $\mu_1<\mu<\mu_0$ there is a limit cycle surrounding p which tends to p as μ tends to μ_0. An application is given for generalized differential equations of growth.

W. R. Utz (Columbia, Mo.)

Deng, Yao-hua [Teng, Yao-hua]; Luo, Ding-jun [Lo, Ting-chun] 1385

A qualitative study of the integral curves of the differential equation

$$\frac{dy}{dx}=\frac{q_{00}+q_{10}x+q_{01}y+q_{20}x^2+q_{11}xy+q_{02}y^2}{p_{00}+p_{10}x+p_{01}y+p_{20}x^2+p_{11}xy+p_{02}y^2}.$$

III. The number of limit cycles of equations of type I.

Acta Math. Sinica **14** (1964), 119–127 (*Chinese*); *translated as Chinese Math.* **5** (1964), 129–138.

Part II (by Yeh) appeared in same Acta **12** (1962), 60–67 [MR **27** #5963]. It is shown that the system $x'=-y+dx+lx^2+xy+ny^2$, $y'=x$ has at most one limit cycle if $l=0$, or if $n=0$, or if $4ln=1$.

W. A. Coppel (Canberra)

Wing, G. Milton 1386

Invariant imbedding and the asymptotic behavior of solutions to initial value problems.

J. Math. Anal. Appl. **9** (1964), 85–98.

In the last few years it has been shown that, in a series of papers, the invariant-imbedding technique is a very powerful tool of treating consistently various kinds of transport processes of current interest [cf. Bellman and Kalaba, Proc. Nat. Acad. Sci. U.S.A. **47** (1961), 336–338; MR **23** #A3886; Wing, *An introduction to transport theory*, Wiley, New York, 1962; MR **27** #5580].

The aim of the present paper is to apply the invariant-imbedding technique to the following problem: Assuming that one knows the behavior for large t of the fundamental solutions to the equation given by

$$\frac{d^2u}{dt^2}+u=f(t)u,$$

the author finds the asymptotic behavior of that solution satisfying initial conditions yielded by $u(0)=C_1$ and $u'(0)=C_2$. Making use of Gronwall's inequality, the solution $u(t; x, \theta)$ may be written in the form

$$u(t; x, \theta)=A(x, \theta)\cos(t-x-\theta-\psi(x, \theta))+\zeta(t; x, \theta),$$

where, in the limit $t\to\infty$, $\zeta(t; x, \theta)$ tends uniformly to zero in x and θ, and the solutions of the partial differential equations for $A(x, \theta)$ and $\psi(x, \theta)$ are provided. Furthermore, the solution of the equation $d^2y/dt^2-y=f(t)y$ is also discussed.

S. Ueno (Kyoto)

Forbat, N.; Sonnenschein, J. 1387

Sur des conditions rigoureuses de stabilité asymptotique d'une vibration périodique d'un système non linéaire.

J. Mécanique **2** (1963), 437–440.

Sufficient conditions for the asymptotic stability of a periodic solution $\bar{q}(t)$ of a differential equation of the form

$$\text{(E)} \qquad q''=F(q, q', t)$$

are obtained. In (E), F is continuously differentiable in a domain D of the (q, q')-plane and continuous and periodic in t with the same period T as $\bar{q}(t)$. In the usual way, the stability conditions for $\bar{q}(t)$ are obtained by investigating the asymptotic stability of the trivial solution $q^*(t)\equiv 0$ of the linear homogeneous variational equation obtained from (E) by putting $q^*=q-\bar{q}$. This variational equation has the form

$$q^{*\prime\prime}+A(t, \bar{q}(t), \bar{q}'(t))q^{*\prime}+B(t, \bar{q}(t), \bar{q}'(t))q^*=0,$$

where A and B are periodic in t.

It is assumed that $\bar{q}(t)$ is known only as the limit of a sequence of functions $\{\bar{q}^{(n)}(t)\}$ of period T whose derivatives $\{\bar{q}^{(n)\prime}(t)\}$ converge uniformly towards $\bar{q}'(t)$. Suppose

$$A^{(n)}=A(t, \bar{q}^{(n)}(t), \bar{q}^{(n)\prime}(t)),$$
$$B^{(n)}=B(t, \bar{q}^{(n)}(t), \bar{q}^{(n)\prime}(t)).$$

Then, by choosing a sufficiently large value of n, it is always possible to satisfy

$$A^{(n)}-\varepsilon \leqq A \leqq A^{(n)}+\varepsilon,$$
$$B^{(n)}-\delta \leqq B \leqq B^{(n)}+\delta,$$

where ε and δ are positive and arbitrarily small. If, for such a value of n, the quantities

$$l_1^{(n)}=\inf A^{(n)}, \quad l_2^{(n)}=\sup A^{(n)},$$
$$m_1^{(n)}=\inf B^{(n)}, \quad m_2^{(n)}=\sup B^{(n)}, \quad t\in[0, T],$$

are calculated, it is shown that $\bar{q}(t)$ is an asymptotically stable solution of (E) if $l_1^{(m)}$, $l_2^{(m)}$, $m_1^{(n)}$, $m_2^{(n)}$ satisfy a pair of simple inequalities.

B. Porter (Newcastle upon Tyne)

Favard, J. 1388

Sur les équations différentielles scalaires presque périodiques.

J. Math. Pures Appl. (9) **43** (1964), 87–97.

The author considers the equation $\dot{x}=f(x, t)$, where x, t are real, f is defined on a strip $\alpha<x<\beta$ of finite or infinite width and is Lipschitzian (uniformly with respect to t) and uniformly almost periodic in t for every closed strip of finite width contained in the domain of f. It is further assumed that there exist α_1, β_1, μ, ν such that $\alpha<\alpha_1\leqq\mu<\nu\leqq\beta_1<\beta$ and such that for any ξ, $\mu\leqq\xi\leqq\nu$, the solution x_ξ with $x_\xi(0)=\xi$ is entirely contained in the strip $\alpha_1\leqq x\leqq\beta_1$; let $m(\xi)$, $M(\xi)$ denote the infimum and supremum of x_ξ. Then, if m, M are strictly increasing at ξ_0 and if x_{ξ_0} is stable on the left or on the right, then x_{ξ_0} is almost periodic and its module of Fourier exponents is contained in the module of f for the strip $\alpha_1\leqq x\leqq\beta_1$. Although the proof makes use of the author's well-known method of considering, along with the equation, the closure of the set of translates, the condition uses only the equation itself. The situation is contrasted by the author with the better information available in the periodic case [cf. Massera, Duke Math. J. **17** (1950), 457–475; MR **12**, 705], and he remarks that his use of the order structure

of the real line precludes a significant extension to vector equations (which explains the emphasis on "scalar" in the title). *J. J. Schäffer* (Montevideo)

Borisovič, Ju. G. 1389

The Schauder-Tihonov fixed-point principle and periodic solutions of differential equations. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 434–437.

Author's summary: "A theorem on the fixed points of a weak continuous mapping is applied for the proof of the existence of periodic solutions of differential equations in a Banach space. Bounded solutions are also studied. Applications to integro-differential equations are indicated."

Proskuriakov, A. P. [Proskurjakov, A. P.] 1390

Construction of periodic solutions of quasilinear autonomous systems with several degrees of freedom for particular cases.

Prikl. Mat. Meh. **27** (1963), 1128–1134 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1734–1744.

The systems of differential equations considered are those of the form

$$(*)\quad \sum_{k=1}^{n}(a_{ik}\ddot{x}_k+c_{ik}x_k) = \mu F_i(x_1, \cdots, x_n, \dot{x}_1, \cdots, \dot{x}_n, \mu) \qquad (i = 1, 2, \cdots, n)$$

in which the functions F_i are analytic in all their arguments within some region; the constants a_{ik}, c_{ik} satisfy $a_{ik}=a_{ki}$, $c_{ik}=c_{ki}$; and μ is a small parameter. The main object of the paper is to illustrate how the small-parameter method may be used to construct periodic solutions of (*), given that the frequency equation $\det(c_{ik}-\omega^2 a_{ik})=0$ has multiple roots. In the first part of the paper the author treats the case when the roots are all positive. In the second part he explains what modifications are needed to carry the methods over to the case in which one of the roots is zero. The results obtained generalize earlier results of the author [Prikl. Mat. Meh. **24** (1960), 734–737; ibid. **24** (1960), 1103–1109; ibid. **26** (1963), 358–364].

J. O. C. Ezeilo (Ibadan)

Faizibaev, È. F. 1391

An investigation of non-stationary processes in an oscillatory system which is essentially non-linear. (Russian)

Ukrain. Mat. Ž. **15** (1963), 223–227.

Generalizing the bilinear approximation method of E. I. Ergin [J. Appl. Mech. **23** (1956), 635–641; MR **18**, 515], the author proposes a method of approximating the nonlinear equation by a suitable continuous piecewise linear equation. The author's method includes Ergin's method as a special case.

The author considers the equation (1) $x''+F(x)=P(t)$ ($'=d/dt$, $F(-x)=-F(x)$), where $F(x)=kx+f(x)$ and $f(x)$ is nonlinear. {Reviewer's note: The assumption $F(-x)=-F(x)$ is not written explicitly in the paper.} The author considers a continuous piecewise linear function $g(x)$ such that

$$g(x) = k_ix+\sum_{j=1}^{i}x_j(k_{j-1}-k_j) \qquad (x_i \leqq |x| \leqq x_{i+1})$$
$$(i = 0, 1, 2, \cdots, n-1;\ k_0 = k),$$

where $0=x_0<x_1<x_2<\cdots<x_{n-1}<x_n=a$, and he determines (x_i, k_i) $(i=1, 2, \cdots, n-1)$ so that $\partial\Delta/\partial x_i=\partial\Delta/\partial k_i=0$ $(i=1, 2, \cdots, n-1)$, where $\Delta=\int_0^a[g(x)-F(x)]^2\,dx$. The explicit form of the equation by which the (x_i, k_i) are determined is given. The author points out that Ergin's result is a special case where $n=2$. After having determined $g(x)$ in the above way, the author replaces (1) by the piecewise linear equation $x''+g(x)=P(t)$.

The method is exemplified by the equation $x''+(x-x^3/6+x^5/120)=R\sin m\omega t$ and, for $n=3$, $a=R/|1-m^2|$, the values of (x_1, k_1) and (x_2, k_2) are calculated.

M. Urabe (Fukuoka)

Bondar, M. G. [Bondar′, N. G.] 1392

Approximate integration of equations of vibrations of nonlinear systems close to conservative. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 332–336.

Author's summary: "The author considers the approximate solution of the equation of vibrations of nonlinear systems close to conservative. The equation may describe free vibrations of nonlinear systems in the presence of viscous friction or autovibrations. The solution is obtained by means of the variable time scale method, which essentially consists in the fact that by introducing a variable time scale the nonlinear differential equation is reduced to a linear equation with constant coefficients."

Coppel, W. A. 1393

On the stability of ordinary differential equations.

J. London Math. Soc. **39** (1964), 255–260.

The author shows that the necessary and sufficient condition for the existence of a bounded solution of the vector differential equation $\dot{x}=A(t)x+f(t)$ for every bounded continuous f is the existence of supplementary projections P_0, P_1 and a positive number K such that $\int_0^t\|U(t)P_0U^{-1}(s)\|\,ds+\int_t^\infty\|U(t)P_1U^{-1}(s)\|\,ds\leqq K$ for all $t\geqq 0$, where U is the principal operator solution. (As usual, continuity of A and of f are assumed unnecessarily.) This result generalizes one of J. L. Massera and the reviewer (for A in the space M) [Ann. of Math. (2) **67** (1958), 517–573; MR **20** #3466; Math. Ann. **139** (1960), 287–342; MR **22** #8181], and reduces to a theorem of R. E. Bellman [Ann. of Math. (2) **49** (1948), 515–522; MR **10**, 121] in the case of stability. The proof of the necessity is strictly finite-dimensional. This result is used to generalize slightly part of a theorem in the first-quoted paper of Massera and the reviewer concerning the asymptotic stability of the trivial solution of the non-linear equation $\dot{x}=A(t)x+f(t, x)$, where $\|f(t, x)\|\leqq\gamma\|x\|$, $\gamma K<1$.

J. J. Schäffer (Montevideo)

Coppel, W. A. 1394

Perturbation of periodic solutions: The degenerate case. (Russian summary)

Qualitative methods in the theory of non-linear vibrations (Proc. Internat. Sympos. Non-linear Vibrations, Vol. II, 1961), pp. 190–194. *Izdat. Akad. Nauk Ukrain. SSR, Kiev,* 1963.

The perturbation problem under discussion concerns the differential system (1) $x'=A(t)x+b(t)+\mu h(t, x, \mu)$, where A is an $n\times n$ matrix, x, b, h are n-vectors, and μ is a small

parameter. It is also assumed that A, b, h are periodic in t of period T, continuous, and h has also continuous first derivative h_x with respect to x, and h_x is, of course, an $n \times n$ matrix. It is assumed that $x = A(t)x + b(t)$ has a known periodic solution $p(t)$ of period T. In the present degenerate case under discussion it is assumed that the system (2) $z' = -A'(t)z$ has periodic solutions of period T. (I) A necessary condition in order that system (1) possess periodic solutions close to $p(t)$ for all $|\mu| \neq 0$ sufficiently small is that $\int_0^T z'(t)h[t, p(t), 0]\, dt = 0$ for periodic solutions $z(t)$ of (2) of period T. (II) If condition (I) is satisfied, and $y(t) = 0$ is the only solution of the system (3) $y' = A(t)y$ for which $\int_0^T z'(t)h[t, p(t), 0]y(t)\, dt = 0$ for every periodic solution $z(t)$ of (2) of period T, then system (1) has a unique periodic solution of period T close to $p(t)$ for all $|\mu| \neq 0$ sufficiently small. The author obtains this elegant formulation of conditions for the perturbation problem in the degenerate case by means of a suitable implicit-function theorem. For previous results on the same problem see E. A. Coddington and N. Levinson [*Contributions to the theory of nonlinear oscillations*, Vol. II, pp. 19–35, Princeton Univ. Press, Princeton, N.J., 1952; MR **14**, 981], K. O. Friedrichs [Proc. Sympos. Nonlinear Circuit Analysis (New York, 1953), pp. 56–57, Polytech. Inst. Brooklyn, Brooklyn, N.Y., 1953; MR **16**, 700], and J. Kurzweil [Czechoslovak Math. J. **5** (**80**) (1955), 517–531; MR **18**, 309]. *L. Cesari* (Ann Arbor, Mich.)

Hale, Jack K. 1395

Periodic and almost periodic solutions of functional-differential equations.

Arch. Rational Mech. Anal. **15** (1964), 289–304.

The author considers functional-differential equations in which the right members are periodic or almost periodic in the independent variable, and obtains theorems on the existence, uniqueness, and stability of periodic or almost periodic solutions. These theorems generalize known results for ordinary differential equations and differential-difference equations. The paper gives a clear picture of the methods used and the relationships among the various theorems, and since the basic definitions and notations are explained, is quite self-contained.

The basic result is this theorem (abbreviated statement): Suppose $f(t, \varphi)$ is continuous, linear in φ, bounded on $-\infty < t < \infty$ for each φ, and the linear system (1) $\dot z(t) = f(t, z_t)$ is uniformly asymptotically stable. If $h(t)$ is continuous and bounded on $-\infty < t < \infty$, the solutions of (2) $\dot z(t) = f(t, z_t) + h(t)$ are bounded for $t \geqq t_0$. If $f(t, \varphi)$ and $h(t)$ are almost periodic in t uniformly with respect to φ, then there is a unique asymptotically stable, almost periodic solution of (2).

One of the applications is the following theorem. Assume that $f(t, \varphi)$ is linear in φ and system (1) is uniformly asymptotically stable. Assume that $X(t, 0, 0) = 0$ and that for $-\infty < t < \infty$, $\|\varphi\| \leqq \rho$, $\|\psi\| \leqq \rho$, $0 \leqq \mu \leqq \mu_0$,

$$|X(t, \varphi, \mu) - X(t, \Psi, \mu)| < \eta(\rho, \mu)\|\varphi - \Psi\|,$$

where $\eta(0, 0) = 0$. If $f(t, \varphi)$ and $X(t, \varphi, \mu)$ are almost periodic in t uniformly with respect to φ, μ, then there is a unique uniformly asymptotically stable almost periodic solution of (3) $\dot x(t) = f(t, x_t) + X(t, x_t, \mu)$ for μ and the initial value φ sufficiently small, and this solution approaches zero as $\mu \to 0$.

Other applications are to rapidly oscillating systems of the form $\dot x(t) = f(x_t) + X(\omega t, x_t) + kg(\omega t)$, where ω is a large parameter, and to certain coupled linear systems with the coupling involving a small parameter μ.

{Misprints: In the second line of Corollary II.3, replace $\dot z(t) = f(\mu, z_t)$ by $\dot z(t) = \mu f(\mu, z_t)$. In (2.8), replace $f(\mu, \mu\omega(t, \mu))$ by $f(\mu, \mu\omega_t)$. In (2.17), replace $\dot V_{(2.16)}$ and $\dot W_{(2.17)}$ by $\dot V_{(2.15)}$ and $\dot W_{(2.16)}$.} *K. L. Cooke* (Claremont, Calif.)

Stokes, A. 1396

On the stability of a limit cycle of an autonomous functional differential equation.

Contributions to Differential Equations **3** (1964), 121–139.

This paper gives an analogue for functional-differential equations of a well-known theorem [cf. E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, p. 323, McGraw-Hill, New York, 1955; MR **16**, 1022] concerning orbital asymptotic stability of a periodic solution of an autonomous ordinary differential equation.

Consider the autonomous functional-differential equation (1) $\dot w(t) = f(w_t)$ (the notation is that used by J. K. Hale, who discussed periodic solutions of non-autonomous equations [see #1395 above]), and suppose that there is a non-constant periodic solution $v(t)$ of period ω. Then $V = \{v_c : 0 \leqq c \leqq \omega\}$ is a closed trajectory in the "phase space" C of continuous functions. Using the Fréchet derivative, the author defines the linear variational functional-differential equation of (1) with respect to v_t. A closed trajectory V in C is said to be orbitally asymptotically stable if there exists a neighborhood N of V such that φ in N implies that the distance between V and the solution $w_t(\varphi)$ starting from φ at time zero tends to zero as $t \to \infty$. It has asymptotic phase if also there is a constant $c = c(\varphi)$ such that $\|w_t(\varphi) - v_{t+c}\| \to 0$ as $t \to \infty$. Theorem: Assume that (1) possesses a closed trajectory V described by v_t where v_t is of period ω in t. Assume that in the related linear variational functional-differential equation, the characteristic multiplier 1 has multiplicity one, and that the remaining characteristic exponents all have negative real parts. Then V is orbitally asymptotically stable with asymptotic phase. (The terms characteristic multiplier and exponent, and multiplicity, are defined in the paper.)

The author also shows that if the characteristic multipliers λ_i for which $|\lambda_i| = 1$ have simple elementary divisors (defined in the paper), then the nonlinear equation has a stable initial manifold, which is a homeomorphic image of a ball in a subspace of C, this subspace being stable for the related linear equation.

K. L. Cooke (Claremont, Calif.)

Ljapunov, A. M. [Ляпунов, А. М.] 1397

★A study of one of the special cases of the problem of stability of motion [Исследование одного из особенных случаев задачи об устойчивости движения].

Izdat. Leningrad. Univ., Leningrad, 1963. 116 pp. 0.52 r.

In this review $[u, v, \cdots]_k$ stands for a power series in $u, v, \cdots$, holomorphic at the origin and with no terms of degree $< k$. It may also denote (clear from the context) a vector whose components behave as stated relative to the components of $u, v, \cdots$. The letters z, P represent an n-vector and an $n \times n$ constant stable matrix (all characteristic roots have negative real parts). x, y denote scalars.

In his opus magnum [Thesis, Kharkov, 1892; French transl., *Problème général de la stabilité du mouvement*, Ann. Fac. Sci. Univ. Toulouse (2) **9** (1907), 203–474; photolithoprinted, Princeton Univ. Press, Princeton, N.J., 1947; MR **9**, 34] Liapunov investigated in full the stability problem of the system (1) $\dot{y}=Y(y,z)$, $\dot{z}=Pz+Z(y,z)$, where Y and Z are $[y,z]_2$. There is a unique solution $u(y)$ of $Pu+Z(y,u)=0$ and the substitution $z\to z+u$ reduces (1) to $\dot{y}=gy^m+\cdots$, $m>1$, $\dot{z}$ as before. The motion is asymptotically stable if m is odd and $g<0$, and unstable otherwise.

Now in 1893 Liapunov published a large paper entitled Investigation of one of the special cases of the stability of motion" [Mat. Sb. **17** (1893), 253–333; also his *Collected works* (Russian), Vol. II, Izdat. Akad. Nauk SSSR, Moscow, 1956; MR **27** #5980]. In this paper he stated that he proposed to discuss the stability of a system (2) $\dot{x}=y+X(x,y,z)$, $\dot{y}=Y(x,y,z)$, $\dot{z}=Pz+Z(x,y,z)$, where X, Y, Z are $[x,y,z]_2$, but actually the paper only dealt with the preliminary problem of (3) $\dot{x}=y+X(x,y)$, $\dot{y}=Y(x,y)$, where X, Y are $[x,y]_2$. In fact, by a series of quite difficult transformations Liapunov succeeded in treating this case completely.

At the time of the recent publication of the *Complete works* there was discovered an extensive manuscript of Liapunov under the same title as the large paper in which he dealt with the full problem of (2) but was unable to terminate the research. Presumably this was the reason why the manuscript was withheld from publication. The present monograph presents this manuscript preceded by an excellent summary by V. P. Bassov.

Let $Y(x,y,0)=Y_0(x)+Y_1(x)y+W(x,y)y^2$, $Z(x,y,0)=Z_0(x)+Z_1(x)y+U(x,y)y^2$, where the coefficients are holomorphic in x or x, y at the origin. If the functions other than U, W are not identically 0, let their expansions be $Y_0(x)=gx^m+g_1x^{m+1}+\cdots$ $(m\geqq 2)$; $Y_1(x)=ax^\alpha+a_1x^{\alpha+1}+\cdots$ $(\alpha>1)$; $Z_0(x)=hx^p+\cdots$, $Z_1(x)=kx^\beta+\cdots$. Here g, g_1, a, a_1 are scalar constants and h, k are constant n-vectors. By transformations not affecting the form of the system one may obtain this situation: (a) If $Y_0(x)\equiv Y_1(x)\equiv 0$, then also $Z_0(x)=Z_1(x)\equiv 0$; (b) if $Y_0(x)\equiv 0$, $Y_1(x)\not\equiv 0$, then $\beta>\alpha$; (c) if $Y_0(x)\not\equiv 0$, then $p>m$ and also $\beta>\alpha$ if $\alpha<m$ and $\beta\geqq m$ if $\alpha\geqq m$. In addition, one may always assume that $X(x,0,0)\equiv 0$. Liapunov distinguishes these four cases: (I) $Y_0(x)\equiv 0$; (II) m even; (III) m odd, $g>0$; (IV) m odd, $g<0$. For the cases (I), (II), (III), and (IV) with $\alpha<m$ and α even, Liapunov was able to conclude the treatment along the same lines as his 1893 paper, always using his stability theorems, except that when it came to instability his own theorem on instability had to be replaced by what amounted to using the Četaev generalization. It is actually the full treatment of (IV) that Liapunov was unable to carry out to completion. It still remains as an open problem. [Additional reference: Četaev, *Stability of motion* (Russian), 2nd ed., GITTL, Moscow, 1955; MR **17**, 1087].

S. Lefschetz (Providence, R.I.)

Thomas, Johannes 1398

Über die Invarianz der Stabilität bei einem Phasenraum-Homöomorphismus.

J. Reine Angew. Math. **213** (1963/64), 147–150.

Let P and Q be metric spaces with respective metrics ρ and σ. Suppose that $y=f(t,x)$, $t\geqq t_0$, represents a family of homeomorphisms from a compact subspace $X(t)\subset P$ onto a compact subspace $Y(t)\subset Q$. Let $\delta_1(t,\varepsilon)>0$ be such that $\rho(x_1,x_2)<\delta_1(t,\varepsilon)$, x_1, $x_2\in X(t)$, implies that $\sigma(f(t,x_1),f(t,x_2))<\varepsilon$. Let $\delta(t,\varepsilon)$ be the supremum of all such $\delta_1(t,\varepsilon)$. The author shows that if $\Delta(t_0,\varepsilon)=\inf\{\delta(t,\varepsilon):t\geqq t_0\}$ is positive for all $\varepsilon>0$, then the continuous transformation $y=f(t,x)$ preserves stability properties (in the Liapunov sense). *K.-T. Chen* **(New Brunswick, N.J.)**

Antipenko, V. I. 1399

A new method of determining Lyapunov functions. (Ukrainian)

Avtomatika **1964**, *no.* 2, 71–75.

The author considers a system of first-order autonomous differential equations which, in general, are non-linear. In order to investigate the asymptotic stability of the system one has to know its Lyapunov function. The author proposes a new method for finding a general form of the Lyapunov function for the system considered. In his method there are no a priori conditions imposed on the Lyapunov function which, instead of being constructed, is derived from a special system of differential equations derived in turn from the original system. The author illustrates his method on two examples, in each example the system consisting of two equations. The reviewer feels that for large systems this method may not be so easy to apply. Nevertheless, the author's work is a step in the right direction. Up to now the construction of a Lyapunov function was mostly an art—not a very satisfactory state of affairs. *T. Leser* (Aberdeen, Md.)

Cronin, Jane 1400

★Fixed points and topological degree in nonlinear analysis.

Mathematical Surveys, No. 11.

American Mathematical Society, Providence, R.I., 1964. xii+198 *pp.* $10.00.

This very interesting monograph contains a systematic treatment of fixed-point and local degree theory as it applies to differential equations and related topics. The author has endeavored to make the treatment as self-contained as possible and has repeatedly discussed topics amply dealt with in existing texts, for example, existence theorems of ordinary differential equations, Liapunov stability theory, and (perhaps too extensively) topology. She has also included many noteworthy results merely stated without proof, and has given ample references to many more. Another worthwhile feature consists in repeatedly pointing out why certain obvious approaches to this or that problem were really "out of bounds". Altogether we have here a most valuable and most readable presentation of a highly important topic in modern analysis. A detailed examination of the four chapters of the monograph follows.

The first chapter consists of a development of the topology of finite Euclidean sets required later, particularly centering around the notion of local degree. In its simplest aspect it goes back to Kronecker but "considerable sophistication" has set in since his day.

Let D be a bounded domain in oriented Euclidean R^n and let B be its boundary. One must first define n-cycles of $\bar{D}$ mod B, or rather their topological homology class. To that end take an ε-neighborhood N_ε of B and let K

be a simplicial complex of mesh $<\varepsilon$ covering $\bar{D}$. The chain sum of the positively oriented n-simplexes of K is an n-cycle mod $\bar{N}_\varepsilon$, γ_ε. If $\varepsilon' < \varepsilon$, two such cycles $\gamma_{\varepsilon'}'$, $\gamma_{\varepsilon'}''$ are $\sim$ mod $\bar{N}_\varepsilon$. The γ_ε are elements of a topological homology class Γ of n-cycles of $\bar{D}$ mod B. Let now f be a mapping $\bar{D} \to R^n$ and let $q \notin fB$, hence also $\notin f\bar{N}_\varepsilon$ for ε small enough. It follows that the Kronecker index $KI(f\gamma_\varepsilon, q) = \nu$ is well-defined and has topological character. This is the degree $\nu(fD, q)$. Its basic property is: $\nu \neq 0$ implies that there is a $p \in D$ such that $fp = q$.

In Chapter II the main topic is Poincaré's bifurcation equation for the quasi-linear n-vector system (1) $\dot{x} = A(t)x + \mu f(x, t, \mu) + g(t)$, with period T. Resonance occurs when $\dot{x} = A(t)x$ has solutions of period T. The discussion is quite complete, stability included, and especially when $n = 2$ and the components of f are polynomials in those of x. We merely mention the following topics which are amply treated in the chapter, each requiring a special adaptation of the general techniques: (a) almost periodic solutions of almost periodic systems; (b) periodic solutions of autonomous systems; (c) periodic systems with large nonlinearities (main contributions, for $n = 2$, due to Gomory).

In Chapter III the topology adapted to infinite-dimensional spaces is discussed. Progress has only been possible for normed linear spaces and the subclass of complete spaces—Banach spaces. However, although only partial, the results have been most useful and important. The main theorems treated are: (I) Schauder fixed-point theorem. Let $\mathcal{N}$ be a normed linear space, K a compact, closed, convex subset. Then a compact mapping $T: K \to K$ always has a fixed point. (II) Schaefer's theorem. Let T be a compact mapping $\mathcal{N} \to \mathcal{N}$, and let $\lambda_1 \in [0, 1]$. Then $\lambda_1 T$ has a fixed point or else the set of fixed points of λT, $\lambda \in [0, 1]$ is unbounded. (III) The Leray-Schauder local degree. (IV) Banach fixed-point theorem. First a definition. Let E be a subset of a Banach space $\mathcal{B}$. A mapping $T: E \to B$ is a contraction whenever there is a $c \in (0, 1)$ such that if $x, x' \in E$, then $\|Tx - Tx'\| < c\|x - x'\|$. The fixed-point theorem asserts that if $E \subset \mathcal{B}$ is closed and T is a contraction $E \to E$, then it has a unique fixed point in E. (This is the basic property in the proof of the existence theorem for ordinary differential equations by successive approximations.)

In Chapter IV, the last, the techniques developed in the preceding chapter are applied, together with abundant analysis of the following questions: (I) Existence of solutions $\varphi(s)$ of a nonlinear integral equation $\varphi(s) + \int_a^b K(s, t, \varphi(t))\,dt = f(s)$. (II) Elliptic differential equations in two dimensions: $A(x, y, z, p, q)z_{xx} + B(\cdots)z_{xy} + C(\cdots)z_{yy} = 0$, $p = z_x$, $q = z_y$, where A, B, C are well-behaved in their five variables, and x, y is in a bounded Euclidean plane domain. The problem is the obvious extension of the classical Dirichlet problem. The extensive work of Leray-Schauder and of Nirenberg is dealt with in some detail. (III) Parabolic and hyperbolic equations.

The monograph concludes with quite an extensive bibliography. *S. Lefschetz* (Providence, R.I.)

Infante, E. F.; Clark, L. G. 1401

A method for the determination of the domain of stability of second-order nonlinear autonomous systems.

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 315–320.

The authors consider systems of the form $\dot{x}_1 = f_1(x_1, x_2)$ and $\dot{x}_2 = f_2(x_1, x_2)$. In most cases such systems do not have a set of time-independent solutions. The authors modify the right-hand sides of the differential equations by adding new functions of x_1 and x_2 which satisfy certain conditions. The new differential equations then will have a set of time-independent integrals. A simple geometrical test is suggested to find under what conditions the trajectories of the original system cross the set of time-independent integrals $\hat{h}(x_1, x_2) = c$ of the modified set. It is obvious that these curves $\hat{h}$ (when properly chosen) can serve as Lyapunov functions. Examples illustrate the method, which seems attractive for two-dimensional problems. *I. Flügge-Lotz* (Stanford, Calif.)

Vlasov, I. M. 1402

Criteria for boundedness of solutions of systems of ordinary differential equations. (Russian)

Izv. Vysš. Učebn. Zaved Matematika **1964**, *no.* 3 (40), 39–42.

The author considers the unperturbed system (1) $x' = f(t, x)$ and the perturbed system (2) $y' = f(t, y) + g(t, y)$. Assuming that all the solutions $x(t)$ of (1) are bounded, he obtains some criteria for the boundedness of solutions $y(t)$ of (2). The results depend basically on the following inequality:

$$|y(t)| \leq |x(t)| + \int_\tau^t \|K(t, s, y(s))\|\, |g(s, y(s))|\, ds,$$

where $K(t, \tau, x) = \partial\phi(t, \tau, x)/\partial x$, $\phi(t, \tau, x)$ being a solution of (1) through (τ, x). *V. Lakshmikantham* (Aurangabad)

Klotter, K.; Kreyszig, E. 1403

On a nonlinear vibrating system having infinitely many limit cycles.

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 321–324.

The paper contains an analytical investigation of the limit cycles of the system given by

$$\ddot{y} - \left(\frac{ak^2}{2} \cos ay\right)\dot{y}|\dot{y}| + \sum_{m=0}^{n} c_m y^{2m+1} = 0$$

with $c_n > 0$; $c_m \geq 0$ when $m < n$. Approximate analytic solutions are compared to exact numerical solutions for $n = 0, 1, 2$; $a = 1$ and several values of k^2.

I. Flügge-Lotz (Stanford, Calif.)

Protasov, V. I. 1404

On the solutions of linear equations of infinite order in generalized derivatives. (Russian)

Dokl. Akad. Nauk SSSR **154** (1964), 1273–1275.

Dans cette note l'auteur présente, sans donner des démonstrations, quelques résultats intéressants concernant l'existence et l'analyticité des solutions pour l'équation différentielle linéaire d'ordre infini dans le domaine complexe

$$(1) \qquad \sum_{n=0}^{\infty} a_n D^n y(z) = f(z),$$

où a_n sont des constantes complexes et la fonction $f(z)$ est supposée analytique dans un disque centré dans $z = 0$. Les dérivées $D^n y(z)$ sont considérées dans le sens généralisé suivant: Soit $y(z) = \sum_{k=0}^{\infty} c_k z^k$ une fonction analytique

dans un disque centré dans $z=0$ et soit α_k, $k=1, 2, 3, \cdots$, une suite de nombres complexes fixée avec $\alpha_k \neq 0$; par $D^n y(z)$ on entend la fonction définie par la série suivante (si cette série est convergente dans un certain disque): $D^n y(z) = \sum_{k=n}^{\infty} (\alpha_{k-n}/\alpha_k) c_k z^{k-n}$.

L'équation (1) avec des dérivées généralisées dans le même sens a été considérée aussi par Ju. N. Frolov [Vestnik Moskov. Univ. Ser. I Mat. Meh. **1960**, no. 4, 3–13; MR **23** #A2754] mais par des méthodes différentes.

L'auteur cherche les solutions de l'équation (1) parmi les fonctions analytiques en supposant que la série qui intervient dans (1) est absolument convergente dans un disque. Si on suppose aussi que la fonction $f(z)$ est analytique, alors les développements de $f(z)$ et $y(z)$ seront de la forme $f(z) = \sum_{j=0}^{\infty} b_j z^j$, $y(z) = \sum_{k=0}^{\infty} c_k z^k$.

L'auteur pose l'équation (1) sous la forme matricielle

$$(2) \qquad \begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ & a_0 & a_1 & \cdots \\ & & a_0 & \cdots \\ & & & \cdots \end{bmatrix} \begin{bmatrix} \frac{c_0}{\alpha_0} \\ \frac{c_1}{\alpha_1} \\ \frac{c_2}{\alpha_2} \\ \vdots \end{bmatrix} = \begin{bmatrix} \frac{b_0}{\alpha_0} \\ \frac{b_1}{\alpha_1} \\ \frac{b_2}{\alpha_2} \\ \vdots \end{bmatrix}.$$

La transformation linéaire (2) qui applique les "points" $(c_0, c_1, c_2, \cdots)$ dans les "points" $(b_0, b_1, b_2, \cdots)$ est traitée par l'auteur comme transformation dans les espaces des suites, considérés par G. Köthe et O. Toeplitz [J. Reine Angew. Math. **171** (1934), 193–226].

L'auteur affirme qu'à l'aide des théorèmes, qu'il a obtenus, sur les transformations de ces espaces on peut démontrer des théorèmes d'existence pour l'équation (1). Dans sa note l'auteur énonce quatre tels théorèmes. Nous citerons le théorème 1.

Soit $\varphi(t)$ la fonction caractéristique de l'équation (1), c'est-à-dire $\varphi(t) = \sum_{n=0}^{\infty} a_n t^n$. Théorème 1 : Si

$$\liminf_{n\to\infty} n^{1/\rho} \sqrt[n]{|\alpha_n|} = (\underline{\sigma} e\rho)^{1/\rho} > 0,$$

$$\limsup_{n\to\infty} n^{1/\rho} \sqrt[n]{|\alpha_n|} = (\bar{\sigma} e\rho)^{1/\rho},$$

et la fonction $\varphi(t)$ est entière ayant l'ordre de croissance ρ et le type σ, alors si la fonction $f(z)$ est analytique dans $|z| < R$, avec $R > (2\sigma/\underline{\sigma})^{1/\rho}$, il existe une solution analytique dans le disque $|z| < [R^\rho \underline{\sigma}/\bar{\sigma} - \sigma/\bar{\sigma}]^{1/\rho}$, le rayon de convergence pouvant être plus grand. En particulier, si $f(z)$ est une fonction entière, alors il existe une solution dans la classe des fonctions entières. Les autres théorèmes ont des énoncés analogues. *M. Reghiş* (Timişoara)

Valeev, K. G. 1405

Linear differential equations with sinusoidal coefficients and stationary retardations of the argument. (Russian. English summary)

Qualitative methods in the theory of non-linear vibrations (Proc. Internat. Sympos. Non-linear Vibrations, Vol. II, 1961), pp. 100–119. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, 1963.

Two types of problem are considered: (I) The system of differential equations

$$(1) \qquad \sum_{k=0}^{n} (A_{0k} + e^{\omega t} A_{-1k} + e^{-\omega t} A_{1k}) \frac{d^k Y(t)}{dt^k} = \Phi(t);$$

$Y(t)$ is an m-vector, A_{jk} an $m \times m$ matrix, A_{0n} the unit matrix E, and ω is a pure imaginary. The initial values of Y and its first $n-1$ derivatives are prescribed. (II) The system of differential equations with constant delays

$$(2) \qquad \sum_{q=-1}^{1} e^{-q\omega t} \sum_{k=0}^{n} \int_{-h}^{0} dA_{qk}(\vartheta) \frac{d^k Y(t+\vartheta)}{dt^k} = \Phi(t),$$

with Y and its first $n-1$ derivatives prescribed on $[-h, 0]$.

For each problem the Laplace transform $F(p)$ of the solution $Y(t)$ is shown to satisfy a matrix difference equation

$$(3) \qquad \sum_{q=-1}^{1} L_q(p+q\omega) F(p+q\omega) = R(p)$$

with determinate $L_q(p)$, $R(p)$.

The equation (3) is solved by means of a matrix series which converges, however, only in one half of the complex p-plane. Its analytic continuation is effected by transforming into a matrix continued fraction, using an ingenious and apparently novel technique whereby the terms of the series are shown to be isomorphic to a certain class of generalised numbers $[\chi, \sigma]$, and use is then made of the properties of these numbers. Having effected the analytic continuation, a transcendental equation emerges whose roots give the pole of the image $F(p)$, and hence the characteristic exponents of the original solution, thus determining the stability of the problem. Finally, the inverse transformation into the solution of the original problem is given.

There are several examples of applications to typical problems. *F. M. Arscott* (London)

Gumowski, Igor 1406

Sur les solutions périodiques de l'équation de Cherwell-Wright.

C. R. Acad. Sci. Paris **258** (1964), 416–419.

The author suveys known results concerning the functional-differential equation

$$\dot{y}(t) = -\alpha[1+y(t)]y(t-1).$$

He further discusses briefly some of these results from the point of view of standard techniques employed in nonlinear mechanics. *G. S. Jones* (Baltimore, Md.)

Lakshmikantham, V. 1407

Functional differential systems and extension of Lyapunov's method.

J. Math. Anal. Appl. **8** (1964), 392–405.

The author extends some of his results on stability and boundedness of solutions of ordinary differential systems [Michigan Math. J. **9** (1962), 311–320] to a delay-differential system. The technique involves dominating a Lyapunov function of the solutions by the maximal solution of a related scalar ordinary differential equation. The notation is rather involved, and is further complicated by a number of apparent misprints. *R. D. Driver* (Albuquerque, N.M.)

Li, Xun-jing [Li, Hsun-ching] 1408

On the absolute stability of systems with time lags.

Acta Math. Sinica **13** (1963), 558–573 (*Chinese*); *translated as Chinese Math.* **4** (1964), 609–626.

Consider the difference-differential system

$$dx(t)/dt = Ax(t) + Bx(t-\tau) + Rf[Cx(t) + Dy(t)],$$
$$dy(t)/dt = f[Cx(t) + Dy(t)],$$

where $x(t)$ is an n-vector, $y(t)$ is an m-vector, $f(z)$ is a continuous m-vector-valued function of an m-vector z with the properties that $f_j(z)$ depends only on z_j and $f(0) = 0$, $\tau > 0$ is a constant, and A, B, C, D, and R are constant matrices with appropriate dimensions. The author obtains conditions on A, B, C, D, and R such that the trivial solution will be absolutely stable when the feedback function f lies in the class such that $0 < z_j f_j(z) \leqq h_j z_j^2$ ($z_j \neq 0$, $j = 1, \cdots, m$), where the h_j's are given positive constants. The results extend those obtained by V. M. Popov [Avtomat. i Telemeh. **22** (1961), 961–979; MR **24** #A3394] for an ordinary differential system ($B = 0$).

R. D. Driver (Albuquerque, N.M.)

Trenogin, V. A. 1409

Existence and asymptotic behavior of the solution of the Cauchy problem for a first-order differential equation with small parameter in a Banach space. (Russian)

Dokl. Akad. Nauk SSSR **152** (1963), 63–66.

The Cauchy problem in a real Banach space E for the equation

$$\varepsilon \frac{dy}{dt} = F(y, \varepsilon), \qquad 0 < t \leq T, \ \varepsilon > 0,$$
$$y(0) = 0,$$

is considered. Here $F(y, \varepsilon)$ has a uniformly convergent Fréchet power series expansion for $\|y\|^2 + \varepsilon^2$ sufficiently small. The term B of order zero is allowed to be a densely defined unbounded operator with closed range, but the other terms in the expansion are bounded (multilinear) operators. (I) Suppose that B^{-1} exists, that B generates a strongly continuous semigroup e^{-Bt}, $t \geqq 0$, with $\|e^{-Bt}\| \leqq e^{-\beta t}$, $\beta > 0$, and that $dz/dt + Bz = 0$, $z(0) = 0$, has only the trivial solution. Then for ε sufficiently small there is a unique solution y to the Cauchy problem, and an asymptotic form is given. (II) If 0 is a simple eigenvalue of B, B^* and if the hypotheses in (I) are satisfied on the range of B, then the existence of the solution is obtained under some additional bifurcation conditions.

R. G. Bartle (Urbana, Ill.)

Skripnik, V. P. 1410

Systems with transformed argument. Boundary-value problems and the Cauchy problem. (Russian)

Mat. Sb. (N.S.) **62 (104)** (1963), 385–396.

On généralise les systèmes différentiels à l'argument retardé en considérant une classe plus large de systèmes appelés par l'auteur systèmes à l'argument transformé. Avec les notations

$$f(t, \xi_{jk}) = f(t, \xi_{11}, \cdots, \xi_{m1}, \cdots, \xi_{1p}, \cdots, \xi_{mp}),$$
$$f(t, x_j[\varphi_{jk}(t)]) = f(t, x_1[\varphi_{11}(t)], \cdots, x_m[\varphi_{m1}(t)], \cdots, x_1[\varphi_{1p}(t)], \cdots, x_m[\varphi_{mp}(t)]),$$

on a le système

$$(1) \qquad x_i'(t) = f_i(t, x_j[\varphi_{jk}(t)]), \qquad i = 1, 2, \cdots, m.$$

Pour définir les solutions d'un tel système on considère les fonctions φ_{jk} définies et continues dans $[a, b]$. En désignant $E_{jk} = \varphi_{jk}([a, b])$, $E_j = \bigcup_{k=1}^p E_{jk} \cup [a, b]$, on appelle solution sur $[a, b]$ du système (1) toute fonction vectorielle $x(t)$ jouissant des propriétés suivantes: (1) La composante $x_i(t)$ de la fonction $x(t)$ est définie pour chaque $t \in E_i$; (2) Cette composante est dérivable pour chaque $t \in [a, b]$; (3) La fonction $x(t)$ vérifie identiquement pour $t \in [a, b]$ le système (1).

Le théorème fondamental (Théorème 1) est le suivant: Si (1) Les fonctions f_i sont continues pour $t \in [a, b]$ et $|\xi_{jk} - x_{j0}| \leqq R_j$ ($R_j > 0$); (2) Les fonctions f_i sont lipschitziennes, c'est-à-dire

$$|f_i(t, \xi_{jk}^1) - f_i(t, \xi_{jk}^0)| \leqq k_i(t) \sum_{j=1}^m \sum_{k=1}^p |\xi_{jk}^1 - \xi_{jk}^0|$$

pour $t \in [a, b]$, $|\xi_{jk}^0 - x_{j0}| \leqq R_j$, $|\xi_{jk}^1 - x_{j0}| \leqq R_j$, où $k_i(t)$ sont des fonctions continues dans $[a, b]$; (3) $a_i \in [a, b]$; (4) $L < 1$ où $L = \sup_{t \in [a,b]} \sum_{j=1}^m \sum_{k=1}^p |\int_{a_j}^{\varphi_{jk}(t)} k_j(\tau)\, d\tau|$; (5) $M_i + MK_i/(1-L) \leqq R_i$, où $M_i = \sup_{t \in [a,b]} |\int_{a_i}^t f_i(\tau, x_{jk})\, d\tau|$, $x_{jk} = x_{j0}$, $M = \sup_{t \in [a,b]} \sum_{i=1}^m \sum_{k=1}^p |\int_{a_i}^{\varphi_{ik}(t)} f_i(\tau, x_{jk})\, d\tau|$, $K_i = \sup_{t \in [a,b]} |\int_{a_i}^t k_i(\tau)\, d\tau|$. Alors le problème aux limites (multiponctuel) $x_i(a_i) = x_{i0}$ ($i = 1, 2, \cdots, m$) pour le système (1) a une solution $x(t)$, dans $[a, b]$, uniquement déterminée dans la classe des fonctions vectorielles $\{\varphi\}$, continues dans $[a, b]$, telles que $|\varphi_i(t) - x_{i0}| \leqq R_i$ pour $t \in [a, b]$ ($\varphi_i(t)$ étant les composantes du vecteur $\varphi(t)$).

Dans ce théorème on suppose que $\varphi_{jk}([a, b]) \subset [a, b]$. Le cas général (Théorème 5) se réduit à celui-ci après avoir défini les prolongements admissibles des fonctions φ_{jk} en dehors du segment $[a, b]$.

La méthode de démonstration est toujours une adaptation directe de la méthode des approximations successives de Picard. En particulier, si $a_i = t_0 \in [a, b]$ ($i = 1, 2, \cdots, m$), on obtient de ce théorème l'existence et l'unicité de la solution du problème (avec des données initiales) de Cauchy. On peut préciser ce résultat si on impose aux fonctions φ_{jk} des conditions supplémentaires (Théorèmes 2 et 3).

On donne aussi une méthode de réduire des problèmes aux limites plus générales au problème multiponctuel considéré dans le Théorème 1. Finalement, l'auteur examine quelques exemples.

M. Reghiş (Timişoara)

Halanaĭ, A. [Halanay, A.] 1411

Absolute stability of certain non-linear control systems with lag. (Russian)

Avtomat. i Telemeh. **25** (1964), 290–301.

Popov [Acad. R. P. Romîne Stud. Cerc. Energet. **9** (1959), 119–135; ibid. **9** (1959), 647–680; Avtomat. i Telemeh. **22** (1961), 961–979; MR **24** #A3394; ibid. **23** (1962), 3–24; MR **28** #3212] has used the method of Fourier transforms in deciding the asymptotic behavior of solutions of equations of the form $\sigma(t) = z(t) + \int_0^t k(t-a) f(\sigma(a))\, da$. In this paper Popov's methods are first applied to systems of the form $\dot{x}(t) = \int_{-\infty}^0 x(t+s)\, d\eta(s) + bf(\sigma(\ ))$, where x is a vector, η is a matrix and the argument of σ is t or $t - \tau$ and $\sigma = xc$. "Critical" systems of the form $y = \int_{-\infty}^0 y(t+s)\, d\eta(s) + b\xi(t)$, where either $\dot{\xi}(t) = f(\sigma(t))$ and $\sigma = yc - r\xi$, or else $\ddot{\xi}(t) = f(\sigma(t))$ and $\sigma = yc + p\xi + q\dot{\xi}$, are also treated. Popov's original work for scalar equations is repeated for easy reference, and the systems treated here are in each case reduced to Popov's equations.

C. S. Coleman (Claremont, Calif.)

Corduneanu, Constantin 1412

Sur une équation fonctionnelle liée à la théorie de la stabilité.

C. R. Acad. Sci. Paris **256** (1963), 56–58.

Consider a system (S) $\dot{x}=f(t,x)$, where t is real and x, f are elements of a real Banach space E and f is continuous on Δ: $t \geqq 0$, $\|x\| < \infty$. It is assumed that $f(t,x)$ satisfies a Lipschitz condition in x on every closed subspace of Δ. Let $V(t,x)$ be a real function defined on Δ. Let $V'(t,x)$ be the derivative of $V(t,x)$ along the solution of (S), that is, for $x=F(t;t_0,x_0)$, $V'(t,x)=dV(t,F(t;t_0,x_0))/dt$, where $F(t;t_0,x_0)$ is a solution through (t_0,x_0). The author considers a functional equation (1) $V'(t,x)=\omega(t,x,V(t,x))$ with a condition (2) $V(0,x)=\mu(x)$, where $\omega(t,x,V)$ is continuous on $\Delta\times R$. The following theorem is obtained: Under the conditions above, if every solution of (S) is defined for all $t\geqq 0$ and $\omega(t,x,V)$ satisfies a Lipschitz condition with respect to (x,V) on every set $B\times R$, where $B\subset\Delta$ is bounded, and if $\mu(x)$ is continuous on E uniformly on every bounded subspace, then there exists a unique continuous solution of (1) which satisfies (2). This solution is defined on Δ. *J. Yoshizawa* (Zbl **107**, 101)

PARTIAL DIFFERENTIAL EQUATIONS

See also 1115, 1400, 1551, 1558, 1559, 1560, 1773, 1851.

Hornich, H. 1413

★Existenzprobleme bei linearen partiellen Differentialgleichungen.

Mathematische Forschungsberichte, Bd. IX.

VEB Deutscher Verlag der Wissenschaften, Berlin, 1960. 57 *pp.* *DM* 8.50.

O. Perron nel 1929 osservò che il teorema di Peano sull'esistenza di almeno una soluzione $y=y(x)$ dell'equazione differenziale $y'=f(x,y)$ con $f(x,y)$ continua in un insieme aperto G, che in un punto (x_0,y_0) di G soddisfa la condizione $y(x_0)=y_0$ non può estendersi alle equazioni alle derivate parziali. L'autore, che nel periodo 1954–1959 [Rend. Sem. Mat. Univ. Padova **23** (1954), 333–339; MR **16**, 480; Monatsh. Math. **59** (1955), 34–42; MR **16**, 825; Univ. e Politec. Torino Rend. Sem. Mat. **14** (1954/55), 33–37; MR **17**, 620; Jber. Deutsch. Math. Verein. **58** (1956), Abt. 1, 103–109; MR **19**, 420; Österreich. Ing.-Arch. **10** (1956), 195–197; MR **19**, 424] ha approfondito il problema sull'inesistenza o sull'esistenza di soluzioni di equazioni lineari alle derivate parziali, in questa sua ampia memoria considera le equazioni nella funzione incognita $U(x,y)$ (1) $\cos\alpha(x,y)\partial U/\partial x+\sin\alpha(x,y)\partial U/\partial y=f(x,y)$ con $\alpha(x,y)$, $f(x,y)$ uniformemente continue in un insieme aperto e connesso G. Una funzione $U(x,y)$ è chiamata dall'autore una soluzione generalizzata della (1) se è continua in G e in ogni punto $P\equiv(x,y)\in G$ ha una derivata nella direzione uscente da P e formante con l'asse x l'angolo $\alpha(x,y)$, e tale derivata è uguale ad $f(x,y)$. L'autore considera l'equazione differenziale ordinaria (2) $dx/dy=\cos\alpha(x,y)/\sin\alpha(x,y)$ associata alla (1) e sulle sue soluzioni L: $x=x(s)$, $y=y(s)$ che la soddisfano quasi ovunque fa le seguenti ipotesi. La prima, di carattere topologico, assicura la non esistenza di soluzioni chiuse della (2) appartenenti al campo $G+\mathfrak{F}G$. La seconda ipotesi è di carattere metrico. Definite le così dette soluzioni complesse come composte opportunamente con un numero finito o un'infinità di archi L, per quelle congiungenti due punti suppone che la loro lunghezza risulti uniformemente limitata rispetto a tale coppia di punti. In queste ipotesi l'autore dimostra che la (1) ammette una soluzione generalizzata e successivamente, grazie ad alcuni esempi, prova che se una o l'altra delle precedenti ipotesi non sono verificate il teorema può essere falso. *G. Sansone* (Zbl **93**, 286)

Hornich, Hans 1414

Huygenssche Differentialgleichungen im R_n.

Math. Z. **84** (1964), 268–270.

A type of equation is constructed whose solutions tend to become singular at one point because of focusing. By excluding solutions with this type of singularity one gets something superficially resembling Huygens' Principle. *P. Ungar* (New York)

Ghermănescu, M. Michel 1415

Fonctions harmoniques (p, q)-conjuguées.

Mathematica (Cluj) **4** **(27)** (1962), 241–246.

In this note the author considers systems of partial differential equations of the form

$$(1)\qquad \begin{aligned} v_x &= au_x+bu_y,\\ v_y &= cu_x+du_y, \end{aligned}$$

where a, b, c, and d are given real functions which are defined and analytic in a closed region; he wishes to determine conditions under which u and v satisfy the same linear partial differential equation of the second order. If u and v are assumed to be sufficiently smooth, then he finds that u satisfies

$$L(u) = cu_{xx}+(d-a)u_{xy}-bu_{yy}+(c_x-a_y)u_x+(d_x-b_y)u_y = 0,$$

and that v satisfies

$$M(v) = cv_{xx}+(d-a)v_{xy}-bv_{yy}+(\bar{a}_y-\bar{c}_x)v_x+(\bar{b}_y-\bar{d}_x)v_y = 0,$$

where $\Delta=ad-bc$ (assumed to be non-zero), $\Delta\bar{a}=d$, $\Delta\bar{b}=-b$, $\Delta\bar{c}=-c$, and $\Delta\bar{d}=a$. He says that the equations $L(u)=0$ and $M(v)=0$ above are "équations-conséquentes" of the original system (1). The author gives necessary and sufficient conditions that an elliptic system (1) have a unique "équation-conséquente"; he also gives necessary and sufficient conditions that the "équation-conséquente" be the Laplace equation. The author concludes by making an application of this final result to a problem due to Poincaré concerning harmonic functions. *M. Reade* (Ann Arbor, Mich.)

Ames, W. F.; de la Cuesta S., H. 1416

Solutions of a class of linear partial differential equations with variable coefficients.

J. Math. and Phys. **42** (1963), 301–306.

In this paper the authors study the criteria under which equations of the form

$$\frac{\partial T}{\partial t}-\nabla\phi\cdot\nabla T = k\nabla^2 T$$

are solvable by separation of variables. By setting

$T = h\exp[-\phi/2k]$ the authors reduce this equation to a Schrödinger-type equation,

$$\frac{\partial h}{\partial t} + k\left\{\left(\frac{\nabla\phi}{2k}\right)^2 + \frac{\nabla^2\phi}{2k}\right\}h = k\nabla^2 h.$$

They then investigate the criteria under which the coefficient of h has the property of "separateness", which they give as the condition

$$\left(\frac{\nabla\phi}{2k}\right)^2 + \frac{\nabla^2\phi}{2k} = \sum_i \frac{X_i(x^i)}{g_{ii}},$$

where the x^i are orthogonal coordinates, and g_{ii} is the covariant component of the metric tensor.

R. P. Gilbert (College Park, Md.)

Lavruk, B. [Ławruk, B.] 1417

On the unique solvability of a general boundary-value problem for homogeneous linear systems of second-order differential equations of elliptic type with constant coefficients in a half-space. (Russian)

Ann. Polon. Math. **14** (1963), 85–95.

Der Autor untersucht eine allgemeine Randwertaufgabe für den linearen Differentialoperator zweiter Ordnung vom elliptischen Typus

(1) $$A\left(\frac{\partial}{\partial x}\right) = \sum_{i,j=1}^{n} A_{ij}\frac{\partial^2}{\partial x_i \partial x_j},$$

deren Koeffizienten quadratische reelle konstante Matrizen der Ordnung p sind. Es wird gezeigt, daß das Differentialgleichungssystem

(2) $$A\left(\frac{\partial}{\partial x}\right)u(x) = 0 \qquad (x = (x_1, \cdots, x_n);\ x_n > 0)$$

in der Matrizenklasse $u(x)\to 0$ für $|x|\to\infty$ ($|x| = (\sum_{i=1}^n x_i^2)^{1/2}$) eine eindeutige, für $x_n > 0$ zweimal stetig differenzierbare Lösung besitzt, die der Randbedingung

(3) $$\lim_{x_n\to 0^+}\begin{pmatrix} B_I\left(\frac{\partial}{\partial x}\right) \\ -B_{II}\end{pmatrix}u(x) = 0$$

genügt. Hierbei sind $B_I(\partial/\partial x) = \sum_{i=1}^n B_{Ii}(\partial/\partial x_i)$ und B_{II} reelle konstante Matrizen mit

$$\det\begin{pmatrix} B_{In} \\ B_{II}\end{pmatrix} \neq 0 \qquad (I + II = p),$$

von denen außerdem gefordert wird, daß ihre Grenzwerte bezüglich $x' = (x_1, \cdots, x_{n-1}, 0)$

$$\lim_{x_n\to 0^+}\sum_{i=1}^n B_{Ii}\left(\frac{\partial u(x)}{\partial x_i}\right) \quad \text{und} \quad \lim_{x_n\to 0^+} B_{II}\left(\frac{\partial u(x)}{\partial x_n}\right)$$

gleichmäßig existieren.

Für $I = p$ und $II = 0$ geht diese Randwertaufgabe in eine Randwertaufgabe mit schiefer Richtungsableitung über. Im Falle $I = 0$ und $II = p$ dagegen ergibt sich eine Randwertaufgabe vom Dirichletschen Typus.

Der Beweis der oben formulierten Lösungseigenschaft wird unter Verwendung der Ergebnisse einer früheren Arbeit des Autors durchgeführt [siehe dieselben Ann. **13** (1963), 67–91; MR **26** #6557]. *W. Schmidt* (Simferopol)

Mihaĭlov, L. G. 1418

Elliptic equations with singular coefficients. (Russian)

Dokl. Akad. Nauk SSSR **139** (1961), 552–555.

This is a brief statement of results proved and extended in the author's paper [Izv. Akad. Nauk SSSR Ser. Mat. **26** (1962), 293–312; MR **26** #2723].

R. K. Juberg (Minneapolis, Minn.)

Kohn, J. J. 1419

Non-coercive estimates. (French summary)

Les Équations aux Dérivées Partielles (Paris, 1962), pp. 47–52. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1963.

This is an expository paper in which the author presents some of the results in his more recent papers [Ann. of Math. (2) **78** (1963), 112–148; MR **27** #2999; ibid. **79** (1964), 450–472]. This expository paper was presented before its author had obtained his excellent results on the smoothness of the solutions up to the boundary which are presented in Part II of the paper cited above. Still simpler presentations of these results will be presented in a paper by the author and Nirenberg and in the reviewer's forthcoming book, *Multiple integrals in the calculus of variations*. *C. B. Morrey, Jr.* (Berkeley, Calif.)

Liang, Shi-Ting 1420

The regularity of the solution of elliptic equations.

Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **4 (52)** (1961), *no.* 1–2, 57–68 [1963].

The author generalizes results of the reviewer's paper [Math. Z. **72** (1959/60), 146–164; MR **22** #11200] concerning equations of the form

(*) $$\int_G [v_{,\alpha}(a^{\alpha\beta}u_{,\beta} + b^\alpha u + e^\alpha) + v(c^\alpha u_{,\alpha} + du + f)]\,dx = 0, \qquad v \in H_{20}{}^1(G),$$

in which the coefficients $a^{\alpha\beta}$, b^α, and $e^\alpha \in C_\mu{}^0(\bar G)$, G is of class $C_\mu{}^1$, and the $a^{\alpha\beta}$ satisfy the uniform ellipticity condition. The author has shown that if the c^α and $d \in L_p(G)$ with $p = n/(1-2\mu)$ and $f \in L_q(G)$ with $q > 2n/(n+2)$ ($x = x^1, \cdots, x^n$) and satisfies

$$\int_{B(x_0,r)\cap G} |f(x)|\,dx \leqq Lr^{n-1+2\mu} \qquad (0 < \mu < \tfrac{1}{2}),$$

then any solution u of (*), where $u \in H_{20}{}^1(G)$, is also in $C_\mu{}^1(\bar G)$. The reviewer notes that 2μ may be replaced by μ in the requirements listed above. These results and more general ones obtained routinely by the reviewer will appear in the reviewer's forthcoming book, *Multiple integrals in the calculus of variations.*

C. B. Morrey, Jr. (Berkeley, Calif.)

Li, Ming-zhong [Li, Ming-chung] 1421

An existence theorem and a representation formula for generalized solutions of second-order elliptic differential equations.

Acta Math. Sinica **14** (1964), 7–22 (*Chinese*); *translated as Chinese Math.* **5** (1964), 8–24.

Analog zur Theorie der verallgemeinerten analytischen Funktionen im Sinne von I. N. Vekua baut der Autor eine Theorie zur Untersuchung der Lösungseigenschaften eines partiellen Differentialgleichungssystems zweiter

rdnung vom elliptischen Typus auf. Das Differentialeichungssystem

$$a_{i1}u_{xx}+a_{i2}v_{xx}+2b_{i1}u_{xy}+2b_{i2}v_{xy}+c_{i1}u_{yy}+c_{i2}v_{yy}+d_{i1}u_x$$
$$+d_{i2}v_x+e_{i1}u_y+e_{i2}v_y+f_{i1}u+f_{i2}v = g_i \qquad (i = 1, 2)$$

vom elliptischen Typus, wenn die Determinante

$$|a_{ik}+2b_{ik}\lambda+c_{ik}\lambda^2| \qquad (i, k = 1, 2)$$

r alle nicht verschwindenden λ-Werte ungleich Null ist. In Analogie zum Falle eines elliptischen Differentialeichungssystem erster Ordnung führt der Autor das ystem (1) zunächst auf eine komplexe partielle Differentialgleichung zweiter Ordnung zurück. Mit Hilfe niger singulärer Integraloperatoren ergibt sich schließlich, ie auch bei I. N. Vekua [*Verallgemeinerte analytische unktionen* (Russisch), Fizmatgiz, Moscow, 1959; MR **21** 7288; Englische Übersetzung, Pergamon, London, 1962; R **27** #321; Deutsche Übersetzung, Akademie-Verlag, erlin, 1963; MR **28** #1312], eine singuläre Integraleichung, die dem ursprünglichen Differentialgleichungsystem äquivalent ist. Unter Verwendung der Eigenhaften genannter singulärer Integraloperatoren wird zeigt, daß die verallgemeinerte Lösung der partiellen ifferentialgleichung zweiter Ordnung hier von zwei wöhnlichen analytischen Funktionen abhängig ist. Es ird eine Darstellungsformel für die Lösungen des ystems (1) angeben, die etwa der Darstellung zweiter Art r verallgemeinerte analytische Funktionen im Vekuahen Sinne entspricht.

Abschließend wird eine Randwertaufgabe der Form

$$\alpha_1(t)(u_x-v_y)+\beta_1(t)(u_y+v_x) = \gamma_1(t),$$
$$(2) \qquad \alpha_2(t)u+\beta_2(t)v = \gamma_2(t)$$
$$\text{mit } \alpha_k^2+\beta_k^2 = 1 \ (k = 1, 2),\ |t| = 1,$$

ir das Differentialgleichungssystem (1) betrachtet. Es ird gezeigt, daß die Existenz von Lösungen dieser andwertaufgabe von den Vorzeichen der Indices

$$n_k = \frac{1}{2\pi}\Delta_{|t|=1}\arg\lambda_k(t), \qquad k = 1, 2, \quad \lambda_k = \alpha_k+i\beta_k,$$

bhängig ist.

Sind beide Indices n_1 und n_2 nicht negativ, so ist die andwertaufgabe (1), (2) stets lösbar, und ihre Lösung ängt von $2(n_1+n_2+1)$ beliebigen Konstanten ab.

Im Falle $n_1<0$, $n_2\geqq 0$ ist die Randwertaufgabe nur ann lösbar, wenn $-2n_1-1$ Integralbedingungen erfüllt ind. Die Lösung ist dann von $2n_2+1$ beliebigen Kontanten abhängig.

Entsprechend ergibt sich im Falle $n_1\geqq 0$, $n_2<0$, daß ie Randwertaufgabe nur dann lösbar ist, wenn $-2n_2-1$ eelle Zusatzbedingungen erfüllt sind. Die Lösung ist von n_1+1 beliebigen Konstanten abhängig.

Falls $n_1<0$, $n_2<0$, so ist die Randwertaufgabe nur ann lösbar, wenn $-2(n_1+n_2+1)$ reelle Zusatzbedingngen erfüllt sind. In diesem Falle ist die Lösung indeutig bestimmt.

Der Autor gibt für alle Fälle explizite Lösungsformeln n.

W. Schmidt (Simferopol)

antscher, Lothar 1422

Grundlösungen von elliptischen Differentialgleichungen für den ganzen Raum.

J. Reine Angew. Math. **214/215** (1964), 350–359.

In connection with an earlier paper [same J. **207** (1961), 129–178; MR **24** #A321] the author considers in a bounded domain C of R^n the elliptic operator

$$Eu = \sum a_{pq}\frac{\partial^2 u}{\partial x_p\partial x_q}+\sum b_p\frac{\partial u}{\partial x_p}+cu-\kappa^2 hu,$$

with convenient smoothness assumptions on the coefficients a_{pq}, b_p, c, h, κ being a parameter. On "mollifying" the coefficients, the given operator E is approximated by an operator E' defined in the whole of R^n. Using now a parametrix approach, a certain two-sided fundamental solution of E' is constructed. Further, the question of convergence of the solutions is investigated; in particular, it is shown that the solution of the first boundary problem in a subdomain T of C for E' converges uniformly to the corresponding solution for E.

J. Peetre (Lund)

Marcinkowska, H. 1423

On the differentiability of weak solutions of certain non-elliptic equations. II.

Studia Math. **23** (1963), 85–103.

This is a continuation of the author's earlier paper with the same title [same Studia **22** (1962/63), 91–107; MR **26** #492] in that the basic class of differential operators under consideration and the methods used are the same. The results are extended in several directions. In particular, the author considers solutions which are periodic with respect to some variables and satisfy boundary conditions of Dirichlet type with respect to the others.

R. S. Freeman (College Park, Md.)

Oskolkov, A. P. 1424

A class of solutions of boundary-value problems for linear elliptic equations with three independent variables in an unbounded region. (Russian. English summary)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **19** (1964), *no.* 1, 150–156.

Author's summary: "The class of linear elliptic partial differential equations whose coefficients are regular at infinity, and the absolute terms satisfy the inequality $|f(x)|\leqq|x|^{-2}\delta(|x|)$, where δ is a Dini function, is considered. The principal boundary-value problems with homogeneous and inhomogeneous limiting conditions at infinity are solved for these equations in the unbounded domain."

Parasjuk, L. S. 1425

Radiation conditions for certain elliptic differential equations which are degenerate on the boundary of the region. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 441–443.

For equations of the type

$$\sum_{i=1}^{n-1}\frac{\partial^2 U}{\partial x_i^2}+x_n^\alpha\frac{\partial^2 U}{\partial x_n^2}+CU = 0,$$

$$\sum_{i=1}^{n-1}\frac{\partial^2 U}{\partial x_i^2}+\frac{\partial}{\partial x_n}\left(x_n^\alpha\frac{\partial U}{\partial x_n}\right)+CU = 0,$$

where $C=\kappa^2>0$ is a constant, which degenerate at the

boundary $x_n = 0$ ($n \geqq 3$), the author gives an explicit form for the source functions for the fundamental boundary-value problems. The radiation conditions are similar to those of the Helmholtz equation.

A. J. Lohwater (Providence, R.I.)

Miranda, Carlo 1426

Equazioni lineari ellittiche di tipo non variazionale. (English summary)

Rend. Sem. Mat. Fis. Milano **33** (1963), 206–219.

This is an expository paper concerned with the existence and regularity properties of the solutions in $H^{2,2}(\Omega)$ of second-order elliptic equations which are not in divergent form:

$$(*)\quad \sum a_{ik}\frac{\partial^2 u}{\partial x^i \partial x^k} + \sum b_i \frac{\partial u}{\partial x^i} + cu = f \qquad (x = x^1, \cdots, x^n),$$

$$m_0 \sum \lambda_i^2 \leqq \sum a_{ik}\lambda_i\lambda_k \leqq m_1 \sum \lambda_i^2.$$

In case the a_{ik} are continuous on a smooth domain Ω, many results are known. However, the author treats many cases where the $a_{ik} \in H_n^{\ 1}$; this condition does not quite imply their continuity but does make it possible to put (*) in divergence form. If the a_{ik} are allowed to be in $H^1_{n-\varepsilon}$, then A. D. Alexandrov has shown that the expected existence and regularity theory does not necessarily hold. The reviewer did not find a reference to the work of H. O. Cordes on the solutions of such equations [Math. Ann. **138** (1959), 155–178; MR **22** #143; Proc. Sympos. Pure Math., Vol. IV, pp. 157–166, Amer. Math. Soc., Providence, R.I., 1961; MR **26** #4033].

C. B. Morrey, Jr. (Berkeley, Calif.)

Vaĭnberg, B. R. 1427

The existence and uniqueness over the entire plane of certain elliptic equations. (Russian)

Dokl. Akad. Nauk SSSR **142** (1962), 14–16.

This is a particular example of the author's more general result [Dokl. Akad. Nauk SSSR **144** (1962), 958–961; MR **26** #472]. *R. K. Juberg* (Minneapolis, Minn.)

Fedosov, B. V. 1428

Asymptotic formulae for eigenvalues of the Laplace operator for a polyhedron. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 536–538.

The author considers $-\Delta U = k^2 U$ in an m-dimensional polyhedral domain Π with the conditions $U=0$ or $\partial U/\partial n = 0$ on the boundary. Letting $n(k)$ denote the number of eigenvalues not exceeding k^2, he shows that for $0 \leqq p \leqq m-1$ and $k \to \infty$,

$$\frac{1}{\Gamma(p+1)}\int_0^k (k-t)^p\, dn(t) = \sum_{l=1}^m a_l \frac{\Gamma(l+1)}{\Gamma(p+l+1)} k^{p+l} + O(k^{m-1}),$$

where the a_l are constants depending on the geometry of Π. The results extend those of the author for polygons [same Dokl. **151** (1963), 786–789; MR **28** #335] and are obtained by similar methods, i.e., a Green's function for the wave equation in Π and a Tauberian theorem. A similar program using the heat equation was carried out for polygons by P. B. Bailey and F. H. Brownell [J. Math. Anal. Appl. **4** (1962), 212–239; MR **26** #2748].

R. E. L. Turner (Madison, Wis.)

Volevič, L. R. 1429

On hypo-elliptic systems with variable coefficients. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 1262–1265.

A class of hypo-elliptic systems with variable coefficients is exhibited. Writing the systems in the form $P(x, D)u = f$, where $x = (x_1, \cdots, x_n)$, $D = (-i\partial/\partial x_1, \cdots, -i\partial/\partial x_n)$, while u, f are m-vectors of functions and $P(x, D)$ an $m \times m$ matrix of partial differential operators defined in a domain Ω of R^n, and using Hörmander's notation, the assumptions read: (A) For each $x \in \Omega$ the polynomial $Q(x, \xi) = \det P(x, \xi)$ is hypo-elliptic. (B) For each x', $x'' \in \Omega$ the inequality $\tilde{Q}(x', \xi) \leqq C\tilde{Q}(x'', \xi)$ holds. (C) There are integers $s_1, \cdots, s_m$, $t_1, \cdots, t_m$, with $\sum_{i=1}^m (t_i - s_i) = 1$, and a real $\sigma > 0$ such that for each $x \in \Omega$ the inequality $|P_{ij}^{(\alpha)}(x, \xi)| \leqq C|\tilde{Q}(x, \xi)|^{t_j - s_i}(1+|\xi|)^{-\sigma|\alpha|}$ holds, with $P_{ij}(x, \xi) \equiv 0$ if $t_j - s_i < 0$. The proof of the hypo-ellipticity, which is briefly outlined, follows closely a paper by the reviewer in the case of one single equation [Comm. Pure Appl. Math. **14** (1961), 737–744; MR **26** #1610].

J. Peetre (Lund)

Zachmanoglou, E. C. 1430

The decay of solutions of the initial-boundary value problem for the wave equation in unbounded regions.

Arch. Rational Mech. Anal. **14** (1963), 312–325.

The decay of solutions of the wave equation in many dimensions and in the exterior of reflecting bodies is considered. The results of the reviewer [Comm. Pure Appl. Math. **15** (1962), 349–361; MR **27** #1696] showing pointwise decay like $1/t^2$ for the homogeneous equation and the convergence to the Sommerfeld solution (limiting amplitude principle) for the non-homogeneous equation with harmonic forcing term are extended to these restrictive conditions. Principally, the decay rate is extended to any dimension greater than 2 and to infinite star-shaped bodies. The limiting amplitude principle is extended to infinite star-shaped bodies, but the space dimension is 3 and the initial data do not have to be compact, but have restrictions on their growth at infinity. The complicated identities of the reviewer [loc. cit.] are reduced to a more elegant and manageable form.

Exponential decay and the limiting amplitude principle can be obtained for odd dimensions and finite bodies using results of Lax, the reviewer and Phillips [ibid. **16** (1963), 477–486; MR **27** #5033], which are valid for all situations where the Huygens principle holds.

C. S. Morawetz (New York)

Friedlander, F. G. 1431

On the radiation field of pulse solutions of the wave equation. II.

Proc. Roy. Soc. Ser. A **279** (1964), 386–394.

In a previous paper [same Proc. **269** (1962), 53–65; MR **26** #455] the author discussed the solution $u(x, t)$ of the homogeneous wave equation $u_{tt} = \Delta u$ for $t \geqq r \geqq a$ under the assumption that u vanishes for $t \leqq r$ in $r \geqq a$ and showed, in particular, that $ru(r\xi, t) \to f(\xi, t-r)$ as $r \to \infty$, where $x \equiv (x_1, x_2, x_3) = r\xi$, $r = |x|$ and ξ is a unit vector. In the

present paper the Laplace transform of u with respect to t is assumed to exist. This implies that the Laplace transform of f with respect to $s=t-r$ also exists and is an analytic function of ξ. On the basis of this and other assumptions two results are proved. "The first is a unique continuation property: if two radiation fields $f_1(\xi, s)$ and $f_2(\xi, s)$ are equal to each other for all s and for any set of unit vectors ξ that is an open set on the unit sphere, then $f_1=f_2$ for all ξ. Furthermore, an integral representation of the Laplace transform of u in terms of the Laplace transform of f will be obtained, which generalizes Weyl's integral representation of a diverging spherical wave in terms of plane waves with complex propagation vectors."

S. Datta Majumdar (Calcutta)

Avila, Geraldo S. S.; Keller, Joseph B. 1432

The high-frequency asymptotic field of a point source in an inhomogeneous medium.

Comm. Pure Appl. Math. **16** (1963), 363–381.

Asymptotic solutions of the wave equation $\Delta u+k^2n^2(x)u=\delta(x)$ for large k are derived in the form $u(x,k)=v(x,k)\exp ikS(x)$, where $S(x)$ satisfies the ray equation $(\nabla S)^2=n^2$. It is shown that in spaces of odd dimensions a single asymptotic expansion for $v(x,k)$ exists. In spaces of even dimensions separate expansions near the origin and in the outer region have to be used. These have to be matched. This has been done in the paper for two-dimensional waves.

L. J. F. Broer (Eindhoven)

Bakel'man, I. Ja.; Guberman, I. Ja. 1433

The Dirichlet problem for an equation with a Monge-Ampère operator. (Russian)

Sibirsk. Mat. Ž. **4** (1963), 1208–1220.

The authors prove an existence theorem for a solution $z(x,y)$ to the problem (*) $rt-s^2=g(x,y,z,p,q)$ in a convex Jordan region $\Omega\subset E$, where z is required to be convex toward the positive direction of the z-axis, and to assume given boundary values $z|_\Gamma=L(x,y)$ on the boundary Γ of Ω. Here g is defined for all $(x,y)\in\Omega$, $0\leqq z\leqq R_0$, $-\infty<p$, $q<+\infty$, and is continuous, non-negative, and of the form $g=R(p,q)f(x,y,z,p,q)$, where $0\leqq R(p,q)\leqq c(1+p^2+q^2)^k$, $c,k>0$, and $f(x,y,z,p,q)\leqq a[\rho(x,y)]^\lambda$ $(a,\lambda>0)$, where ρ is the distance of (x,y) from Γ. Also it is assumed that $f_R(x,y)=\operatorname{Sup} f(x,y,z,p,q)$ is L-integrable in Ω, where Sup is taken for $0\leqq z\leqq R_0$, $-\infty<p$, $q<+\infty$, and that $0\leqq L(x,y)\leqq R<R_0$ for a convenient number R. On the convex boundary curve Γ it is assumed that every point Q of Γ belongs to an arc Γ_Q of Γ sufficiently small, containing Q as an interior point such that for every two points Q', Q'' of Γ_Q and corresponding tangents t', t'', we have $\theta/l\geqq bl^\nu$, $b,\nu>0$, $\theta=\langle t,t'\rangle$, where l is the arclength of QQ'. This allows zero curvature at points of Γ. A more specific hypothesis is made: $k\leqq 1+(\nu+2)^{-1}+2^{-1}\lambda$. The authors first prove a theorem of compactness for convex surfaces $P: z=z(x,y)$, $(x,y)\in\Omega$, z convex. Then existence theorems are given for the Dirichlet problem for the equation $rt-s^2=R(p,q)\phi(x,y)$. Finally, the existence theorem for equation (*) is obtained by the known Leray process based on the Schauder fixed-point theorem. The present existence theorem extends previous ones of I. Ja. Bakel'man [Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **13** (1958), no. 1, 25–38; MR **20** #3384] and of A. V. Pogorelov [*On Monge-Ampère equations of elliptic type* (Russian), Izdat. Har'kovsk. Gos. Univ., Kharkov, 1960; MR **23** #A1137].

L. Cesari (Ann Arbor, Mich.)

Aleksandrov, A. D. 1434

Uniqueness conditions and bounds for the solution of the Dirichlet problem. (Russian. English summary)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **18** (1963), *no.* 3, 5–29.

With $u_i=\partial u/\partial x_i$, $u_{ik}=\partial^2u/\partial x_i\partial x_k$ consider the differential equation (*) $\sum_{i,k=1}^n a_{ik}u_{ik}=\varphi$ in a bounded domain G of the $(x_1,\cdots,x_n)$-space. The coefficients a_{ik} and φ may depend on u and the u_i, but only such solutions u of (*) are considered for which the matrix a_{ik} has no negative eigenvalues. The results hold under either one of the following assumptions on u. (1) u has generalized second derivatives in the sense of Sobolev whose nth power is summable over every compact subset of G.

$$\text{(2)}\qquad \lim_{x'\to x}\,(|\operatorname{grad} u(x)-\operatorname{grad} u(x')|/|x-x'|)<\infty,$$

which implies that the second derivatives u_{ik} exist a.e. in the ordinary sense. With $u(x)$ we associate the maximal convex function $w_1(x)\leqq u(x)$ and the minimal concave function $w_2(x)\geqq u(x)$, both defined in the convex hull H of G. For each point y in H we consider all linear functions $\sum p_i(x_i-y_i)+w_1(y)\leqq w_1(x)[\sum p_i(x_i-y_i)+w_2(y)\geqq w_2(x)]$ and for a subset E of H we denote as the normal image $\psi(w_i,E)$ of w_i in E the totality of points p in the $(p_1,\cdots,p_n)$-space obtained from those linear functions at the points $y\in E$, $|p|=1$.

The theorems refer to m-flats L $(1\leqq m\leqq n)$ through the origin. It is always assumed that L is the flat $x_{m+1}=\cdots=x_n=0$.

a_L is the determinant of principal minor $i\leqq m$, $k\leqq m$ of a_{ik} formed with $u_j=0$ for $j>m$. The principal lemma, from which almost all results follow, is this: Let a solution u of (*) for the flat L satisfy the inequality $a_L{}^{-1/m}\varphi\leqq P_L(x_1,\cdots,x_m)Q_L(u_1,\cdots,u_m)$, where $P_L\geqq 0$, $Q_L\geqq 0$. Then (**) $\int_{G_L}P_L{}^m\,dx_1\cdots dx_m\geqq m^m\int_{L\psi(\bar w_1,G)}Q_L{}^{-m}\,du_1\cdots du_m$, where G_L is the projection of G on L and $L\psi(w_1,G)$ is the intersection of L with the normal image $\psi(w_1,G)$. If $a_L{}^{-1/m}\varphi\geqq -P_LQ_L$, then (**) holds with w_2 instead of w_1.

Assume the solution u of (*) satisfies the hypothesis of the first part of this lemma in each compact subset D of G, that $P_L{}^m$ is summable on D_L and $a_L{}^{-m}$ is not summable in any neighborhood of the origin. Then $u(x)$ attains its greatest lower bound on the boundary in the sense that a sequence x^i of points in G exists tending to the boundary with $u(x^i)\to\inf_G u$. Under the hypotheses of the second part, $u(x)$ attains its least upper bound on the boundary. We mention only one application. With $A(u)=\sum a_{ik}u_{ik}+\sum b_iu_i+cu$ let $A(u)=f$ be a linear equation. Put $b_L=(\sum_{i=1}^m b_i{}^2)^{1/2}$. If $c\leqq 0$ and in each compact $D\subset G$ the inequality $a_L{}^{-1/m}b_L\leqq P_L(x_1,\cdots,x_m)$, where $P_L\geqq 0$ depends on L and D, $P_L{}^m$ is summable in D. Then the following three statements hold. If $A(u)\leqq 0$ and somewhere $u(x)<0$, then $u(x)$ attains its greatest lower bound on the boundary. If $A(u)\leqq A(v)$ and $u\geqq v$ on the boundary, then $u\geqq v$ everywhere. The Dirichlet problem has at most one solution. The assertion is the best possible in the sense that it is no longer true if the summability of $P_L{}^m$ is replaced by that of $P_L{}^{m-\epsilon}$.

There are many more theorems, but their formulation is so involved that they cannot be given here.

H. Busemann (Los Angeles, Calif.)

Miller, Keith 1435

Three circle theorems in partial differential equations and applications to improperly posed problems.

Arch. Rational Mech. Anal. **16** (1964), 126–154.

For complex-valued functions analytic and single-valued on an annulus, we know Hadamard's celebrated three-circle theorem. In this paper, analogous three-circle theorems are found for harmonic functions on a disc, but with respect to L_2 or even weaker bounds on the inner and outer circles. Analogous three-line theorems are also found for periodic solutions of the heat equation and for periodic solutions of the Cauchy problem for Laplace's equation. The main results are as follows.

(I) Let $u=u(r,\theta)$ be harmonic in the disc $r<1$, and define $|u|_r=\sup_\theta|u(r,\theta)|$, $\|u\|_r=\{(2\pi)^{-1}\int_0^{2\pi}u(r,\theta)^2\,d\theta\}^{1/2}$ $(r<1)$ and $\|u\|_1=\lim_{r\uparrow 1}\|u\|_r$ $(\leqq\infty)$. Since such u has a Fourier expansion in the disc, there correspond, for each integer $N>0$, a low-order part u_N and a high-order part u_N^∞ of u defined by

$$(1)\qquad u_N(r,\theta)=\frac{a_0}{\sqrt{2}}+\sum_1^N(a_n\cos n\theta+b_n\sin n\theta)r^n,$$
$$u_N^\infty(r,\theta)=\sum_{N+1}^\infty(a_n\cos n\theta+b_n\sin n\theta)r^n,$$

and we have $u=u_N+u_N^\infty$. Let Γ denote the class of functions $u(r,\theta)$ harmonic in the disc $r<1$ and satisfying $\|u\|_a\leqq m=a^\alpha$ and $\|u\|_1\leqq 1$ $(0<a<1)$, and let $M(r)$ denote the supremum of $|u|_r$ over Γ. Let $L(N,r)$ denote the maximum of $|u_N|_r$ over the class of low-order functions satisfying $\|u_N\|_a\leqq m$, and $H(N,r)$ the maximum of $|u_N^\infty|_r$ over the class of high-order functions satisfying $\|u_N^\infty\|_1\leqq 1$. Then, for any $u\in\Gamma$, (i) $\|u\|_r\leqq m^{\log r/\log a}=r$ for $a\leqq r\leqq 1$; (ii) $M(r)\leqq L([\alpha],r)+H([\alpha],r)\leqq 2M(r)$ and $L([\alpha],r)+H([\alpha],r)<r^\alpha[(2\sum_0^{[\alpha]}(a/r)^{2n})^{1/2}+(2(1-r^2)^{-1})^{1/2}]$ for $a<r<1$. Similar results are obtained also for partial derivatives of $u\in\Gamma$.

(II) Consider solutions $u=u(x,t)$ of the heat equation $u_{xx}=u_t$ $(t>0)$ which are periodic in x with period 2π, define $|u|_t$ and $\|u\|_t$ analogously to the definition of $|u|_r$ and $\|u\|_r$ in (I), and put $\|u\|_0=\lim_{t\downarrow 0}\|u\|_t$. Again write

$$(2)\quad u(x,t)=\frac{a_0}{\sqrt{2}}+\sum_1^\infty(a_n\cos nx+b_n\sin nx)e^{-n^2t}=u_N+u_N^\infty,$$

where these low- and high-order parts are defined analogously to (1). Let Γ denote the class of solutions $u(x,t)$ of the heat equation which are periodic in x with period 2π and satisfy $\|u\|_a\leqq m=e^{-\alpha^2a}$ and $\|u\|_0\leqq 1$, and let $M(t)$ denote the supremum of $|u|_t$ over Γ. $L(N,r)$ and $H(N,r)$ are also defined similarly to those in (I). Then, for any $u\in\Gamma$, (i) $\|u\|_t\leqq m^{t/\alpha}=e^{-\alpha^2t}$ for $0\leqq t\leqq a$; (ii) $M(t)\leqq L([\alpha],t)+H([\alpha],t)\leqq 2M(t)$ and $L([\alpha],t)+H([\alpha],t)<m^{t/\alpha}[(2\sum_0^{[\alpha]}e^{-2n^2(a-t)})^{1/2}+(2\sum_0^\infty e^{-2n^2t})^{1/2}]$ for $0<t<a$.

(III) Consider functions $u(x,y)$ harmonic on an infinite strip $0<y<a$, periodic in x with period 2π and satisfying $\int_0^{2\pi}u(x,y)\,dx=0$ for each y (the last condition is not an essential restriction). Such functions have a representation of the form

$$(3)\qquad u(x,y)=\sum_1^\infty(A_{n,y}\cos nx+B_{n,y}\sin nx).$$

$|u|_y$ and $\|u\|_y$ are defined similarly to $|u|_t$ and $\|u\|_t$ in (II). Let $D^{-1}u_y$ denote the termwise x-integral of u_y. The author proves (i) if $\|u\|_0\leqq m=e^{-\alpha a}$, $\|D^{-1}u_y\|_0\leqq m$ and $\|u\|_a\leqq 1$, then

$$|u|_y<m^{1-y/a}\left[\left(\frac{5}{2}\sum_0^{[\alpha-1]}e^{-2ny}\right)^{1/2}+m^{2y/a}\left(\frac{2}{1-e^{-2y}}\right)^{1/2}+\left(\frac{2}{1-e^{-2(a-y)}}\right)^{1/2}\right]$$

for $0<y<a$; (ii) if $\|\,|\mathrm{grad}\,u|\,\|_0\leqq m=e^{-\alpha a}$ and $\|u\|_a\leqq 1$, then

$$|u|_y<m^{1-y/a}\left[\frac{2\pi}{\sqrt{6}}+\left(\frac{2}{1-e^{-2(a-y)}}\right)^{1/2}\right]$$

for $0<y<a$. Next, consider functions u harmonic on the whole symmetric strip: $-a<y<a$. For u of the form (3) on $-a<y<a$, we write $u=u_N+u_N^\infty$, where these low- and high-order parts are defined analogously to (1). Let $L(N,y)$ denote the maximum of $|u_N|_y$ over the class of low-order functions satisfying $\|u_N\|_0\leqq m$ and $\|D^{-1}(u_N)_y\|_0\leqq m$, and let $H(N,y)$ denote the maximum of $|u_N|_y$ over the class of high-order functions satisfying $\|u_N^\infty\|_a^2+\|u_N^\infty\|_{-a}^2\leqq 2$. Then, (iii) if $\|u\|_0\leqq m=e^{-\alpha a}$, $\|D^{-1}u_y\|_0\leqq m$ and $\|u\|_a^2+\|u\|_{-a}^2\leqq 2$, then

$$|u|_y<L([\alpha],y)+H([\alpha],y)<m^{1-|y|/a}\left[\left(\frac{a}{2}\sum_0^{[\alpha-1]}e^{-2n|y|}\right)^{1/2}+\left(\frac{10}{1-e^{-2(a-|y|)}}\right)^{1/2}\right]$$

for $-a<y<a$.

In the latter half of this paper, the author gives some applications of the above results to numerical methods for improperly posed problems in partial differential equations: harmonic and analytic continuation, backward solution of the heat equation, and the Cauchy problem for Laplace's equation.

In the last section it is suggested that (i) the method of this paper may be extended to the case of several space variables, (ii) the periodicity restriction in (II) and (III) may be removed by the use of the Fourier integral in place of the Fourier series, and (iii) the general methods of this paper should apply to parabolic and elliptic equations with non-constant coefficients where separation of variables allows an eigenfunction expansion of the solution.

S. Itô (Tokyo)

Kružkov, S. N. 1436

The Cauchy problem in the large for non-linear equations and for certain first-order quasilinear systems with several variables. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 743–746.

The author proves the existence and uniqueness of a generalized solution for the equation $u_t+f(u_x)=0$, $x=(x_1,\cdots,x_n)$, $t\geqq 0$, with the Cauchy data $u(0,x)=u_0(x)$. It is assumed that u_0 is bounded and Lipschitz, while f is twice continuously differentiable and $0<\lambda(u_x)|\xi|^2\leqq$

$\sum f_{ij}(u_x)\xi_i\xi_j$. The generalized solution u is bounded, Lipschitz, and satisfies

$$\frac{u(t, x+l)-2u(t, x)+u(t, x-l)}{|l|^2} \leq \frac{C}{t}$$

for every $l \neq 0$. Moreover, u is the limit as $\varepsilon \to 0^+$ of the solution u^ε of the same Cauchy problem for $u_t + f(u_x) = \varepsilon\Delta u$. If $f(0) = f_i(0) = 0$, $i = 1, \cdots, n$, then $u(t, x) \to \mu = \inf u_0(x)$ as $t \to +\infty$.

{Reviewer's note. The author bounds u^ε, u_x^ε, u_t^ε uniformly by using the maximum principle. The existence of a limit u satisfying the equation $u_t + f(u_x) = 0$ almost everywhere then can be proved without assuming convexity of f; see the reviewer's paper [J. Math. Mech. **13** (1964), 987–1008].} *W. H. Fleming* (Providence, R.I.)

Mosolov, P. P. 1437

The Dirichlet problem for partial differential equations. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1960**, *no.* 3 (16), 213–218.

Let $P(s, t)$ be a polynomial of degree p with respect to s and t and also with respect to each variable separately. The problem considered is to find a smooth function $u = u(x, y)$ satisfying (1) $P(\partial^2/\partial x^2, \partial^2/\partial y^2)u = f$ in $R = \{(x, y): 0 < x < 1, 0 < y < a^{-1}; a \geq 1\}$, where f is in $C_0^\infty(R)$, and having the homogeneous boundary values (2) $\partial^{2j}u/\partial n^{2j} = 0$, $0 \leq j \leq p-1$, on the boundary of R. The author shows that the problem (1), (2) has a unique solution, for almost all values of a, that is provided by the formula $u(x, y) = \sum_{m,n=1}^\infty d_{mn} \sin \pi nx \sin \pi may$, where $d_{mn} = c_{mn}/P((\pi n)^2, (\pi ma)^2)$ and c_{mn} is the corresponding Fourier coefficient of f. This follows directly from the Lemma: (1) The set $\{a: P((\pi n)^2, (\pi ma)^2) = 0;\ m, n$ arbitrary integers$\}$ is denumerable; (2) $P((\pi n)^2, (\pi ma)^2)$ can approach zero as m, n approach infinity faster than any power of $1/mn$ only when a belongs to a certain set of linear measure zero. *R. K. Juberg* (Minneapolis, Minn.)

Lehman, R. Sherman 1438

Algebraic properties of the composition of solutions of partial differential equations.

Pacific J. Math. **13** (1963), 157–168.

Consider a homogeneous linear partial differential equation with constant coefficients in two independent variables and of order ρ:

$$(1) \qquad Lu(P) = 0.$$

Let B_i be certain bilinear forms in u and its derivatives of order $< \rho$, evaluated at $P-Q$, and in v and its derivatives of order $< \rho$, evaluated at Q, such that for any domain G

$$(2) \quad \iint_G [u(P-Q)Lv(Q) - v(Q)Lu(P-Q)]\, dx_Q dy_Q =$$

$$\int_{\partial G} [B_1(u(P-Q), v(Q))\, dx_Q + B_2(u(P-Q), v(Q))\, dy_Q].$$

Using an arbitrary choice of the B_i for which (2) holds, H. Lewy [J. Math. Mech. **8** (1959), 185–192; MR **21** #3672] defined the composition of two solutions of (1) by setting

$$(3) \quad u \circ v\,(P) = \int_0^P [B_1(u(P-Q), v(Q))\, dx_Q + B_2(u(P-Q), v(Q))\, dy_Q].$$

The author gives first a simpler proof of Lewy's theorem that (3) is a solution of (1) and then investigates commutativity and associativity of (3). The symmetric composition and the covariant one, introduced by the author, are shown to be commutative. In the case of second-order equations it is shown that covariant composition is associative for parabolic equations, whereas for elliptic and hyperbolic ones none of the compositions is associative. *J. Szarski* (Kraków)

Tersenov, S. A. 1439

A problem with data on a line of degeneration for systems of equations of hyperbolic type. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 285–288.

For the system

$$u_x + v_y + au + bv = f_1,$$
$$yu_y + v_x + cu + dv = f_2$$

the author defines two expressions involving u and v, respectively, whose limits as $y \to +0$ can be prescribed, and these data determine the solution. For

$$y^n u_x + v_y + au + bv = f_1,$$
$$u_y + v_x + y^m cu + dv = f_2,$$

u and v may themselves be prescribed under some restrictions. *P. Ungar* (New York)

Baranovskiĭ, F. T. 1440

Differential properties of the solution of a mixed problem for a strongly degenerate hyperbolic equation. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 3 (40), 17–27.

The author studies generalized solutions of the degenerate hyperbolic equation

$$(1) \quad \varphi(t)u_{tt} - \sum_{i,j=1}^n (a_{ij}u_{x_i})_{x_j} + \sum_{i=1}^n b_i u_{x_i} + b_0(t)u_t + cu = f,$$

where $\varphi(0) = 0$, and the coefficients a_{ij}, b_i, c, and the free term f depend on $(t, x_1, \cdots, x_n)$ in the cylinder $Q_l: x \in D$, $0 \leq t \leq l$. Here D is a bounded domain in x-space with boundary Γ. The paper is a continuation of an earlier study by the author [same Izv. **1960**, no. 3 (16), 30–42; MR **26** #474], in which it was shown that the solution of the mixed boundary-value problem for (1) has generalized (in the sense of Sobolev) first derivatives in Q_l and generalized second derivatives in any subdomain $Q_\delta = D_\delta \times [0, l]$, where the points of $D_\delta \subset D$ are at least a distance δ away from Γ $(\delta > 0)$.

The following assumptions are made with respect to the coefficients of (1): $a_{ij} = a_{ji}$ and $\sum_{i,j=1}^n a_{ij}\eta_i\eta_j \geq \lambda \sum_{i=1}^n \eta_i^2$ for some positive constant λ; $\varphi(t) \in C^k$, and for some real α and sufficiently small t, there exist positive constants $C_1^{(p)}$ and $C_2^{(p)}$ such that

$$C_1^{(p)}t^{\alpha-p} \leqq \varphi^{(p)}(t) \leqq C_2^{(p)}t^{\alpha-p} \qquad (p = 0, 1, \cdots, k).$$

The a_{ij} are in C^k, while b_0, the b_i, c, and f are in C^{k-1}. The derivatives of f with respect to t satisfy

$$\int_0^l\int_D (\partial^i f/\partial t^i)^2 t^{(i-k+1)(2-\alpha)}\,dx\,dt < \infty \quad (i = 0, 1, \cdots, k-1).$$

The author's result is that under the above assumptions, for $1 \leqq \alpha < 2$, $k = [n/2]+3$, and $b_0(t) - \frac{1}{2}\varphi'(t) > \lambda_0 > 0$ $(0 \leqq t \leqq l)$, the generalized solution has continuous second derivatives in an arbitrary subdomain of Q_l.

The analysis is accomplished by transforming equation (1) to a form with $\varphi \equiv 1$. The author gives an example to show that the condition $b_0 - \frac{1}{2}\varphi' > \lambda_0 > 0$ is essential. Further references are: G. Hellwig [Math. Z. **61** (1954), 26–46; MR **16**, 828], the author [Leningrad. Gos. Ped. Inst. Učen. Zap. **166** (1958), 227–254; MR **21** #5071] and A. V. Bicadze [Proc. Third All-Union Math. Conf. (Moscow, 1956), Vol. 2, p. 9, Izdat. Akad. Nauk SSSR, Moscow, 1956; MR **20** #6973b]. *W. S. Loud* (Darmstadt)

Karp, V. N. 1441

Application of the wave-domain method to the solution of a problem of forced non-linear periodic vibrations of a string. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1961**, *no.* 6 (25), 51–59.

The paper deals with the hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = \Phi(x, t) + \mu f\left(x, t, u, \frac{\partial u}{\partial t}\right),$$

where $u(0, t) = u(1, t) = 0$, $u(x, 0) = (x, 1)$, $\partial u/\partial t|_{t=0} = \partial u/\partial t|_{t=1}$, a is an odd number and μ is a small parameter. Assuming that $\Phi(x,t) = \Phi(x, t+1) = \Phi(1-x, t)$, $f(x, t, b, c) = f(x, t+1, b, c) = f(1-x, t, b, c)$, $|u|$ and $|\partial u/\partial t| < C$, where $C = 3\sup[|\Phi|, |\Phi_x'|, |\Phi_t'|]$, $|\mu| < \min\{1, 1/24R(1+2L), C/6P\}$, where $R = \max(M, N)$,

$$\left|f'\left(x_1, t_1, u_1, \frac{\partial u_1}{\partial t}\right) - f'\left(x_2, t_2, u_2, \frac{\partial u_2}{\partial t}\right)\right| \leqq$$
$$M\left[|x_1 - x_2| + |t_1 - t_2| + |u_1 - u_2| + \left|\frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial t}\right|\right]$$

but f' is to be replaced by f_x', f_t', f_u', $f'_{\partial u/\partial t}$, $N = \sup[|f_x'|, |f_t'|, |f_u'|, |f'_{\partial u/\partial t}|]$, $L = \max\{C, \frac{3}{2}[N(1+C)+P]\}$, $|f| < P$, existence and uniqueness theorems are proved for the closed domain $0 \leqq x \leqq 1$, $0 \leqq t \leqq 1$.

Şt. I. Gheorghiţă (Bucharest)

Palamodov, V. P. 1442

On systems of differential equations with constant coefficients. (Russian)

Dokl. Akad. Nauk SSSR **148** (1963), 523–526.

Consider the general system of linear equations with constant coefficients $\sum_{j=1}^m p_{ij}(D)u_j = w_i$ $(i = 1, \cdots, k)$, where $D = (i\partial/\partial x_1, \cdots, i\partial/\partial x_n)$ and the p_{ij} are polynomials. A theorem is stated giving necessary and sufficient conditions for solvability of this system in certain spaces, including spaces of functions of exponential growth. A second theorem is presented which gives the general form of the solution of the system when $w = 0$. The theorems formulated permit one to carry over to arbitrary systems many results known for one equation, in particular, to investigate problems on hypo-ellipticity and hyperbolicity of such systems. A counterexample is given to a statement of L. Ehrenpreis [Studia Math. (Ser. Specjalna) Zeszyt 1 (1963), 35–36; MR **26** #2735].

R. C. Gilbert (Fullerton, Calif.)

P'janzina, L. Ja. 1443

Approximation of the solutions of boundary-value problems for parabolic and hyperbolic equations by solutions of the Cauchy problem. (Russian)

Dokl. Akad. Nauk SSSR **154** (1964), 1276–1279.

Starting with the parabolic equation

$$(1)\quad a(x, t)\frac{\partial u}{\partial t} + \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(b_{ij}(x, t)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^n c_i(x, t)\frac{\partial u}{\partial x_i} + d(x, t)u = \phi(x, t)$$

in $\Omega \times [0, T]$, Ω a bounded region in E^n,

$$a(x, t) \leqq \bar{a} < 0,$$

$$\lambda_1 \sum_{i=1}^n \xi_i^2 \geqq \sum_{i,j=1}^n b_{ij}(x, t)\xi_i\xi_j \geqq \lambda_1 \sum_{i=1}^n \xi_i^2$$

and the initial condition

$$(2)\qquad u(x, t)|_{t=0} = f(x) \quad \text{for } x \text{ in } \Omega,$$

a new problem is obtained, as follows: Roughly speaking, each coefficient in (1) is extended to all of $E^n \times [0, T]$, then averaged at each point (x, t) over a ball of small radius, h, giving new coefficients $a^h(x, t)$, $b_{ij}^h(x, t)$, etc. Also, $f(x)$ is extended to all of E^n, and "mollified", using the same parameter h. Consider the solution $u^h(x, t)$ of the equation

$$(3)\quad a^h(x, t)\frac{\partial u^h}{\partial t} + \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(b_{ij}^h(x, t)\frac{\partial u^h}{\partial x_j}\right) + \sum_{i=1}^n c_i^h(x, t)\frac{\partial u^h}{\partial x_i} + d^h(x, t)u^h = \phi^h(x, t)$$

with

$$(4)\qquad u^h(x, t)\Big|_{t=0} = f^h(x) \quad \text{for all } x.$$

Then, it is asserted, as $h \to 0$, u^h approaches, in the mean, on every compact subset of $\Omega \times [0, T]$, a weak solution of one of the classical boundary-value problems for (1) and (2), which problem depending on the details of the above extension and averaging process. A similar result is described for the hyperbolic equation

$$(5)\quad a(x, t)\frac{\partial^2 u}{\partial t^2} + \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(b_{ij}(x, t)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^n c_i(x, t)\frac{\partial u}{\partial x_i} + g(x, t)\frac{\partial u}{\partial t} + d(x, t)u = \phi(x, t)$$

in $\Omega \times [0, T]$, where a and the b_{ij} are as above, with the initial conditions

$$(6)\quad u(x, t)\Big|_{t=0} = f_0(x), \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = f_1(x) \quad \text{for } x \text{ in } \Omega.$$

No proofs are given. *Stanley Kaplan* (New York)

Kalašnikov, A. S. 1444

The Cauchy problem in the class of increasing functions for equations of non-stationary filtration type. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1963**, *no.* 6, 17–27.

The equation is

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}\varphi(t, x, u).$$

Assume that $u_1 < u_2$ implies $\varphi_u(t, x, u_1) \leqq \varphi_u(t, x, u_2)$ and that $\varphi(t, x, u) \leqq M[\varphi_u(t, x, u)]^p$ for some constants M, p. If two weak solutions $u_1(t, x), u_2(t, x)$ satisfy (1) $\varphi_u(t, x, u_i) \leqq M_1(1+x^2)$ and $u_1(0, x) = u_2(0, x)$, then $u_1 \equiv u_2$.

For the existence theorem φ is restricted by several inequalities, all of which generalize properties of $\varphi \equiv u^\mu$, $\mu > 1$. If, in addition, $u_0(x) \geqq 0$ for all x and

$$(2) \qquad \varphi_u(0, x, u_0(x)) \leqq M_1(1+x^2)^{1-\varepsilon}, \qquad \varepsilon > 0,$$

then a weak solution with initial values u_0 exists for all t. If $\varepsilon = 0$, it exists in a finite t-interval but may blow up as $t \to T$.

The author also shows one may not relax the bounds (1) and (2). *P. Ungar* (New York)

Žitomirskiĭ, Ja. I. 1445

On a theorem of Liouville type. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1961**, *no.* 1 (20), 66–76.

Consider the linear system of equations parabolic in the sense of Petrovskiĭ:

$$(1) \qquad \frac{\partial u(x, t)}{\partial t} = \sum_{r=0}^{p} P_r\left(x, \frac{\partial}{\partial x}\right)u(x, t),$$

where $x = (x_1, \cdots, x_n)$, $u(x, t) = \{u_1(x, t), \cdots, u_N(x, t)\}$, $P_r(x, \partial/\partial x)$ contains derivatives of order $p-r$ with coefficients depending on x. Let $A_r(x)$ denote an arbitrary coefficient of the operator $P_r(x, \partial/\partial x)$. It is assumed that: (a) the coefficients $A_0(x)$ and their derivatives up to order $p+1$ are continuous and bounded in R^n; (b) the coefficients $A_r(x)$, $1 \leqq r \leqq p$, are continuous together with their first derivatives and further satisfy the inequalities

$$|D_x^m A_r(x)| \leqq K\left(\sum_{s=1}^{n} |x_s|^{k_r} + 1\right)$$

in R^n, where K is a positive constant, $k_r < r(p-1)^{-1}$, and $m = 0, 1$.

It is shown that the null solution is the only regular solution of (1) in the half-space $t \leqq T$ subject to the condition

$$(2) \qquad |u(x, t)| \leqq \psi(t)\exp\left(h\sum_{s=1}^{n}|x_s|^k\right),$$

where h is a non-negative constant, $k < p(p-1)^{-1}$, and $\psi(t)$ satisfies a certain growth condition in a neighborhood of $-\infty$. The proof proceeds from the basic estimates of the fundamental solution matrix of (1) (c.f., e.g., the author [same Izv. **1959**, no. 1 (8), 55–74; MR **26** #4060], S. D. Èĭdel'man [Ukrain. Mat. Ž. **8** (1956), 191–207; MR **18**, 215]). This result extends and generalizes that of S. D. Èĭdel'man and B. Ja. Lipko [Mat. Sb. (N.S.) **40 (82)** (1956), 273–280; MR **18**, 657], who assumed the coefficients in (1) to be bounded and also that $k = 1$ in (2).

R. K. Juberg (Minneapolis, Minn.)

Manacorda, Paola 1446

La disuguaglianza di Harnack per l'equazione del calore.

Boll. Un. Mat. Ital. (3) **19** (1964), 31–39.

Author's summary: "Starting from a theorem of Pini and Hadamard, which is proved here in another way, relating to an inequality of Harnack type for positive solutions of the heat equation, the author determines the form and the properties of the constant appearing in the inequality."

Şt. I. Gheorghiţă (Bucharest)

Kamynin, L. I.; Maslennikova, V. N. 1447

Boundary estimates for the solution of the third boundary-value problem for a parabolic equation. (Russian)

Dokl. Akad. Nauk SSSR **153** (1963), 526–529.

In this paper an a priori estimate for the solution of the third boundary-value problem for a parabolic equation of the second order and for systems of such equations is established. In case of a bounded domain the $(2+\alpha)$-norm $|u|_{2+\alpha}$ of the solution is estimated in terms of the α-norm of the right-hand side, of the $(2+\alpha)$-norm of the initial condition and of the $(1+\alpha)$-norm of the boundary condition. If the domain Q in question is not bounded, then $|u|_{2+\alpha}$ taken over a part of Q is estimated in a similar manner, but in addition $\sup u$ taken over all of Q is used to estimate $|u|_{2+\alpha}$. It should be mentioned that $(k+\alpha)$-norms in the parabolic case differ in a natural manner from the norms used in the theory of elliptic equations. *R. Výborný* (Prague)

Kusano, Takaŝi 1448

On the maximum principle for quasi-linear parabolic equations of the second order.

Proc. Japan Acad. **39** (1963), 211–216.

In this paper Nirenberg's strong maximum principle for parabolic equations is generalized to quasilinear parabolic equations of the form

$$(1) \quad \sum a_{ij}(x, t, u, \operatorname{grad} u)\frac{\partial^2 u}{\partial x_i \partial x_j} - \frac{\partial u}{\partial t} = f(x, t, u, \operatorname{grad} u),$$

$$\operatorname{grad} u = \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right).$$

The author assumes that (i) $f(x, t, u, 0) \geqq 0$ for $u \geqq 0$, and (ii) $f(x, t, u, p)$ satisfies Lipschitz conditions with respect to u and p and that there is a lower semicontinuous function h such that

$$\sum a_{ij}(x, t, u, p)\xi_i\xi_j \geqq h(x, t, u, p)\sum \xi_i^2$$

for every u, p and for (x, t) belonging to the domain in question.

From the maximum principle some uniqueness theorems for the first boundary-value problem for the equation (1) are deduced. *R. Výborný* (Prague)

Kusano, Takaŝi 1449

Remarks on some properties of solutions of some boundary value problems for quasi-linear parabolic and elliptic equations of the second order.

Proc. Japan Acad. **39** (1963), 217–222.

In this paper the author generalizes results of Oleĭnik,

Hopf and the reviewer that the normal derivative of a solution of an elliptic or parabolic equation at the point where it assumes its maximum or minimum value is non-zero in case of a quasilinear elliptic or parabolic equation. Then he deduces uniqueness theorems for some boundary-value problems with oblique derivative.

R. Výborný (Prague)

Milicer-Grużewska, H. 1450

Propriété limite de la matrice du potentiel spécial de simple couche d'un système parabolique d'équations.

Ann. Polon. Math. **14** (1963/64), 239–268.

Detailed proofs are given for the results previously announced [Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **9** (1961), 429–433; MR **26** #4059].

J. Elliott (New York)

Ventcel', T. D. 1451

Some quasi-linear parabolic systems with increasing coefficients. (Russian)

Dokl. Akad. Nauk SSSR **140** (1961), 284–286.

The existence of solutions in the large for mixed initial- and boundary-value problems are considered for systems of the form:

$$(*)\qquad \varepsilon\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}+\frac{\partial\varphi(u,v)}{\partial x},\qquad \varepsilon\frac{\partial^2 v}{\partial x^2}=\frac{\partial v}{\partial t}+\frac{\partial\psi(u,v)}{\partial x}$$

when the first-order system obtained by setting $\varepsilon=0$ is assumed to be hyperbolic. The corresponding initial and boundary conditions are: $u(x,0)=u_0(x)$, $v(x,0)=v_0(x)$; $u(x_1,t)=u(x_2,t)=v(x_2,t)=v(x_1,t)=0$.

Suppose $F=F(u,v)$ is a solution of the equation

$$(**)\qquad \varphi_v F_{uu}-(\varphi_u-\psi_v)F_{uv}-\psi_u F_{vv}=0$$

satisfying the conditions:

$$\text{(a)}\qquad F_{uu}\xi^2+2F_{uv}\xi\eta+F_{vv}\eta^2\geqq\mu(u,v)(\xi^2+\eta^2),\qquad \mu>0,$$

for all u, v (convexity condition);

$$\text{(b)}\qquad F(0,0)=F_u(0,0)=F_v(0,0)=0.$$

The type of the equation (**) is the same as that of the first-order system associated to (*). In fact, the discriminant of the operator in (**) is just the discriminant of the quadratic equation whose roots determine the type of the first-order system. Upon multiplying the equations in (*) by F_u and F_v, respectively, adding and integrating the result over $R=\{(x,t): x_1\leqq x\leqq x_2,\ 0\leqq t\leqq T\}$, it follows by the usual manipulations that

$$(***)\qquad \int_{x_1}^{x_2}F(u(x,T),v(x,T))\,dx\leqq\int_{x_1}^{x_2}F(u_0(x),v_0(x))\,dx.$$

From (***), together with certain additional assumptions on φ and ψ (too detailed to describe here), and a bound on $\max(|\partial u/\partial x|,\ |\partial v/\partial x|)$ in R presented in a previous paper [same Dokl. **117** (1957), 21–24; MR **21** #205], the author proves the a priori boundedness of u and v in the class of continuous functions. The existence of a solution for all time then follows from a theorem given in the previous paper cited above.

The author also describes a class of functions of the form $\varphi=\varphi(v)$, $\psi=\psi(u)$ to which this theory applies, where $\varphi(s)$ and $\psi(s)$ behave essentially like $|s|^p$.

R. K. Juberg (Minneapolis, Minn.)

Isakova, E. K. 1452

On the Cauchy problem for parabolic equations with a small parameter. (Russian)

Dokl. Akad. Nauk SSSR **148** (1963), 271–273.

This article deals with the behaviour, for small ε, of solutions $u^\varepsilon(x,t)$ of the parabolic equation of order $2p$

$$L_\varepsilon u^\varepsilon=L_\varepsilon^{\,1}u^\varepsilon+L_\varepsilon^{\,2}u^\varepsilon+L_0u^\varepsilon=0,$$

where $L_0=\partial/\partial t+\sum_{i=1}b_i(x,t)\partial/\partial x_i+c(x,t)$ is the "unperturbed" first-order operator, and

$$L_\varepsilon^{\,1}=(-1)^p\sum_{s\leq|i|\leq 2p}a_i(x,t)\varepsilon^{|i|/2p}D^i,$$

$$L_\varepsilon^{\,2}=\varepsilon^{s/2p}\sum_{1\leq|i|\leq s-1}a_i(x,t)D^i\qquad(1\leqq s\leqq 2p-1)$$

give the "perturbation" terms. All coefficients are assumed C^∞, and the initial-value problem is considered for data which has a discontinuity across a hyperplane. The main results, given without proof, indicate that as $\varepsilon\to0$, u^ε approaches the solution u^0 of the unperturbed equation everywhere except at the characteristic discontinuity surface for u_0, and that, furthermore, u^ε may be expanded in a perturbation series in ε, where the successive terms are obtained through consideration of what the author calls "$2p$-parabolic boundary-layer" terms. This article is an extension of the author's previous work for the case $p=1$ [same Dokl. **117** (1957), 935–938; MR **21** #203; ibid. **119** (1958), 1077–1080; MR **21** #204].

Stanley Kaplan (New York)

Il'in, A. M. 1453

On the fundamental solution of a parabolic equation. (Russian)

Dokl. Akad. Nauk SSSR **147** (1962), 768–771.

It is known [see S. D. Èĭdel'man, Mat. Sb. (N.S.) **53** (95) (1961), 73–136; MR **24** #A925] that if the coefficients $a_{ij}(t,x)$ in the parabolic equation

$$(1)\qquad \frac{\partial u}{\partial t}=\sum_{i,j=1}^{n}a_{ij}(t,x)\frac{\partial^2 u}{\partial x_i\partial x_j}$$

are bounded, continuous in t uniformly in x, and satisfy a Hölder condition in the space variables $x=(x_1,x_2,\cdots,x_n)$, then there is a fundamental solution $G(t,\tau,x,\xi)$ to the Cauchy problem for (1), with certain regularity properties. In particular, $0<G(t,\tau,x,\xi)<K(\delta)$ whenever $t-\tau>\delta>0$, or, putting it differently, if $u(x,t)$ is any regular solution of (1) for $t>\tau$, with $u(x,\tau)=\phi(x)$, then, since

$$(2)\qquad u(t,x)=\int_{E^n}G(t,\tau,x,\xi)\phi(\xi)\,d\xi,$$

it follows that

$$(3)\qquad |u(t,x)|\leqq K(\delta)\int_{E^n}|\phi(\xi)|\,d\xi\quad\text{for } t-\tau>\delta.$$

In the present paper the author constructs an example to show that the Hölder condition cannot be dropped, i.e., he constructs a function $a(t,x)$, continuous in x and t, and satisfying $\frac{1}{2}\leqq a(t,x)\leqq\frac{3}{2}$ such that for the equation (in one space variable) $\partial u/\partial t=a(t,x)\partial^2u/\partial x^2$ the estimate (3) does not hold for all solutions. The author also gives a completely analogous example for the Dirichlet problem for the elliptic equation in two variables

$$(4)\qquad a(x,y)\frac{\partial^2u}{\partial x^2}+2b(x,y)\frac{\partial^2u}{\partial x\partial y}+c(x,y)\frac{\partial^2u}{\partial y^2}=0.$$

Stanley Kaplan (New York)

Golec', B. I.; Èĭdel'man, S. D. 1454

Some properties of linear systems with several space coordinates. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 575–580.

Authors' summary: "A number of results are given describing the properties of solutions of systems of differential equations with many space coordinates and with essentially different properties in different coordinates. Some parabolic systems of a fractional positive kind are considered; correct solvability (initial values in the classical sense being satisfied) is established for some parabolic systems of a positive kind and variable coefficients in classes of quickly increasing functions. In conclusion the result of a study of Cauchy's problem is given for some equations of a non-positive kind."

Èĭdel'man, S. D.; Lipko, B. Ja. 1455

Boundary-value problems for parabolic systems in regions of general type. (Russian)

Dokl. Akad. Nauk SSSR **150** (1963), 58–61.

The present article continues the work of one of the authors [S. D. Èĭdel'man, same Dokl. **142** (1962), 812–814; MR **28** #3263; ibid. **149** (1963), 792–795; MR **26** #5297]. In the previous articles, a fundamental solution matrix in a half-space was constructed for general linear constant-coefficient inhomogeneous boundary-value problems for homogeneous, constant-coefficient systems of equations, parabolic in the sense of Petrovskiĭ, and of arbitrary even order. This half-space fundamental solution was then used to prove the existence of solutions to variable-coefficient boundary-value problems for parabolic systems with variable coefficients in convex cylindrical domains; the fact that this fundamental solution might have singularities in the other half-space prevented an extension of the results to more general domains. In the present article, a modification of the previous method is sketched which removes this difficulty, and existence theorems are stated for smooth non-convex and even non-cylindrical domains. *Stanley Kaplan* (New York)

Bugrov, Ja. S. 1456

Differential properties of the solutions of a certain class of differential equations of higher order. (Russian)

Mat. Sb. (N.S.) **63 (105)** (1964), 59–121.

In this paper the regularity problem for the equation

$$(-1)^{l+1}\frac{\partial^{2l}U}{\partial x^{2l}}+(-1)^{s+1}\frac{\partial^{2s}U}{\partial y^{2s}} = 0$$

is considered. The paper contains proofs of results announced earlier [Dokl. Akad. Nauk SSSR **148** (1963), 259–261; MR **26** #6625]. *R. Výborný* (Prague)

Wiener, Klaus 1457

Zur Lösung einer partiellen Differentialgleichung vom gemischten Typ.

Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe **12** (1963), 259–264.

The equation $K(y)u_{xx}+u_{yy}=0$ where $K(y)=1$ for $y>0$ and $K(y)=-a^2$ for $y<0$ was first studied by the reviewer [Arch. Rational Mech. Anal. **10** (1962), 267–272; MR **25** #4256]. The solvability there was discussed by the use of singular integral equations. Subsequently v. Wolfersdorf [ibid. **13** (1963), 386–391; MR **27** #1720] arrived at similar conclusions by reducing the problem to two Riemann-Hilbert problems. The paper under review studies the equation by returning to the method of Bizadse [*Zum Problem der Gleichungen vom gemischten Typus*, VEB Deutscher Verlag Wiss., Berlin, 1957; MR **20** #1087]. The resultant singular integral equations are reduced to Fredholm equations of the second kind. *H. Hochstadt* (Brooklyn, N.Y.)

von Wolfersdorf, Lothar 1458

[v. Wolfersdorf, Lothar]

Über das singuläre elliptische Neumannproblem der Tricomi-Gleichung.

Math. Nachr. **25** (1963), 255–292.

The existence and uniqueness of a solution of the Tricomi equation in canonical form $u_{xx}+u_{yy}+(3y)^{-1}u_y=0$ is proved for domains bounded by a curve in the elliptic region and the axis $y=0$, with the boundary values of $\partial u/\partial n$ given. (These must, of course, satisfy a certain integral relation.) The solution is represented as a singular integral involving the fundamental solution which is expressible in terms of hypergeometric functions. *C. S. Morawetz* (New York)

D'jačenko, V. F. 1459

Uniqueness conditions for the continuous solution for the discontinuity decomposition problem for a system of three equations. (Russian)

Dokl. Akad. Nauk SSSR **153** (1963), 1245–1248.

Hyperbolic systems of conservation laws $\partial u_i/\partial t+\partial f_i(u)/\partial s=0$ $(i=1, 2, 3)$, for which the eigenvalues λ_i of $\partial f_i/\partial u_j$ are everywhere real and distinct, are considered. Let $l_i(u)$ be the eigenvector with eigenvalue λ_i. Cauchy problems with $u=u^-=\text{const}$ for $t=0$, $s<0$ and $u=u^+=\text{const}$ for $t=0$, $s>0$ are considered with the added restriction that u depend only on s/t and be a continuous function of s/t. Such solutions are unique when there are only two dependent variables but not when there are three. The example given even satisfies the conditions $l_i\cdot\nabla\lambda_i\neq 0$, known to be necessary for uniqueness of weak solutions. *P. Ungar* (New York)

Merzljakova, G. D. 1460

An application of the method of finite differences to the solution of boundary-value problems on Riemann surfaces. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 507–509.

The author assumes a Riemann surface G with boundary given as a bounded finite-sheeted plane region with smooth boundary curves. He describes procedures for setting up finite-difference approximations for elliptic boundary-value problems in G. Conditions for existence of a unique solution and for convergence of an iterative process for determining the solution are stated, with outlines of proof. *W. Kaplan* (Ann Arbor, Mich.)

Finn, R. 1461

On the inclination of a minimal surface $\varphi(x, y)$.

Colloq. Math. **11** (1963/64), 195–201.

This paper announces and discusses a number of results in the general theory of equations of minimal surface type. Full proofs appear elsewhere [Arch. Rational Mech. Anal. **14** (1963), 337–375; MR **28** #336].
J. B. Serrin (Minneapolis, Minn.)

Sobolevskiĭ, P. E. 1462
An investigation of the Navier-Stokes equations by methods of the theory of parabolic equations in Banach spaces. (Russian)
Dokl. Akad. Nauk SSSR **156** (1964), 745–748.

The author extends his previous studies [same Dokl. **155** (1964), 50–53; MR **28** #4258] of the time-dependent Navier-Stokes system. The setting is now the closure in L_q $(1<q<\infty)$ of smooth solenoidal vectors vanishing on the boundary. (Previously he considered only the case $q=2$.) As before, the reduction is made to an integral equation whose solution is a function of t with range in the above-mentioned Banach space. Several results from the theory of semigroups and parabolic equations in Banch spaces are needed. The results are analogous to those the author obtained previously; however, the greater generality allows higher-dimensional problems to be treated. *P. C. Fife* (Minneapolis, Minn.)

Carroll, R. W.; Glick, A. J. 1463
On the Ginzburg-Landau equations.
Arch. Rational Mech. Anal. **16** (1964), 373–384.

In this paper the authors investigate the equations given by Ginzburg and Landau that describe the behavior of superconducting materials in magnetic fields and at temperatures closed to the transition temperature. That is, they investigate the equations

$$-\operatorname{curl}^2 \mathbf{A} = |\psi|^2\mathbf{A}+\frac{i}{2k}(\bar\psi\nabla\psi-\psi\nabla\bar\psi),$$

$$-\frac{\nabla^2\phi}{k^2}+\frac{i}{k}\{\mathbf{A}\cdot\nabla\psi+\operatorname{div}(\psi\mathbf{A})\}+\mathbf{A}\cdot\mathbf{A}\psi = \psi(1-|\psi|^2),$$

$$\mathbf{n}\cdot\left(\frac{i\nabla}{k}+\mathbf{A}\right)\psi|_S = 0,$$

where S is the boundary of a closed region $\Omega\subset R^3$ with exterior normal $\mathbf{n}$ and $\mathbf{A}$ is not generally assumed real. They assume also certain natural surface requirements, i.e., that $\mathbf{n}\times(\mathbf{n}\times\mathbf{A})$ and $\mathbf{n}\cdot\operatorname{curl}\mathbf{A}$ are continuous. In addition, they have

$$\mathbf{n}\times\operatorname{curl}\mathbf{A} = \mu(\mathbf{n}\times H_0\mathbf{k}) \text{ on } S,$$

(where $\mu H=\operatorname{curl}\mathbf{A}$), the external field is assumed to be $\mathbf{H}_e=H_0\mathbf{k}$, and $\mu_e=1$ (in appropriate units). The authors transform these equations into a more suitable form for investigation, namely,

$$a(\boldsymbol\alpha,\boldsymbol\beta) = L(\boldsymbol\gamma,\boldsymbol\beta),$$

where

$$L(\boldsymbol\gamma,\boldsymbol\beta) = \int_\Omega\phi\bar\chi-\frac{2i}{k}\int_\Omega(\mathbf{w}\cdot\nabla\phi)\bar\chi-\int_\Omega|\phi|^2\mathbf{w}\cdot\bar{\mathbf{u}}$$
$$-\int_\Omega(\mathbf{w}\cdot\mathbf{w}+|\phi|^2)\phi\bar\chi-\frac{i}{2k}\int_\Omega(\bar\phi\nabla\phi-\phi\nabla\bar\phi)\cdot\bar{\mathbf{u}}$$
$$+\frac{i}{k}\int_S\phi\mathbf{w}\cdot\mathbf{n}\bar\chi-\int_S\bar{\mathbf{u}}\cdot\mu(\mathbf{n}\times H_0\mathbf{k}),$$

$$a(\boldsymbol\alpha,\boldsymbol\beta) = \frac{i}{k^2}\int_\Omega\nabla\psi\cdot\nabla\bar\chi+\int_\Omega\operatorname{curl}\mathbf{A}\cdot\operatorname{curl}\bar{\mathbf{u}}.$$

They show the existence of a unique solution $\boldsymbol\alpha=T(\boldsymbol\gamma)\in V$ $V_1\times V_2$, where Ω is a Sobolev set, and V_1, V_2 are suital defined subspaces of $\mathbf{H}'(\Omega)$ and $H'(\Omega)$, respective Furthermore, they then show that under certain co ditions T is a contraction mapping and there is a fi point $\boldsymbol\alpha=T(\boldsymbol\gamma)$ which is a generalized solution to Ginzburg-Landau equations.
R. P. Gilbert (College Park, M

FINITE DIFFERENCES AND FUNCTIONAL EQUATIO
See also 1175, 1410, 1412.

Chakrabarti, S. C. 14
On higher differences—a few identities.
Indian J. Mech. Math. **2** (1964), 13.

As a sample of the results obtained we cite the followin

$$\sum_{p=0}^{n}(-1)^p(ha^{n-1})_{n-p}(ha^{p-1-x})_p a^{px}\cdot{}^nS_p = (a^{x+n-1})_n,$$

where

$$(h)_p = (h-1)(ha^{-1}-1)\cdots(ha^{-p+1}-1)$$

and ${}^nS_p$ is the sum of the products of $1, a, \cdots, a^{n-1}$ tak p at a time. *L. Carlitz* (Durham, N.(

Morrison, J. A. 14
A certain functional-difference equation.
Duke Math. J. **31** (1964), 445–448.

The functional-difference equation in question is t following:

$$(\alpha-x)(\beta-\alpha)^{n-1}g_n(x) = \alpha(\beta-x)^n g_{n-1}(\alpha)-x(\beta-\alpha)^n g_{n-1}(x$$
$$n = 1, 2, \cdot$$

with $g_0(x)=1$ and $0<\alpha<\beta$. It arose in a study of a te phone traffic system with inputs from two sources, ma by the author's colleague, J. P. Runyon, who conjectur that

$$g_n(\alpha) = \sum_{r=0}^{n-1}\binom{n-1}{r}\binom{n}{r}\beta^{n-r}\alpha^r/(r+1).$$

The author proves the conjecture (without determinati of $g_n(x)$) by deft use of the generating function for Jaco polynomials. *J. Riordan* (Murray Hill, N.J

Carlitz, L. 14
A functional-difference equation.
Duke Math. J. **31** (1964), 449–453.

The author addresses himself to the determination of t function $g_n(x)$ satisfying the functional-difference equati of the preceding review [#1465]. Writing

$$g_n(x) = \sum_{r=0}^{n-1}A_r^{(n)}w^r, \qquad w = (x-\beta)/(\alpha-\beta),$$

$$\varphi_{rk} = \sum_{j=0}^{M}\binom{r}{j}\binom{k}{j}\alpha^j\beta^{k-j}, \qquad M = \min(r,k),$$

he finds

$$A_0^{(n)} = \beta^n,$$

$$A_r^{(n)} = \beta(\varphi_{r,n-1} - \varphi_{r-1,n-1}) - \alpha \sum_{s=1}^{r-1} g_{r-s}(\alpha)\varphi_{s-1,n-r+s-1},$$

$$r = 1, 2, \cdots, n-1,$$

and also

$$A_r^{(n+r+1)} = \sum_{s=0}^{r-1} \binom{r+n-s}{r} \beta^{n-s} B_{rs}$$

with $B_{r0} = \alpha g_r(\alpha)$, $r = 1, 2, \cdots$, and

$$B_{r,s-1} = B_{r-1,s} + (\alpha-\beta)B_{r-1,s-1} + (-1)^{s+1}\binom{r}{s}\alpha\beta^s g_{r-1}(\alpha),$$

$$r = 2, 3, \cdots.$$

Reviewer's remark: The author's question as to possible simplification of the sum in his expression for $A_r^{(n)}$ is answered in the affirmative by the following result

$$A_r^{(n)} = \sum_{j=1}^{r} \frac{1}{j} \binom{n-1}{j-1}\binom{r-1}{j-1}(n-r)\beta^{n-j}\alpha^j.$$

J. Riordan (Murray Hill, N.J.)

Coifman, Ronald 1467

Sur l'équation fonctionnelle d'Abel-Schröder et l'itération continue.

C. R. Acad. Sci. Paris **258** (1964), 5324–5325.

This note consists of the statement of two theorems and a few remarks regarding them and their proofs. Necessary and sufficient conditions are given that the functional equation

$$f(\varphi(x)) = \varphi(ax-1), \qquad 0 < a \leqq 1,$$

have a solution, where rather stringent conditions are put upon both f and φ. *T. Fort* (Columbia, S.C.)

SEQUENCES, SERIES, SUMMABILITY

See also 1184, 1185, 1311, 1342, 1498, 1534.

Jasek, B. 1468

Complex series and connected sets. I, II.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 583–585; *ibid.* **11** (1963), 649–652.

The limit points of series $\sum m_k z_k$ with (z_k) a given complex sequence and the m_k chosen from some given set are studied. Among other results a necessary and sufficient condition is stated in order that a sequence (z_k) have the property that the family of limit sets of all series of the form $\sum \pm z_k$ coincide with the continuum of the complex plane extended by adjoining a circle at infinity.

A. Dvoretzky (Jerusalem)

Weiss, George H. 1469

Calculation of certain multiple generating functions.

J. Res. Nat. Bur. Standards Sect. B **68B** (1964), 13–15.

Let M_j denote the jth greatest of the set of integers $n_1, n_2, \cdots, n_k$. The reviewer [Portugal. Math. **21** (1962), 201–207] has evaluated the sum

$$F_j(\{x\}) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} M_j(n_1, \cdots, n_k) x_1^{n_1} \cdots x_k^{n_k}.$$

In a later paper [Duke Math. J. **30** (1963), 191–201; MR **26** #6063] he has discussed the more general function

$$G_j(\{x\}; s) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} e^{-sM_j(n_1,\cdots,n_k)} x_1^{n_1} \cdots x_k^{n_k};$$

the notation is that of the present paper. The present author derives the result for $F_j(\{x\})$ by another method which is also applicable to evaluate such generating functions as

$$H_j(\{x\}) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} (M_j(n_1, \cdots, n_k))^A x_1^{n_1} \cdots x_k^{n_k}.$$

He also obtains the recurrence

$$G_{j+1}(\{x\}; s) = \sum_{r=1}^{k} (G_j(\{x\} - x_r; s) + (1-k)G_j(\{x\}; s)).$$

L. Carlitz (Durham, N.C.)

Gorst, Ju. G.; Elin, M. V. 1470

A property of almost-convergent sequences. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 712–716.

The authors give a simple sufficient condition in order that A-lim $s_n = B$-lim s_n, where A and B are two regular methods which sum translatively a given bounded sequence s_n. They prove that if m_1 is the subset of the space m of bounded sequences on which all translative (in m) regular methods are consistent, then m_1 is exactly the set of all almost-convergent sequences.

G. G. Lorentz (Syracuse, N.Y.)

Erdős, P.; Piranian, G. 1471

Laconicity and redundancy of Toeplitz matrices.

Math. Z. **83** (1964), 381–394.

Let $A = (a_{n,k})$, $n, k = 0, 1, \cdots$, be a Toeplitz summability matrix with $\|A\| = \sup_n \sum_k |a_{n,k}|$ finite. A is conservative if, on setting $t_n = \sum_k a_{n,k} s_k$ $(n = 0, 1, \cdots)$, $\lim t_n$ exists whenever $\lim s_k$ exists. If, in addition, $\lim t_n = \lim s_k$ always holds, A is regular. A matrix B is said to be a submatrix of A, $B \subset A$, if each row of B is a row of A and infinitely many rows of A are not rows of B. Let (A) be the bounded convergence field of A, consisting of all bounded sequences s for which As is convergent. Clearly if $B \subset A$, then $(B) \supset (A)$. A matrix A is said to be redundant if it contains a submatrix B for which $(A) = (B)$. If A is not redundant, it is called laconic. Two typical results from this paper are the following. Let $\alpha(du)$ be a finite measure on $[0, 1]$, and let $a_{n,k} = C_{n,k} \int_0^1 u^k (1-u)^{n-k} \alpha(du)$ for $0 \leqq k \leqq n$, and $a_{n,k} = 0$ if $k > n$. The matrix A is then a Hausdorff matrix. It is shown that A is laconic if and only if $\alpha(\{1\}) \neq 0$. Again let p_n, $n = 0, 1, \cdots$, be absolutely convergent, and let $a_{n,k} = p_{n-k}$ if $0 \leqq k \leqq n$, and $a_{n,k} = 0$ if $k > n$. The matrix A is then a little Nörlund matrix. To indicate the dependence of A on the sequence p we write A_p. Let $p(z) = \sum_n p_n z^n$. The authors conjecture that if $p(z)$ is a polynomial, then A_p is laconic if and only if $p(z) = q(z^k)$ for some positive integer k, where q is of degree two and A_q is laconic. (The condition is sufficient.)

Also all laconic matrices A_q where $q(z)$ is of degree at most two and $q(0) \neq 0$ are identified: q must be of the form $c(z-a)^2$, where $|a| = 1$.

I. I. Hirschman, Jr. (St. Louis, Mo.)

APPROXIMATIONS AND EXPANSIONS

See also 1292, 1341, 1496, 1730, 1753.

Campbell, L. L. 1472

A comparison of the sampling theorems of Kramer and Whittaker.

J. Soc. Indust. Appl. Math. **12** (1964), 117–130.

The sampling or interpolation theorem due to Whittaker says that if

$$(1) \qquad f(t) = \int_{-\Omega}^{\Omega} g(\omega)e^{i\omega t}\,dt$$

and if $g(\omega) \in L_2(-\Omega, \Omega)$, then

$$(2) \qquad f(t) = \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{\Omega}\right)\frac{\sin(\Omega t - n\pi)}{\Omega t - n\pi}.$$

Kramer has shown that this result is an example of a generalized sampling theorem associated with the solution $K(x, \lambda)$ of a Sturm-Liouville system consisting of the differential equation

$$(3) \qquad \sum_{k=0}^{n} p_{n-k}(x)\frac{d^k u}{dx^k} + \lambda\rho(x)u = 0,$$

with corresponding two-point boundary conditions. Here any function $F(\lambda)$, having a certain integral representation in terms of the eigenfunctions $K(x, \lambda_k)$ of the Sturm-Liouville system, also has the representation

$$(4) \qquad F(\lambda) = \sum F(\lambda_n)S_n(\lambda),$$

where the λ_n are the eigenvalues of the Sturm-Liouville system, and the $S_n(\lambda)$ are the normalized Fourier coefficients for the expansion of $K(x, \lambda)$.

This paper is concerned with the comparison of the result of Whittaker with that of Kramer and shows that in many cases the generalized theorem gives no improvement over the special theorem. The cases in (3) where $n = 1$ or 2 are treated in detail, with suggestions about a method of extending the result to cases when $n > 2$.

F. M. Stein (Fort Collins, Colo.)

Yamanouchi, Jiro 1473

Some computational methods for best uniform polynomial approximations. (Japanese)

Sûgaku **15** (1963), 40–48.

The author gives a computational method, called the "folding method", to obtain the best-fit uniform polynomial approximation. Suppose, for example, that an even function $f(x)$ is given in $-2u \leqq x \leqq 2u$ with Chebyshev expansion

$$f(x) = \sum_{i=0}^{\infty} t_{2i}2c_{2i}, \qquad x = 2u\cos\theta,\ 0 \leqq \theta \leqq \pi,$$

where $c_n = \cos n\theta$. We shall have a best-fit uniform polynomial approximation of $f(x)$ in the form

$$f(x) \sim \sum_{i=0}^{N-1}(t_{2i} + A_{2i})2c_{2i},$$

where the absolute error is

$$E(x) = \sum_{i=1}^{N} A_{2N-2i}2c_{2N-2i} - \sum_{\lambda=0}^{\infty} t_{2N+2\lambda}2c_{2N+2\lambda}.$$

The deviation points are near $\theta_0 = m\pi/2N$ ($m = 1, \ldots, 2N-1$), where $c_{(2n+1)2N\pm 2j} = c_{2N-2j}$. Using this property the term $2c_{2N+2\lambda}$ on the right-hand side of $E(x)$ is "folded" up into $2c_{2N-2\bar{\lambda}}$ ($0 \leqq \bar{\lambda} \leqq N$), and the parameters A_{2i} are determined successively by the conditions that the absolute values of the extremals of $E(x)$ are the same. This is the main idea of the "folding method". Similar considerations are possible also for an odd function. The author gives actual computational methods and simple examples for trigonometric functions.

S. Hitotumatu (Tokyo)

Sendov, Blagowest 1474

Approximation bezüglich einer Hausdorffschen Metrik mittels algebraischer Polynome.

Wiss. Z. Techn. Univ. Dresden **12** (1963), 121–122.

Author's summary: "Die Approximation gegebener Funktionen durch Polynome wird für eine Hausdorffsche Metrik untersucht, deren Abstandsdefinition nicht nur die Ordinatendifferenzen benutzt. Es werden damit Näherungen erzielt, die steil ansteigende Stellen der approximierten Funktion nicht mehr überbewerten. Weiter wird das asymptotische Verhalten des Abstandes zwischen der vorgelegten Funktion und dem optimalen Approximationspolynom in Abhängigkeit von dessen Grad angegeben."

G. Meinardus (Clausthal-Zellerfeld)

Babušková, Renata 1475

Eine Bemerkung zur Tschebyscheffschen Approximation der Funktion $\frac{\sin x}{x}$.

Wiss. Z. Techn. Univ. Dresden **12** (1963), 111–112.

The author presents a numerically stable method of calculating the coefficients of the Tschebyscheff polynomial expansion of the function $\sin x/x$. The method is a simple adaptation of the matrix elimination method and overcomes the numerical instability arising from the use of recurrence formulae for the coefficients. Ten-decimal-place calculations by both the stable and unstable methods are presented for comparison with the exact values.

R. B. Grafton (Providence, R.I.)

Parodi, Maurice 1476

Sur quelques propriétés des zéros des polynomes combinaisons linéaires à coefficients constants de polynomes récurrents.

Bull. Sci. Math. (2) **87** (1963), 1ière partie, 43–52.

Suppose that polynomials $\pi_n(x)$ satisfy the recurrence

$$\pi_n(x) + (\alpha_n - \beta_n x)\pi_{n-1}(x) + \gamma_n\pi_{n-2}(x) = 0 \qquad (n \geqq 2),$$

$$\pi_0(x) = 1,\ \pi_1(x) = -\alpha_1 + \beta_1 x.$$

Set $Z_n(x) = \pi_n(x) + a_1\pi_{n-1}(x) + \cdots + a_{n-1}\pi_1(x) + a_n$. The author displays a matrix M_n whose eigenvalues are the zeros of Z_n and applies the Gershgorin theorem to M_n. In this way he is able to localize the zeros of Z_n. Applications are made to linear combinations of Gegenbauer, Legendre, and electro-spherical polynomials. For example,

if π_n are the Legendre polynomials and if $\sum_{k=1}^{n} |a_k| < 1$, then all the zeros of Z_n are in the unit circle. Application is also made to the Landau-Montel problem which considers all the a_k with the exception of a_{n-1} as fixed.

P. J. Davis (Providence, R.I.)

Moldovan, Elena 1477

Observations sur la suite des polynomes de S. N. Bernstein d'une fonction continue.

Mathematica (Cluj) **4 (27)** (1962), 289–292.

Es sei $B_n[f; x]$ das n-te Bernsteinsche Polynom der im Intervall $0 \leqq x \leqq 1$ stetigen Funktion $f(x)$. Dann wird durch elementare Schlüsse gezeigt: Gilt für $0 \leqq x \leqq 1$ die Ungleichung $f(x) \geqq B_n[f; x]$ (bzw. $f(x) \leqq B_n[f; x]$), $n = 1, 2, \cdots$, so ist $f(x)$ für $0 \leqq x \leqq 1$ nicht-konvex (bzw. nicht-konkav). Diese Aussagen werden noch für ein beliebiges kompaktes Intervall $[a, b]$ formuliert.

G. Meinardus (Clausthal-Zellerfeld)

Koromysličenko, V. D. 1478

The direct and inverse problem of the algebraic minimax. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 15–22.

This is a continuation of two previous papers by the same author [Ukrain. Mat. Ž. **13** (1961), no. 3, 59–74; MR **25** #5322; ibid. **14** (1962), 145–159; MR **26** #1668] and establishes necessary and sufficient conditions for a problem of the algebraic minimax. On a bicompact Hausdorff space G let there be given $n+1$ continuous linearly independent functions $\{\varphi_j(x)\}_0^n$ and a continuous function $f(x)$. The author's object is to study the problem of finding a generalized polynomial $F(a^*; x) = \sum_0^n a_j^* \varphi_j(x)$ for which

$$(1)\qquad \inf_{a_j} \sup_{x \in G} (F(a; x) - f(x)) = \sup_{x \in G} (F(a^*; x) - f(x)),$$

$$(2)\qquad \sum_{j=0}^{n} \alpha_j^{(i)} a_j^* = \alpha_i, \qquad i = 1, \cdots, p \quad (p \leqq n),$$

where one assumes that the relations (2) are linearly independent. The polynomial $F(a; x)$ whose coefficients satisfy the relations (2) are called admissible. Given $F(a; x)$ and $f(x)$, a point $x_0 \in G$ for which $\max_{x \in G}(F(a; x) - f(x)) = F(a; x_0) - f(x_0)$ is called a maximum point.

The main result consists of the following: In order that the admissible polynomial $F(a^*; x)$ be a solution of problem (1) it is necessary and sufficient that at least for a system of maximum points $\{x_s\}_p^{p+h-1}$, $h \leqq n+2-p$, one of the following two conditions be satisfied: (1) for the linearly independent vectors $\varphi(x_s) = (\varphi_0(x_s), \cdots, \varphi_n(x_s))$, $s = p, \cdots, p+h-1$, corresponding to the maximum points, the conditions of the analogous theorem in the papers mentioned above be fulfilled; (2) there occurs a linear dependence, in the restricted sense, of the vectors $\varphi(x_s)$:

$$\sum_{s=p}^{p+h-1} c_s \varphi(x_s) = 0, \qquad c_s > 0,\ s = p, \cdots, p+h-1.$$

By the inverse of the algebraic minimax problem we mean the problem of finding the linearly independent relations for which the polynomial taken is a solution of the problem (1) for a given function $f(x)$. Relative to this, the author gives the following theorem: For any polynomial $F(a; x)$ and function $f(x)$ one can select p ($\leqq n+1$) linearly independent relations such that the parameters of this polynomial satisfy these relations and the polynomial $F(a; x)$ is the solution of problem (1). A numerical example is worked out in detail. {There are several misprints.}

D. D. Stancu (Cluj)

Stoer, Josef 1479

A direct method for Chebyshev approximation by rational functions.

J. Assoc. Comput. Mach. **11** (1964), 59–69.

Bei der numerischen Behandlung der Tschebyscheffschen Approximation einer stetigen Funktion $f(x)$ durch rationale Funktionen $R(x) = P_l(x)/Q_m(x)$ in einem reellen kompakten Intervall muß man bei einem iterativen Vorgehen nicht-lineare Gleichungssysteme der Art

$$f(x_\nu) - R(x_\nu) = (-1)^\nu \lambda g(x_\nu), \qquad \nu = 0, 1, \cdots, l+m+1,$$

lösen. Hier ist $g(x)$ eine positive Gewichtsfunktion. Der Verfasser gibt für den Fall, daß der Zählergrad mindestens gleich den Nennergrad ist, wichtige vorschläge zur Lösung dieser Systeme. Im Vordergrund steht die Auffassung, daß die obigen Gleichungssysteme aus rationalen Interpolationsaufgaben entspringen, wobei die Größe λ iterativ ermittelt werden muß. Es empfiehlt sich, um den interpolatorischen Charakter der Aufgabe numerisch nutzbar zu machen, die rationale Funktion $R(x)$ in der Form

$$R(x) = e_0 + e_1(x - y_0) + \cdots + e_{l-m}(x - y_0) \cdots (x - y_{l-m+1}) + \frac{e_{l-m+1}(x - y_0)|}{|\ 1} + \frac{x - y_{l-m+1}|}{|\ e_{l-m+2}} + \cdots + \frac{x - y_{l+m-1}|}{|\ e_{l+m}}$$

zu schreiben, wobei meist $y_\nu = x_\nu$, $\nu = 0, 1, \cdots, l+m-1$, gesetzt wird. Dadurch ist es möglich, einfache Verfahren der rationalen Interpolation (in den Punkten x_μ, $\mu = 0, 1, \cdots, l+m$) zur Berechnung der Koeffizienten e_μ zu verwenden. Die Größe λ wird am Anfang gleich Null gesetzt und dann nach der Newtonschen Methode oder nach der Methode der Regula Falsi verbessert un zu einer Lösung des ursprünglichen nicht-linearen Gleichungssystems zu gelangen. Die Konvergenz der Iteration wird nicht untersucht.

Die iterative Verbesserung der Punkte x_ν wird ebenfalls beschrieben. Sodann wird die Methode an einem numerischen Beispiel illustriert. Die Arbeit schließt mit einem ALGOL-Programm zur rationalen Approximation mit $l \geqq m$ im Normalfall.

G. Meinardus (Clausthal-Zellerfeld)

Goldstein, A. A. 1480

On the stability of rational approximation.

Numer. Math. **5** (1963), 431–438.

The delicacy of the existence problem for rational approximation is well known (the ratio of incorrect proofs to correct proofs is still the order of 1). A natural approach when confronted with non-existence phenomena is to enlarge the set of approximating functions so as to recover existence. This is done in this paper for a quite general setting and in an intuitively pleasing way. An obvious question to ask is: Are best approximations in the enlarged set the limit of approximations in the original set? The answer in this case is "no" and in order to

analyze the cause of this, the concepts of "stable" and "unstable" approximations are introduced.

To be precise we have: B is an arbitrary set, N and D bounded maps of B to real Euclidean space E_n, $[\ ,\ ]$ the inner product on E_n, f a bounded real function on B, $\|x\| = \max|x_i|$ for $i = 1, 2, \cdots, n$; $\Omega = \{N(b) - f(b)D(b) | b \in B\}$. Further, $R_1 = \{x \in E_n | [D(b), x] > 0, b \in B\}$, $R_2 = \{x \in E_n | [D(b), x] \geqq 0$ for $b \in B$ and $x \notin R_1$ and $[D(b), x] = 0 \Rightarrow [N(b), x] = 0\}$, $H(S)$ the convex hull of the set $S \subset E_n$. Consider for all $b \in B$ and $\|x\| = 1$ the two systems of inequalities:

$$\mathrm{I}(M) \qquad |[N(b) - f(b)D(b), x]| \leq M[D(b), x],$$

$$\mathrm{I}'(M) \qquad |[N(b) - f(b)D(b), x]| < M[D(b), x].$$

It is shown under the hypothesis that Ω is n-dimensional that there is a least number M_0 and a greatest number M_* such that $\mathrm{I}(M)$ and $\mathrm{I}'(M)$ are consistent, respectively. Set $S(M, b) = N(b) - (f(b) + M)D(b)$, $S_1(M) = \bigcup_{b \in B} S(M, b)$, $S_M = S_1(M) \cup -S(-M)$. The system $\mathrm{I}(M)$ is said to be M-stable if given $\varepsilon > 0$ there exists $x(\varepsilon) \in R$, satisfying $\mathrm{I}(M + \varepsilon)$. The principal result is the theorem: Assume Ω is n-dimensional and for some $\varepsilon > 0$ the system $[D(b), x] \geqq \varepsilon$, $b \in B$ is consistent. Further, assume that $H(S_M)$ is closed for all M. The system $\mathrm{I}(M)$ is M-stable if and only if $M \geqq M_*$. Furthermore, the following statements are equivalent: (1) $M_0 = M_*$; (2) the system $\mathrm{I}(M)$ is M_0-stable; (3) there exists a unique M such that 0 of E_n belongs to the boundary of $H(S_M)$. A computational algorithm of the linear programming variety is described to compute a sequence (M^k, x^k) with $M^k \to M_*$ and $x_k \in R_1$. Upper and lower bounds on M_* are available for each k, and they converge monotonically to M_*.

It may be distasteful to some to have "approximations" which are undefined (i.e., 0/0) for some $b \in B$. Nevertheless, the results of this paper appear to exhibit the true nature of the general existence problem for rational functions. Note that in certain applications one may want to modify the condition $b \in B$ in the definition of R_1 and R_2 to $b \in B_0$, where $B_0 \supset B$. The analysis and proofs may be carried over with minor modifications.

J. R. Rice (Warren, Mich.)

Schönhage, A. 1481

Optimale Punkte für Differentiation und Integration.
Numer. Math. **5** (1963), 303–331.

Es sei $L_n(x)$ das Interpolationspolynom von höchstens $(n-1)$-tem Grad für die Funktion $f(x)$ zu den Stützstellen x_ν mit $x_1 < x_2 < \cdots < x_n$. Der Verfasser stellt sich die Aufgabe, die Stützstellen derart zu wählen, daß mit einer Belegung $p(x)$ der Ausdruck $\int_a^b p(x)(f'(x) - L_n'(x))^2\,dx$ klein wird (Die Striche bedeuten die Ableitungen). Zunächst wird ein hinreichendes Kriterium dafür angegeben, daß für Funktionen $f(x)$ mit stetiger n-ter Ableitung $f^{(n)}(x)$ eine Abschätzung der Form

$$f'(x) - L_n'(x) = \frac{f^{(n)}(\xi)}{n!} H_n'(x)$$

mit

$$H_n(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$$

für eine Zwischenstelle ξ gilt. Die obige Formel ist für solche Werte von x richtig, für welche

$$\left(H_n'(x) - \frac{H_n(x)}{x - x_1}\right)\left(H_n'(x) - \frac{H_n(x)}{x - x_n}\right) \geqq 0$$

ist, eine Bedingung, die sich auch geometrisch interpretieren läßt.

Eine Differentiationsformel $f'(x) \to L_n'(x)$ wird optimal genannt bezüglich einer Belegung $p(x)$, wenn die Stützstellen so gewählt sind, daß das Integral $\int_a^b p(x)(H_n'(x))^2\,dx$ einen möglichst kleinen Wert erhält. Der Zusammenhang mit orthogonalen Polynomen zur Belegung $p(x)$ liegt jetzt auf der Hand. Ausführlich wird dann der Fall der Jacobischen Belegungsfunktion $(1-x)^\alpha(1+x)^\beta$ im Intervall $[-1, +1]$ mit $\alpha > -1$, $\beta > -1$ behandelt. Die optimalen Punkte der Differentiation stehen in engem Zusammenhang mit Interpolationsquadraturen der Art, daß bei n Stützstellen der Fehler der Quadraturformel für alle Polynome vom Grade $\leqq 2n - 3$ verschwindet. Hierbei ergibt sich eine interessante strukturelle Kennzeichnung der Jacobi-Belegung.

Die Frage nach der Konvergenz der Differentiationsformeln wird für die Jacobi-Belegung ausführlich untersucht: Sind $L_n(x)$ die Interpolationspolynome zu den optimalen Punkten der Differentiation bezüglich der Jacobi-Belegung, so gilt für jede stetig differenzierbare Funktion $f(x)$ die Beziehung

$$\lim_{n \to \infty} \int_{-1}^{+1} (1-x)^\alpha (1+x)^\beta (f'(x) - L_n'(x))^2\,dx = 0.$$

Der Beweis dieser Tatsache ist schwierig.

Für die Intervalle $[0, \infty)$ und $(-\infty, +\infty)$ bestimmt der Autor die optimalen Punkte der Differentiation bei der Laguerre- bzw. bei der Hermite-Belegung. Die Arbeit schließt mit einer Anwendung der Ergebnisse auf eine Eigenwertaufgabe bei einer gewöhnlichen Differentialgleichung zweiter Ordnung.

G. Meinardus (Clausthal-Zellerfeld)

Sul'din, A. V. 1482

Wiener measure and its applications to approximation methods. II. (Russian)
Izv. Vysš. Učebn. Zaved. Matematika **1960**, *no.* 5 (18), 165–179.

Ähnlich wie in der vorhergehenden Arbeit des Verfassers [dieselben Izv. **1959**, no. 6 (13), 145–158; MR **28** #722] wird eine Approximation $\sum_{i=1}^n C_i F_i(x)$ für ein lineares Funktional $F(x)$ über dem Raum C der in $[0, 1]$ stetigen Funktionen x mit $x(0) = 0$ dann als "beste Approximation" bezeichnet, wenn $\int_C (F(x) - \sum_{i=1}^n C_i F_i(x))^2\, d_w x$ als Funktion der C_i ein Minimum ist; dabei ist $d_w x$ das Wienersche Maß im Raum C, die linearen Funktionale F_i werden als vorgegeben betrachtet. Durch Anwendung dieses Prinzips ergeben sich in direktem Vorgehen eine Reihe interessanter Näherungsformeln. Auch Nebenbedingungen der Art, daß die Approximation für einen vorgeschriebenen m-dimensionalen Unterraum von C exakt sein soll ($m \leqq n$), können berücksichtigt werden. Es werden unter anderem explizite "beste" Formeln für folgende Aufgabenstellungen hergeleitet: Näherung mit Hilfe von Polynom-Funktionalen $\int_0^1 x(t)\, d(t^k)$, Fourier-Funktionalen $\int_0^1 x(t)\, d\,\frac{\sin k\pi t}{\cos k\pi t}$, und δ-Funktionalen $x(t_i)$ (= diskrete Approximation). Ferner wird der Fall behandelt, wo F selbst eine δ-Funktion ist, also die Approximation von stetigen Funktionen; dabei ergibt sich eine Formel der Art $x(u) \approx \sum_{i=1}^n C_i(u) x(t_i)$, die im obigen Sinne die bestmögliche für Funktionen $x \in C$ ist.

H. J. Stetter (Zbl **107**, 50)

Kripke, Bernard R. 1483

Best approximation with respect to nearby norms.

Numer. Math. **6** (1964), 103–105.

The following theorem is established. Let V be a vector space of finite dimension, W a subspace of V, $f \in V$. Let ρ_k, $k=1, 2, \cdots$, be seminorms on V such that for each $v \in V$, $\lim_{k\to\infty} \rho_k(v)=\rho(v)$, where ρ is a norm. Let $p_k \in W$ be the ρ_k-best approximation to f. Then (i) every subsequence of $\{p_k\}$ has a ρ-convergent subsequence; (ii) $\rho(f-p_k)\to\inf\{\rho(f-w)|w \in W\}$; (iii) every ρ-cluster point of $\{p_k\}$ is a ρ-best approximation to f; (iv) If f has a unique ρ-best approximation p out of W_1, then $\rho(p-p_k)\to 0$. It is pointed out that several known and apparently dissimilar results are simple consequences of this theorem for specific choices of V, W, ρ, and $\{\rho_k\}$.

J. R. Rice (Warren, Mich.)

Walsh, J. L. 1484

The convergence of sequences of rational functions of best approximation.

Math. Ann. **155** (1964), 252–264.

Let E be a bounded closed set whose complement in the extended z-plane is a domain possessing a Green's function $g(z)=g(z, \infty)$. Let $f(z)$ be analytic in $F(\rho)=E_U\{z|g(z)<\log\rho\}$ $(\rho>1)$, except for a finite number, ν, of poles, counted according to their multiplicity. The author investigates the approximation of $f(z)$ by rational functions $R_{nk}(z)=P(z)/Q(z)$, where P is a polynomial of degree $\leqq n$, Q a polynomial of degree $\leqq k$. Theorem 1: There is a sequence $\{R_{n\nu}(z)\}$ such that

$$(1)\qquad \limsup_{n\to\infty} \max_{z\in E} |f(z)-R_{n\nu}(z)|^{1/n} \leqq 1/\rho.$$

Theorem 2: If E is bounded by an analytic Jordan curve and if the sequence $R_{n\nu}(z)$ is such that (1) holds, then $R_{n\nu}(z)$ has precisely ν poles in $\{z|\ 0<g(z)<\rho\}$ which approach the poles of $f(z)$ as $z\to\infty$. Also $|f(z)-R_{n\nu}(z)|\to 0$ uniformly in any closed set $G\subset F(\rho)$ which does not contain any poles of $f(z)$. The paper also contains further, more detailed results about the mode of convergence of such sequences $R_{n\nu}$. *W. H. J. Fuchs* (Ithaca, N.Y.)

Rice, John R. 1485

On the computation of L_1 approximations by exponentials, rationals, and other functions.

Math. Comp. **18** (1964), 390–396.

The author studies L_1 approximation on an interval by $F(A, x)$, a non-linear function of the parameters $A=(a_1, \cdots, a_n)$. For each A there is a sign function s, $s(x)=1$ or -1 with at most n changes, such that $\int s\,\partial F/\partial a_i=0$, $1\leqq i\leqq n$. Under certain conditions, A_0 is an optimal choice in approximating f if $F(A_0, x)=f(x)$ at the points where s changes sign. From this theory the author develops a neat method for computing A_0.

W. G. Strang (Cambridge, Mass.)

Garkavi, A. L. 1486

A criterion for the element of best approximation. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 472–476.

The criterion of the title is given in Theorem 2: Let M be a convex subset of a (real or complex) normed linear space B. A point y_0 in M is a nearest point in M to a point x in B $(x\notin M)$ if and only if for each y in M there exists a continuous linear functional $f=f_y$ on B such that $f(x-y_0)=\|x-y_0\|$, $\operatorname{Re} f(y_0-y)\geqq 0$ and f is an extreme point of the unit ball S_{B^*} of B^*. The author's proof uses the following well-known result (Theorem 1): If L is a subspace of B and φ an extreme point of S_{L^*}, then there exists an extreme point of S_{B^*} which extends φ.

R. R. Phelps (Seattle, Wash.)

FOURIER ANALYSIS

See also 1287, 1288, 1292, 1365, 1437, 1545.

Leindler, L. 1487

Unconditional convergence of trigonometrical series. I. (Russian. German summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 151–156.

Let $f(x)\in L^2(0, 2\pi)$ be a 2π-periodic function. The Fourier series of $f(x)$ is said to be unconditionally convergent if every rearrangement of it converges almost everywhere. The author proves Theorem 1: Let $E_k(f)$ denote the distance in L^2 between f and the trigonometric polynomial of order $k-1$ which best approximates f. Let $\gamma(t)$ be a non-negative function such that $\sum_2^\infty [k(\ln k)\gamma(k)]^{-1}<\infty$. If $\sum_2^\infty k^{-1}(\ln k)\gamma(k)[E_k(f)]^2<\infty$, the Fourier series of f converges unconditionally. Theorem 2: Let $\rho(t)$ be a non-negative function such that $\int_0^{2\pi} \{t[\ln(t^{-1}+1)]\rho(t)\}^{-1}\,dt<\infty$. If

$$\int_0^{2\pi}\int_0^{2\pi} t^{-1}[\ln(t^{-1}+1)]\rho(t)[f(x+t)-f(x-t)]^2\,dx\,dt<\infty,$$

the Fourier series of f converges unconditionally. These theorems generalize results obtained by Ul'janov [Izv. Akad. Nauk SSSR Ser. Mat. **22** (1958), 515–542; MR **24** #A1562] for the special case $\gamma(t)=\rho(t)=|\ln|\ln t||^{1+\varepsilon}$ with $\varepsilon>0$. It is asserted that Theorem 1 is valid for arbitrary orthonormal systems with a corresponding definition of $E_k(f)$, and that the hypotheses of Theorem 2 ensure also the unconditional convergence of the conjugate series.

H. J. Landau (Murray Hill, N.J.)

Tomić, M. 1488

A convergence criterion for Fourier series.

Proc. Amer. Math. Soc. **15** (1964), 612–617.

A positive function L defined on $[0, c]$ is called slowly varying at 0 if for every $\eta>0$ there exists $t_\eta\in(0, c)$ such that $t^\eta L(t)$ is non-decreasing and $t^{-\eta}L(t)$ is non-increasing on $(0, t_\eta)$. The main result of this paper is the following theorem: Let f be a periodic function with period 2π and Lebesgue integrable on $[-\pi, \pi]$, and let ϕ_x be defined by $\phi_x(t)=f(x+t)+f(x-t)-2f(x)$. If, for a fixed $x_0\in[-\pi, \pi]$, $\lim_{t\to+0}\phi_{x_0}(t)=0$ and ϕ_{x_0} is slowly varying at 0, then the Fourier series of f at x_0 converges to $f(x_0)$.

This interesting convergence test is in a certain sense close to the well-known Dirichlet-Jordan test. From the hypothesis that ϕ_{x_0} is slowly varying at 0 it follows that ϕ_{x_0} is of bounded variation on every interval $[y, c]$ with $0<y<c\leqq t_\eta$ but not necessarily on $[0, c]$, so that the

Dirichlet-Jordan method cannot be used here. However, it can be shown that the above hypothesis implies that

$$\int_y^c t^{-\eta} |d\phi_{x_0}(t)| = o(y^{-\eta}\phi_{x_0}(y)) \qquad (y \to +0)$$

for some $\eta > 0$, and this asymptotic property of the variation of ϕ_{x_0} is sufficient for the proof of convergence tests of the above type for Fourier series.

The author uses a different method based on other properties of a slowly varying function. He uses also implicitly the following property of the integral modulus of continuity: $\liminf_{h \to +0} \omega_1(h; f)/h \geqq c > 0$, unless f is constant a.e. on $[-\pi, \pi]$. *R. Bojanic* (Columbus, Ohio)

Schulz, Gerhard 1489

Nichtklassische Orthogonalsysteme von Polynomen. (Russian, English, and French summaries)

Wiss. Z. Humboldt-Univ. Berlin Math.-Natur. Reihe **11** (1962), 731–737.

Author's summary: "Nichtklassische Orthogonalsysteme von Polynomen entstehen durch Orthogonalisierung der Funktionenfolge $x^0, x^1, x^2, \cdots$ in der natürlichen Reihenfolge, und zwar unter Verwendung der Belegungsfunktion $p(x) = x^{2\nu}$ mit ganzzahligem positivem ν in bezug auf ein beliebiges endliches Integrationsintervall $\langle a, b \rangle$. Sie werden in Analogie zu den klassischen Orthogonalsystemen von Polynomen als n-te Differentialquotienten dargestellt. Sie sind Lösungen einer homogenen linearen Differentialgleichung 2. Ordnung der Fuchsschen Klasse, deren vier Koeffizienten sich aus einem linearen Gleichungssystem berechnen lassen. Ihre Anwendung ist bei der Reihenentwicklung stetiger Funktionen der Form $g(x)/x^\nu$ von Vorteil."

Liedl, Roman 1490

Über eine spezielle Klasse von stark multiplikativ orthogonalen Funktionssystemen.

Monatsh. Math. **68** (1964), 130–137.

Für ein beliebiges Funktionensystem $\{f_n(x)\}_1^\infty$ $(-\infty < x < \infty)$ wird $w_0(x) \equiv 1$ und $w_n(x) = f_{n_1+1}(x) \cdots f_{n_m+1}(x)$ gesetzt, wobei $n = 2^{n_1} + \cdots + 2^{n_m}$ $(0 \leqq n_1 < \cdots < n_m)$ die dyadische Darstellung von $n \geqq 1$ bedeutet. Ist $\int_{-\infty}^\infty w_i(x) w_j(x)\, d\mu = \delta_{i,j}$, dann wird nach G. Alexits $\{f_n(x)\}$ stark multiplikativ genannt und ist $\{w_n(x)\}$ das von $\{f_n(x)\}$ erzeugte W-System. Das vollständige, orthonormierte System $\{g_n(x)\}_0^\infty$ wird ein $\mathfrak{G}$-System genannt, falls für alle x und für alle i, j $g_i(x) g_j(x) = g_{k(i,j)}(x)$ gilt. (I) Für $\mathfrak{G}$-Systeme ist $\mu(x)$ eine stetige, monoton wachsende Funktion von der Schwankung 1. (II) Das stark multiplikative orthogonale System $\{f_n(x)\}$, dessen W-System normiert ist und für das $|f_n(x)| = 1$ mit Ausnahme von x-Werten vom μ-Mass 0 für alle n gilt, ist über $L_\mu^2(-\infty, \infty)$ ein vollständiges W-System dann und nur dann, wenn für jedes $\mathfrak{J} = (a, b) \subseteq (-\infty, \infty)$, $\lim_{m\to\infty} 2^{-m} \sum_{\nu=1}^{2^m} \varepsilon_\nu^{(m)} \eta_\nu^{(m)} = 0$ gilt, wobei $\varepsilon_\nu^{(m)} = 2^m \mu(\mathfrak{M}_\nu^m \cap \mathfrak{J})$ und $\eta_\nu^{(m)} = 2^m \mu(\mathfrak{M}_\nu^m - \mathfrak{J})$ sind, weiterhin $\mathfrak{M}_\nu^m$ $(\nu = 1, \cdots, 2^m)$ die Menge von x bezeichnet, für die die Vorzeichensfolge $(f_1(x), \cdots, f_m(x))$ dieselbe ist.

K. Tandori (Szeged)

Vermes, Robert 1491

On Wronskians whose elements are orthogonal polynomials.

Proc. Amer. Math. Soc. **15** (1964), 124–126.

Let $Q_r(x)$ be a sequence of polynomials orthogonal with respect to a positive continuous weight function $q(x)$ on the finite interval $[a, b]$. Let $W(n, l, x)$ denote the Wronskian of the functions $Q_n(x), Q_{n+1}(x), \cdots, Q_{n+l-1}(x)$. It is proved that the zeros of $W(n, l, x)$ are contained in $S[(a, b); \pi/l]$, where $S[(a, b): \theta]$ denotes the region comprised of all points at which the interval $[a, b]$ subtends an angle of at least θ. *S. Karlin* (Stanford, Calif.)

Newman, D. J. 1492

The closure of translates in l^p.

Amer. J. Math. **86** (1964), 651–667.

Given a sequence $\{c_n\} \in l^q$, $1 \leqq q \leqq \infty$, the spectrum of $\{c_n\}$ is the support of $\sum_{-\infty}^\infty c_n e^{inx}$, where the trigonometric series is to be interpreted, say, as a distribution on the circle group. Call a closed subset S of the circle "q-negligible" if there is no sequence in l^q with spectrum in S other than the null sequence.

This paper is concerned with restrictions of the "size" of q-negligible sets. (The author treats "p-spanning" sets; by duality, "p-spanning" is equivalent to "q-negligible", where $1/p + 1/q = 1$, and the present terminology simplifies matters.) Three cases may be disposed of immediately: 1-negligible $\Leftrightarrow$ nowhere dense, 2-negligible $\Leftrightarrow$ measure zero, ∞-negligible $\Leftrightarrow$ empty; the last is Wiener's Tauberian Theorem. For $q > 2$, if S is the support of a sequence in l^q, then S must have positive α-capacity for all $\alpha < 2/q$. The relation between capacity and Hausdorff dimension leads to the result: for $q > 2$, Hausdorff dimension $< 2/q \Rightarrow$ q-negligible. In § 2 of this paper the author gives a slight strengthening: strong $2/q$-measure $0 \Rightarrow q$-negligible.

It is generally believed that, except in the cases $q = 1, 2, \infty$, "size" conditions cannot be necessary and sufficient for q-negligibility and that certain arithmetical properties of sets must play a role. In § 3 the author gives a powerful theorem in this direction. Let S^k denote the set of all sums of k elements from S. Theorem: S^k has measure $0 \Rightarrow S$ is $2k$-negligible. As a corollary one gets the existence of a set of dimension 1 which is q-negligible for all $q < \infty$. (Note that if $q > r$, then q-negligible $\Rightarrow$ r-negligible.)

In § 3 the author proves that for each $q < 2$ there are q-negligible sets of positive measure. {This is not "surprising", but the proof shows that there is an S which has positive measure and is q-negligible for all $q < 2$. Thus the author establishes more than he claims; the full strength of his result is, perhaps, more than what one would have expected, and the comparatively complicated proof probably cannot be simplified in any essential way.} Since q-negligibility is a local property of sets, we see that given $\varepsilon > 0$ there is a set S which is q-negligible for all $q < 2$ and whose complement $\tilde{S}$ has measure $< \varepsilon$. Thus a refinement of the notion of measure is called for to attack the problem. The author considers the quantities $R_n(S) = \sum_{k > n} (b_k - a_k)$, where $(a_1, b_1), (a_2, b_2), \cdots$ are the components of $\tilde{S}$ arranged in order of decreasing length. For example, $R_0(S)$ is the measure of $\tilde{S}$. The sequence $\{R_n(S)\}$ converges rapidly to 0 if S is "well-packed". Theorem: For $1 < q < 2$, $\beta > (2-q)/(2q-2)$, and S a closed set of positive measure, $R_n(S) = O(n^{-\beta}) \Rightarrow S$ is not q-negligible. Indeed, under these hypotheses, the characteristic function of S has its Fourier coefficients in l^q. *C. S. Herz* (Ithaca, N.Y.)

Sharma, P. L. 1493
On the sequence of Fourier coefficients.
Proc. Amer. Math. Soc. **15** (1964), 337–340.
Defining a method of summability which imposes a regular Toeplitz method on the $(C, 1)$ method, the author says that $\{s_n\}$ is summable $A\cdot(C, 1)$ to s if $\sum_{n=1}^{\infty} a_{nm}\sigma_n \to s$ as $m\to\infty$, where $A=(a_{nm})$ is a regular Toeplitz matrix and σ_r denotes the first Cesàro mean of s_n. He proves that, if $f(t)\sim \frac{1}{2}a_0+\sum (a_n \cos nt+b_n \sin nt)$, if $f(x+t)-f(x-t)-l=o\{\log(1/t)\}^{-1}$ as $t\to 0^+$, and if $\lim_{n\to\infty}\sum n^r|a_{nm}-a_{n+1,m}|=0$ for some r with $0<r<1$, then $\{n(b_n \cos nx - a_n \sin nx)\}$ is summable $A\cdot(C, 1)$ to l/π. *P. Heywood* (Edinburgh)

Narendra, K. Govil 1494
On the summability of a class of Fourier series.
Math. Student **31** (1963), 7–12 (1964).
The following theorem is proved. If (i) (Λ) is a regular triangular matrix such that for $0<r<1$,

$$\sum k^r|\lambda_{n,k}-\lambda_{n,k+1}| = O(1),$$

and (ii) $f(t)$ is a Lebesgue integrable function over $(0, \pi)$ and

$$\Phi(t) = \int_0^t \phi_x(u)\,du = o\left[|t| \log \frac{1}{|t|}\right],$$

where $\phi_x(u)=|f(x+u)+f(x-u)-2f(x)|$ for some x, then at the point $t=x$, the Fourier series of $f(t)$ is summable (Λ) to $f(x)$.

This result extends that of the reviewer [Proc. Amer. Math. Soc. **11** (1960), 994–998; MR **22** #11252] and has also recently been obtained by Tomić [Acad. Serbe Sci. Arts Glas Cl. Sci. Math. Natur. **254** (1963), 25–34; MR **27** #2781]. *G. M. Petersen* (Swansea)

Livšic, M. I. 1495
Uniqueness of a trigonometric series expansion for summability methods. (Russian. English summary)
Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, *no.* 3, 15–24.
Let T be a regular, row-finite matrix summability method and

$$(1) \qquad \frac{a_0}{2}+\sum_{n=1}^{\infty} (a_n \cos nx+b_n \sin nx)$$

a trigonometric series with $\lim_{n\to\infty} a_n=\lim_{n\to\infty} b_n=0$. Let σ_k $(k=1, 2, \cdots)$ denote the T-means of (1). The following theorems are proved: (i) Let P be a closed set of uniqueness with respect to ordinary convergence and let the series (1) be T-summable almost everywhere on $[-\pi, \pi]$ to a function $f\in L[-\pi, \pi]$. Furthermore, in every interval $(a, b)\subset[-\pi, \pi]$ adjacent to P let either $|\sigma_k(x)|<A$ for every $x\in(a, b)$ (A a constant) or let $\{\sigma_k(x)\}$ be uniformly convergent on every set $E\subset[c, d]\subset(a, b)$ in which σ_k is convergent. Then (1) is the Fourier series of f. (ii) Let (1) be T-summable on $[-\pi, \pi]$ to an $f\in L[-\pi, \pi]$ except on a set of first category which is of measure zero. Then (1) and the Fourier series of f are equiconvergent in a certain perfect set. (iii) Let (1) be T-summable almost everywhere on $[-\pi, \pi]$ to an $f\in L[-\pi, \pi]$. Furthermore, let the partial sums $s_n(x)$ of (1) fulfill the condition $\lim \sup_{n\to\infty} |s_n(x)|<\infty$ for all $x\in[-\pi, \pi]$ except possibly on a countable set E. Then (1) is the Fourier series of f. *G. Goes* (Chicago, Ill.)

Lebedeva, L. P. 1496
Singular integrals similar to the Weierstrass integral and a summability method for Fourier series related to them. (Russian)
Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 3 (40), 75–83.
Es sei $f(x)\in C_{2\pi}$, $\{\beta_n\}$ eine Folge reeller Zahlen und

$$W_n(\beta_n; f; x) = \int_0^{2\pi} e^{-\beta_n \sin^2(t/2)} f(t+x)\,dt \Big/ \int_0^{2\pi} e^{-\beta_n \sin^2(t/2)}\,dt.$$

Dann ist die Bedingung $\lim \beta_n=+\infty$ notwendig und hinreichend dafür, daß $W_n(\beta_n; f; x)$ für $n\to\infty$ gleichmäßig bezüglich x gegen $f(x)$ strebt. Ist $\omega(\delta)$ der Stetigkeitsmodul von $f(x)$, so gilt

$$|W_n(\beta_n; f; x)-f(x)| \leqq 2\omega\left(\frac{2}{\sqrt{(\pi\beta_n)}}+O(\beta_n^{-3/2})\right),$$

wobei sich die Ordnung dieser Abschätzung nicht verbessern läßt. Existiert $f''(x)$ in einem festen Punkt x, so gilt $W_n(\beta_n; f; x)=f(x)+f''(x)\beta_n^{-1}+o(\beta_n^{-1})$. Außerdem werden asymptotische Abschätzungen der Kolmogorov-Nikol'skii-Konstanten durchgeführt (vgl. auch den von Autor [Vestnik Leningrad. Univ. **14** (1959), no. 1, 134–139; MR **21** #272] behandelten analogen Fall).

Bildet man mit den Konstanten $\rho_k^{(n)}=e^{-k^2/\beta_n}=W_n(\beta_n; \cos kt; 0)+O(\beta_n^{-1})$ die Polynome $D_n(\beta_n; f; x)=A+\sum_{k=1}^{n} \rho_k^{(n)}(a_k \cos kx+b_k \sin kx)$, wobei a_k, b_k die Fourier-Koeffizienten von $f(x)\in C_{2\pi}$ sind, so sind die Bedingungen $\beta_n\to+\infty$, $e^{-n^2/\beta_n} \ln n=O(1)$ notwendig und hinreichend dafür, daß $D_n(\beta_n; f; x)$ für $n\to\infty$ gleichmäßig bezüglich x gegen $f(x)$ strebt. Dieses Summierungsverfahren ist im Falle $\beta_n\to+\infty$ permanent.
L. Berg (Halle)

Askey, Richard; Hirschman, I. I., Jr. 1497
Mean summability for ultraspherical polynomials.
Math. Scand. **12** (1963), 167–177.
Ce mémoire, très dense, a pour object d'étendre aux moyennes de Cesàro d'une série de polynômes ultrasphériques des résultats obtenus précédemment (et essentiellement dus à Pollard) et relatifs aux sommes ordinaires de Fourier de la série considérée. Posant

$$\hat{f}(n) = \int_{-1}^{+1} f(x)P_n(x)\,dx,$$

$$S_N f(x) = \sum_0^N (n+\tfrac{1}{2})\hat{f}(n)P_n(x),$$

où $P_n(x)$ désigne le $n^{\text{ième}}$ polynôme de Legendre (les démonstrations et les calculs pour les polynômes ultrasphériques sont tout à fait analogues), Pollard a montré que si $\frac{4}{3}<p<4$, il existe nécessairement une constante $A(p)$ telle que $\|S_N f\|_p\leqq A(p)\|f\|_p$, où $\|f\|$ désigne la norme de f dans $L_p(-1, +1)$.

Si l'on pose

$$S_N^{\alpha} f(x) = (A_N^{\alpha})^{-1}(n+\tfrac{1}{2})\hat{f}(n)A_{N-n}^{\alpha}P_n(x)$$

avec $A_n^{\alpha}=\binom{n+\alpha}{n}$, alors on montre que si $4/(3+2\alpha)<p<4/(1-2\alpha)$, il existe ici aussi une constante $A(p, \alpha)$ telle que, si $0\leqq\alpha\leqq\frac{1}{2}$, $\|S_N^{\alpha} f\|_p\leqq A(\alpha, p)\|f\|_p$.

Il y a beaucoup d'autres inégalités analogues. Le mémoire se termine en donnant l'état des connaissances

actuelles relativement aux polynômes orthogonaux classiques où ces questions sont loin d'être résolues.
R. Campbell (Caen)

Varshney, O. P. 1498
On Iyengar's Tauberian theorem for Nörlund summability.
Tôhoku Math. J. (2) **16** (1964), 105–110.
The author proves a theorem stated by K. S. K. Iyengar [Proc. Indian Acad. Sci. Sect. A **18** (1943), 113–120; MR **5**, 65], which allows one to obtain a scale of convergence tests for Fourier series, of which the simplest is a test due to Hardy and Littlewood [A. Zygmund, *Trigonometric series*, Vol. I, p. 63, second edition, Cambridge Univ. Press, New York, 1959; MR **21** #6498].
S. Ikehara (Tokyo)

Wells, James 1499
Restrictions for Fourier-Stieltjes transforms.
Proc. Amer. Math. Soc. **15** (1964), 243–246.
Let G be a locally compact abelian group with Haar measure dx, and let Γ be the dual group. The algebra of Fourier transforms of summable functions on G will be denoted by A. If μ is a Radon measure on G, its Fourier-Stieltjes transform $\hat\mu$ is defined by $\hat\mu(y)=\int_G (x, y)\, d\mu(x)$, where (x, y) denotes the value of the character y at the point $x \in G$. H. Helson proved that a function φ defined on Γ is the Fourier-Stieltjes transform of a Radon measure if and only if $\varphi\hat f \in A$ for every $\hat f \in A$ [Ark. Mat. **2** (1953), 475–487; MR **15**, 327]. The present author generalizes the result of Helson as follows: Let $F \subset \Gamma$ be closed, let φ be a function on Γ and let $\varphi|F$ denote the restriction of φ to F; a necessary and sufficient condition that $\varphi|F=\hat\mu|F$ for some Radon measure μ is that $\varphi A|F \subset A|F$. Helson's proof of his theorem rested on a criterion for Fourier-Stieltjes transforms due to I. J. Schoenberg for the reals [Bull. Amer. Math. Soc. **40** (1934), 277–278] and to W. F. Eberlein for general locally compact abelian groups [Duke Math. J. **22** (1955), 465–468; MR **17**, 281]. The present author uses a slight variant of the Schoenberg criterion to obtain his generalized version of Helson's theorem. *G. Maltese* (College Park, Md.)

Essén, Matts 1500
Studies on a convolution inequality.
Ark. Mat. **5**, 113–152 (1964).
Let μ be a positive regular measure with total mass one on a locally compact Abelian group G. This paper is concerned with continuous solutions φ of the convolution inequality $\varphi-\varphi*\mu \geqq 0$. The author gives necessary and sufficient conditions on the measure μ in order that this inequality have a non-trivial solution, i.e., a continuous solution with strict inequality holding on a set of positive Haar measure. Although these conditions are too complicated to give here, they are equivalent to the condition that there exist a neighborhood $\hat 0$ of zero in the dual group $\hat G$ and a constant C such that

$$\int_{\hat 0} \operatorname{Re}\left\{\frac{1}{1+\varepsilon-\hat\mu(\hat x)}\right\} d\hat x \leqq C \quad \text{for all } \varepsilon > 0,$$

where $\hat\mu$ is the Fourier-Stieltjes transform of μ.
In the particular case where G is R^n or Z^n, $n=1, 2$, and the closed subgroup $G(\mu)$ generated by the elements of the support of μ is all of G, the author gives conditions on the moments of μ in order that the convolution inequality have a bounded non-trivial solution. When $n \geqq 3$ and $G(\mu)=G$, bounded non-trivial solutions always exist. Most of the paper is concerned with the relationship between growth properties of a solution φ and the function $\varphi-\varphi*\mu$. Results of this type are too numerous to give in detail. The following result, however, is typical. Suppose $G=G(\mu)=R^1$ with the moments of μ satisfying certain conditions. Let φ be a solution of the convolution inequality. Then (a) $\varphi(x)=O(|x|^\alpha)$, $0 \leqq \alpha \leqq 1$ implies $\varphi-\varphi*\mu \in L^1(R^1)$ and (b) $\varphi(x)=o(|x|^\alpha)$, $0 \leqq \alpha \leqq 1$ implies $\varphi-\varphi*\mu=0$.
L. L. Helms (Urbana, Ill.)

INTEGRAL TRANSFORMS, OPERATIONAL CALCULUS

Ghosh, P. K. 1501
A note on Laplace transform of distributions.
Bull. Calcutta Math. Soc. **53** (1961), 193–195.
The author extends L. Schwartz's theory of the Laplace transform of his kind of distributions via the Fourier transform of a distribution T multiplied by $\exp(-x\xi)$ with $x=(x_1, \cdots, x_n)$ and $\xi=(\xi_1, \cdots, \xi_n)$. He considers other types of test functions already being discussed in the books of Gelfand and Schilow [*Verallgemeinerte Funktionen*, I, VEB Deutscher Verlag Wiss., Berlin, 1960; MR **22** #5889; II, 1962; MR **26** #6765], namely, the following spaces: (1) K_p $(p>1)$: the set of all functions $\varphi(x)$ satisfying the conditions

$$|D^q\varphi(x)| \leqq c_1(q, \varphi)\cdot\exp[-c_2(q, \varphi)\cdot|x|^p]$$

with constants c_1 and $c_2>0$ and $|x|=\sqrt{(x_1^2+\cdots+x_n^2)}$. Here D^q means any derivative of the type

$$\partial^{q_1+\cdots+q_n}/\partial x_1^{q_1}\cdots\partial x_n^{q_n}$$

with $q=(q_1, \cdots, q_n)$ and $q_j \geqq 0$. (2) Z^p $(p \geqq 1)$: the set of all functions $\varphi(x)$ which may be continued analytically to entire functions of $z=(z_1, \cdots, z_n)$ with $z_j=x_j+iy_j$ $(j=1, \cdots, n)$, such that for any polynomial $P(z)$ one has the estimate

$$\int_{-\infty}^{\infty} |P(x+iy)\cdot\varphi(x+iy)|^2\, dx \leqq c_1(P,\varphi)\cdot\exp[c_2(P,\varphi)\cdot|y|^p].$$

(3) $Z_p{}^p$ $(p>1)$: the set of all functions $\varphi(x)$ of one variable x which may be continued to entire functions of $z=x+iy$ satisfying the inequalities

$$|\varphi(z)| \leqq c_1(\varphi)\cdot\exp[c_2(\varphi)\cdot|z|], \qquad c_j(\varphi) > 0.$$
$$|\varphi(x)| \leqq c_3(\varphi)\cdot\exp[-c_4(\varphi)\cdot|x|^p],$$

The Fourier transform $\mathfrak{f}$ of a continuous linear functional (T, φ) on one of these spaces with a certain topology is defined in the usual way by $(\mathfrak{f}T, \mathfrak{f}\varphi)=(T, \varphi(-x))$ for all φ. Now it is possible to define Fourier and Laplace transforms of rapidly increasing functions such as $\exp(cx)$, $\exp(ax^2)$ with $a>0$, etc. *E. Meister* (Riegelsberg)

Ghosh, P. K. 1502
Mikusinski's operational calculus and divergent integrals containing a parameter.
Bull. Calcutta Math. Soc. **54** (1962), 79–82.

This paper contains the evaluation of

$$\int_0^\infty \sin x(\tau+t)\,dx, \quad \int_0^\infty \cos x(\tau+t)\,dx \quad \text{and} \quad \int_{-\infty}^\infty e^{itx}\,dx$$

by the methods of Mikusiński's operational calculus.

J. L. Griffith (Kensington)

Wimp, Jet 1503

A class of integral transforms.

Proc. Edinburgh Math. Soc. (2) **14** (1964/65), 33–40.

The author obtains an inversion formula for the integral transform

$$(1)\quad g(x) = \int_0^\infty G^{m,n+2}_{p+2,q}\left(t\left|\begin{matrix}1-\nu+ix,\ 1-\nu-ix,\ a_p\\ b_q\end{matrix}\right.\right)f(t)\,dt.$$

The inversion formula is

$$(2)\quad f(x) = \frac{i}{\pi}\int_0^\infty te^{-\nu\pi i}\{e^{\pi t}\Lambda(xe^{\pi i}|\nu+it,\ \nu-it) - e^{-\pi t}\Lambda(xe^{\pi i}|\nu-it,\ \nu+it)\}g(t)\,dt,$$

where

$$\Lambda(z|\alpha,\beta) = G^{q,\ m,p-n+1}_{p+2,q}\left(z\left|\begin{matrix}-a_{n+1},\ -a_{n+2},\ \cdots,\ -a_p,\ \alpha,\ -a_1,\ -a_2,\ \cdots,\ -a_n,\ \beta\\ -b_{m+1},\ -b_{m+2},\ \cdots,\ -b_q,\ -b_1,\ -b_2,\ \cdots,\ -b_m\end{matrix}\right.\right).$$

The known special cases of (1) and (2) are the Kontorovitch-Lebedev transform pair [A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Tables of integral transforms*, Vol. II, McGraw-Hill, New York, 1954; MR 16, 468] and the generalised Mehler transform pair [P. L. Rosenthal, Ph.D. Dissertation, Oregon State Univ., Corvallis, Ore., 1961]. Three more special cases are also given. The proofs of the results are formal.

B. R. Bhonsle (Jabalpur)

Bhise, V. M. 1504

Some finite and infinite series of Meijer-Laplace transform.

Math. Ann. **154** (1964), 267–272.

Under suitable conditions the author obtains some finite and infinite series for the Meijer-Laplace transform, defined for $\operatorname{Re}(s) > 0$ by

$$\Phi\left(s;\begin{matrix}\alpha_1,\cdots,\alpha_m\\ \eta_1,\cdots,\eta_m,\rho\end{matrix}\right) = s\int_0^\infty G^{m+1,0}_{m,m+1}\left(st\left|\begin{matrix}\eta_1+\alpha_1,\ \cdots,\ \eta_m+\alpha_m\\ \eta_1,\ \cdots,\ \eta_m,\ \rho\end{matrix}\right.\right)f(t)\,dt,$$

where G denotes Meijer's G-function. The following are typical examples:

$$\sum_{r=0}^{\infty}\frac{(-1)^{n+rn}C_r}{\Gamma(1+\eta_1-\eta_m-\alpha_m+r)}\Phi\left(s;\begin{matrix}\alpha_1-r,\ \alpha_2,\ \cdots,\ \alpha_m\\ \eta_1+r,\ \eta_2,\ \cdots,\ \eta_m,\ \rho\end{matrix}\right) = \frac{1}{\Gamma(1+\eta_1-\eta_m-\alpha_m+n)}\Phi\left(s;\begin{matrix}\alpha_1,\ \cdots,\ \alpha_{m-1},\ \alpha_m-n\\ \eta_1,\ \cdots,\ \eta_m,\ \rho\end{matrix}\right),$$

$$\sum_{r=0}^{\infty}\frac{(-h)^r}{r!}\Phi\left(s;\begin{matrix}\alpha_1,\ \cdots,\ \alpha_{m-1},\ \alpha_m-r\\ \eta_1,\ \cdots,\ \eta_m,\ \rho\end{matrix}\right) = (1-h)^{\eta_m+\alpha_m}\Phi\left(\frac{s}{1-h};\begin{matrix}\alpha_1,\ \cdots,\ \alpha_m\\ \eta_1,\ \cdots,\ \eta_m,\ \rho\end{matrix}\right).$$

From these he deduces formulae such as

$$\sum_{r=0}^{\infty}\frac{(-1)^{n+rn}C_r}{\Gamma(1-a_1+b_\delta+r)}G^{\alpha,\beta}_{\gamma,\delta}\left(x\left|\begin{matrix}a_1-r,\ a_2,\ \cdots,\ a_\gamma\\ b_1,\ \cdots,\ b_\delta\end{matrix}\right.\right) = \frac{1}{\Gamma(1-a_1+b_\delta+n)}G^{\alpha,\beta}_{\gamma,\delta}\left(x\left|\begin{matrix}a_1,\ \cdots,\ a_\gamma\\ b_1,\ \cdots,\ b_{\delta-1},\ b_\delta+n\end{matrix}\right.\right)$$

for $\gamma \geqq \beta \geqq 1$, $\delta > \alpha$, $\operatorname{Re}(b_\delta+n+1) > 0$, and

$$\sum_{r=0}^{\infty}\frac{(-h)^r}{r!}G^{\alpha,\beta}_{\gamma,\delta}\left(x\left|\begin{matrix}a_1,\ \cdots,\ a_\gamma\\ b_1,\ \cdots,\ b_{\delta-1},\ b_\delta+r\end{matrix}\right.\right) = (1-h)^{-b_\delta}G^{\alpha,\beta}_{\gamma,\delta}\left((1-h)x\left|\begin{matrix}a_1,\ \cdots,\ a_\gamma\\ b_1,\ \cdots,\ b_\delta\end{matrix}\right.\right)$$

for $\beta \leqq \gamma$, $\alpha < \delta$, $|h| < 1$.

P. Heywood (Edinburgh)

Saxena, R. K. 1505

Relation between Whittaker transform and modified $\chi_{\nu,k,m}$-transform.

Math. Ann. **154** (1964), 301–306.

The modified $\chi_{\nu,k,m}$-transform is defined for $\xi > 0$ by

$$\chi_{\nu,k,m}[f(t);\xi] = \int_0^\infty G^{2,1}_{2,4}\left(\xi t\left|\begin{matrix}k-m-\frac12-\frac12\nu,\ -k+m+\frac12+\frac12\nu\\ \frac12\nu,\ \frac12\nu+2m,\ -\frac12\nu,\ -\frac12\nu-2m\end{matrix}\right.\right)f(t)\,dt.$$

The author proves that if $f(t) \in L(0,\infty)$, $\chi_{\nu,k,m}[f(t);\xi] \in L(0,\infty)$, $\operatorname{Re}(\mu) > -1$, $\operatorname{Re}(s) \geqq s_0 > 0$, $\operatorname{Re}(\lambda \pm \delta) < \frac12$, $\operatorname{Re}(\mu+\frac12\nu+\frac54\pm\delta+m\pm m) > 0$ and $\operatorname{Re}(\mu+\lambda+k-m-\frac12\nu+\frac14) > 1$, then

$$\int_0^\infty (2st)^{-1/4}W_{\lambda,\delta}(2st)t^\mu f(t)\,dt = \int_0^\infty k(s,\xi)\chi_{\nu,k,m}[f(t);\xi]\,d\xi,$$

where the left-hand side is the Whittaker transform of $t^\mu f(t)$ and where $k(s,\xi)$ denotes $\{\xi^{\mu+1}\Gamma(\frac12-\lambda-\delta)\Gamma(\frac12-\lambda+\delta)\}^{-1}$ multiplied by the G-function

$$G^{3,3}_{5,4}\left(\frac{2s}{\xi}\left|\begin{matrix}\lambda+\frac34,\ -\mu-\frac12\nu,\ -\mu-\frac12\nu-2m,\ -\mu+\frac12\nu,\ -\mu+\frac12\nu+2m\\ \frac14+\delta,\ \frac14-\delta,\ \frac12-\mu+m-k+\frac12\nu,\ -\frac12-\mu+k-m-\frac12\nu\end{matrix}\right.\right).$$

This result is used to evaluate the Whittaker transforms of the functions $t^{\mu+\rho-1}$, $t^{\mu+\rho-1}(t+\beta)^{-\sigma}$, and $t^{\mu+\rho-1}J_\nu(2\sqrt{t})$.

P. Heywood (Edinburgh)

Berg, Lothar 1506

Zu den Grundlagen der Operatorenrechnung.

Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe **11** (1962), 983–987.

As a basis of an operational calculus for functions on the half-line $t \geqq 0$ the Duhamel product

$$f(t)\cdot g(t) = \frac{d}{dt}\int_0^t f(\tau)g(t-\tau)\,d\tau$$

may be used in place of the convolution $f * g$, the basis used by Mikusiński. Then an adequate set of functions that constitute a ring under that product and ordinary addition is to be considered. The author shows that such a set consists of all functions $f(t)$ for which, on each interval $(0,\alpha)$, the Fourier coefficients

$$\alpha^{-1}\int_0^\alpha f(\tau)\exp(\pm 2\pi i n\tau/\alpha)\,d\tau$$

are of order $O(n^{\varepsilon-1})$ for each fixed $\varepsilon>0$, as $n\to\infty$, and that ring is essentially the most general one. The set is also characterized by the requirement that $p\int_0^\alpha e^{-p\tau}f(\tau)\,d\tau$ be of order $O(p^\varepsilon)$ for each fixed positive α and ε, as $|p|\to\infty$ when $\operatorname{Re}(p)\geqq 0$.

R. V. Churchill (Ann Arbor, Mich.)

Mikolás, M. 1507

Generalized Euler sums and the semigroup property of integro-differential operators.

Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **6** (1963), 89–101.

Let $f(x)$ be a real-valued function of a real variable x, put

$$_a\Delta_n{}^\mu f(u)|x = {}_a\Delta_n{}^\mu f|x = \sum_{p=0}^{n-1}(-1)^p\binom{\mu}{p}f\left(x-p\frac{x-a}{n}\right).$$

The function f is called $_aD^\mu$-derivable at x if

$$_af^{(\mu)}(u)|x = {}_af^{(\mu)}|x = \lim_{n\to\infty}\left(\frac{u-a}{n}\right)^{-\mu}{}_a\Delta_n{}^\mu f(u)|x$$

exists. The reviewer [Comment. Math. Helv. **27** (1953), 140–150; MR **14**, 957] gave necessary conditions for the existence of the $_aD^\mu$ derivative of f and some instances when $_aD^{\mu_1}{}_aD^{\mu_2}={}_aD^{\mu_1+\mu_2}$ holds.

If $_af^{(\mu_2)}|u$ is known for rational $(u-a)/(x-a)$ and if one of the expressions $_a({}_af^{(\mu_2)}|u)^{(\mu_1)}|x$, $_af^{(\mu_1+\mu_2)}|x$ exists, then the author gives necessary and sufficient conditions that the other one exists also and that both are equal.

He calls $f(u)$ uniformly $_aD^\mu$-derivable in $(a,x]$ if the sequence $\left(\frac{u-a}{n}\right)^{-\mu}{}_a\Delta_n{}^\mu f|u$ $(n=1,2,\cdots)$ converges uniformly in $(a,x]$. Using this definition, he gives necessary and sufficient conditions that the existence of one of the expressions

$$_a({}_af^{(\mu_2)}u)^{(\mu_1)}|x, \quad {}_af^{(\mu_1+\mu_2)}|x$$

implies the existence of the other one. He then shows that if $f(u)$ is uniformly $_aD^{\mu_0}$-derivable in $(a,x]$, then it is $_aD^\mu$-derivable at x for $\mu<\mu_0$.

Applications are given for the solution of generalised differential equations, and the extension to complex order μ is discussed.

C. F. Moppert (Melbourne)

INTEGRAL EQUATIONS

See also 1400, 1555a–b, 1556, 1561, 1562, 1752, 1754.

Mikhlin, S. G. [Mihlin, S. G.] 1508

★Integral equations and their applications to certain problems in mechanics, mathematical physics and technology.

Second revised edition. Translated from the Russian by A. H. Armstrong. A Pergamon Press Book.

The Macmillan Co., New York, 1964. xiv+341 *pp.* **$12.50.**

The original Russian edition [Fizmatgiz, Moscow, 1959] was reviewed in MR **23** #A490. It should, however, be mentioned that the present book is more comprehensive than the one to which the mentioned review applies. It contains not only an account of the general Fredholm and Hilbert-Schmidt theories, and a study of certain singular equations that involve the Cauchy principal part of integrals, but it has six chapters on applications of integral equations and also a bibliography of forty-two items. As a matter of fact, the applications constitute the major part of the book, about 200 pages out of a total of 339 pages. Methods for approximate solutions of Fredholm's equation, and basic problems in mathematical physics, hydrodynamics and elasticity are considered. In the front of the book it is stated that the book is a revised translation of the Russian, *Lectures on linear integral equations* [see MR **23** #A490 cited above], but it includes more than that book, and corresponds more to the 1949 book by the same author [*Integral equations and their applications to some problems of mechanics, mathematical physics and engineering* (Russian), second edition, GITTL, Moscow, 1949; MR **12**, 712; Engl. transl., Pergamon, New York, 1957; MR **19**, 428].

H. P. Thielman (Alexandria, Va.)

Fox, Charles 1509

The solution of an integral equation.

Canad. J. Math. **16** (1964), 578–586.

Consider the integral equation (*) $g(a)=\int_0^\infty f(ab)f(b)b^\alpha\,db$ for $a\geqq 0$ and known g subject to the necessary condition $g(1/a)=ag(a)$. (*) has statistical applications for $\alpha=1$, stemming from the fact that if $\xi\geqq 0$ and $\eta\geqq 0$ are independent with common density f, then the density of ξ/η is g. A simple substitution reduces (*) to the case $\alpha=0$; this reduction is understood below. Using $\hat{}$ to denote the Mellin transform, one finds that if $g\in L^2[0,\infty)$, then $f\in L^2[0,\infty)$ is a solution if and only if $g\hat{}(s)=f\hat{}(s)f\hat{}(1-s)$. Given such a solution f, others can be constructed by the rule $f\to k[f]=\int_0^\infty k(ab)f(b)\,db$ if k is a Fourier kernel in the sense that, on the line Q: $\operatorname{Re}(s)=1/2$, $k\hat{}(s)=k\hat{}(1-s)^*$ is of modulus 1 (the author's statement of this condition is not complete). It is proposed that if $g\in L^2[0,\infty)$ and if $g\hat{}=0$ has no roots on the line Q, then all solutions of the reduced problem belonging to $L^2[0,\infty)$ can be obtained in this way from a single solution; this is correct for real solutions (the condition on $g\hat{}$ is then superfluous), though the proof advanced is incomplete. An infinite number of solutions can still exist under the extra conditions $0\leqq f, g$ imposed in statistical applications; for instance, both $e^{\xi/2}f_1(e^\xi)=(2\pi\xi^3)^{-1/2}e^{-1/2\xi}$ and $e^{\xi/2}f_2(e^\xi)=(2\pi)^{-1}\int\cos(\xi\gamma)e^{-\sqrt{|\gamma|}}\,d\gamma$ are non-negative, and $\int_0^\infty f_1(ab)f_1(b)\,db=\int_0^\infty f_2(ab)f_2(b)$, as can be verified by taking Mellin transforms. A note on a related case is promised for the near future [private communication].

H. P. McKean, Jr. (Cambridge, Mass.)

Čalenko, P. Ĭ. 1510

On the solution of integral equations with degenerate kernel. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 306–310.

Author's summary: "A method is presented for solving the integral equation

$$\varphi(x) = f(x)+\lambda\int_a^b\sum_{i=1}^n a_i(x)b_i(s)\varphi(s)\,ds$$

with a degenerate kernel with λ not coinciding with any eigenvalue. It is shown that in a number of cases the method described involves less computation work than the method generally used."

Romanov, M. F. 1511
Solution of a class of Fredholm integral equations. (Russian. English summary)
Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **19** (1964), *no.* 1, 156–159.

Author's summary: "The author solves in explicit form the integral equation

$$\int_0^T \left(R(\tau-t)+\sum_{k=1}^{r} X_k(t)Y_k(\tau)\right)\varphi(\tau)\,d\tau = f(t),$$

where

$$R(\tau) = \sum_{\lambda=1}^{\nu}\left(\sum_{k=0}^{n_\lambda} A_{k\lambda}\tau^k\right)e^{-i\omega_\lambda\tau} \quad (\tau \leqq 0),$$

$$= \sum_{\lambda=1}^{\nu'}\left(\sum_{k=0}^{n_\lambda'} A_{k\lambda}'\tau^k\right)e^{-i\omega_\lambda'\tau} \quad (\tau \geqq 0),$$

$\operatorname{Im}\omega_\lambda > 0$, $\operatorname{Im}\omega_\lambda' < 0$. The functions $X_k(t)$, $Y_k(\tau)$ and the constants $A_{k\lambda}$, $A_{k\lambda}'$ are arbitrary."

Aleksandrov, V. M. 1512
On the approximate solution of a class of integral equations. (Russian. Armenian summary)
Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk **17** (1964), *no.* 2, 3–8.

On traite d'une manière approximative, à l'aide des polynômes de Čebyšev, l'équation intégrale de la forme

$$-\int_{-1}^{1}\varphi(\xi)\ln\frac{|x-\xi|}{\lambda}\,d\xi = \pi f(x)-\int_{-1}^{1}\varphi(\xi)F\left(\frac{x-\xi}{\lambda}\right)d\xi, \quad |x| \leqq 1,$$

où $\lambda\in(0,\infty)$ est un paramètre arbitraire et $F(k)$ est une fonction paire, ainsi que $F(k)\in H_p^{\alpha}(-2/\lambda, 2/\lambda)$, $\alpha>0$, $p\geqq 2$; $H_p^{\alpha}(-\beta,\beta)$ est l'espace des fonctions dont la dérivée d'ordre p vérifie la condition de Hölder avec l'indice α pour $-\beta\leqq x\leqq\beta$. On utilise les résultats obtenus pour résoudre, à l'aide de la machine électronique à calcul "Ural", le problème de l'action d'un poinson sur une couche élastique, qui repose sans frottement sur une base rigide. *P. P. Teodorescu* (Bucharest)

Röhrl, Helmut 1513
Ω-degenerate singular integral equations and holomorphic affine bundles over compact Riemann surfaces. I.
Comment. Math. Helv. **38** (1963), 84–120.

The author studies singular integral equations on a system S of (not necessarily disjoint) curves on a compact Riemann surface X. His aim is to translate the problems on S into problems of holomorphic function theory on X. The integrals involved are of the form $\int_S K(x_1,x_2)\Omega(x_2,x_1)$ where K is a (matrix) kernel and Ω is a suitable generalization of the Cauchy kernel $dx_1/(x_1-x_2)$. Moreover, he considers in the first place only degenerate kernels, i.e., K of the form $\sum_\nu B_\nu(x_1)C_\nu(x_2)$. He first shows that, under appropriate hypotheses, the solutions of the singular integral equation

$$(1) \quad A(x_1)f(x_1)+\frac{1}{\pi i}\int_S K(x_1,x_2)f(x_2)\Omega(x_2,x_1) = 0$$

correspond bijectively to the solutions of a certain transmission problem. He then shows that the solutions of this transmission problem correspond to the holomorphic sections of a certain holomorphic vector bundle W over X. The vector bundle W is determined explicitly by A and K and a formula is given for the degree of W (i.e., its first Chern class) in terms of A and K. The adjoint equation of (1) leads to the bundle $W^*\otimes K$, where W^* is the dual bundle of W and K is the canonical line bundle on X. Thus the Riemann-Roch theorem for W yields at once the formula for the index of the operator (A, K) in the equation (1).

One of the advantages claimed for this treatment is that it lends itself to the study of families of operators depending differentiably or holomorphically on a parameter. The author gives a number of results in this direction. Finally he shows how to pass from degenerate K to the general case. *M. F. Atiyah* (Oxford)

Erdélyi, A. 1514
An integral equation involving Legendre functions.
J. Soc. Indust. Appl. Math. **12** (1964), 15–30.

The integral equation in question is

$$(*) \qquad \int_a^x (x^2-t^2)^{\lambda/2}P_\nu^{-\lambda}\left(\frac{x}{t}\right)g(t)\,dt = f(x),$$
$$\lambda > -1,\ \nu \geqq -\tfrac{1}{2},\ 0 < a \leqq x \leqq b.$$

By using fractional integrals and derivatives, the author derives the solution in the form

$$g(x) = (2x)^{\nu+1}I_{x^2}^{-\nu-1}I_x^{\nu-\lambda}f(x),$$

where I^α is the operator of the fractional integration defined for suitable f, by

$$I_{x^n}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x^n-t^n)^{\alpha-1}f(t)nt^{n-1}\,dt \quad \text{for } \alpha > 0,$$
$$I_{x^n}^{0}f(x) = f(x),$$
$$I_{x^n}^{\alpha}f = g \quad \text{if } f = I_{x^n}^{-\alpha}(g) \quad \text{for } \alpha < 0.$$

Some explicit analytic forms of this solution are also given. A particular case of (*) has been considered by Buschman [SIAM Rev. **5** (1963), 232–233; MR **27** #5098], who used certain integrals involving the product of Legendre functions for obtaining a solution of (*). But the method that is given here is more direct and leads to the solution under more liberal conditions. Moreover, this method illustrates the utility of fractional integrals as a tool in applied mathematics. An analogous solution of the integral equation

$$\int_x^\infty (t^2-x^2)^{\lambda/2}P_\nu^{-\lambda}\left(\frac{x}{t}\right)g(t)\,dt = f(x),$$
$$\lambda > -1,\ \nu \geqq -\tfrac{1}{2},\ 0 < a \leqq x,$$

is given. *K. N. Srivastava* (Bhopal)

Berikašvili, N. A. 1515
On the index of systems of singular integral equations on two-dimensional manifolds. (Russian. Georgian summary)
Soobšč. Akad. Nauk Gruzin. SSR **34** (1964), 257–264.

The author extends to arbitrary two-dimensional manifolds a formula of Vol'pert [Dokl. Akad. Nauk SSSR **142** (1962), 776–777; MR **24** #A2215] for the index of an elliptic system of singular integral equations. The proof uses far less topology than the general formula of Atiyah and Singer [Bull. Amer. Math. Soc. **69** (1963), 422–433; MR

28 #626], which applies to operators on arbitrary vector bundles over an orientable manifold of any dimension. The connection between the two formally different results is given in a brief expository paper of Singer [Proc. Joint Soviet-American Symposium on Partial Differential Equations (Novosibirsk, 1963)].

R. T. Seeley (Waltham, Mass.)

Zeragija, P. K. 1516

Solution of a non-linear integro-differential equation by the method of upper and lower functions. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **34** (1964), 3–10.

The non-linear integro-differential equation

$$(1)\qquad y''(x) = F\left[x, \lambda \int_a^x K(x, t, y(t), y'(t))\, dt\right]$$

is considered. The problem is to find a function $y(x)$ which has continuous second-order derivatives in the closed interval $[a, b]$, satisfies equation (1) in this interval and the boundary conditions $y(a)=y_0$, $y'(a)=y_0'$, where a, b, y_0, and y_0' are given numbers. Sequences of upper and lower functions are constructed, for certain conditions on the functions F and K, with the aid of some functional inequalities. It is shown that these sequences converge to the solution function of equation (1) for any given finite interval $[a, b]$. *H. P. Thielman* (Alexandria, Va.)

Berezovskiĭ, A. A. 1517

Non-linear integral equations for stationary boundary-value problems of heat radiation. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 295–299.

Author's summary: "The stationary boundary problem of heat radiation for an arbitrary body G, part of the surface of which S_1 is kept at a given temperature $u(P)=T(P)$, $P\in S_1$, and the other part S_2 radiates heat by the Stefan-Boltzmann law, $\partial u(P)/\partial n_P = -hu^4(P)+u_0$, $P\in S_2$, is reduced to a nonlinear integral equation of the Hammerstein type

$$u(P) = f(P) - h\int_{S_2} u^4(Q)k(P, Q)\, dS_Q, \quad P\in G+S, \quad Q\in S_2.$$

For an approximate solution of this equation use is made of the method of successive approximations, the Bubnov-Galerkin method and Ju. D. Sokolov's method of averaging functional corrections [Ukrain. Mat. Ž. **9** (1957), 394–412; MR **21** #2888]. Of all approximate solutions the one having a physical sense is selected."

FUNCTIONAL ANALYSIS

See also 1216, 1284, 1347, 1352, 1358, 1360, 1409, 1433, 1486, 1492, 1500, 1610, 1753, 1887, 1901, 2101.

Martin, William Ted; Segal, Irving (Editors) 1518

★Analysis in function space.

Proceedings of a Conference, Dedham, Massachusetts, June 9–13, 1963.

The M.I.T. Press, Cambridge, Mass., 1964. vi+218 pp. $6.00.

This volume contains 13 contributions representing the formal part of the program mentioned above, and a summary by R. E. Peierls of the conference. The papers will be reviewed individually.

Köthe, Gottfried 1519

Homomorphismen von (F)-Räumen.

Math. Z. **84** (1964), 219–221.

Mit den Bezeichnungen wie im Buch des Verfassers [*Topologische lineare Räume*, I, Springer, Berlin, 1960; MR **24** #A411] wird die folgende Verallgemeinerung des ursprünglich für Banach-Räume formulierten Satzes von Banach-Schauder bewiesen: E und F seien tonnelierte Räume mit je einem abzählbaren Fundamentalsystem absolutkonvexer kompakter Mengen. Eine stetige lineare Abbildung A von E in F ist dann und nur dann ein topologischer Homomorphismus, wenn $A(E)$ folgenabgeschlossen ist. *G. Goes* (Chicago, Ill.)

Bernau, S. J. 1520

A note on (M)-spaces.

J. London Math. Soc. **39** (1964), 541–543.

The author gives a proof of the following theorem. For a normed linear lattice S, if $\|x\vee y\| = \mathrm{Max}\{\|x\|, \|y\|\}$ for $x\wedge y=0$, then this equality holds for every $x, y \geqq 0$. The proof is very complicated. But to the reviewer this theorem is trivial. Indeed, it is obvious in the case where S is sequentially continuous as a linear lattice. Applying the cut extension in the reviewer's book [*Modern spectral theory*, Maruzen, Tokyo, 1950; MR **12**, 419], the general case is reduced to this case. *H. Nakano* (Detroit, Mich.)

Edelstein, Michael 1521

On non-expansive mappings of Banach spaces.

Proc. Cambridge Philos. Soc. **60** (1964), 439–447.

Let a Banach space X be strictly convex, i.e., the unit sphere of X contains no line segment. A mapping f of X into X is called non-expansive if $\|f(p)-f(q)\| \leqq \|p-q\|$. A $y\in X$ is said to belong to the f-closure X^f of X if there exists a point $x\in X$ such that, for a certain sequence $\{n_i\}$ of positive integers, $\{f^{n_i}(x)\}$ converges to y. It is proved that if X^f is non-empty and if the linear variety spanned by $\{f^n(x)\}$ is finite-dimensional, then f has at least one fixed point. An example is given which shows that this no longer holds for an infinite-dimensional linear variety spanned by the $\{f^n(x)\}$, even for Hilbert space X. However, if an $x\in X$ exists such that $\{n^{-1}\sum_{k=1}^n f^k(x)\}$ contains a subsequence which is weakly convergent to some $y\in X$, then $f(y)=y$. *K. Yosida* (Tokyo)

Kolumban, I. [Kolumbán, I.] 1522

On the uniqueness of the extension of linear functionals. (Russian)

Mathematica (*Cluj*) **4** (**27**) (1962), 267–270.

The following consequence of the Hahn-Banach theorem was proved by the reviewer [Trans. Amer. Math. Soc. **95** (1960), 238–255; MR **22** #3964]: Suppose that M is a linear subspace of the normed linear space E. Then every bounded linear functional on M has a unique norm-preserving extension to E if and only if the annihilator $M^\perp$ of M in E^* has the unique nearest point property.

(This latter means that to each element f of E^* there exists a unique point in $M^\perp$ which is nearest to f.)

By making appropriate alterations in the definitions involved and using the same methods, the author proves this result for spaces E in which the norm is not necessarily symmetric. *R. R. Phelps* (Seattle, Wash.)

Thorp, E. O. 1523

Finite dimensional normed spaces.

J. London Math. Soc. **39** (1964), 211–214.

The following two theorems are proved: A normed linear space X is finite-dimensional if and only if TX is closed whenever T is a continuous linear mapping of X into a normed space. A normed space X is finite-dimensional if and only if, for every nested collection $\{M_a\}$ of dense linear manifolds in X, $\bigcap M_a$ is dense in X.

R. R. Phelps (Seattle, Wash.)

Barbeau, E. J. 1524

Two results on semi-algebras.

Quart. J. Math. Oxford Ser. (2) **15** (1964), 69–76.

A type n semi-algebra A is a cone of non-negative real-valued functions on some space, closed under multiplication and the transformation $f \to f^n(1+f)^{-1}$. The author first proves that the identity function can be adjoined to A whilst preserving the type n condition and also whilst preserving closure in a given topology of A. Secondly, he gives a set of necessary and sufficient algebraic conditions for a uniformly closed cone A of non-negative continuous functions on a compact space to be an upper semi-lattice, i.e., such that $f, g \in A$ implies $\max(f, g) \subset A$. He gives examples to show that the adjunction of the identity function need not preserve the property either of being or of not being a semi-lattice.

J. A. H. Shepperd (Manchester)

Garkavi, A. L. 1525

On the uniqueness of the L-problem of moments. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 553–570.

The uniqueness question of the title admits an immediate reformulation as follows: Suppose that L_n is an n-dimensional ($1 \leqq n < \infty$) linear subspace of the Banach space B. Find necessary and sufficient conditions for L_n to have property PU: Every linear functional on L_n has a unique norm-preserving extension to B. This problem has previously been treated by the reviewer [Trans. Amer. Math. Soc. **95** (1960), 238–255; MR **22** #3964] and the author [Mat. Sb. (N.S.) **62 (104)** (1963), 104–120; MR **27** #6076]. Both the methods and results of this earlier work are applied to obtain characterizations of PU subspaces for arbitrary Banach spaces as well as for classical Banach spaces of real functions. The main result is Theorem 1: Let S_{B^*} denote the unit ball of B^*, $\operatorname{ext} S_{B^*}$ the set of extreme points of S_{B^*}. Suppose that $L_n \subset B$ is generated by $x_1, x_2, \cdots, x_n$. Then L_n has property PU if and only if whenever $x \in L_n$, $x \neq 0$, and $f_1, \cdots, f_r$ form a linearly independent subset of $\operatorname{ext} S_{B^*} \cap \{f: f \in S_{B^*}, f(x) = \|x\|\}$, then $r \leqq n$ and the matrix $\|f_i(x_k)\|$ ($i = 1, \cdots, r$; $k = 1, \cdots, n$) has rank r. For classical real spaces like $C(S)$ (S compact Hausdorff), L_1 and L_∞ (over a σ-finite measure space) and $B(S)$ (bounded functions on a set S), the set $\operatorname{ext} S_{B^*}$ has well-known concrete characterizations; these lead to immediate applications of Theorem 1. Theorem 2 characterizes PU subspaces in terms of weak differentiability of the norm "along a subspace".

It is known [the reviewer, loc. cit.] that the space $L_\infty[0, 1]$ contains no finite-dimensional PU subspace. The author observes that it is unknown, however, whether a separable Banach space necessarily contains a finite-dimensional PU subspace. (It is asserted that the answer is unclear even for $C(S)$, S compact metrizable, but in this case it is easily seen that (for infinite S), there exist PU subspaces $L_n \subset C(S)$ for each $1 \leqq n < \infty$.) The paper concludes with the result (Theorem 11) that the space $L_1(T, \Sigma, \mu)$ contains a Čebyšev subspace of dimension n if and only if (T, Σ, μ) has at least n atoms.

R. R. Phelps (Seattle, Wash.)

Gudiev, A. H. 1526

Imbedding theorems for certain classes of abstract functions. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 1014–1017.

The author deals with spaces of functions in n variables which are Lebesgue integrable with mixed powers together with a number of their derivatives, and gives imbedding theorems for such spaces. Theorem 3 is given here as an example. Let $W_p{}^l(R^n)$ be the Sobolev space and let $L^l_{(p_1,p_2)}(R^n)$ be the space of functions $f(x)$ possessing generalized derivatives $D^\alpha f(x)$ with

$$\|f\|_{L^l_{(p_1,p_2)}} = \left\{\int_{R^{n-s}}\left[\int_{R^s}\sum_{|\alpha|=1}|D^\alpha f(x)|^{p_1}dx_{n-s}\cdots dx_n\right]^{p_2/p_1}dx_1\cdots dx_s\right\}^{1/p_2} < \infty,$$

where $x = (x_1, \cdots, x_n)$. When r is an integer, let $W^r_{(p_1,p_2)}(R^n)$ be the space of functions $f(x)$ such that $f \in L^0_{(p_1,p_2)}(R^n)$ and $f \in L^r_{(p_1,p_2)}(R^n)$ (an appropriate definition for non-integer r is also given). A norm in $W^r_{(p_1,p_2)}(R^n)$ is given by

$$\|f\|_{W^r_{(p_1,p_2)}} = \|f\|_{L^0_{(p_1,p_2)}} + \|f\|_{L^r_{(p_1,p_2)}}.$$

Theorem 3 says that if $1 \leqq m \leqq n-1$, $1 < p < p_1 < p_2 < \infty$, $r = l-(n-1)/p + m/p_1 + (n-m-1)/p_2$, then $W_p{}^l(R^{n-1})$ may be continuously imbedded into $W^r_{(p_1,p_2)}(R^{n-1})$. This theorem generalizes some results of S. L. Sobolev and S. M. Nikol'skiĭ. {The paper contains a number of misprints.} *J. Musielak* (Poznań)

Henkin, G. M. 1527

Imbedding the space of s-smooth functions of n variables into a space of sufficiently smooth functions of fewer variables. (Russian)

Dokl. Akad. Nauk SSSR **153** (1963), 57–60.

Let I^n be the unit cube in R^n and let $C^{(s)}(I^n)$ be the space of all s-times continuously differentiable real or complex functions on I^n with norm equal to the sum of the suprema of the derivatives of order at most s. If $p > 0$, let $\langle p\rangle$ denote the closest integer on the right of p. Theorem 1: If $n > m$, and $s > \langle n/m\rangle(1 + \frac{1}{2}\langle n/m\rangle)p$, then there exists a continuous isomorphism of $C(I^n)$ into $C(I^m)$ which maps $C^{(s)}(I^n)$ into $C^{(p)}(I^m)$. Theorem 2: If $s < (n/m)p$, then there is no isomorphism of $C(I^n)$ into $C(I^m)$ which maps $C^{(s)}(I^n)$ into $C^{(p)}(I^m)$. *R. G. Bartle* (Urbana, Ill.)

Srinivasan, T. P. 1528
Doubly invariant subspaces.
Pacific J. Math. **14** (1964), 701–707.

Let X be a locally compact Hausdorff space, m a regular Borel measure on X, $\mathscr{H}$ a separable Hilbert space. Let $L_{\mathscr{H}}^p$ denote the space of weakly measurable $\mathscr{H}$-valued functions whose norms belong to $L^p(dm)$. Let A be a weak-* dense subspace of L^∞. A closed subspace $\mathscr{M}$ of $L_{\mathscr{H}}^p$ (assumed weak-* closed if $p=\infty$) is called "doubly invariant" if it is invariant under multiplication by all functions in A. This terminology originates in the classical case: X the circle, m Lebesgue measure, $\mathscr{H}$ one-dimensional, A the set of trigonometrical polynomials; now $L_{\mathscr{H}}^2$ is isomorphic to (double-ended) l^2, and double invariance means invariance under both left and right shifts. A "measurable range function" is a weakly measurable function P on X whose values are orthogonal projections on $\mathscr{H}$. Such a P defines an operator $\hat{P}$ on $L_{\mathscr{H}}^p$ by $(\hat{P}f)(x)=P(x)f(x)$. In the classical case, a theorem attributed to Wiener asserts that every doubly invariant subspace is of the form $C_E L^2$, where C_E is the characteristic function of the measurable set E. The author gives an extremely short proof of Wiener's theorem, which works word for word in the more general case when X is arbitrary and m is a finite measure. He gives also a proof of the vector version of Wiener's theorem: Every A-invariant subspace of $L_{\mathscr{H}}^2$ is of the form $\hat{P}L_{\mathscr{H}}^2$ for some (essentially uniquely determined) measurable range function P. For $p\neq 2$, he proves only the scalar case of this theorem. The author extends his results from the circle to the line by a method which is applicable to the case of m σ-finite.

A. Browder (Berkeley, Calif.)

Hasumi, Morisuke; Srinivasan, T. P. 1529
Doubly invariant subspaces. II.
Pacific J. Math. **14** (1964), 525–535.

With the same notations as in the previous review [#1528], the theorem that every A-invariant subspace of $L_{\mathscr{H}}^p$ is of the form $\hat{P}L_{\mathscr{H}}^p$ for some measurable range function P, determined essentially uniquely, is proved in full generality. A further weakening of the hypothesis is also made; the authors show that if A is a conjugate-closed algebra of bounded continuous functions such that $A|K$ is dense in $L^2(dm|K)$ for every compact K, then A is weak-* dense in $L^\infty(dm)$. The authors also prove the following: Let $\hat{T}$ be a continuous (weak-* continuous if $p=\infty$) operator on $L_{\mathscr{H}}^p$, commuting with multiplication by functions in A. Then $\hat{T}$ operates pointwise, i.e., there is a function $T(x)$ defined a.e. on X whose values are operators on $\mathscr{H}$, uniformly bounded and weakly measurable, such that $(\hat{T}f)(x)=T(x)f(x)$ a.e. (a.e. on compact sets if $p=\infty$). This theorem was known for $L_{\mathscr{H}}^2$, but existing proofs required the assumption that $L_{\mathscr{H}}^2$ be separable. *A. Browder* (Berkeley, Calif.)

Albrycht, Jerzy 1530
The theory of L^p spaces with mixed norm for finitely additive set functions. (Polish)
Uniw. Adama Mickiewicza w Poznan. Prace Wydz. Mat. Fiz. Chem. Ser. Mat. No. 1 (1963), 51 pp.

This article contains the proofs of results announced earlier [Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 681–685; MR **28** #4326], as well as some additional results. It is first shown that the linear space K_ν of all real-valued set functions on the product of n Boolean algebras of sets, finitely additive in each variable separately and with bounded variation, is a conditionally complete vector lattice. Then a normed space $V^p\subset K_\nu$, $p=(p_1,\cdots,p_n)$, of set functions is considered, with the natural definition of p norm, and its properties are studied; the results generalize some theorems of S. Bochner [Ann. of Math. (2) **40** (1939), 769–799; MR **1**, 110], S. Leader [ibid. (2) **58** (1953), 528–543; MR **15**, 326], and A. Benedek and R. Panzone [Duke Math. J. **28** (1961), 301–324; MR **23** #A3451]. Then a space R^p of set functions, whose density has a finite iterated Riemann integral of mixed power, is shown to be dense in V^p. Finally, the space L^p is related to V^p.

R. Carroll (Urbana, Ill.)

Il'in, V. P. 1531
Some inequalities for differentiable functions of several variables.
Dokl. Akad. Nauk SSSR **135** (1960), 779–782 (*Russian*), *translated as Soviet Math. Dokl.* **1** (1961), 1310–1313.

The author proves a number of inequalities for functions belonging to the space $W_p^{(l)}$ of S. L. Sobolev [*Some applications of functional analysis in mathematical physics* (Russian), Izdat. Leningrad. Gos. Univ., Leningrad, 1950; MR **14**, 565] in a region D of n-dimensional space, where l is an arbitrary positive real number, and $p\geqq 1$.

Y. M. Chen (Zbl **103**, 83)

Ding, Shia-shi 1532
Some properties of a class of functional spaces and their applications.
Acta Math. Sinica **10** (1960), 316–360 (*Chinese*); *translated as Chinese Math.* **1** (1962), 328–373.

Let E_n be n-dimensional Euclidean space, and let $D_p^{(l)}$ be the space of all functions φ on E_n such that $\varphi\in L_p(E_n)$ and $r^l\varphi\in L_p(E_n)$, where $r=(x_1^2+\cdots+x_n^2)^{1/2}$, $l>0$ and $1\leqq p<\infty$. The author obtains various criteria for the imbeddability of $D_p^{(l)}$ in spaces of the form $L_q(E_n)$ and $D_q^{(m)}$; the natural relation between $D_p^{(l)}$ and the Sobolev spaces $W_q^{(m)}$ is then discussed. Extension of the imbedding results are made by replacing L_p by an Orlicz space, and complete proofs are given of theorems announced earlier by the author [Sci. Record (N.S.) **1** (1957), 315–318; MR **20** #6653; ibid. (N.S.) **2** (1958), 59–63; MR **21** #2907]. Imbedding theorems are also given for a class of sequence spaces. *D. E. Edmunds* (Cardiff)

Ding, Xia-xi [Ding, Shia-shi] 1533
A note on my paper "On some properties of a class of functional spaces and their applications."
Acta Math. Sinica **12** (1962), 107–108 (*Chinese*); *translated as Chinese Math.* **3** (1963), 113–114.

The author generalises an imbedding theorem given in the paper reviewed above [#1532]. *D. E. Edmunds* (Cardiff)

Robinson, Abraham 1534
On generalized limits and linear functionals.
Pacific J. Math. **14** (1964), 269–283.

The following continuous linear functionals F on the space

m of all bounded sequences $\sigma=\{s_n\}$ of real numbers are examined: $F(\sigma)=\text{s.p.}(\sum_{n=1}^{\infty}\alpha_n s_n)$, where α_n are elements (satisfying some additional hypotheses) of a non-standard model R of the field of real numbers, $\sum_{n=1}^{\infty}$ is taken in R, and s.p. is an abbreviation for "the standard part". A necessary and sufficient condition is formulated for the functional F to have the property $F(\sigma)=\lim_{n\to\infty}s_n$ for every convergent sequence σ. Also the case is examined when F has the properties: $F(\sigma)\geqq 0$ for all non-negative σ and $F(\{s_n\})=F(\{s_{n+1}\})$. It is proved that there exists a non-standard model R such that every continuous linear functional on m is of the above form. This non-standard model can be constructed by the ultraproduct method.

R. Sikorski (Warsaw)

Grisvard, P. 1535

Identités entre espaces de traces.

Math. Scand. **13** (1963), 70–74.

The following identity between "trace spaces" is proven:

$$T_j{}^m(p_0,\alpha_0,A_0;p_1,\alpha_1,A_1)=T_0{}^1(p_0,\alpha_0+j,A_0;p_1,\alpha_1+j-m-1,A_1),$$

with equivalence of norms, where A_0, A_1 are Banach spaces and $m,j,p_0,\alpha_0,p_1,\alpha_1$ are parameters, m, j integers; $m>j\geqq 0$; $1\leqq p_0,p_1\leqq\infty$; $1/p_0+\alpha_0+j>0$; $1/p_1+\alpha_1+j<m$. This result was, by a different method, first obtained by Lions and the reviewer (cf. Inst. Hautes Études Sci. Publ. Math. No. 19 (1964), 5–68, in particular, Chapter VI [MR **29** #2627], where also the definition of these spaces can be found), except for a certain exceptional case of the parameters ($1/p_0+\alpha_0=\text{integer}\leqq 0$); this remaining exceptional case is thus now also settled.

J. Peetre (Lund)

Gobert, J. 1536

Espaces nucléaires et problèmes aux limites.

Bull. Soc. Roy. Sci. Liège **33** (1964), 5–12.

Soit Ω un ouvert borné à frontière régulière dans l'espace euclidien E_n et désignons par $V(\Omega)$ l'espace de Hilbert des fonctions qui sont dans L_2 ainsi que leurs dérivées premières (au sens des distributions), muni du produit scalaire $(u,v)_{V(\Omega)}=(u,v)_{L_2}+\sum_{i=1}^n(D_{x_i}u,D_{x_i}v)_{L_2}$. Soit $\mathring{V}$ l'adhérence dans $V(\Omega)$ des fonctions indéfiniment différentiables à support compact dans Ω. Soit N_p l'espace de Hilbert des fonctions $u\in L_2$ qui satisfont à $\Delta^j u\in L_2$, $D_{x_i}\Delta^j u\in L_2$ $(i=1,\cdots,n;\ 0\leqq j\leqq p)$ et soit à (1) $\Delta^j u\in\mathring{V}$ $(0\leqq j\leqq p-1)$, soit à

$$(2)\qquad(\Delta^{j+1}u,v)_{L_2}+\sum_{i=1}^n(D_{x_i}\Delta^j u,D_{x_i}v)_{L_2}=0$$

pour tout $v\in V$ et $0\leqq j\leqq p-1$, muni du produit scalaire

$$(u,v)_{N_p}=\sum_{j=1}^p\left[(\Delta^j u,\Delta^j v)_{L_2}+\sum_{i=1}^n(D_{x_i}\Delta^j u,D_{x_i}\Delta^j v)_{L_2}\right].$$

L'auteur montre que l'espace $N_\infty=\bigcap_{p\geq 1}N_p$ muni de la famille de semi-normes $\|u\|_{N_p}$ est nucléaire.

J. Horváth (Nancy)

Sz.-Nagy, Béla; Foiaş, Ciprian 1537

Une caractérisation des sous-espaces invariants pour une contraction de l'espace de Hilbert.

C. R. Acad. Sci. Paris **258** (1964), 3426–3429.

The authors announce that all proper closed subspaces invariant under a contraction T acting on a Hilbert space are obtainable by factorizations of the characteristic function of T introduced in the authors' previous paper [same C. R. **256** (1963), 3236–3238; MR **26** #6781] if T satisfies a suitable condition. The authors also indicate that the condition is satisfied if $T^n h$ does not converge strongly to zero for a vector h and $T^{*n}k$ does not converge strongly to zero for a vector k, that is, $T\in C_{11}$ in the sense of the authors [ibid. **256** (1963), 3413–3415; MR **26** #6782].

M. Nakamura (Osaka)

Gil de Lamadrid, Jesús 1538

Some simple applications of the closed graph theorem.

Proc. Amer. Math. Soc. **15** (1964), 509–510.

Let E be a Banach under a norm $\|\ \|_1$ and F be a linear subspace of E such that F is a Banach space under a second norm $\|\ \|_2\geqq k\|\ \|_1$ for some $k>0$. Let T be a bounded linear operator on E into E such that $TF\subsetneqq F$. Then [cf. A. Wilansky and K. Zeller, Michigan Math. J. **6** (1959), 349–357; MR **22** #1804], by the closed graph theorem, it is proved that the restriction of T to F is continuous as an operator on F into F. This implies, as a particular case, the following situation: Let T be a bounded linear operator on $L^2(0,1)$ into itself such that if $x(t)\in L(0,1)$ is a continuous function, so is $(Tx)(t)$. Then the restriction of T to $C[0,1]$ is a bounded linear operator on $C[0,1]$ into itself.

K. Yosida (Tokyo)

Berberian, S. K. 1539

The numerical range of a normal operator.

Duke Math. J. **31** (1964), 479–483.

The closure of the numerical range of a hyponormal operator is conjectured to coincide with the closed convex span of its spectrum. This is suggested by a new proof of the known result for normal operators.

L. de Branges (Lafayette, Ind.)

Jakubov, S. Ja. 1540

Hilbert-Schmidt theory for J-symmetrizable operators acting in a Banach space. (Russian. Azerbaijani summary)

Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Mat. Tehn. Nauk **1961**, *no.* 1, 39–48.

Let E be a complex Banach space and let A be a linear (generally speaking, unbounded) operator acting in E with domain $D(A)$ dense in E. Suppose there exists a linear positive operator J acting from E to E^* such that $D(J)\supset D(A)\cup R(A)$ and $(x,JAy)=\overline{(y,JAx)}$ for all $x,y\in D(A)$ (i.e., A is J-symmetrizable). Denoting the resolvent of A by $R(\lambda,A)$, suppose that one of the following conditions is satisfied: if $(x,JAx)\geqq 0$ for all x, then $R(\lambda,A)$ converges weakly to zero as $\lambda\to+\infty$; if $(x,JAx)\leqq 0$ for all x, then $R(\lambda,A)$ converges weakly to zero as $\lambda\to-\infty$; $R(\lambda,A)$ converges weakly to zero as $|\lambda|\to\infty$. Under the additional assumptions that $JA\neq 0$, that the spectrum of A excluding the zero point consists solely of proper values of finite multiplicity having no finite limit point other than zero, and that $(x,Jx)\neq 0$ for all proper elements x of A, a Hilbert-Schmidt theory is developed for A, i.e., the following are proved: existence of at least one non-zero proper value, reality of the proper values, orthogonality of the proper elements corresponding to different proper

values (in the sense that $(x_i, Jx_k)=0$), extremal properties of the proper values and proper elements with regard to the functional (x, JAx), and expansion of Ax in terms of the proper values and proper elements. Two theorems give criteria for the weak or strong convergence of this expansion. *R. C. Gilbert* (Fullerton, Calif.)

Sine, R. C. 1541

Spectral decomposition of a class of operators.

Pacific J. Math. **14** (1964), 333–352.

The paper is devoted to an analysis of conditions which imply a spectral decomposition for operators in Banach spaces. Let X be a complex Banach space and let T be a bounded linear operator $T: X \to X$. Before going into the details given by the author we consider the structures at hand on which one may place restrictive hypotheses. First, there is the spectrum $\sigma(T)$ of T. Here, a stipulation clearly called for is that it should be nowhere dense, although that alone will not be sufficient. Another condition is that T should not have a point spectrum unless X is reflexive, in which case the point spectrum can be handled by an ergodic type of argument. Next, consider the operational calculus $\phi: f \to f(T)$, where f is analytic on some neighborhood of $\sigma(T)$. Let $\mathscr{F}(T)$ denote the class of analytic functions f for which ϕ is defined. Considered as continuous functions over $\sigma(T)$, it will be desirable to introduce a norm into the class and complete it in this norm, thus obtaining an algebra A of functions. In the classical case (self-adjoint and unitary operators) the completion yields the algebra of all continuous functions on $\sigma(T)$, namely $C(\sigma(T))$. (It is wise to replace $\sigma(T)$ by an overset M. For example, in the case of unitary operators, M is the unit circle.) However, it is known that other algebras play a central role in spectral analysis, for example, the algebra W_0 of absolutely convergent trigonometric series. Another condition to consider on A is that it should be dense in $C(M)$. We now have a homomorphism $\phi: A \to B(X)$, the algebra of all operators on X. A condition to be imposed here is that ϕ be bounded. In order to obtain projections (not necessarily bounded), a delicate hypothesis on the ideals of A seems called for. The one here invoked by the author can be described in the language of spectral synthesis as follows: A is an N^* algebra if each ideal I with a one-point hull is an ideal of spectral synthesis. This means that if I is contained in only one maximal ideal, I is maximal. The condition is used as follows. Suppose $f \in A$ and $f(t)=0$ for $t \in M$. Then f can be approximated in the norm of A and, hence uniformly, by functions g in A which vanish in a neighborhood of t. This leads to projections. Assuming for the sake of convenience that $\sigma(T) \subset [0, 1]$, call the projections $P(t)$. In order to obtain an integral representation for T, the author hypothesizes the uniform bound of the $P(t)$. Then, writing $S=\int t\,dP(t)$, it may be shown that $T-S$ is quasi-nilpotent. The development outlined above includes: the classical operator theory in Hilbert space; the weakly periodic operators considered by the reviewer (here called operators of bounded growth); the well-bounded operators of Smart [J. Austral. Math. Soc. **1** (1959/60), 319–333; MR **23** #A3462]; the spectral operators of Dunford.

We give some details of the development. The sets M of the complex plane which are considered are R-sets, that is, are compact, nowhere dense, and the uniform closure of the family $R(M)$ of rational functions with poles off M is $C(M)$. A is assumed regular. $\mathscr{F}(M)$ is dense in A. Thus A is inverse closed and the maximal ideal space of A is M. A is called a pattern and its image under ϕ is called a pattern calculus. Important to the development is the extension theorem: Let K be closed in M and suppose f in A does not vanish on K. Then there exists a function g in A which agrees with f on a neighborhood of K and does not vanish on M. This leads to the spectral identification theorem: For f in A, let $S_f=\{t: f(t)\neq 0\}$ and let Cl denote closure (thus Cl S_f is the support of f). Let $x \in X$ and let $\sigma(x)$ denote the spectrum of x. Then $\mathrm{Cl}[\sigma(x) \cap S_f] \subset \sigma(f(T)x) \subset \sigma(x) \cap \mathrm{Cl}\, S_f$. This leads to many important corollaries including the spectral mapping theorem: For $f \in A$, $\sigma(f(T))=f[\sigma(T)]$. The theorem on the closure property states: Let K be closed in M and let X_K denote the manifold of $x \in X$ such that $\sigma(x) \subset K$. Then X_K is a closed subspace of X. Using the N^* hypothesis, the author obtains an approximation which he denotes as the Lorch approximation theorem: For each $\varepsilon>0$ and each $z \in \sigma(T)$ there is a $\delta>0$ such that $|(T-z)x| \leq \varepsilon K|x|$ for vectors x with $\sigma(x)$ lying in a δ-neighborhood of z. Here K is the norm of ϕ. The density theorem states that if z belongs to the continuous spectrum, the set of x such that $z \notin \sigma(x)$ is dense in X. Supposing now that T has a purely continuous spectrum in the unit interval $[0, 1]$, it is shown that there exists a densely defined projection $P(t)$ "reducing" T and having a closed extension. An operator T is said to be of polynomial growth of order q if $|T^n| \leq K|n^q|$ for some K. As mentioned before, if $q=0$, the author calls it of bounded growth. A function $f=\sum c_n \exp(int)$ is said to belong to class W_q if $|c_n n^q| < \infty$. The classes W_q play a role in the study of transformations of polynomial growth (although only W_0 is an N^* algebra). An extension of the approximation theorem is given for transformations of polynomial growth. Finally, making the assumption that the projections $P(t)$ are uniformly bounded (here $\sigma(T) \subset [0, 1]$), the author discusses integral representations. *E. R. Lorch* (New York)

Bartle, Robert G. 1542

Spectral localization of operators in Banach spaces.

Math. Ann. **153** (1964), 261–269.

If T is a bounded linear operator on a complex Banach space X, if $R(\lambda)$ is the resolvent of T, and if x is a vector in X, the function $R(\lambda)x$, where λ is a complex parameter, is an analytic vector-valued function. The study of the properties of this function extended to its maximal domain (which clearly may include the resolvent set of T as a proper subset) has been undertaken by Dunford [Pacific J. Math. **2** (1952), 559–614; MR **14**, 479]. This function will be denoted by $\tilde{x}(\lambda)$. Several studies have been made of the growth of the resolvent $R(\lambda)$ as λ approaches the spectrum $\sigma(T)$ of T. The present author obtains estimates on the growth of $\tilde{x}(\lambda)$ as λ approaches a point of the spectrum $\sigma(x)$ of x, which, as we have just noted above, may be a proper subset of $\sigma(T)$. Thus information is given on the behavior of $\tilde{x}(\lambda)$ for values of λ which may be in $\sigma(T)$. For convenience, T is assumed to have a real spectrum, although it would be sufficient to assume that $\sigma(T)$ lies on a smooth arc. The fundamental assumption on T is the growth condition (G_n): There exists a K such that $|\mathrm{Im}\,\lambda|^n|R(\lambda)| \leq K$ for $0<|\mathrm{Im}\,\lambda| \leq 1$. (Here "Im" stands for "imaginary part of".) The case in which T satisfies (G_1) (this is the classic case in Hilbert space) receives particular attention. The first theorem, from which a good part of the further results flows, is the following.

Let T satisfy (G_n) and let $d>0$. Then there exists $c>0$ such that if a is a real number and x is a vector whose spectrum $\sigma(x)$ does not intersect $(a, a+d)$, then $|\tilde{x}(\lambda)| \leqq 2K(1+d)^n d^{-n}]|x|\ |\lambda-a|^{-n}$ for λ belonging to the half-neighborhood $0<|\lambda-a|<c$, $\operatorname{Re}\lambda>a$. The proof is ingenious but requires only elementary types of estimates. If $n=1$, more precise conclusions can be reached. In this case the author shows that $|\tilde{x}(\lambda)| \leqq 4K|x|\ |\lambda-a|^{-1}$ for all λ satisfying $0<|\lambda-a|<d/3$, $\operatorname{Re}\lambda>a$. The above results are uniform in x. If F is a closed set in the complex plane, let $\mathfrak{M}(F)$ represent the linear manifold of all x such that $\sigma(x)$ lies in F. The author proves that if T satisfies (G_n) and F and G are disjoint closed sets of numbers, then there exists a Q such that for all $x\in\mathfrak{M}(F)$ and $\lambda\in G$ one has $|\tilde{x}(\lambda)| \leqq Q|x|$. Using this he shows: $\mathfrak{M}(F)$ is a closed linear manifold invariant under T and under any operator commuting with T, and the restriction T_F of T to $\mathfrak{M}(F)$ satisfies $\sigma(T_F)\subset\sigma(T)\cap F$. Next he studies elements x having a one-point spectrum: If T satisfies (G_n), $\sigma(x)=\{\lambda\}$ if and only if $(\lambda I-T)^n x=0$. Then the author makes the first step in the direction of obtaining an integral representation of T, which implies that in some sense $(T-\lambda I)x$ should be small if the spectrum of x lies near λ: If T satisfies (G_n) and if $\varepsilon>0$, there exists a $\delta>0$ such that if J is an interval of length less than δ, if $\lambda\in J$, and x is such that $\sigma(x)\subset J$, then $|(\lambda I-T)^n x| \leqq \varepsilon|x|$. If $n=1$, the author obtains a theorem of a type given by the reviewer and recently by Leaf [ibid. **13** (1963), 141–155; MR **27** #589; see also the preceding review of the article by R. C. Sine, #1541]. Let T satisfy (G_1) and let $\varepsilon>0$. If J is an interval of length at most $\varepsilon/3K$, if $\lambda\in J$, and if $\sigma(x)\subset J$, then $|(\lambda I-T)x| \leqq \varepsilon|x|$. Finally the author obtains an estimate on the "angle" between the manifolds $\mathfrak{M}(F_1)$ and $\mathfrak{M}(F_2)$, where F_1 and F_2 are disjoint intervals, in terms of their distance apart. In general, the angle becomes small when the distance does. However, the introduction of a uniform boundedness (away from zero) condition makes possible the development of a complete projection theory. *E. R. Lorch* (New York)

Chacon, R. V.; Krengel, U. 1543

Linear modulus of a linear operator.

Proc. Amer. Math. Soc. **15** (1964), 553–559.

Simplifying and extending a theorem of N. Dunford and the reviewer [cf. J. Rational Mech. Anal. **5** (1956), 129–178; MR **17**, 987], the authors prove that if T is a bounded operator of an L_1-space with underlying measure space S, and if we put

$$|T|f(s) = \sup_{|g|\leqq f} |Tg|(s), \qquad s\in S,$$

for $f\geqq 0$, then $|T|$ extends to a linear operator of norm no larger than the norm of T. Various algebraic and topological properties of this "modulus operator" and of the space of linear operators on an L_1-space are also discussed. It is shown that $|T|$ and T admit the same subsets of the measure space S as invariant subsets. A number of generalized ergodic theorems are given as applications. *J. T. Schwartz* (New York)

Kužel', O. V. [Kužel', A. V.] 1544

Spectral decomposition of quasi-unitary operators of arbitrary rank in a space with indefinite metric. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 430–433.

Author's summary: "A triangular model is constructed of an arbitrary K_r-operator $(0<r<\infty)$, non-expanding on a negative subspace, acting in a space with an indefinite metric Π_κ, and the condition of completeness is obtained for a system of radical vectors of a K_r-operator non-expanding throughout the whole of the space Π_κ."

Mlak, W. 1545

Some prediction theoretical properties of unitary dilations.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 37–42.

Let H be a complex Hilbert space, let T be a contraction in H, and let U be the (minimal) unitary dilation of H, operating in the dilation space $K\supset H$ [see, e.g., I. Halperin, Duke Math. J. **28** (1961), 563–572; MR **24** #A2853]. Let S be any closed subspace of H, let $M_+(S)$ be the span of U^nS $(n=0,1,2,\cdots)$, and $R_+(S)=\bigcap_0^\infty U^nM_+(S)$. The author gives some sufficient conditions for $R_+(S)=\{0\}$. For example, it suffices that $T^{*n}f\to 0$ for all $f\in S$. Some of the more complicated conditions contain the Radon-Nikodym derivative of $(E(\lambda)f,f)$, where $E(\lambda)$ is the spectral function of U. Integrals of the logarithms of such derivatives occur in prediction theory [cf. H. Helson and D. Lowdenslager, Acta Math. **99** (1958), 165–202; MR **20** #4155]. *N. G. de Bruijn* (Eindhoven)

Kurpel', N. S. 1546

On the approximate solution of non-linear operator equations by the method of Ju. D. Sokolov. (Russian)

Ukrain. Mat. Ž. **15** (1963), 309–314.

The method concerns the solution of the equation (*) $u=Tu$, where u denotes an element of a Hilbert space H and T is a nonlinear operator which is a contraction in the whole of H or in some sphere $R_M=[\|u\|\leqq M]$. In the method under consideration the solution u^* of (*) (in H, or in R_M) is obtained as the limit of the successive approximations $u_n=T(u_{n-1}+\alpha_n)$, $\alpha_n=P(u_n-u_{n-1})$, $n=1,2,\cdots$, where u_0 is arbitrary, and P is a projector operator. By manipulation one sees that each α_n must be determined as the solution of the equation (**) $\alpha_n=PT(u_{n-1}+\alpha_n)-Pu_{n-1}$, and P must be chosen so that the equations (**) are of easy solution, and still certain inequalities are satisfied. The author gives error bounds and particular criteria assuring that $T: R_M\to R_M$ so that the method can be applied. An example is worked in detail. For previous papers on the same subject see Ju. D. Sokolov [same Ž. **9** (1957), 394–412; MR **21** #2888; ibid. **10** (1958), 419–433; MR **21** #977]; E. A. Černyšenko [ibid. **6** (1954), 305–313; MR **17**, 665; Dopovidi Akad. Nauk Ukraïn. RSR **1956**, 10–12; MR **17**, 901], and A. Ju. Lučka [Ukrain. Mat. Ž. **13** (1961), no. 1, 39–52; MR **28** #5307; Dopovidi Akad. Nauk Ukraïn. RSR **1962**, 1149–1153; MR **26** #1720]. *L. Cesari* (Ann Arbor, Mich.)

Fell, J. M. G. 1547

The structure of algebras of operator fields.

Acta Math. **106** (1961), 233–280.

Consider a locally compact Hausdorff space T (the base space). For each $t\in T$ let A_t be a C^*-algebra. A function $t\to x(t)$ with each $x(t)$ in A_t is called an operator field. By a full algebra of operator fields on T is meant a set A of

operator fields on T such that (i) A is a *-algebra under all the pointwise algebraic operations, (ii) for each x in A the mapping $t \to \|x(t)\|$ is continuous and vanishes at infinity, (iii) for each $t \in T$, the set $\{x(t) | x \in A\}$ is dense in A_t, (iv) A is complete in the norm $\|x\| = \sup \|x(t)\|$. Clearly A is a C^*-algebra. Of particular interest is a maximal full algebra—all operator fields $x(t)$ such that $t \to \|x(t)\|$ is continuous and vanishes at infinity on T. Let $\hat{A}$ be the dual space of A—the family of all irreducible *-representations of A in the hull-kernel topology. If each A_t is a simple dual C^*-algebra (the algebra of all compact operators on some Hilbert space), then $\hat{A}$ is Hausdorff. If, in addition, A is maximal, then $\hat{A}$ is homeomorphic to T.

Next, let A be an arbitrary C^*-algebra. The question discussed is whether A is isomorphic to a full algebra of operator fields on some T. In general, this is impossible if all the A_t are required to be primitive. However, it is shown that this is always possible if it is only required that the A_t be primitive on a dense subset of T.

A C^*-algebra A is called homogeneous of order n if every irreducible *-representation of A has the same finite dimension n. Let M_n be the algebra of all complex matrices of order n and let G_n be the group of all automorphisms of M_n of the form $a \to u^{-1}au$, where u is unitary. For these algebras A, A is isomorphic to the family of all continuous cross-sections of a fibre bundle B which vanish at infinity, where B has base space $\hat{A}$, fibre space M_n and group G_n.

A C^*-algebra A is said to have a continuous trace if it is a GCR-algebra whose dual space $\hat{A}$ is Hausdorff where, for each T in $\hat{A}$ there is an element a in A and a neighborhood U of T such that, for all S in U, S_a is a one-dimensional projection. In this case the mapping $S \to \mathrm{Tr}(S_a)$ is continuous on $\hat{A}$ for any $a \in A$ for which $S \to \mathrm{rank}(S_a)$ is a bounded map on $\hat{A}$. Every GCR-algebra has a composition series all of whose quotients are C^*-algebras with continuous trace. A theory is developed for the construction of algebras with continuous trace from algebras A_t of all compact operators on a Hilbert space H_t.

In the final section the author studies in great detail the structure of the C^*-algebra of the group $\mathrm{SL}(2, C)$. It is closely akin to an algebra of continuous functions on a suitable space with values in the algebra of compact operators on a fixed separable, infinite-dimensional Hilbert space. For details, reference must be made to the paper. *B. Yood* (Eugene, Ore.)

Takesaki, Masamichi 1548

On the unitary equivalence among the components of decompositions of representations of involutive Banach algebras and the associated diagonal algebras.

Tôhoku Math. J. (2) **15** (1963), 365–393.

From the introduction: "Let

$$(1) \qquad \varphi = \int_\Gamma^\oplus \varphi(\gamma)\, d\mu(\gamma)$$

be an irreducible decomposition of a representation φ of an involutive Banach algebra $\mathfrak{B}$ over a measure space (Γ, μ). As shown by several authors [Glimm, Ann. of Math. (2) **73** (1961), 572–612; MR **23** #A2066; Mackey, *The theory of group representations* (Lecture notes, Summer, 1955), Dept. of Math., Univ. Chicago, Chicago, Ill., 1955; MR **19**, 117; Trans. Amer. Math. Soc. **85** (1957), 134–165; MR **19**, 752; the author, Tôhoku Math. J. (2) **15** (1963), 79–95; MR **26** #6810; etc.], this decomposition cannot be regarded as a decomposition of the unitary equivalence class of φ into the unitary equivalence classes of $\varphi(\gamma)$ except for some fairly nice cases, since this decomposition is determined only up to unitary equivalence. Some representations can be decomposed in two ways that have no common components as in Mackey [loc. cit.], and two representations of different types can be decomposed into the direct integrals of the same components as in Takesaki [loc. cit]. Therefore one asks what determines the unitary equivalence relation among the components $\{\varphi(\gamma) : \gamma \in \Gamma\}$ of (1). We show in § 1 that the algebraic relation between the commutant $\varphi(\mathfrak{R})' = \mathbf{M}$ of $\varphi(\mathfrak{B})$ and the associated diagonal algebra $\mathbf{A}$ determines completely the unitary equivalence $\mathfrak{R}$ among the components $\varphi(\gamma)$. We can regard $\mathfrak{R}$ as an algebraic invariant of the couple $(\mathbf{M}, \mathbf{A})$. Guichardet [Bull. Soc. Math. France **89** (1961), 77–101; MR **25** #2460] used $\mathfrak{R}$ for the characterization of discrete von Neumann algebras. We study $\mathfrak{R}$ in more general situations. In § 2 we give the definitions of simplicity, smoothness and complete roughness of $\mathbf{A}$ in $\mathbf{M}$ using $\mathfrak{R}$. In § 3 we reduce the study of smooth maximal abelian subalgebras to that of simple ones. In § 4 we show relations between simple or completely rough maximal abelian subalgebras and regular, semi-regular or singular ones as defined by Dixmier [Ann. of Math. (2) **59** (1954), 279–286; MR **15**, 539]. In § 5 we give examples of factors of type II and type III with simple maximal abelian subalgebras and completely rough ones simultaneously respectively."

B. Yood (Eugene, Ore.)

Bear, H. S. 1549

A strict maximum theorem for one-part function spaces and algebras.

Bull. Amer. Math. Soc. **70** (1964), 642–643.

Having generalized Gleason's concept of "part" from function algebras to function spaces [Proc. Amer. Math. Soc. (to appear)], the author now proves the following. Let B be a separating subspace of $\mathscr{C}(X, R)$ containing the unit function. Suppose that $X - \Gamma$ (where Γ is the Shilov boundary) is a single part (= "monomeric" function space), then only constant functions in B assume extreme values on $X - \Gamma$. *R. Arens* (Los Angeles, Calif.)

de Leeuw, K.; Mirkil, H. 1550

Algebras of differentiable functions in the plane.

Ann. Inst. Fourier (*Grenoble*) **13** (1963), *fasc.* 2, 75–89.

Proofs are presented for the results which were announced earlier by the authors [Bull. Amer. Math. Soc. **68** (1962), 411–415; MR **25** #4342]. *P. Civin* (Eugene, Ore.)

Meyers, Norman G.; Serrin, James 1551

$H = W$.

Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 1055–1056.

Let Ω be an open set in R^n and for an integer $m \geqq 0$ and $1 \leqq p < \infty$ let $W^{m,p}(\Omega)$ denote the Banach space of all complex-valued functions u on Ω with distribution derivatives up to order m in $L^p(\Omega)$ and norm $\|u\| = \sum_{|\alpha| \leq m} \|D^\alpha u\|_p$. Let $H^{m,p}(\Omega)$ denote the closure in $W^{m,p}(\Omega)$ of $C^\infty(\Omega) \cap W^{m,p}(\Omega)$. The authors prove (without further assumptions on Ω) that $H^{m,p}(\Omega) = W^{m,p}(\Omega)$. *R. R. Phelps* (Seattle, Wash.)

Putnam, C. R. 1552
Commutators, absolutely continuous spectra, and singular integral operators.
Amer. J. Math. **86** (1964), 310–316.

Let A be a bounded self-adjoint operator on a Hilbert space H with the spectral measure $E(\ )$. Let B be a bounded operator, and let $C=AB-BA$. Let H_a be the subspace of H consisting of all x with $\|E(\lambda)x\|^2$ absolutely continuous in λ. Let M_B be the smallest subspace of H reducing $\operatorname{Im}(B)=(B-B^*)/2i$ and containing the range of C. Let N be the smallest subspace reducing A and containing M_B, $\operatorname{sp}(A)$ the spectrum of A, and d_B the diameter of $\operatorname{sp}(\operatorname{Im}(B))$. The following theorem is proved: If C is self-adjoint with $C\geqq 0$ or $C\leqq 0$, then

(I) $$\pi\|C\| \leqq (d_B/2) \operatorname{meas} \operatorname{sp}(A),$$

(II) $$N\subset H_a.$$

As a corollary it follows that $\|C\|\leqq(2/\pi)\|A\|\,\|B\|$, which is a refinement of a result proved earlier by the author [Acta Math. **106** (1961), 215–232; MR **24** #A3514] and answers a question raised there. This theorem is applied to a singular integral operator of the form

$$Ax(t) = h(t)x(t)+(1/i\pi)\phi(t)\int_a^b \phi^*(s)(s-t)^{-1}x(s)\,ds.$$

T. Kato (Berkeley, Calif.)

Berezanskiĭ, Ju. M. 1553
Spaces with negative norm. **(Russian)**
Uspehi Mat. Nauk **18** (1963), *no.* 1 (109), 63–96.

The author gives a systematic exposition of the ideas underlying certain methods recently used in the theory of boundary-value problems and elsewhere. He starts with a Hilbert space H_0 containing a dense vector subspace H_+, which is itself a Hilbert space under a different inner product, the two norms being related by the inequality $\|u\|_0\leqq\|u\|_+$ for $u\in H_+$. He calls H_+ a "space with positive ('pozitivnaja') norm". {Reviewer's remark: Perhaps this might be less embarrassingly translated as "space of positive order".} For $f\in H_0$ and $u\in H_+$ let $B(f,u)=(f,u)_0$; we can also write $B(f,u)=(If,u)_+$, where I is a continuous linear mapping of H_0 into H_+. We can then define a new inner product in H_0 by the equation $(f,g)_-=(If,g)_0=(If,Ig)_+$, and form the completion H_- of H_0 with respect to it. The space H_- is a "space with negative ('negativnaja') norm". It turns out that H_0 is dense in H_- and that there is a canonical isometry $\mathbf{I}$ mapping H_- onto H_+. The space H_- can be regarded as a space of distributions with H_+ as space of test functions. The isometry $\mathbf{I}$ has a canonical factorization $\mathbf{I}=J\mathbf{J}$, where $\mathbf{J}$ and J are isometries from H_- onto H_0 and from H_0 onto H_+ respectively.

As a first example, the author takes $L_2(G)$ as H_0 and the Sobolev space $W_2^1(G)$ as H_+, G being a domain in Euclidean space with a smooth boundary. In this case, H_- is the completion of $L_2(G)$ under the inner product

$$(f,g)_- = \int_G\int_G K(x,y)f(y)\overline{g(x)}\,dx\,dy,$$

where K is the Green's function for the operator $u\to -\Delta u+u$ under the boundary condition $\partial u/\partial\nu=0$.

The chain $H_+\subset H_0\subset H_-$ can be extended to a doubly infinite chain

$$\cdots\subset H_2\subset H_1\subset H_0\subset H_{-1}\subset H_{-2}\subset\cdots,$$

H_{-k} being a space of distributions over H_k, and there is a corresponding sequence of canonical isometries.

The author sets up tensor product machinery and examines its compatibility with other operations. The following kernel theorem is proved: If H_+ is separable and its embedding in H_0 is quasi-nuclear (i.e., the embedding operator belongs to the Hilbert-Schmidt class), then every continuous bilinear form on $H_0\times H_0$ can be represented by a generalized kernel $\Phi\in H_-\otimes H_-$. The theorem has a kind of converse.

The results are illustrated by applications to various Sobolev spaces and generalizations of Sobolev spaces. Brief indications are given of other ways in which structures of this kind arise naturally, and of extensions beyond the Hilbert space case.

F. Smithies (Cambridge, England)

Gilbert, Richard C.; Kramer, Vernon A. 1554
Trace formulas for powers of a Sturm-Liouville operator.
Canad. J. Math. **16** (1964), 412–422.

Let H_0 be the mth power of the operator defined in $L^2(0,\pi)$ by the differential operator $-d^2/dx^2$ and the boundary conditions $u(0)=u(\pi)=0$. Let $p(x)$ be a real-valued C^{2m-2} function defined on $[0,\pi]$ such that $p^{(j)}(0)=p^{(j)}(\pi)=0$ for j odd and less than $2m-4$. Let H_1 be the mth power of the operator $-d^2/dx^2+p(x)$ defined by the same boundary conditions. Let $\mu_n=n^{2m}$ be the eigenvalues of H_0, and let λ_n be those of H_1 arranged in increasing order. Let $\lambda_n(\varepsilon)=\mu_n+\varepsilon\mu_n^{(1)}+\varepsilon^2\mu_n^{(2)}+\cdots$ be the eigenvalue of $H_0+\varepsilon(H_1-H_0)$ corresponding to μ_n by perturbation. Then it is proved, by contour integration, that

$$\sum_{n=1}^{\infty}(\lambda_n-\mu_n-\mu_n^{(1)}-\cdots-\mu_n^{(s)}) = 0 \quad \text{if } s\geqq 2m.$$

K. Yosida (Tokyo)

Koppelman, Walter 1555a
Spectral multiplicity theory for a class of singular integral operators.
Trans. Amer. Math. Soc. **113** (1964), 87–100.

Pincus, Joel David 1555b
On the spectral theory of singular integral operators.
Trans. Amer. Math. Soc. **113** (1964), 101–128.

A general method of diagonalizing self-adjoint singular integral transformations in $L^2(a,b)$ is obtained by the authors independently. The transformation L considered is of the form

$$L: x(\lambda)\to A(\lambda)x(\lambda)+(\pi i)^{-1}\int_a^b k(\lambda)\bar{k}(\mu)x(\mu)(\mu-\lambda)^{-1}\,d\mu,$$

where the coefficients $A(\lambda)$ and $k(\lambda)$ are bounded, the interval is finite, and the integral is a Cauchy principal value. (The conditions imply that $XL-LX$ has one-dimensional range, where X is multiplication by x in $L^2(a,b)$.) Additional differentiability conditions are made on the coefficients, but these seem to be a convenience in calculation rather than a limitation of the method. As in previous work the solution is expressed in terms of the function

$$E(l,z) = \exp\left\{\frac{1}{2\pi i}\int_a^b \log\frac{f(\mu)-l+k(\mu)}{f(\mu)-l+k(\mu)}\frac{d\mu}{\mu-z}\right\},$$

but it differs from Koppelman [same Trans. **97** (1960), 35–63; MR **22** #9865] and Pincus [Ph.D. Dissertation, New York Univ., New York, 1959] in a complete treatment of spectral multiplicity. (The chief new ingredient is a partial fraction decomposition due to Pincus.) Some of the calculations are valid whenever L and X are given, bounded, self-adjoint transformations in a Hilbert space and $XL-LX$ has one-dimensional range. Does the Koppelman-Pincus method apply in this more general situation?

L. de Branges (Lafayette, Ind.)

Gegelia, T. G. 1556

Properties of n-dimensional singular integrals in the space $L_p(S;\rho)$. (Russian)

Dokl. Akad. Nauk SSSR **139** (1961), 279–282.

The author considers the integral operator

$$K_\phi(x) = \int_S^* k(y,x)\phi(y)\,dSy,$$

where S is an m-dimensional manifold with certain smoothness properties in the Euclidean space E_{m+1}, $k(y,x)=\prod_{i=1}^{m+1}(x_i-y_i)^{\lambda_i}[\sigma(y,x)]^{-\lambda}$, with

$$\sigma^2(y,x) = \sum_{i,j=1}^{m+1} A_{ij}(y)(x_i-y_i)(x_j-y_j),$$

the λ_i are non-negative integers, $\lambda-m=\sum\lambda_i$ is a positive odd number, and where the quadratic form corresponding to the symmetric matrix $\|A_{ij}(y)\|$ is positive definite at each point y of S. The asterisk denotes the principal value of the integral. The first theorem states that K_ϕ is a bounded operator taking the space $L_p(S)$ into itself. This theorem is then used, together with known results, to establish similar theorems for the space $L_p(S;\rho)$, where $\rho(y)$ is, e.g., of the form $\rho(y)=r^{-\alpha}(y,z)$, $0<\alpha<m$, $z\in S$, or $\rho(y)=r^\alpha(y,z)$, $0<\alpha<m(p-1)$, or where $\rho(y)$ is a product of such functions. Here $r(y,z)$ denotes the Euclidean distance between y and z.

D. H. Hyers (Los Angeles, Calif.)

Zaidman, S. 1557

A global existence theorem for some differential equations in Hilbert spaces.

Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 1019–1022.

Consider, on the real line $(-\infty,\infty)$, the differential equation

$$(1) \qquad u' = Au+f,$$

where A is a closed densely defined operator in a Hilbert space H. It is assumed that the resolvent $R(\lambda, A^*)$ of the adjoint A^* of A, for some j, s and M, is "(j,s)-bounded by M" [cf. Agmon and Nirenberg, Comm. Pure Appl. Math. **16** (1963), 121–239; MR **27** #5142] on a doubly infinite sequence of vertical lines $\operatorname{Re}\lambda=\sigma_n$ with $\sigma_n\to\infty$ as $n\to\infty$ and $\sigma_n\to-\infty$ as $n\to-\infty$. This means that $\|R(\lambda,A^*)\|\leqq M$ when $\operatorname{Re}\lambda=\sigma_n$, except on j intervals of length s (depending on n). Then, says the main result of the note, (1) has a (weak) solution $u\in L^2_{\text{loc}}$ for any right-hand side $f\in L^2_{\text{loc}}$. The proof, which is outlined in its main features, depends on an approximation device which has already been used by Malgrange [Ann. Inst. Fourier (Grenoble) **6** (1955/56), 271–355; MR **19**, 280] and Calderón [Fluid Dynamics and Applied Mathematics (Proc. Sympos., Univ. Maryland, 1961), pp. 147–195, Gordon and Breach, New York, 1962; MR **27** #6010] in more concrete cases but goes actually back to Mittag-Leffler. For more details, see the author, Ricerche Mat. **13** (1964), 56–69; ibid. **13** (1964), 118–134 [MR **29** #3910].

J. Peetre (Lund)

Morel, Henri 1558

Introduction de poids dans l'étude de problèmes aux limites.

Ann. Inst. Fourier (Grenoble) **12** (1962), 299–413.

Si tratta di un ampio e interessante studio dei problemi ai limiti, in particolare di quello di Dirichlet, per operatori lineari ellittici e di alcuni problemi di evoluzione, svolto in classi di funzioni "dotate di peso". Sia Ω un aperto di R^n, $\dot\Omega$ la sua frontiera, $(h)=(h_0,h_1,\cdots,h_k)$ una funzione (peso) definita in Ω a valori in R_+^{k+1}, con $h_k^{-1}\in\mathscr{L}^\infty_{\text{loc}}$ e, se $i<k$, con $h_i^{-1}\in\mathscr{L}^\infty_{\text{loc}}$ oppure $h_i\equiv 0$. Sia $(p)=(p_0,\cdots,p_k)$ con $p_i\geqq 1$ e poniamo

$$(1) \qquad \|u\|_{(h),(p)} = \sum_{i=0}^k\left(\int_\Omega h_i\sum_{|\alpha|=i}|D^\alpha u|^{p_i}\,dx\right)^{1/p_i}.$$

Si indicherà con $W^0_{(h),(p)}$ lo spazio di Banach completamento di $\mathscr{D}(\Omega)$ (spazio delle funzioni indefinitamente differenziabili e a supporto compatto in Ω) rispetto alla norma (1): e con $W_{(h),(p)}$ lo spazio delle distribuzioni u su Ω con derivate d'ordine $\leqq k$ misurabili e tali che (1) sia finita, munito della topologia la meno fine tale che l'iniezione in $\mathscr{D}'(\Omega)$ (spazio delle distribuzione su Ω) e la seminorma $u\to\|u\|_{(h),(p)}$ siano continue; porremo, se $p_i=2$, $W^0_{(h),(p)}=W^0_{(h)}$ e $W_{(h),(p)}=W_{(h)}$, e allora si tratta di spazi di Hilbert. Il primo problema che viene posto è quello di sapere quando $W^0_{(h),(p)}$ è uno spazio (necessariamente normale) di distribuzioni su Ω. Esso è ampiamente studiato nel § 2 de cap. I, che risulta essere la parte più interessante del lavoro, sopratutto per alcuni notevoli lemmi di maggiorazione, quali il lemma 2 di pag. 333; la generalizzazione di una classica disuguaglianza di Hardy, il lemma 3 di pag. 343 e la proposizione di pag. 348–349; ci limitiamo per brevità a riportare il lemma 3: sia $A=\sum_{i,j=1}^n a_{ij}D^iD^j$, a_{ij} costanti reali, $\sum_{i,j=1}^n a_{ij}\xi_i\xi_j>0$ e $a(u,v)=\sum_{i,j=1}^n\int_\Omega a_{ij}D^iuD^j\bar v\,dx$ sia una forma sesquilineare hermitiana su $\mathscr{D}(\Omega)\times\mathscr{D}(\Omega)$; sia h una funzione reale positiva, due volte differenziabile con continuità in Ω, tale che $Ah\geqq 0$; allora, se $a>-1$ per ogni $\varphi\in\mathscr{D}(\Omega)$ si ha:

$$(2) \qquad \int_\Omega h^a|\text{grad}_A h|^2|\varphi|^2\,dx \leqq \frac{4}{|a+1|^2}\int_\Omega h^{2+a}|\text{grad}_A\varphi|^2\,dx$$

dove $|\text{grad}_A v|^2=\sum_{i,j=1}^n a_{ij}D^ivD^j\bar v$; se è $Ah\leqq 0$ e $a<-1$ la (2) è ancora valida; se $Ah=0$, $a=-1$, si ha invece

$$(3) \qquad \int_\Omega h^{-1}|\text{grad}_A h|^2|\varphi|^2\,dx \leqq 4\int_\Omega h|\log h|^2|\text{grad}_A\varphi|^2\,dx.$$

Inoltre se Ω ammette un pezzo Γ di frontiera differenziabile con continuità tale che da una parte di Γ e in un suo intorno non ci sono altri punti di $\dot\Omega$, se h si annulla su Γ come la distanza da Γ, allora nelle (2) e (3) le costanti a secondo membro sono le migliori possibili. Utilizzando i lemmi suddetti vengono, sempre nel § 2, ottenuti diversi criteri per stabilire se $W^0_{(h),(p)}$ è uno spazio di distribuzioni su Ω, tra i quali è da rilevare in particolare il seguente (teorema 3 di pag. 364): se h è una funzione sottoarmonica positiva e continua non costante nell'aperto Ω connesso e $(h)=(0,h^\alpha)$, allora, se $\alpha>1$, $W^0_{(h)}$ è uno spazio di distribuzioni e il risultato è ancora vero se h è sopraarmonica e $\alpha<1$. Il secondo problema studiato nel § 3 del cap. I è quello di caratterizzare, nel caso che $W^0_{(h),(p)}\subset\mathscr{D}'(\Omega)$ e dunque $W^0_{(h),(p)}\subset W_{(h),(p)}$, gli elementi di $W^0_{(h),(p)}$ mediante certe condizioni di annullamento su $\dot\Omega$ (il "modo" di annullarsi dipendendo ovviamente dal peso (h)). Si hanno

così i teoremi di tracce e di caratterizzazione del n. 1, § 3 che vengono applicati successivamente per l'interpretazione delle condizioni ai limiti nel problema di Dirichlet. Si arriva infine allo studio dei problemi di Dirichlet e di Neumann per operatori lineari ellittici, svolto nel § 4. Essi vengono trattati mediante il metodo delle proiezioni ortogonali negli spazi di Hilbert $W_{(h)}$, utilizzando la coercività di: certe forme sesquilineari o più in generale un teorema di esistenza per equazioni funzionali di J. L. Lions [*Équations différentielles opérationnelles et problèmes aux limites*, Springer, Berlin, 1961; MR **27** #3935]. Dopo una impostazione e una discussione di carattere generale sono ottenuti diversi interessanti risultati per il problema di Dirichlet relativo agli operatori $-\Delta+\alpha$, $-\Delta-(\alpha/x_n)D^n$, $\sum_{i=1}^n (D^i)^4$, $-\Delta-(a/\rho)(\partial/\partial\rho)$.

Particolarmente notevoli sono i seguenti teoremi. Premettiamo la definizione: sia E uno spazio di distribuzioni d'ordine $-k$ su Ω e v una funzione positiva e k volte differenziabile con continuità in Ω; si indica con vE lo spazio immagine di E nell'applicazione biunivoca $T\to vT$ di $\mathscr{D}'^k(\Omega)$ su $\mathscr{D}'^k(\Omega)$ munito della topologia indotta dall'applicazione stessa. Si ha allora il Teorema 1: Sia θ una funzione armonica positiva non costante nell'aperto Ω connesso; siano a e b costanti tali che $|a-b|<1$, $a+b\neq 1$; allora $W^0_{(0,\theta^{a+b})}$ e il suo duale $(W^0_{(0,\theta^{a+b})})'$ sono spazi di distribuzioni e l'operatore Δ stabilisce un isomorfismo di $\theta^a W^0_{(0,\theta^{a+b})}$ su $\theta^{-b}(W^0_{(0,\theta^{a+b})})'$; se $a+b<1$ e $\theta=h_T$; essendo h_T la funzione di Martin relativa al punto T di $\dot\Omega$, T a distanza finita, $\dot\Omega$ essendo una varietà regolare di dimensione $n-1$ una volte differenziabile con continuità, allora $\theta^a W^0_{(0,\theta^{a+b})}$ è uno spazio di funzioni "nulle" su $\dot\Omega$ nel senso di una opportuna convergenza integrale a zero (cf. def. 1 pag. 380) e verificanti su ogni compatto π del complementare di T in $\dot\Omega$ una condizione di Lipschitz integrale (cf. def. 2, pag. 381) di esponente $(1+(a-b))/2$, questo valore essendo il migliore possibile. Inoltre se é $a=0$, $b=1$ oppure $a=1$, $b=0$ e se $W^0_{(0,\theta)}\subset\mathscr{D}'(\Omega)$ (ciò che avviene se $\theta=h_T$) allora esiste un sottospazio $\mathscr{N}$ di $W^0_{(0,\theta)}$ tale che Δ applica $\theta^a\mathscr{N}$ su $\theta^{-b}(W^0_{(0,\theta)})'$, potendo in certi casi mancare l'unicità della soluzione del relativo problema di Dirichlet. L'altro teorema da segnalare è il Teorema 3: se $\Omega=\{x|x\in R^n, x_n>0\}$ e se $\alpha<1$, $a+b\neq 1$, $-1<a-b<1-2\alpha$, allora l'operatore $-\Delta-(\alpha/x_n)D^n$ è un isomorfismo di $x_n{}^a\, W^0_{(0,x_n{}^{a+b})}$ su $x_n{}^{-b}(W^0_{(0,x_n{}^{a+b})})'$, le funzioni del primo spazio essendo "nulle" su $\dot\Omega$ e verificando una condizione di Lipschitz integrale su ogni compatto di $\dot\Omega$ e anche su tutto $\dot\Omega$ di esponente $(1+(a-b))/2$. Questo ultimo risultato relativo all'operatore $-\Delta-(\alpha/x_n)D^n$, che è dotato di singolarità nel coefficiente di D^n, é particolarmente significativo anche in relazione ai risultati di M. Schechter [Comm. Pure Appl. Math. **13** (1960), 321–328; MR **22** #3872], A. Huber [Proc. Conf. Differential Equations, pp. 147–155, Univ. Maryland Book Store, College Park, Md., 1956; MR **18**, 650], e P. I. Lizorkin [Dokl. Akad. Nauk SSSR **139** (1961), 1052–1055; MR **25** #315] sullo stesso problema.

Nel cap. II del lavoro, molto più breve a meno significativo del cap. I, viene iniziato lo studio delle equazioni di evoluzione mediante la teoria dei semigruppi, introducendovi il concetto di "peso"; le note definizione di R. S. Phillips sugli operatori dissipativi vengono generalizzate per operatori a coefficienti costanti $P(D)$ in opportuni concetti di operatori h-Ω-dissipativi, rispetto all'aperto Ω e al peso h, funzione misurabile positiva in Ω con $h^{-1}\in\mathscr{L}^\infty_{\text{loc}}$ e vengono dati alcuni criteri semplici di h-Ω-dissipatività per certi operatori quali l'operatore di Laplace ($P(D)$ è detto h-Ω-dissipativo se $\operatorname{Re}\int_\Omega [P(D)\varphi]\bar\varphi h\,dx\leq 0$, $\forall\varphi\in\mathscr{D}(\Omega)$). *E. Magenes* (Pavia)

Morel, Henri 1559

Introduction de poids dans l'étude de problèmes aux limites.

Les Équations aux Dérivées Partielles (*Paris*, 1962), *pp.* 123–128. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1963.

This is an exposition of previous work by the author [#1558]. *M. Schechter* (New York)

Boman, Jan 1560

On the propagation of analyticity of solutions of differential equations with constant coefficients.

Ark. Mat. **5**, 271–279 (1964).

Let $P(D)$ be a partial differential operator with constant coefficients and Ω an open set in $\mathbf{R}^n$. For real d write $\Omega_d=\{x;\,x\in\Omega,\,x_n>d\}$. F. John [Comm. Pure Appl. Math. **13** (1960), 551–585; MR **24** #A317] and Malgrange [Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **3 (51)** (1959), 433–440; MR **28** #4396] have proved that if u is a distribution in Ω such that $P(D)u\in C^\infty(\Omega_d)$ and $u\in C^\infty(\Omega_d-F)$, where F is a compact subset of Ω, then $u\in C^\infty(\Omega_d)$ [see also Hörmander, *Linear partial differential operators*, Theorem 3.6.2, Academic Press, New York, 1963; MR **28** #4221]. Using this result the author shows that if $P(D)u$ is real analytic in Ω_d and u is real analytic in Ω_d-F, then u is real analytic in Ω_d. More generally, he proves the following result. Let $\{L_k\}$ be an increasing sequence of positive numbers such that $k\leqq L_k$ and assume that $L_{2k}\leqq CL_k$ for some constant C ($k=1,2,\cdots$). For any open subset ω of $\mathbf{R}^n$ denote by $C^L(\omega)$ the set of all functions $f\in C^\infty(\omega)$ such that for every compact subset K of ω there exists a constant A such that $|D^\alpha f(x)|\leqq A^kL_k{}^k$ if $|\alpha|=k$ ($k=1,2,\cdots$) and $x\in K$. (The choice $L_k=k$ yields the class of real analytic functions.) If $P(D)u\in C^L(\Omega_d)$ and $u\in C^L(\Omega_d-F)$, then $u\in C^L(\Omega_d)$. *J. Horváth* (Nancy)

Pettineo, Benedetto 1561

Equazioni funzionali negli spazi di Hilbert e teoria fredhölmiana.

Atti Accad. Sci. Lett. Arti Palermo Parte I (4) **20** (1959/60), 117–185 (1961).

This somewhat discursive paper deals with some questions concerning bounded linear operators with closed range on a separable (not at all necessary) Hilbert space $\mathbf{H}$ into itself. Let A be a bounded linear self-adjoint operator of $\mathbf{H}$ into itself. Denote by $R(A)$ the range of A and by $N(A)$ the null space of A. The principal result of the first part of the paper is: $R(I-\lambda A)$ is closed if and only if (*) $A=\sum_{i=1}^\infty \lambda_i P_i$, where the P_i are mutually orthogonal projections and the λ_i form a countable discrete set of real numbers; A is completely continuous if and only if it has the form (*) and the $R(P_i)$ are finite-dimensional. These results are obtained in the course of a discussion of the Fredholm alternative for solutions of integral equations of the third kind.

The second part of the paper (Chapter III) contains the result that a bounded linear operator B of $\mathbf{H}$ into itself has the property that $R(I-\lambda B)$ is closed and $\dim N(I-\lambda B)$ is finite for all λ with non-zero real part if $B+B^*$ is completely continuous. The same conclusions hold for all λ if

and only if B is completely continuous. These results depend on the fact that the perturbation of a bounded linear operator on **H** with closed range and finite nullity by a completely continuous operator also has closed range and finite nullity. Application is made to certain singular integral operators.

The author gives elementary, if somewhat lengthy, proofs of all these operator results. No mention is made of several authors who have treated such problems of operators with closed range even in the more general setting of Banach spaces and/or unbounded operators; e.g. Dieudonné [Bull. Sci. Math. (2) **67** (1943), 72–84, MR **7**, 124], Yood [Duke Math. J. **18** (1951), 599–612; MR **13**, 355], Gohberg and Kreĭn [Uspehi Mat. Nauk **12** (1957), no. 2 (74), 43–118; MR **20** #3459; Amer. Math. Soc. Transl. (2) **13** (1960), 185–264; MR **22** #3984], and Kato [J. Analyse Math. **6** (1958), 261–322; MR **21** #6541].

C. R. DePrima (Pasadena, Calif.)

Pettineo, Benedetto 1562

Equazioni funzionali negli spazi di Hilbert e teoria fredhölmiana.

Atti Accad. Sci. Lett. Arti Palermo Parte I (4) **20** (1959/60), 239–255 (1961).

This is a verbatim reprinting of Chapter III of the paper reviewed above [#1561].

C. R. DePrima (Pasadena, Calif.)

Dolph, C. L. 1563

Positive real resolvents and linear passive Hilbert systems.

Ann. Acad. Sci. Fenn. Ser. A I No. 336/9 (1963), 39 *pp.*

This paper is partly original and partly expository, with more than 100 references. In § 1 (the introduction) a broad review of the relationship between the theory of dissipative operators and that of linear passive systems [as formulated by König and Meixner, Math. Nachr. **19** (1958), 265–322; MR **21** #7410] is presented, including comprehensive discussion of related problems in physics and engineering, much in the same style as in the earlier paper of the author [Bull. Amer. Math. Soc. **67** (1961), 1–69; MR **25** #5612]. In § 2, the admittance or black-box concept in physics is discussed. § 3 deals with linear passive systems and the related theory of positive real functions. § 4 is an exposition of the results on dissipative operators in Hilbert space, mostly due to Phillips, including a new proof of the representation theorem for the resolvent of a dissipative operator [the author and Penzlin, Ann. Acad. Sci. Fenn. Ser. A I No. 263 (1959); MR **22** #1818]. § 5 is devoted to the discussion of a system defined by the differential equation $du/dt - Au = f(t)$, where A is a dissipative operator and where f is supposed to be the input and u the output. A sufficient condition is given for this system to be a linear passive system for appropriate choice of the classes of inputs and outputs. There are two appendices on the Dolph-Penzlin theorem and on a theorem of Greenstein.

T. Kato (Berkeley, Calif.)

Rusev, Petr 1564

Similarity transformations of metric spaces and the concept of ε-entropy of a metric space. (Bulgarian)

Fiz.-Mat. Spis. Bŭlgar. Akad. Nauk. **6** (**39**) (1963), 37–39.

This note is concerned with ε-entropy for totally bounded metric spaces in the sense of Kolmogorov: see the review of A. N. Kolmogorov and V. M. Tihomirov [Uspehi Mat. Nauk **14** (1959), no. 2 (86), 3–86; MR **22** #2890] for definitions and notation. Consider metric spaces W and Z with metrics ρ and r, respectively. A mapping ϕ of W onto Z is called a similarity if $r(\phi(a), \phi(b)) = \lambda\rho(a, b)$ for all $a, b \in W$, where λ is a positive constant, called the modulus of ϕ. If A is a totally bounded subset of W, it is obvious that $H_\varepsilon(\phi(A)) = H_{\varepsilon/\lambda}(A)$ for all $\varepsilon > 0$ if ϕ is a similarity. The author proves a partial converse, as follows. Let W be a metric space such that for all $a, b \in W$, there is a midpoint $x: \rho(a, x) = \rho(x, b) = \frac{1}{2}\rho(a, b)$. {The author seems unaware that this property was introduced in 1928 by Karl Menger [Math. Ann. **100** (1928), 75–163].} Suppose that ψ is a one-to-one mapping of W onto Z such that every pair of points a, b in W admits a midpoint x for which $\phi(x)$ is a midpoint between $\phi(a)$ and $\phi(b)$. Suppose also that $\lim_{\varepsilon\to 0}[H_\varepsilon(\phi(A)) - H_{\varepsilon/\lambda}(A)] = 0$ for all totally bounded subsets A of W. Then ϕ is a similarity with modulus λ.

Edwin Hewitt (Seattle, Wash.)

CALCULUS OF VARIATIONS

See also 1116, 1593, 1652, 1731, 2125, 2131.

Graves, Lawrence M. 1565

Extensions of the lemma of Haar in the calculus of variations.

Bull. Amer. Math. Soc. **65** (1959), 319–321.

Siano (1) $L(v) = \int_G \sum_i A_i D^{(i)}v\,dx$ un funzionale lineare, G un insieme aperto e connesso, $D^{(i)}v$ operatori differenziali definiti per v in C^∞ a supporto compatto su G, A_i localmente integrabili su G. Definite le medie integrali

$$M_{x_j h_j} f(u) = \int_0^{h_j} f(x_p, \cdots, x_j + s, \cdots, x_m)\,ds,$$

e gli incrementi

$$\Delta_{x_j h_j} f(x) = f(x_1, \cdots, x_j + h_j, \cdots, x_m) - f(x_1, \cdots, x_j, \cdots, x_m),$$

d'una funzione localmente integrabile, posto $M_h{}^i = \prod_{j=1}^n M_{x_j}{}^{i_j}{}_{h_j}$ dove $\prod$ indica il prodotto operativo, e, analogamente, posto $\Delta_h{}^i = \prod_{j=1}^n A^{i_j}_{x_j h_j}$. 1. Posto

$$H_h A = \sum_i (-1)^{|i|}\Delta_h{}^i M_h{}^{m-i} A_i, \quad \text{con } |i| = \sum_{j=1}^n i_j,$$

si ha allora che la forma (1) si annulla per tutte le funzioni v allora e allora soltanto che $H_h A = 0$ per tutti i parallelipipedi di estremi x e $x + h$ contenuti in G eccetto per quelli nei quali almeno uno dei vertici $(x_j - i_j h_j)$, con $0 \leqq i_j \leqq m_j$ appartenga a un insieme di misura nulla. 2. Se G è di forma semplice (non precisata) e sa a è un punto di G, posto

$$I_{x_j} f(x) = \int_{a_j}^{x_j} f(x_1, \cdots, x_{j-1}, t, x_{j+1}, \cdots)\,dt$$

e definito l'operatore $I^i = \prod_{j=1}^n I_{x_j}{}^{i_j}$, indi l'operatore $RA = (-1)^{|i|} I^{m-i} A_i$, si afferma che $\Delta_h{}^m RA = H_h A$. La condizione $H_h A = 0$, q.o. del teorema precedente prende la forma seguente: RA e uguale, quasi ovunque in G, alla somma di

n funzioni $c_k(x)$, con $c_k(x)$ polinomi di grado non maggiore di m_k in x_k, con coefficienti localmente integrabili rispetto alle rimanenti variabili. L'autore fornisce qualche indicazioni ma non le dimostrazioni complete.

E. Baiada (Zbl **89**, 86)

Graves, Lawrence M. 1566

The Weierstrass condition for multiple integral variational problems involving higher derivatives.

Proc. Amer. Math. Soc. **11** (1960), 750–756.

L'auteur étend, au cas où l'intégrande $f(x, z, D)$ dépend des dérivées Dz d'ordre supérieur, une méthode antérieurement développée [Duke Math. J. **5** (1939), 656–660; MR **1**, 19] pour le cas des dérivées du premier ordre.

Th. Lepage (Zbl **100**, 98)

GEOMETRY

Wylie, C. R., Jr. 1567

★Foundations of geometry.

McGraw-Hill Book Co., New York, 1964. viii + 338 *pp.* $8.50.

Die Zielsetzung des Buches läßt sich am besten mit den folgenden Sätzen des Vorworts beschreiben: "This book has been written primarily for students preparing to become teachers of secondary school mathematics, although it should also be of interest to practicing teachers and to undergraduates majoring in mathematics who wish to review or to extend their background in geometry. Its purpose is to present a careful axiomatic development of certain important parts of elementary euclidean and non-euclidean geometry, and, in so doing, to acquaint the student with the axiomatic method as a general pattern of thought." Der Verfasser hat in der School Mathematics Study Group (SMSG) mitgearbeitet; daher trägt der von ihm gewählte Aufbau der Geometrie Züge, die man aus SMSG-Veröffentlichungen kennt. Nach einer didaktisch sehr geschickten Einführung in die axiomatische Methode (Ch. 1) wird die dreidimensionale euklidische Geometrie axiomatisch begründet (Ch. 2). Als undefinierte Grundbegriffe dienen dabei: Punkt, Gerade, Ebene, Streckenmaß, Winkelmaß, Flächenmaß. Ausdrücklich als undefinierte Grundbegriffe aufgeführt werden allerdings nur Punkt, Gerade, Ebene (S. 46). Daß es sich bei den drei Maßbegriffen ebenfalls um solche handelt, wird leider dadurch verschleiert, daß Postulate 8 (entsprechend Postulaten 13 und 18 für die beiden andern Maßbegriffe) zu je zwei verschiedenen Punkten A, A' die Existenz einer auf der Menge aller Punktmengen $\{X, X'\}$ definierten reellwertigen Funktion fordert, die für $\{X, X\}$ den Wert 0, für $\{X, X'\}$ mit $X \neq X'$ einen positiven Wert und für $\{A, A'\}$ den Wert 1 hat; anschließend heißt es dann: "The number which corresponds to a pair of points in accordance with postulate 8 we call the measure of the distance between the points relative to the unit pair $\{A, A'\}$." Die Zwischen-Beziehung wird mittels der Streckenmaße, die Orthogonalität mittels der Winkelmaße definiert. Das 125 Seiten starke Kapitel entwickelt auf diesen Grundlagen die wesentlichsten Teile der elementaren Geometrie, mit Ausnahme der räumlichen Inhaltsmessung. Die neuerdings auch in der elementaren Geometrie üblich gewordenen Hilfsmittel Vektoren und Spiegelungen werden nicht verwendet. Das in vielen Lehrbüchern für den von A nach B führenden Vektor benutzte Zeichen $\overrightarrow{AB}$ bedeutet in diesem Buch den von A über B führenden Strahl (= Halbgerade). Besondere Beachtung verdienen die beiden letzten Kapitel (Ch. 3, 4). In ihnen wird analog zur Behandlung der dreidimensionalen euklidischen Geometrie eine axiomatische Grundlegung der vierdimensionalen euklidischen Geometrie und der ebenen hyperbolischen Geometrie gegeben. Im Ganzen genommen erscheint das Buch als eine sehr gute Einführung in die geometrische Axiomatik. Allerdings setzt der vom Verfasser eingeschlagene Weg die reellen Zahlen voraus, was trotz der damit erreichten Vereinfachung zu bedauern ist.

G. Pickert (Giessen)

Sloss, James M. 1568

The shape of the sides of an arc.

Israel J. Math. **1** (1963), 217–220.

The point p is in the left neighborhood S^+ of an oriented Jordan arc L with end points L_a and L_b if there exists a point $t_0 \in L$ and an open disc D_0 centered at t_0 for which $D_0 - L$ is the union of two disjoint open sets—one $D_0{}^+$ on the left and one on the right, as we traverse L in the prescribed direction—and $p \in D_0{}^+$ [compare N. I. Mushelišvili, *Singular integral equations* (Russian), first edition, OGIZ, Moscow, 1946; MR **8**, 586; second edition, revised, Fizmatgiz, Moscow, 1962; English transl., Dept. of Supply and Development, Aeronaut. Res. Lab., Melbourne, 1949; MR **11**, 523; Noordhoff, Groningen, 1953; MR **15**, 434]. The envelope of the circles centered on L which pass through L_a or L_b permits one to find the shape of S^+. The author formulates geometrically and analytically the conditions that these circles have such an envelope. An example is given.

Z. Nádeník (Prague)

Benz, Walter 1569

Pseudo-Ovale und Laguerre-Ebenen.

Abh. Math. Sem. Univ. Hamburg **27** (1964), 80–84.

Let $\{A, B, C, \cdots\}$ be a set of elements called "spears" and $\{a, b, c, \cdots\}$ a set of subsets called "cycles". The intersections $a \cap b$ with $a \neq b$ and $|a \cap b| > 1$ are called "traces" $\{\varphi, \psi, \cdots\}$. Parallelism is defined as follows. $A \parallel B$ means: $A = B$ or no cycle contains A and B; and $\varphi \parallel \psi$ means: if $A \in \varphi$ [or $A \in \psi$], then $A \parallel B$ for some $B \in \psi$ [$B \in \varphi$]. The system $\mathfrak{S} = \{A, B, C, \cdots; a, b, c, \cdots\}$ is called a Laguerre plane if the following conditions are satisfied: (LI) (a) If $A \neq B$, then there is a unique $\varphi \equiv (AB)$ with $A, B \in \varphi$. (b) If $A \nparallel B \nparallel C$ and A, B, C not on a common cycle, then $(AB) \parallel (BC)$. (LII) If $A \in a$, $B \notin a$ and $A \nparallel B$, then there is a unique b with $A, B \in b$ and $a \cap b = \{A\}$. (LIII) If $A \notin a$, then there is a unique $B \in a$ with $A \parallel B$. (LIV) There exist C, c with $C \notin c$ and $|c| \geqq 3$.

The classical model for this system of axioms is the euclidean plane, with the oriented lines as spears, the oriented circles as cycles, and with $A \in a$ if and only if A and a are tangent with agreeing orientations in the point of contact. This Laguerre plane is "strict" (Laguerre-Ebene im engeren Sinne) in the sense that $|\varphi| = 2$ for every trace φ.

In the present paper an idea of Blaschke [Arch. Math. Phys. (3) **18** (1911), 132–140] is generalized to yield the following construction for a large class of Laguerre planes, including the classical one. Let A be a three-dimensional

affine space, E a plane in A, g a line $\not\parallel E$, and $\mathfrak{C}$ a "pseudo-oval" in E, i.e., for every point $P \in \mathfrak{C}$ there is exactly one line l in E such that $\mathfrak{C} \cap l = \{P\}$. (For example, if a, b are non-parallel lines in E, the set $(a \cup b) - (a \cap b)$ is a pseudo-oval.) Then the spears are the points of the union $\mathfrak{Z}$ of all lines $x \parallel g$ intersecting E in a point of $\mathfrak{C}$, and the cycles are the plane sections of $\mathfrak{Z}$ not containing such a line x. It is also shown that this Laguerre plane is strict if and only if $\mathfrak{C}$ is an oval, i.e., contains no three collinear points.
P. Dembowski (Frankfurt a.M.)

Bonardi, Maria Teresa 1570
Sulle serie ∞^1 di piani appartenenti alla $V_4{}^6$ di C. Segre.
Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **33** (1962), 31–36.

Soient $V_4{}^6 \subset P_8(\mathbf{R})$ la variété de C. Segre produit cartesien de deux plans projectifs réels et V une variété réelle de dimension trois régulièrement plongée $C^{(1)}$ dans un espace projectif réel $P_8(\mathbf{R})$. Dans cette note on démontre la proposition suivante: "Condition nécessaire et suffisante afin que V soit une ∞^1 de plans de la $V_4{}^6$ c'est que tous ses espaces tangents rencontrent quatre plans de P_8, trois par trois indépendants". *F. Gherardelli* (Genoa)

Szamkolowicz, L. 1571
Remarks on finite regular planes.
Colloq. Math. **10** (1963), 31–37.

This paper continues the work of the author [same Colloq. **9** (1962), 245–250; MR **25** #5441]. An ordered pair $\langle A, \circ\rangle$ is an $A_n{}^3$ algebra if it satisfies $a \circ a = a$, $a \circ b = b \circ a$ and $a \circ (a \circ b) = b$. The algebra may thus be viewed geometrically as a Steiner triple system [Bruck, *A survey of binary systems*, p. 58, Springer, Berlin, 1958; MR **20** #76]. If the algebra also satisfies $\sim R_3(a, b, c) \Rightarrow a \circ (b \circ c) = (a \circ b) \circ c$ then it is called a $PA_n{}^3$-algebra. If $(a \circ b) \circ (c \circ d) = (a \circ c) \circ (b \circ d)$, we have an $AA_n{}^3$-algebra. The author derives relations between the number of elements in the algebra and the number of elements in a minimal set of generators in the cases of $PA_n{}^3$-algebras and $AA_n{}^3$-algebras. The relation of these algebras to the study of projective and affine planes is discussed and multiplication tables for an $A_{13}{}^3$- and an $A_{15}{}^3$-algebra are given.
W. Jónsson (Winnipeg, Man.)

André, Johannes 1572
Über projektive Ebenen vom Lenz-Barlotti-Typ III 2.
Math. Z. **84** (1964), 316–328.

Ist K ein angeordneter Schiefkörper und erklärt man in ihm nach Wahl eines positiven Elements $k \neq 1$ eine neue Multiplikation $\circ$ durch $a \circ b = akb$ falls $a, b < 0$, $a \circ b = ab$ sonst, so erhält man bekanntlich eine nichtdesarguessche affine Ebene, indem man die Paare (x, y) mit $x, y \in K$ als Punkte und die Mengen $\{(x, y);\ y = a \circ x + b,\ x \in K\}$, $\{(c, y);\ y \in K\}$ $(a, b, c \in K)$ als Geraden nimmt. Die aus solchen affinen Ebenen durch Hinzufügen einer uneigentlichen Geraden mit ihren Punkten entstehenden projektiven Ebenen werden als verallgemeinerte Moulton-Ebenen bezeichnet. Hauptergebnis der vorliegenden Arbeit ist nun, daß diejenigen verallgemeinerten Moulton-Ebenen, bei denen k im Zentrum von K liegt, unter den projektiven Ebenen gekennzeichnet sind durch die folgende Eigenschaft: Es gibt ein Punkt-Geraden-Paar (u, U) mit $u \notin U$ derart, daß für jedes Punkt-Geraden-Paar (p, L) die (p, L)-Transitivität genau in den Fällen $p = u$, $L = U$ und $p \in U$, $L = pu$ vorliegt und im Falle $p \in U$, $u \in L$, $p \notin L$ die Gruppe der (p, L)-Kollineationen (bei denen also jede Gerade durch p Fixgerade und jeder Punkt auf L Fixpunkt ist) die Menge $U - \{p, U \cap L\}$ in genau zwei Transitivitätsgebiete zerlegt. *G. Pickert* (Giessen)

Parker, E. T.; Killgrove, R. B. 1573
A note on projective planes of order nine.
Math. Comp. **18** (1964), 506–508.

A search carried out on UNIVAC 1206 and SWAC shows that the only projective planes of order 9 having a ternary ring whose additive loop is the elementary abelian group are the four planes known since Veblen and Maclagan-Wedderburn [Trans. Amer. Math. Soc. **8** (1907), 379–388].
P. Dembowski (Frankfurt a.M.)

Levi, F. W. 1574
Order in projective planes.
Calcutta Math. Soc. Golden Jubilee Commemoration Vol. (1958/59), *Part II, pp.* 389–396. *Calcutta Math. Soc., Calcutta,* 1963.

The author discusses the order axioms for projective planes and gives a set of axioms more closely related to the topology of the plane than the usual axioms. The points of a line of a projective plane form a cyclic system, i.e., a set of points satisfying the axioms: (C_1) Every unordered pair of points A, B determines a partition of the points different from A and B into two "segments" without common elements; (C_2) Given four different points of a cyclic system they can always be denoted by A, B, C, D in such a way that $AC \times BD$, $AB \parallel CD$, $AD \parallel BC$ hold. Here $AB \times CD$ is to be read "C and D lie in different segments relative to A and B"; $AB \parallel CD$ is to be read "C and D lie in the same segment relative to A and B". There is also an axiom relating order on one line to that on another.
W. Jónsson (Winnipeg, Man.)

Pickert, Günter 1575
Geometrische Kennzeichnung einer Klasse endlicher Moulton-Ebenen.
J. Reine Angew. Math. **214/215** (1964), 405–411.

Let O, U, V be the origin and the special points on the x and y axes, respectively, of a finite Moulton-plane [W. A. Pierce, Canad. J. Math. **13** (1961), 427–436; MR **23** #A1281] of odd characteristic, then conditions (I), (II), (III) below hold. Conversely, a finite projective plane is a Moulton-plane if there are three non-collinear points, O, U, V, such that: (I) The plane is (V, V) transitive (transitive with respect to V and all lines through V); (II) The group $\Gamma_{O,UV}$ of collineations fixing points of UV and lines through O is commutative; (III) $\Gamma_{O,UV}$ and $\Gamma_{U,OV}$ divide the set of points on OU distinct from O and U into the same two transitivity sets; (IV) $\Gamma_{O,UV}\Gamma_{V,OU}$ is transitive on the points of OV less O and V.

From this a purely algebraic result may be obtained: Let Q be a Veblen-Wedderburn system (left quasi-field) of odd order n. Suppose that there is a group P in the intersection of the right and middle nuclei of Q^* such that P is

commutative and of order $\frac{1}{2}(n-1)$ but P does not contain the multiplicative group of the nucleus of Q. Then Q will permit exactly one composition $(a, b) \to a \cdot b$ such that the elements of Q form a field with respect to the new multiplication and old addition and such that there is a unique automorphism φ of the field with $ab = \varphi(a) \cdot b$. The nucleus of Q is the fixed set under φ and the left nucleus of Q^* is the set of non-zero fixed elements of φ^2.
J. D. Swift (Los Angeles, Calif.)

Seifert, George 1576
An application of covering spaces.
Proc. Iowa Acad. Sci. **70** (1963), 351–353.
Author's summary: "Consider an n-cell containing the origin and suppose its boundary S^{n-1} has the following property: every point on S^{n-1} contains a neighborhood in S^{n-1} such that any ray from the origin intersecting this neighborhood intersects it exactly once. Then it is shown by using the concept of a covering space that any ray from the origin intersects S^{n-1} exactly once."
M. L. Curtis (Houston, Tex.)

CONVEX SETS AND GEOMETRIC INEQUALITIES

Hadwiger, Hugo; Debrunner, Hans 1577
★Combinatorial geometry in the plane.
Translated by Victor Klee. With a new chapter and other additional material supplied by the translator.
Holt, Rinehart and Winston, New York, 1964. vii + 113 *pp.* $3.75.
This book is a translation of the original edition [*Kombinatorische Geometrie in der Ebene*, Inst. Math., Univ. Genève, Geneva, 1960; MR **22** #11310] which was an expanded version of an earlier paper by the same authors [Enseignement Math. (2) **1** (1955), 56–89; MR **17**, 887]. (The two reviews contain detailed descriptions of the material.) At the authors' invitation the translator has added an eleventh chapter ("Further Developments of Combinatorial Geometry", pp. 40–56) that surveys recent developments in problems discussed in the original. For example, many of the problems are generalized to n dimensions. The largest single source of new material is the translator's joint paper with L. Danzer and B. Grünbaum [Proc. Sympos. Pure Math., Vol. VII, pp. 101–180, Amer. Math. Soc., Providence, R.I., 1963; MR **28** #524]. The bibliography has 88 new references (numbered 115–202) to relevant papers published in the last five years. The translation adheres closely to the original text, and the new chapter follows the style of the original in describing related results and suggesting interesting unsolved problems. The authors have good reason to be pleased with the new chapter, the reader must be grateful to the translator for making this excellent book available in English, and the translator should be congratulated for fulfilling his objective (stated in the preface): "··· [to] expose the student to some material that has strong intuitive appeal, is currently of research interest to professional mathematicians, and in which the student himself may discover interesting problems that even the experts are unable to solve".
W. Moser (Montreal, Que.)

Perfect, Hazel; Mirsky, L. 1578
Extreme points of certain convex polytopes.
Monatsh. Math. **68** (1964), 143–149.
Birkhoff's theorem [Univ. Nac. Tucumán Rev. Ser. A **5** (1946), 147–151; MR **8**, 561] states that the extreme points of the doubly stochastic n-by-n matrices or of the convex set

$$S = \left\{X = (x_{ij});\ \sum_{j=1}^{n} x_{ij} = 1,\ \sum_{i=1}^{n} x_{ij} = 1,\ x_{ij} \geqq 0\right\}$$

are the permutation matrices. These are $n!$ in number.
In this paper the authors consider the convex set

$$T = \left\{X = (x_{ij});\ \sum_{j=1}^{n} x_{ij} = 1,\ \sum_{i=1}^{n} \lambda_i x_{ij} = \lambda_j,\ x_{ij} \geqq 0\right\},$$

with $\lambda_i \geqq 0$ for all i. Let $|T|$ be the number of extreme points of T. The authors show, for $n \geqq 3$: (i) if $\lambda_1 = \cdots = \lambda_n > 0$, then $|T| = n!$ (this is Birkhoff's result); (ii) if $\lambda_1 = 0$, $\lambda_2 = \cdots = \lambda_n > 0$, then $|T| = n!$; (iii) if $n = 3$ and $\lambda_1 = 0$, then $|T| = 3!$; (iv) in all other cases $|T| > n!$; and (v) in particular, if $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \cdots = \lambda_n = 1$, then $|T| \geqq (2 - 2n^{-1})n!$.
M. L. Balinski (Princeton, N.J.)

Chakerian, G. D.; Stein, S. K. 1579
On the centroid of a homogeneous wire.
Michigan Math. J. **11** (1964), 189–192.
Let C be a closed convex plane curve. Define the perimeter centroid of C to be the center of mass of C considered as a wire of uniform density. Call a line l passing through a point P of C a normal to C at P provided l is orthogonal to some line of support of C through P. Using purely geometric methods (as opposed to the Fourier series methods employed by other authors to prove certain related results under special smoothness hypotheses on C), it is shown that at least four normals to C pass through its perimeter centroid.
T. A. Botts (Charlottesville, Va.)

Aissen, Michael I. 1580
Quasi-dimensional analysis and approximation formulae.
J. Soc. Indust. Appl. Math. **11** (1963), 342–354.
For real-valued functions f and g defined for some convex sets in euclidean n-space E^n the following analogues of Landau's symbol are defined: $f = \mathbf{O}_c(g)$ [$\mathbf{O}_E(g)$, respectively, $\mathbf{O}_R(g)$] if and only if there is a positive constant M such that $f(D) \leqq Mg(D)$ for every convex body D [n-ellipsoid D, n-rectangle D] for which $f(D)$ and $g(D)$ are defined. If D is a convex body in E^n, there is a vector a and an n-rectangle R such that $R \subset D \subset 4^n D + a$. It follows that if f and g are two translation-invariant and montone functions (one may be increasing, the other decreasing with growing D) at least one of which is positively homogeneous, then $f = \mathbf{O}_R(g)$ implies $f = \mathbf{O}_c(g)$, and so does $f = \mathbf{O}_E(g)$. Hence the three relations are all equivalent.
As applications consider the radius ρ_k of the largest k-sphere contained in D ($k = 1, 2, \cdots, n$). Many functions relevant in geometry or physics are clearly translation-invariant and R- or E-equivalent to a product of powers of these ρ_k. On the other hand, the author shows that two such products which are c-equivalent necessarily coincide. The last two sections are devoted to examples for the case $n = 2$, in particular, to the torsional rigidity.
L. W. Danzer (Göttingen)

Coxeter, H. S. M. 1581
An upper bound for the number of equal nonoverlapping spheres that can touch another of the same size.
Proc. Sympos. Pure Math., Vol. VII, pp. 53–71. *Amer. Math. Soc., Providence, R.I.*, 1963.

Reasons are given for believing that the number of equal nonoverlapping $(n-1)$-spheres that can touch another of the same size (in Euclidean n-space) has the upper bound $2f_{n-1}(x)/f_n(x)$, where the function f is defined recursively by

$$f_n(x) = \frac{1}{\pi}\int_{n-1}^{x} \frac{f_{n-2}(t-2)}{t(t^2-1)^{1/2}}\,dt,$$

with initial conditions $f_0(x)=f_1(x)=1$.

The author summarizes the history of the problem (from the seventeenth century right up to the latest research) with an excellent list of references. The material is arranged so as to be useful to the specialist and interesting to the novice. Some of this information is described in the following paragraphs.

Kepler (1611, misprinted as 1911 on page 71), discovered the cubic close-packing of equal spheres in Euclidean 3-space. To the question "Can a rigid material sphere be brought into contact with 13 other such spheres of the same size?" Gregory said "Yes" and Newton said "No"; 180 years later the issue was settled in favor of Newton.

In any packing of space by equal spheres each sphere is surrounded by a Dirichlet region (or Voronoi polyhedron) consisting of all points which are as near to the center of that sphere as to the center of any other. The density of the packing is defined as the average of the ratio of the volume of a sphere to the volume of the Dirichlet region that surrounds it. Thus Kepler's cubic close-packing is seen to have density $\pi/3\sqrt{2}=0.74048\cdots$.

Schläfli (1852) generalized spherical trigonometry to an arbitrary number of dimensions and defined the remarkable function $f_n(x)$, which enabled him to express the content of a regular simplex in spherical $(n-1)$-space in terms of its dihedral angle. He observed that, when n is odd,

$$f_n(x) = f_{n-1}(x) - \tfrac{1}{3}f_{n-3}(x) + \tfrac{2}{15}f_{n-5}(x) - \cdots,$$

where the coefficients (apart from sign) agree with

$$\tan x = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \cdots.$$

Relations resembling this were given later by Poincaré, Sommerville, Peschl, and Guinand.

Minkowski [J. Reine Angew. Math. **129** (1905), 220–274] proved that in Euclidean n-space with $n=3$, 4, or 5, the densest packing of equal spheres whose centres form a lattice consists of spheres of radius $1/\sqrt{2}$ whose centres have integral Cartesian coordinates with an even sum. Blichfeldt [Math. Z. **39** (1934), 1–15] obtained the corresponding results for $n=6$, 7, and 8. The author described the centres of such spheres (of radius $3/\sqrt{2}$) as having for coordinates nine integers, mutually congruent modulo 3 and satisfying the equations

$$x_1+x_2+x_3 = x_4+x_5+x_6 = x_7+x_8+x_9 = 0 \quad (n=6),$$
$$x_1+x_2+x_3+x_4+x_5+x_6 = x_7+x_8+x_9 = 0 \quad (n=7),$$
$$x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8+x_9 = 0 \quad (n=8).$$

Let $N(\phi)$ denote the maximum number of "caps" ($(n-2)$-spheres) of angular radius ϕ that can be packed on the unit $(n-1)$-sphere. Rankin [Proc. Glasgow Math. Assoc. **2** (1955), 139–144; MR **17**, 523] proved that $N(\phi)=[\sec(\pi-2\phi)]+1$ for $\pi-\operatorname{arcsec} n \leqq 2\phi \leqq \pi$. Davenport and Hajós [Mat. Lapok **2** (1951), 68] proved that $N(\phi)$ cannot take a value between $n+1$ and $2n$, so that $N(\phi)=n+1$ for $\frac{1}{2}\pi < 2\phi \leqq \pi - \operatorname{arcsec} n$. For smaller values of ϕ Rankin obtained inequalities for the value of $N(\phi)$. In the paper under review the author gives reasons for believing that

$$N(\phi) \leqq 2f_{n-1}(x)/f_n(x), \qquad x = \sec 2\phi + n - 2.$$

Leech computed values of $f_n(x)$, which the author uses to establish the following inequalities for $N_n = N(\phi/6)$, which is the maximum number of spheres that can touch another of the same size in Euclidean n-space: $N_1=2$, $N_2=6$, $N_3=12$, $24\leqq N_4\leqq 26$, $40\leqq N_5\leqq 48$, $72\leqq N_6\leqq 85$, $126\leqq N_7\leqq 146$, $240\leqq N_8\leqq 244$.

The author considers the problem in spherical, elliptic, and hyperbolic spaces as well, and concludes with a proof that $2^{(n-1)/2}\pi^{1/2}e^{-1}n^{3/2}$ is an asymptotic expression for the bound named in the title of the paper.

W. Moser (Montreal, Que.)

Usai, Giuseppe 1582
Sviluppi in serie su ellissi e cilindri ellittici. (French summary)
Atti Accad. Ligure **19** (1962), 283–288 (1963).

Given an ellipsoid, expressions in series involving powers of eccentricities are derived for the lateral and total surface areas of the elliptic cylinder circumscribed on the ellipsoid and of the elliptic cylinder inscribed in the ellipsoid whose lateral area is maximal.

P. V. Reichelderfer (Columbus, Ohio)

Zahn, C. T., Jr. 1583
Black box maximization of circular coverage.
J. Res. Nat. Bur. Standards Sect. B **66B** (1962), 181–216.

Given a circular region A of radius R in the euclidean plane, what portion of A can be covered by n circular discs of equal radius r ($r\leqq R$, the discs possibly overlapping and not necessarily contained in A)? Let $F(M_1,\cdots,M_n, r/R)$ be the percentage of area covered, where $M_1,\cdots,M_n$ are the centers of the discs, and $f(n, r/R)$ the maximal value of F when the centers are varied. Using a digital computer (IBM 704) and the methods of "black box maximization" the author searched for "good relative maxima" of F (for $n=2,3,\cdots,10$ and $r/R=i/16$, i integral). Probably for each n and i the highest value found does not differ from $f(n, i/16)$ by more than 0.002. Though some of the configurations calculated by the computer can easily be improved (e.g., Fig. 27 to $f(5,3/8)\geqq 0.7018$ and Fig. 31 to $f(6,3/8)\geqq 0.795$), most of them are surprisingly close to the corresponding exact relative maximal configurations.

Finally some related questions are discussed and a complete solution of the case $n=2$ is given.

L. W. Danzer (Göttingen)

DIFFERENTIAL GEOMETRY

See also 1111, 1113, 1333, 1434, 1461, 1627, 1775, 1983, 1987, 1988, 1995, 1999.

Pylarinos, O. 1584
Sur les surfaces *W* applicables sur des surfaces de révolution.
Ann. Mat. Pura Appl. (4) **63** (1963), 33–69.

If there is a relation $F(K, H)=0$ between the total curvature K and the mean curvature H of a surface in 3-dimensional euclidean space, it is called a surface W. Here the author treats some properties of a surface W_r which is W and is developable on a surface of revolution. Surfaces W_r are classified into two categories W_{r_1}, W_{r_2}. A surface W_{r_1} is defined as one whose curves r_1 developed on the meridians of the surface of revolution are geodesics. The surface W_{r_1} can be generated by a rigid motion of a fixed curve r_1. The methods of finding W_{r_1} and W_{r_2} by integrating certain differential equations are shown. Next it is proved that the surfaces W_{r_2} whose total curvatures are not constant are isothermal surfaces (that is to say, lines of curvatures are isothermal). Finally, some properties of W_r whose relation between K and H are algebraic are proved. *M. Kurita* (Nagoya)

Ding, Chung-hua [Ting, Chun-hua] 1585

On the total curvature of curves. II.

Acta Math. Sinica **13** (1963), 508–514 (*Chinese*); *translated as Chinese Math.* **4** (1964), 553–560.

Author's introduction: "In a previous paper [J. Northwestern Univ. China **1958**, no. 1, 9] the author studied the total curvature of a curve by using broken lines to approximate the curve. The purpose of this paper is to prove a theorem on a curve with a finite number of vertices by an approximation by smooth curves. With the aid of this theorem, Fenchel's theorem and many other known theorems on the total curvature of a smooth curve can be extended to curves with vertices. The method and results in this paper can be extended without trouble to higher-dimensional euclidean spaces. However, for simplicity we shall only consider the case of three-dimensional euclidean spaces. All curves under consideration are sectionally smooth, and have continuous curvature everywhere except at a finite number of points."

Šimko, V. I. 1586

On the structure of non-rigid closed edged surfaces of revolution. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 3 (40), 184–187.

Der Author knüpft an eine Arbeit von B. A. Bublik an [Uspehi Mat. Nauk **18** (1963), no. 2 (110), 121–125; MR **27** #696; betreffs der folgenden Benennungen und Bezeichnung siehe das zitierte Referat] und mittels der Gleichung von Cohn-Vossen beweist er die Existenz der unstarren geschlossenen kantigen Rotationsflächen R_2, welche eine zu der Rotationsachse orthogonale Symmetrieebene besitzen und welche zwei fundamentale Infinitesimalverbiegungen gestatten. Weiter folgen die Konstruktion und die Beispiele solcher Flächen R_2.
Z. Nádeník (Prague)

Lučinin, A. A. 1587

Congruences which are stratified on families of ruled surfaces with constant affine invariants. (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. Geom. Sb. **161** (1962), 65–75.

Dans ce travail, on détermine les congruences de l'espace à trois dimensions, qui sont stratifiées sur des familles de surfaces réglées à invariants affines constants. On utilise la méthode des formes extérieures de Cartan et le repère canonique de Ščerbakov. Tous les cas possibles sont étudiés: celui qui conduit à la solution la plus générale est celui où les surfaces réglées de stratification sont géodésiques affines et la solution dépend alors de deux fonctions arbitraires d'un paramètre. *M. Decuyper* (Lille)

Čepiga, L. G. 1588

On a class of congruences of second-order curves. (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. Geom. Sb. **161** (1962), 39–41.

V. S. Malahovskiĭ [see, e.g., Dokl. Akad. Nauk SSSR **146** (1962), 765–766; MR **27** #1893] studied the congruences of second-order curves, with two focal surfaces degenerating into points, and he constructed their canonical frame $(A_1A_2A_3A_4)$. By Cartan's method of exterior forms and by the theory of systems in involution the author proves the existence (and indicates an arbitrariness in the solution) of any type of these congruences when the surfaces (A_4) degenerate. These special types are also geometrically characterized. *Z. Nádeník* (Prague)

Su, Buchin 1589a

On certain couples of closed Laplace sequences of period four in ordinary space.

Sci. Sinica **13** (1964), 347–374.

Su, Buchin 1589b

On certain couples of closed Laplace sequences of period four in ordinary space.

Acta Math. Sinica **14** (1964), 151–174 (*Chinese*); *translated as Chinese Math.* **5** (1964), 165–193.

Sur une surface N_1, les transformations de Laplace, liées à un réseau conjugué u, v, définissent deux surfaces N_2 (sens v) et N_4 (sens u). Si la transformée dans le sens u de N_4 coïncide avec celle de N_2 dans le sens v, la suite de Laplace est périodique de période 4. L'auteur se propose la détermination des couples de suites fermées $(N_1N_3N_2N_4)$ et $(W_1W_3W_2W_4)$ telles que si N_1 est dans le plan $N_1N_3N_2$, N_3 dans le plan $N_3N_2N_4$ et ainsi de suite, il en soit de même pour les W de même indice, ces deux suites ont au moins en commun la diagonale $N_3N_4=W_3W_4$. Si de plus l'on a $N_1N_2=W_1W_2$ les suites (N) et (W) forment des configurations de Su; dans ce cas à une suite (N) correspond une ∞^1 de (W) et les deux diagonales forment un couple stratifiable. Dans le cas général $N_1N_2\neq W_1W_2$, les diagonales N_1N_2, N_3N_4, W_1W_2 décrivent des congruences W et les foyers P, P' de N_1N_2, Q, Q' de W_1W_2 forment une configuration T de Finikoff $(PQ'QP')$ qui n'est pas une suite fermée de Laplace; étude des congruences W formées par PQ', $P'Q$; les développables PP' et QQ' se correspondent biunivoquement. Démonstration de l'existence de tels couples $(N_1N_3N_2N_4)$, $(W_1W_3W_2W_4)$; à chaque tel couple correspondent des configurations T dont la détermination dépend de quatre fonctions d'un même argument.

{Le deuxième travail revient au même que le premier.}
B. d'Orgeval (Dijon)

Malahovskii, V. S. 1590

On a pair of ruled surfaces in projective space. (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. Geom. Sb. **161** (1962), 11–12.

Dans le travail [mêmes Trudy **160** (1962), 5–14], l'auteur a associé à une congruence de coniques des couples de surfaces réglées. Utilisant ces résultats, il étudie un tel couple, montrant en particulier que ces surfaces appartiennent à une congruence linéaire et que les droites de Demoulin des deux surfaces coïncident deux à deux.

M. Decuyper (Lille)

Vaškas, P. 1591

On the stratification of complexes of lines. (Russian. Lithuanian and French summaries)

Litovsk. Mat. Sb. **3** (1963), *no.* 1, 51–59.

The aim of this paper is to generalize the theory of stratification of complexes of lines by means of linear elements.

The theory of stratification of pairs of complexes is based on the well-known equation $\nabla\rho \equiv d\rho + \rho(\omega_2{}^2 - \omega_1{}^1) - (\rho)^2\omega_2{}^1 + \omega_1{}^2 = 0$ (where ρ denotes the parameter of the point $A_1 + \rho A_2$ on the line $l \equiv A_1A_2$ of a complex (l), $\{A_i\}$ $(i = 1, 2, 3, 4)$ being the moving frame in a three-dimensional projective space) which may be derived geometrically [S. P. Finikov, *Theory of pairs of congruences* (Russian), GITTL, Moscow, 1956; MR **19**, 676]. The author replaces the above equation, which may be written in the form $t^{[1}\nabla t^{2]} = 0$ $(\rho = t^2 : t^1,\ \nabla t^p = dt^p + t^q\omega_q{}^p;\ p, q = 1, 2)$, by the more general one $2t^{[1}\nabla t^{2]} - a^r_{pq\alpha}t^pt^q\omega_r{}^\alpha = 0$ $(\alpha = 3, 4;$ $a^r_{pq\alpha}$ symmetric in $p, q)$.

A. Urban (Prague)

Grincjavičjus, K. [Grincevičius, K.] 1592

On an involutive pair of complexes. (Russian. Lithuanian and German summaries)

Litovsk. Mat. Sb. **3** (1963), *no.* 1, 103–106.

Let the lines $l_1 \equiv A_1A_2$, $l_2 \equiv A_4A_3$ of the moving frame $\{A_i\}$ $(i = 1, 2, 3, 4)$ in a three-dimensional projective space describe a pair of complexes (l_1), (l_2). Then the equations of the pair (l_1), (l_2) may be written in the form

$$\lambda_\alpha{}^p\omega_p{}^\alpha = 0, \qquad [d\lambda_\alpha{}^p + \lambda_\alpha{}^q\omega_q{}^p - \lambda_\beta{}^p\omega_\alpha{}^\beta,\ \omega_p{}^\alpha] = 0,$$

$$\lambda_p{}^\alpha\omega_\alpha{}^p = 0, \qquad [d\lambda_p{}^\alpha + \lambda_p{}^\beta\omega_\beta{}^\alpha - \lambda_q{}^\alpha\omega_p{}^q,\ \omega_\alpha{}^p] = 0$$

$$(p, q = 1, 2;\ \alpha, \beta = 3, 4).$$

The term $\lambda_\alpha{}^p\lambda_p{}^\alpha$ is a relative invariant of the first order. The author studies the pairs of complexes for which $\lambda_\alpha{}^p\lambda_p{}^\alpha = 0$.

A. Urban (Prague)

Kabanov, N. I. 1593

Geometric theory of Carathéodory transformations in the Lagrange problem. (Russian)

Trudy Sem. Vektor. Tenzor. Anal. **11** (1961), 219–240.

The paper deals with local fields of $(n-2)$-dimensional surfaces in an n-dimensional differentiable manifold X_n. It is based on several papers of V. V. Vagner [in particular, Ann. of Math. (2) **49** (1948), 141–188; MR **11**, 253].

The case where X_n is the centro-affine space is discussed first, and the results are then applied to a general manifold X_n with a centro-affine space as local tangent space. It is assumed that the $(n-2)$-tangent planes of the given surfaces do not pass through the local origin and that the Pick density does not vanish. The surfaces are considered as the envelopes of their tangent hyperplanes through the origin, i.e., as given in the form $X^\alpha = t^\alpha(\xi^\lambda, \eta^a)$ in local point coordinates or $y_\alpha = t_\alpha(\xi^\lambda, W, \eta^a)$ in local hyperplane coordinates, where W is a scalar W-density of weight $-2(n-2)^{-1}$. The corresponding hyperplane translations, first introduced by Carathéodory, take the form $X'^\alpha = \varepsilon X^\alpha(1 - \sigma_\beta(\xi^\lambda)X^\beta)^{-1}$ or $y_\alpha' = \varepsilon(y_\alpha - \sigma_\alpha(\xi^\lambda))$ with $\varepsilon = \pm 1$, where the covariant vector field $\sigma_\alpha(\xi^\lambda)$ determines the different translations. The principal result is the establishment of a linear connection $d\eta^a + \Gamma^a = 0$ invariant under these translations, where Γ^a depends linearly on an affinor, a contravariant vector and W-density of weight $-2(n-2)^{-1}$.

H. Busemann (Los Angeles, Calif.)

Rahula, M. 1594

A sequence of derivations on a manifold. (Russian. Estonian and English summaries)

Tartu Riikl. Ül. Toimetised No. 129 (1962), 37–61.

Author's summary: "This paper gives a scheme of construction for successive derivations on the manifold V_n. Let G denote the group of transformations of V_n; G_1 the group of internal automorphisms of G; G_2 the group of internal automorphisms of G_1, etc.; G_i the group of internal automorphisms of G_{i-1}. The one-parametric subgroup of G induces on the manifold V_n a vector field called a derivation of order 0. The one-parametric subgroup of G_1 induces on the V_n a Lie differentiation called a derivation of order 1. The one-parametric subgroup of G_2 induces on the V_n by analogy a derivation of order 2, etc.; the one-parametric subgroup of G_i induces on the V_n a derivation of order i. At the same time each of the derivations of order i is a differentiation over the derivations of order $(i-1)$. A set of the derivations of order i form a Lie algebra L^i with bracket operation and each of the internal automorphisms of the group G^i (i.e., every element of G^{i+1}) assigns an automorphism of L^i. The calculation formulas for the derivations of orders 0, 1 and 2 in the local consideration are also given. These results bear a relation to both the theory of differential invariants and continuation groups of transformations and the theories of adjoint groups, linear connections and also vector fields."

Karrer, Guido 1595

Einführung von Spinoren auf Riemannschen Mannigfaltigkeiten.

Ann. Acad. Sci. Fenn. Ser. A I No. **336/5** (1963), 16 pp.

This is a very clear and elegant exposition of spinors on differentiable manifolds. There is a section on Clifford algebras followed by a study of the Dirac operator. The subject is treated, as it should be, in the framework of modern fibre-bundle theory, and global topological aspects are not ignored.

M. F. Atiyah (Oxford)

Chu, Hsin; Kobayashi, Shoshichi 1596

The automorphism group of a geometric structure.

Trans. Amer. Math. Soc. **113** (1964), 141–150.

In this short paper the authors give a remarkably complete and self-contained discussion of the general question of when the group of automorphisms of a geometric structure on a differentiable manifold is a Lie transformation group. After a rapid historical survey of the known results in this direction they give a proof of the following theorem due to the reviewer. Let G be a group of diffeomorphisms of a

manifold M. Let S be the set of all vector fields X on M which generate global one-parameter groups $\varphi_t = \exp(tX)$ such that each $\varphi_t \in G$. If S generates a finite-dimensional Lie algebra, then G is a Lie transformation group and S is its Lie algebra. They also state a theorem of Bochner which says that the space of vector fields on a compact manifold which satisfies an elliptic system of partial differential equations is finite-dimensional. Using these two theorems they then proceed to give very simple demonstrations of the other results mentioned in their historical survey. Namely, the automorphism groups of the following structures are always Lie transformation groups: the complex analytic structure of a bounded domain (H. Cartan), the metric structure of a Riemannian manifold (Myers and Steenrod), the complex analytic structure of a compact complex manifold (Bochner and Montgomery), the structure given by an affine connection (Nomizu), the structure given by a trivialization of the tangent bundle (Kobayashi), an n-dimensional complex analytic structure with "sufficiently many" square integrable, holomorphic n-forms (Kobayashi), and the almost complex structure of a compact almost complex manifold (Boothby, Kobayashi and Wang). *R. S. Palais* (Waltham, Mass.)

Teleman, Costache 1597

Sur les connexions infinitésimales qu'on peut définir dans les structures fibrées différentiables de base donnée.

Ann. Mat. Pura Appl. (4) **62** (1963), 379–412.

Let X be a differentiable manifold and x_0 a point of X. Let E_{x_0} be the space of piecewise differentiable curves in X starting from x_0 and let Ω_{x_0} be the space of piecewise differentiable loops. With a suitable topology Ω_{x_0} is an H-space, i.e., something like a topological group. E_{x_0} is something like a principal fibre bundle over X with "group" Ω_{x_0}. On the other hand, the universal covering space $\tilde{X}$ of X is a bona fide principal fibre bundle over X with group $\pi_1(X, x_0)$, and there is a natural bundle homomorphism $E_{x_0} \to \tilde{X}$ compatible with the homomorphism of Ω_{x_0} onto $\pi_1(X, x_0)$. The author constructs a representation of Ω_{x_0} onto a topological group, denoted by Γ, which is more faithful than the representation $\Omega_{x_0} \to \pi_1(X, x_0)$ and also a principal fibre bundle $\mathfrak{U}$ over X with group which lies between the two bundles E_{x_0} and $\tilde{X}$, i.e., $E_{x_0} \to \mathfrak{U} \to \tilde{X}$. To construct Γ, the author fixes a tensor field R_{ijkl} such that $R_{ijkl} = R_{klij}$ and $R_{ijkl} = -R_{jikl}$ and defines an element of area

$$a = \sqrt{(\sum R_{ijkl}\, dx^i \wedge dx^j \cdot dx^k \wedge dx^l)}.$$

Using this element of area, he defines the distance $\rho(f, g)$ between two elements $f, g \in \Omega_{x_0}$ which are homotopic to each other by $\rho(f, g) = \inf_\sigma a(\sigma)$, where $a(\sigma)$ denotes the area of a surface σ joining the two curves f and g. By identifying f and g such that $\rho(f, g) = 0$ he obtains a topological group Γ. A similar construction yields $\mathfrak{U}$. The "Lie algebra" of Γ and the structure equations of Γ, as well as representations of Γ, are studied. The author hopes to obtain more complete results for complex manifolds in a future paper. *S. Kobayashi* (Berkeley, Calif.)

Vranceanu, G. [Vranceanu, Georges] 1598

Sur la décomposabilité et les groupes d'holonomie des espaces A_n.

Publ. Math. Debrecen **7** (1960), 150–157.

Let A_n be a space with an affine connection Γ^i_{jk}. If there is a particular system of coordinates such that the components Γ^i_{jk} $(i, j, k \leqq m)$ are functions only of $x^1, x^2, \cdots, x^m$ and the components Γ^i_{jk} $(i, j, k > m)$ are functions only of $x^{m+1}, \cdots, x^n$, the remaining components being null, then A_n determines two spaces A_m, A_{n-m} such that $A_n = A_m \times A_{n-m}$. If A_m, A_{n-m} are invariant in A_n the space A_n is called decomposable. The paper deals with some theorems referring to the decomposability of spaces A_n. Let $M(\Gamma^i_{jkl}, \Gamma^i_{jkl,p}, \cdots)$ be the matrix formed by the tensor of curvature and its covariant derivatives and let M^i, M_j, denote the matrices that one gets when i denotes the line and $j, k, l, \cdots$ the columns, or j the line and $i, k, l, \cdots$ the columns respectively. Theorem 1: If $A_n = A_m \times A_{n-m}$ and M^i, M_j have maximum rank, then A_n is decomposable. Theorem 2: If there do not exist in A_m and in A_{n-m} fields of parallel contravariant (or covariant) vectors, then the group of motions of A_n equals the direct product of the groups of motions of A_m and A_{n-m}. The theorems are generalized to systems of congruences. The results complete and generalize some theorems of F. A. Ficken [Ann. of Math. (2) **40** (1939), 892–913; MR **1**, 88] and A. E. Liber [Dokl. Akad. Nauk SSSR **66** (1949), 1045–1046; MR **11**, 134]. *L. A. Santaló* (Zbl **101**, 144)

Nguyen-Van-Hai 1599

Sur le groupe d'automorphismes de la variété affine des connexions linéaires invariantes d'un espace homogène.

C. R. Acad. Sci. Paris **258** (1964), 3952–3955.

The author studies the space of invariant affine connections of a homogeneous space G/H and the normalizer $N_G(H)$ as a group of automorphisms of the space of invariant connections. Part of the results (Proposition 1) is closely connected with results of Nomizu [Amer. J. Math. **76** (1954), 33–65; MR **15**, 468] and Wang [Nagoya Math. J. **13** (1958), 1–19; MR **21** #6001.] *S. Kobayashi* (Berkeley, Calif.)

Tsukamoto, Yôtarô 1600

A global property of Riemannian spaces of positive curvature. (Japanese)

Sûgaku **15** (1963), 97–101.

In the proof of the so-called sphere theorem, the following theorem is fundamental. Let M be an even-dimensional, simply connected, compact Riemannian manifold whose sectional curvature $\rho(\mu)$ satisfies $0 < \rho(\mu) \leqq 1$ for every plane μ. Then, for every $p \in M$, we have $d(p, C(p)) \geqq \pi$, where $C(p)$ denotes the cut locus of p. The author gives another proof of this theorem of Klingenberg [cf. Ann. of Math. (2) **69** (1959), 654–666; MR **21** #4445; Comment. Math. Helv. **35** (1961), 47–54; MR **25** #2559]. *S. Kobayashi* (Berkeley, Calif.)

Hicks, Noel 1601

Submanifolds of semi-Riemannian manifolds.

Rend. Circ. Mat. Palermo (2) **12** (1963), 137–149.

In the first three sections, the author gives, in the formalism of covariant differentiation $\nabla_X Y$, a review or a generalization of classical material on a semi-Riemannian manifold (pseudo-Riemannian manifold in S. Helgason's book, *Differential geometry and symmetric spaces* [Academic Press, New York, 1962; MR **26** #2986], which is supposed

to be a standard reference for the paper), such as the Riemannian connection, the induced connection for a non-singular submanifold, the Gauss and Codazzi-Mainardi equations, and the notion of geodesic curvature of a curve. A submanifold N of a semi-Riemannian manifold M with metric g is said to be non-singular if the restriction of g to the tangent space $T_x(N)$ at each point x of N is non-singular.

In the theorem in Section 4 which gives a formal generalization of a classical result on surfaces whose points are all umbilic, one has to add the assumption that the connection has zero torsion (as the author has agreed). Then there are discussions of the theorema egregium of Gauss and the lemma of Synge for Riemannian manifolds and a rigidity theorem for submanifolds in euclidean space R^n. *K. Nomizu* (Providence, R.I.)

Barbance, Christiane 1602

Décomposition d'un tenseur symétrique sur un espace d'Einstein.

C. R. Acad. Sci. Paris **258** (1964), 5336–5338.

L'auteur démontre d'abord les deux propositions suivantes. Proposition 1: Sur un espace d'Einstein compact, orientable ($R_{ji}=\lambda g_{ji}$) à $\lambda\leqq 0$, tout 2-tenseur symétrique $\mathbf{T}$ admet une décomposition unique de la forme $\mathbf{T}=\mathbf{t}+\pounds(\xi)\mathbf{g}$, où ξ est une 1-forme orthogonale aux 1-formes harmoniques et $\mathbf{t}$ tel que $\delta\mathbf{t}$ soit harmonique. Proposition 2: Sur un espace d'Einstein compact, orientable à $\lambda\leq 0$, tout 2-tenseur symétrique $\mathbf{T}$ admet une décomposition unique de la forme $\mathbf{T}=\Delta\mathbf{u}+\pounds(\xi)\mathbf{g}+\mathbf{h}$, où ξ est orthogonale aux 1-formes harmoniques et où $\mathbf{h}$ et $\delta\mathbf{u}$ sont harmoniques. Il démontre ensuite le théorème: Sur un espace d'Einstein compact, orientable à $\lambda\leqq 0$, tout 2-tenseur symétrique $\mathbf{T}$ admet une décomposition unique en somme de quatre termes, orthogonaux deux à deux, de la forme

$$\mathbf{T} = \Delta\mathbf{v}+[\mathbf{g}(\Delta-\lambda)\chi+\nabla\nabla\chi]+\pounds(\xi)\mathbf{g}+\mathbf{h},$$

où $\mathbf{v}$ est tel que $\delta\mathbf{v}$ est harmonique et $\|\mathbf{v}\|$ constant, où $\mathbf{h}$ est harmonique et ξ orthogonale aux 1-formes harmoniques. *K. Yano* (Tokyo)

Eells, James, Jr.; Sampson, J. H. 1603

Harmonic mappings of Riemannian manifolds.

Amer. J. Math. **86** (1964), 109–160.

From the introduction: "With any smooth mapping of one Riemannian manifold into another, it is possible to associate a variety of invariantly defined functionals. Each such functional of course determines a class of extremal mappings, in the sense of the calculus of variations, and those extremals, in the very special cases thus far considered, play an important role in a number of familiar differential-geometric theories.

"The present paper is devoted to a rather general study of a functional E of geometric and physical interest, analogous to energy. Our central problem is that of deforming a given mapping into an extremal of E. Following an infinite-dimensional analogue of the Morse critical point theory, we construct gradient lines of E (in a suitable function space), and E is a decreasing function along those lines. With suitable metric and curvature assumptions on the target manifold (assumptions which cannot be entirely circumvented, in view of the examples of sections 4E and 10D), we prove that the gradient lines do in fact lead to extremals (Theorem 11A).

"If $f\colon M\to M'$ is a smooth mapping of manifolds whose metrics are $g_{ij}\,dx^i\,dx^j$ respectively $g_{\alpha\beta}'\,dy^\alpha\,dy^\beta$, then the energy $E(f)$ is defined by the integral

$$E(f) = \frac{1}{2}\int_M g_{\alpha\beta}'\,\frac{\partial f^\alpha}{\partial x^i}\,\frac{\partial f^\beta}{\partial x^j}\,g^{ij} * 1,$$

where the f^α are local coordinates of the point $f(x)$, $*1$ being the volume element of M (assumed compact). Thus $E(f)$ can be considered as a generalization of the classical integral of Dirichlet. The Euler-Lagrange equations for E are a system of non-linear partial differential equations of elliptic type:

$$\Delta f^\alpha+\Gamma_{\beta\gamma}'^{\alpha}\,\frac{\partial f^\beta}{\partial x^i}\,\frac{\partial f^\gamma}{\partial x^j}\,g^{ij} = 0;$$

Δ is the Laplace-Beltrami operator on M and the $\Gamma_{\beta\gamma}'^{\alpha}$ are the Christoffel symbols on M'. Although this system is suggestive of the simple equation $\Delta u+\phi(u)\cdot\mathrm{grad}^2\,u=0$ for one unknown, there is in general very little connection between the two because of the phenomenon of curvature.

"It has been necessary to go into the question of existence of solutions in rather great detail, owing to the want of general results for non-linear systems. Direct methods of the calculus of variations seem to lead to severe difficulties, and that is one reason why we have preferred to approach the problem through the gradient line technique, which amounts to replacing the equations above by a system of parabolic equations whose relation to the elliptic system is analogous to that of Fourier's equation to Laplace's equation. This approach is of independent interest, in any case. Our methods are strongly potential-theoretic in nature. The local equations are first replaced by global equations of essentially the same form, embedding M' in a Euclidean space. A stability theorem is established showing that a solution of the resulting parabolic system does in fact produce a 1-parameter family of mappings of M into M'. Fundamental solutions of Laplace's equation and the heat equation on (compact) manifolds are used to establish a priori derivative estimates and to construct solutions of the parabolic system, the latter being translated into a system of non-linear integro-differential equations of the Volterra type, following the method used by Milgram and Rosenbloom in a linear problem. Curvature enters in a manner not unlike that exploited by Bochner [Univ. e Politec. Torino Rend. Sem. Mat. **15** (1955–56), 225–253; MR **18**, 819].

"Special cases of our extremal mappings go back to the very beginning of differential geometry. E.g., they include geodesics, harmonic functions, and minimal submanifolds. For minimal surfaces they were first studied locally by Bochner [Trans. Amer. Math. Soc. **47** (1940), 146–154; MR **1**, 271] in an explicitly Riemannian context. That work was carried to completion by Morrey [Ann. of Math. (2) **49** (1948), 807–851; MR **10**, 259]. In a report prepared by J. H. Sampson [M.I.T., Cambridge, Mass., Feb. 1954] the subject was taken up from a somewhat different point of view, and other geometrical applications were discussed. Since firm existence proofs were not then available, general publication of the results did not seem warranted. Shortly thereafter, J. Nash and, independently, F. B. Fuller [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 987–991; MR **16**, 284] advanced the same definition as that on which this

article is based, and Fuller described several examples. The problem has also been considered by E. Rauch.

"The contents of the present paper are presented in the following order: Chapter I, The concepts of energy and tension, (1) the energy integral, (2) the tension field, (3) invariant formulation, (4) examples, (5) the composition of maps; Chapter II, Deformations of maps, (6) deformations by the heat equation, (7) global equations, (8) derivative bounds for the elliptic case, (9) bounds for the parabolic case, (10) successive approximations, (11) harmonic mappings."

The authors prove that the Euler-Lagrange operator applied to a map f defines a vector field $\tau(f)$, the tension field of f, which is the contravariant representative of the differential of the energy function E at the point f in the space of smooth mappings from M to M'. f is called harmonic if $\tau(f)=0$. If M' is compact or satisfies certain rather technical conditions, it is shown that every continuous mapping $M\to M'$ is homotopic to a harmonic mapping; here it should be mentioned that the authors show that a harmonic mapping is smooth. If M' is compact and of non-positive Riemannian curvature, then every homotopy class of mappings $M\to M'$ contains a harmonic mapping whose energy is at an absolute minimum. If M has non-negative Ricci curvature, M' has non-positive Riemannian curvature, and if M' is compact or satisfies certain conditions, then (i) every continuous map $M\to M'$ is homotopic to a totally geodesic map, (ii) if M has a point of positive Ricci curvature, then every continuous map $M\to M'$ is nullhomotopic, and (iii) if the Riemannian curvature of M' is everywhere negative, then every continuous map $M\to M'$ is either nullhomotopic or is homotopic to a map of M onto a closed geodesic of M'. A corollary: If a compact smooth manifold M admits a Riemannian metric of non-positive Riemannian curvature, and if ds^2 is a (generally different) Riemannian metric of non-negative Ricci curvature on M, then the Ricci tensor of ds^2 is identically zero. As a special case, the standard result is proved on the homotopy groups of a complete Riemannian manifold of non-positive Riemannian curvature.

The reviewer expects that the results of this paper will become standard tools in proving the vanishing of homotopy groups. *J. A. Wolf* (Berkeley, Calif.)

Klingenberg, Wilhelm 1604

On the number of closed geodesics on a riemannian manifold.

Bull. Amer. Math. Soc. **70** (1964), 279–282.

The author states and sketches the proof of the following theorems: Let $P^m(\lambda)$, $\lambda\in\{1, 2, 4, 8\}$, be the projective space over the reals ($\lambda=1$), the complex numbers ($\lambda=2$), the quaternions ($\lambda=4$) or the Cayley numbers ($\lambda=8$ and $m=2$). Let M be a compact Riemannian manifold for which there exists a diffeomorphism $\Phi: M\to P^m(\lambda)$ satisfying: $c\|d\Phi(X)\|\leq\|X\|<2c\|d\Phi(X)\|$ for some $c>0$ and all tangent vectors X. Let $m=2^k+s$, $0\leq s<2^k$. Then there exist at least $2m-s-3+2\lambda$ closed geodesics on M, none of which is a covering of another. All their lengths are in the interval $[2\pi c, 4\pi c]$. The algebraic number of closed geodesics on M is at least $(m(m+1)/2)(\lambda(\lambda+1)/2)$.

The corresponding result for the m-sphere is due to Al'ber [Uspehi Mat. Nauk **12** (1957), no. 4 (76), 57–124; MR **19**, 751; Amer. Math. Soc. Transl. (2) **14** (1960), 107–172; MR **22** #4072] and the proof of the present theorem is similar to Al'ber's.

J.-P. Meyer (Baltimore, Md.)

Radziszewski, Konstanty 1605

Sur un théorème de Pogorielov. (Polish and Russian summaries)

Ann. Univ. Mariae Curie-Skłodowska Sect. A **16** (1962), 41–51 (1964).

Es wird bewiesen: (1) Ist $\langle A*B\rangle$ eine Geodätische auf einer glatten konvexen Fläche S und $d(M)$ der Abstand des Punktes $M\in\langle A*B\rangle$ von der Tangentenebene an S im Punkt A, dann gilt die Grenzwertformel

$$\lim_{M\to A} d(M):\int_0^{(A*M)} k(s)\,ds = 1,$$

wobei $k(s)$ die integrale Krümmung des Bogens $\langle A*N\rangle\subset\langle A*M\rangle$ und $s=[A*N]$ ist. (2) Diese Grenzwertformel gilt ebenso, wenn $d(M)$ den Abstand des Punktes M der Geodätischen $\langle A*B\rangle$ der glatten konvexen Fläche S von der Tangente $\mathbf{t}(A)$ an $\langle A*B\rangle$ im Punkte A bedeutet. Daraus folgt (3): Der Radiusvektor $\mathbf{r}(s)=\overline{AM}$ mit $s=[A*M]$ einer Geodätischen $\langle A*B\rangle$ auf einer glatten konvexen Fläche S kann in der Nachbarschaft des Punktes A durch die Formel $\mathbf{r}(s)=s\mathbf{t}+\int_0^s k(x)\,dx\cdot\nu$ ausgedrückt werden, wobei $\mathbf{t}$ der Einheitstangentenvektor der Geodätischen im Punkte A und ν ein Vektor ist, der gegen den Einheitsnormalenvektor $\mathbf{n}$ der Fläche S im Punkte A strebt. Die Darstellung (3) der Geodätischen auf einer konvexen Fläche S gilt, wie gezeigt wird, auch noch unter gewissen schwächeren Voraussetzungen. Die Ergebnisse verallgemeinern Sätze von A. W. Pogorielov. *K. Strubecker* (Karlsruhe)

Kagan, F. I. 1606

On two-dimensional Finsler spaces admitting a singular embedding in a three-dimensional affine space with a vector metric. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 1 (38), 46–55.

An affine space A_N together with a field of affinely equivalent hypersurfaces on it is called an affine space with a vector metric. The Finsler space F_n is said to be embedded in A_N if there exists an n-dimensional manifold M_n in A_N such that F_n is isometric with the Finsler space structure on M_n defined by the intersections of the hypersurfaces corresponding to the points of M_n and the tangent spaces of M_n. An embedding is said to be singular if M_n is a singular manifold.

Geometric and analytic necessary and sufficient conditions for a singular embedding of F_2 in A_3 are given in the paper. *B. Cenkl* (Prague)

Srivastava, T. N. 1607

Generalised Bianchi and Veblen identities in a special Kawaguchi space.

Tensor (N.S.) **15** (1964), 87–102.

S. K. Kaul and R. S. Mishra obtained the generalised Veblen identities in Riemannian space based on the non-symmetric connection by using Einstein covariant derivatives [Tensor (N.S.) **8** (1958), 159–164; MR **21** #1625]. The author and Mishra found Bianchi identities

for the curvature tensors in the Kawaguchi space X^n [ibid. (N.S.) **11** (1961), 43–50; MR **23** #A2856]. The author now sets up analogous rules for defining the Einstein covariant derivatives in X^n, and by using these rules, he finds four kinds of formal identities satisfied by the curvature tensors, which are deduced from Bianchi and Veblen identities in X^n. Each kind contains four independent formulae. *T. Ohkubo* (Kumamoto)

Akbar-Zadeh, Hassan; Bonan, Edmond 1608
Structure presque kählérienne naturelle sur le fibré tangent à une variété finslérienne.
C. R. Acad. Sci. Paris **258** (1964), 5581–5582.
Le but de la présente note est de démontrer le théorème: Pour que la structure presque kählérienne naturellement associée à la structure finslérienne soit kählérienne, il faut et il suffit que le champ d'éléments de contact défini par $Dv=0$ soit complètement intégrable. Le même théorème est aussi démontré par le rapporteur et E. T. Davies [Rend. Circ. Mat. Palermo (2) **12** (1963), 211–228; MR **29** #2755]. *K. Yano* (Tokyo)

GENERAL TOPOLOGY
See also 1400, 1564, 1576.

Mendelson, Bert 1609
★Introduction to topology.
College Mathematics Series.
Allyn and Bacon, Inc., Boston, Mass., 1962. ix + 217 *pp.* $11.35.
This textbook covers most of the material usually taught in an undergraduate topology course. The chapters are titled: 1. Theory of sets; 2. Metric spaces; 3. Topological spaces; 4. Connectedness; 5. Compactness. Uncommon is the omission of all separation axioms (and their consequences) save Hausdorff's, and of the completion of metric spaces. Their place is taken by sections on applications of connectedness (fixpoint theorems), and on local connectedness, homotopic paths and identification spaces. The exposition is very careful, rigorous, well-motivated and conservative. The development of the implications in and connections with basic analysis is rare among today's topology books. There are numerous, excellent problems.
J. Mayer-Kalkschmidt (Albuquerque, N.M.)

Henriksen, Melvin; Isbell, J. R. 1610
Averages of continuous functions on countable spaces.
Bull. Amer. Math. Soc. **70** (1964), 287–290.
Notation: $X=(x_1, x_2, \cdots)$ is a countably infinite completely regular space, $A=(a_{ij})$ is a regular (Toeplitz) matrix; if f is a function on X and $\lim_{i\to\infty} \sum_j a_{ij}f(x_j)$ exists, it is denoted by $\varphi_A(f)$. Theorems: If $\varphi_A(f)$ exists for every $f\in C^*(X)$, then there is an absolutely convergent $\sum \alpha_n$ with $\varphi_A(f)=\sum \alpha_n f(x_n)$; if the complement of each $x\in X$ is C^*-embedded [see, e.g., L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N.J., 1960; MR **22** #6994], then no A can sum every $f\in C^*(X)$. The authors call a point $x=x_n\in X$ "heavy" if there is a matrix A such that, for any bounded f continuous at x, $\varphi_A(f)=f(x)$. They state (the proof is to appear elsewhere) that if a functional $\varphi(f)=\sum \alpha_n f(x_n)$ is representable as φ_A, then $\sum \alpha_n=1$ and $\alpha_n=0$ whenever x_n is not heavy. In addition, a theorem of A. Gleason [Illinois J. Math. **2** (1958), 482–489; MR **22** #12509] is applied to show that an extremally disconnected X containing a dense set of power $\mathfrak{m}$ can be embedded in βD, D discrete of power $\mathfrak{m}$. As pointed out by one of the authors, Theorem 1 is a special case of results by A. D. Aleksandrov [cf. Mat. Sb. (N.S.) **9 (51)** (1941), 563–628, § 7; MR **3**, 207; ibid. (N.S.) **13 (55)** (1943), 169–238, §§ 14, 19; MR **6**, 275]; other proofs have been given by L. LeCam [Univ. Calif. Publ. Statist. **2** (1957), 207–236; MR **19**, 128] and V. S. Varadarajan [Mat. Sb. (N.S.) **55 (97)** (1961), 35–100; MR **26** #6342].) *M. Katětov* (Prague)

Ross, K. A.; Stone, A. H. 1611
Products of separable spaces.
Amer. Math. Monthly **71** (1964), 398–403.
This is a well-designed unified treatment of some known theorems on the topological product of separable spaces. These theorems conclude that the product space is separable in certain weak senses of the word and are consequently important only in the case where the index set is uncountable. The reviewer feels that the almost immediate generalizations to dense subsets of these products should have been mentioned, since this includes linear spaces under their weak topology. Also, it would be instructive if the theorem of Mazur, which is referred to here, could somehow be incorporated in its full strength.
H. H. Corson (Seattle, Wash.)

Gleason, Andrew M. 1612
Universal locally connected refinements.
Illinois J. Math. **7** (1963), 521–531.
Each topological space S is associated with a locally connected space S^* and a 1-1 continuous map φ of S^* onto S universal in the following sense: Each continuous map f of a locally connected space A to S factors into $f=\varphi\circ f^*$, where f^* is a continuous map of A into S^*.

Indeed, the lattice L of all topologies on a set S has a subset F consisting of all locally connected topologies. Every intersection of topologies is a topology and every intersection of locally connected topologies is locally connected, so that L is a complete lattice and the meet of any subset of F is in F. For $t\in L$, let $t^*=\min\{f\in F: f\geqq t\}$. If A has topology u and $f: A\to S$ is any function, let $f(u)$ be the greatest topology of S for which f is continuous. If u is locally connected, so is $f(u)$ and hence, if t is a topology of S for which f is continuous, i.e., if $t\leqq f(u)$, then $t^*\leqq f(u)$. Thus (S, t^*) has the required universal property.

The result is also expressed by saying that the injection functor of the category of locally connected spaces in the category of topological spaces has an adjoint.

Similarly, a uniform space S is associated with a uniformly locally connected space S^*.
C. H. Dowker (London)

Proizvolov, V. V. 1613
A bicompact T_1-space. (Russian. English summary)
Vestnik Moskov. Univ. Ser. I Mat. Meh. **1961**, *no.* 6, 17–18.

Let B be an infinite set with the minimal (coarsest) T_1-topology: only finite subsets and B itself are closed. Any T_1-space which is a continuous image of B is either homeomorphic to B or consists of a single point.

C. H. Dowker (London)

Proizvolov, V. V. 1614

The dimensionality of completely regular spaces. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1961,** *no.* 3, 29–30.

There is a very simple completely regular space R for which $\operatorname{ind} R=2$ but $\operatorname{Ind} R=3$. There is a Hausdorff space X and a closed subset A such that all sufficiently small neighbourhoods of A have the same non-empty boundary.

C. H. Dowker (London)

Smirnov, Ju. M. 1615

On transfinite dimension. (Russian)

Mat. Sb. (*N.S.*) **58 (100)** (1962), 415–422.

An infinite-dimensional space is called weakly infinite-dimensional if, for every sequence of pairs of disjoint closed sets A_i, B_i, there exist closed sets C_i respectively separating A_i, B_i such that, for some finite N, $\bigcap_{i\leq N} C_i=\varnothing$. Otherwise the space is strongly infinite-dimensional. It is shown that every infinite-dimensional metric space is itself strongly infinite-dimensional or else has a point whose complement is strongly infinite-dimensional.

Each metric space which has a transfinite great inductive dimension Ind is weakly infinite-dimensional and is the union of some sequence of 0-dimensional subspaces. Under the usual conditions, Ind can be replaced here by ind. If X is a countable union of 0-dimensional subspaces and if a closed map of countable multiplicity maps X onto a complete metric space Y, then Y has a transfinite dimension ind.

C. H. Dowker (London)

Nagata, Jun-iti 1616

A remark on general imbedding theorems in dimension theory.

Proc. Japan Acad. **39** (1963), 197–199.

A star-space is obtained from a set of segments $[0, 1]$ by identifying the zeros and metrizing it so that $\rho(x, y)$ is $x+y$ if the points are in different segments and is $|x-y|$ if they are in the same segment. It is shown that, for a metric space R, $\dim R\leq n$ [respectively, R is a countable sum of 0-dimensional spaces] if and only if R can be imbedded in the subset K_n [K_∞] of some countable product of star-spaces, where a point belongs to K_n [K_∞] if at most n [at most a finite number] of its coordinates are non-zero and rational.

C. H. Dowker (London)

Nagata, Jun-iti 1617

Two theorems for the n-dimensionality of metric spaces.

Compositio Math. **15,** 227–237 (1963).

If R is a metrizable space, $\dim R\leq n$ if and only if R can be metrized so that the boundaries of all ε-neighbourhoods $S_\varepsilon(p)$ of points have dimension not exceeding $n-1$, $\dim B(S_\varepsilon(p))\leq n-1$, and the family $\{S_\varepsilon(p)\}_{p\in R}$ is closure-preserving. Hence $\dim R\leq n$ if and only if R can be metrized so that the boundaries of all ε-neighbourhoods of closed sets have dimension not exceeding $n-1$, $\dim B(S_\varepsilon(F))\leq n-1$.

If R is a metrizable space, $\dim R\leq n$ if and only if R has a base of rank not exceeding $n+1$. This means that, if any $n+2$ of the basic open sets have non-empty intersection, one of these must contain one of the other $n+1$ sets.

C. H. Dowker (London)

Efimov, B. A. 1618

On the weighted structure of diadic bicompacta. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964.** *no.* **2, 3–11.**

The character of a topological space X at the point x, i.e., the least cardinal of neighborhood system of this point, is denoted by $\varepsilon(x)$. Theorem 1: For a dyadic compact space X (i.e., a continuous image of the cartesian product of a number of two-point discrete spaces) such that for every $x\in X$ the character $\varepsilon(x)$ is of the form $\aleph_{\alpha+1}$, where $\alpha=-1, 0, 1, \cdots$, there exists a family $\{U_i\}_{i=1}^\infty$ of open subsets such that $\bigcup_{i=1}^\infty U_i$ is dense in X and the function ε is constant on every U_i.

Theorem 4: The power of a dyadic compact space of weight $\mathfrak{m}$ is equal to $2^{\mathfrak{m}}$ if $\mathfrak{m}=\aleph_{\alpha+1}$ for some $\alpha=0, 1, \cdots$. Moreover, if $\mathfrak{m}=\aleph_\alpha$, α being a limit ordinal, then the power is greater than or equal to $2^{\mathfrak{n}}$ for every $\mathfrak{n}<\mathfrak{m}$ (and obviously not greater than $2^{\mathfrak{m}}$).

The author remarks that in a paper by the reviewer and A. Pełczyński [Colloq. Math. **11** (1963), 55–63; MR **28** #4504] it is proved (but not explicitly stated) that the weight of dyadic compact spaces is preserved under irreducible mappings. Some examples are given.

R. Engelking (Warsaw)

Rinow, W. 1619

Perfekte lokal zusammenhängende Kompaktifizierungen und Primendentheorie.

Math. Z. **84** (1964), 294–304.

The author is interested in perfect, locally connected compactifications of a locally compact space X. X must be locally connected and have only finitely many components, and then the Freudenthal compactification has the desired properties. Further results associate all such compactifications (rather loosely) with uniform structures on X.

J. R. Isbell (New Orleans, La.)

Nieminen, Toivo 1620

On the completion of uniform spaces.

Ann. Acad. Sci. Fenn. Ser. A I No. 333 (1963), 10 *pp.*

The author employs notions from Section 25 of H.-J. Kowalsky, *Topologische Räume* [Birkhäuser, Basel, 1961; MR **22** #12502] to amplify the treatment of completions in Section 35.

J. R. Isbell (New Orleans, La.)

Skornjakov, L. A. 1621

Compact topological spaces. (Russian)

Dokl. Akad. Nauk SSSR **140** (1961), 1263–1266.

If E is a compact space, let Ω be the directed set of symmetric vicinities in its uniform structure. For each $\alpha\in\Omega$, let M_α be a finite α-net in E; for each $p\in E$, $\alpha(p)$ meets M_α. Let M be the disjoint union of the sets M_α.

If $x \in M_\alpha$, $y \in M_\beta$, $x\Delta y$ means that $\alpha(x) \cap \beta(y) \neq \varnothing$. Generalizing from this, a spectrum (Ω, M, Δ) is defined. A fan in the spectrum is a non-empty subset S of M with at most one element s_α in each M_α such that, if s_α exists and $\beta > \alpha$, then s_β exists and, if s_α, s_β exist, then $s_\alpha \Delta s_\beta$. The spectrum determines a uniform structure on the set of fans. Taking the corresponding separated structure and completing, one arrives again, under certain conditions, at a compact space. *C. H. Dowker* (London)

Traylor, D. R. 1622

Concerning metrizability of pointwise paracompact Moore spaces.

Canad. J. Math. **16** (1964), 407–411.

By a Moore space is meant a topological space for which there exists a sequence of collections of open sets satisfying Axiom 0 and the first three parts of Axiom 1 from R. L. Moore's *Foundations of point set theory* [revised edition, Amer. Math. Soc., Providence, R.I., 1962; MR **27** #709]. Main results are the following. Theorem 1: A locally separable Moore space is metrizable if and only if it is pointwise paracompact (also called "metacompact"). Theorem 2: A Moore space is metrizable if and only if the boundary of each open set is strongly screenable [R. H. Bing, Canad. J. Math. **3** (1951), 175–186; MR **13**, 264]. Theorem 3: Let S be a locally peripherally separable Moore space such that, for each open set G in S, the set of points of the boundary B of G which are arcwise accessible from G is dense in B. Then S is metrizable if and only if S is pointwise paracompact. *A. Lelek* (Wroclaw)

Zarelua, A. 1623

A universal bicompactum of given weight and dimension. (Russian)

Dokl. Akad. Nauk SSSR **154** (1964), 1015–1018.

Let τ be an arbitrary infinite cardinal and n an integer, $n \geqq 0$. The author outlines a proof of the existence of a universal bicompact $B_{n\tau}$ of weight $w(B_{n\tau}) = \tau$ and covering dimension $\dim B_{n\tau} = n$; $B_{n\tau}$ is universal in the sense that it contains, as subsets, all completely regular spaces X with $w(X) \leqq \tau$ and $\dim X \leqq n$. For $\tau = \aleph_0$, the statement reduces to a classical result of dimension theory; for general τ it answers a well-known question of P. Alexandroff [General Topology and its Relations to Modern Analysis and Algebra (Proc. Sympos., Prague, 1961), pp. 41–54, Academic Press, New York, 1962; MR **26** #3003]. Partial results were obtained earlier by B. Pasynkov [Dokl. Akad. Nauk SSSR **144** (1962), 1217–1220; MR **27** #6243]. For another complete solution of the problem cf. B. Pasynkov [#1624].

The proof is based on methods used in the study of the ring $C(X)$ of all bounded continuous functions defined on a completely regular space X. In particular, the author obtains his universal bicompact $B_{n\tau}$ from a certain normed ring $R_{n\tau}$ by taking for $B_{n\tau}$ the space of all closed ideals $J \subseteq R_{n\tau}$ for which $R_{n\tau}/J$ is isomorphic and isometric to the reals. The construction of $R_{n\tau}$ is too complicated to be fully described here. $R_{n\tau}$ is a normed ring of weight τ and algebraic dimensional rank $\operatorname{ard} R_{n\tau} = n$. Moreover, whenever for a space X, $w(X) = \tau$ and $\operatorname{ard} C(x)$ $(= \dim X) = n$, then $R_{n\tau}$ admits continuous homomorphic mappings onto some regular subring of the ring $C(X)$. (For a normed ring G, ard G is defined as follows. Given a subset $\Phi \subseteq G$, one considers all the elements $g \in G$ for which $g^2 = g\varphi$ for some φ taken from the subring generated by Φ. Let the subring generated by all such elements g be denoted by $G(\Phi)$. Then ard G is the minimal positive integer n having the property that every $n+1$ elements of G admit arbitrarily close approximations by elements $g_1, \cdots, g_{n+1}$ taken from a subring of the form $G(\Phi)$, Φ being a set consisting of n elements only.) *S. Mardešić* (Zagreb)

Pasynkov, B. 1624

On universal bicompacta of given weight and dimension. (Russian)

Dokl. Akad. Nauk SSSR **154** (1964), 1042–1043.

The author proves, independently of A. Zarelua [#1623 above], the existence of a universal bicompact $\Pi_{n\tau}$ of weight τ and dimension n. In comparison with Zarelua's work, the present solution is much simpler and more elementary in the sense that it uses only purely topological means.

The main tool in the proof is the factorization theorem, due to the reviewer [Illinois J. Math. **4** (1960), 278–291; MR **22** #7101], according to which every mapping $r: P \to R$ of a bicompact P onto a bicompact R can be factored into mappings $p: P \to Q$, $q: Q \to R$, $r = q \circ p$, $Q = p(P)$, with $\dim Q \leqq \dim P$ and weight $w(Q) \leqq w(R)$. The author considers all completely regular spaces X_α with $w(X_\alpha) \leqq \tau$, $\dim X_\alpha \leqq n$, and assigns to each X_α an imbedding $f_\alpha: X_\alpha \to I^\tau$ into the product I^τ of τ copies of the line segment I. The maps admit extensions $\bar{f}_\alpha: \beta(X_\alpha) \to I^\tau$ to corresponding Čech compactifications $\beta(X_\alpha)$. Next, the author considers the disjoint sum Y of all the spaces $\beta(X_\alpha)$ and the map $f: Y \to I^\tau$ induced by the maps $\bar{f}_\alpha$. Finally, f is extended to $\bar{f}: \beta(Y) \to I^\tau$ and the factorization theorem is applied to $\bar{f}$. It yields a bicompact $Q = \Pi_{n\tau}$, which is readily seen to be universal. (A similar procedure was applied by the reviewer [op. cit.] in proving the existence of a compactification which increases neither weight nor dimension.)

Developing and using analogous techniques, the author also gives a new proof of the theorem of J. Nagata [J. Reine Angew. Math. **204** (1960), 132–138; MR **25** #4493] asserting the existence of a universal metric space of weight τ and dimension n. *S. Mardešić* (Zagreb)

Bing, R. H. 1625

The simple connectivity of the sum of two disks.

Pacific J. Math. **14** (1964), 439–455.

Let K_1 be a curve in the disk $D = \{(\rho, \theta): \rho \leqq 2\}$ consisting of two circles $C(r) = \{(\rho, \theta): \rho \leqq r\}$ $(r = 1, 2)$ and the spiral $S = \{(\rho, \theta): \rho = (2 + e^\theta)/(1 + e^\theta)\}$ which connects $C(1)$ and $C(2)$. Let B_1 be a continuum obtained from $D \times \{0, 1\}$ by the identification of the points $(x, 0)$ and $(x, 1)$ for every $x \in K_1$. The author shows that B_1 is simply-connected (Theorem 1). However, if one replaces the circles $C(1)$ and $C(2)$ in K_1 by a bundle of concentric circles, like the Cartesian product of a circle and the Cantor set, and connects each pair of adjacent circles in this bundle with a spiral-like S, then for such a modified curve K_2 the resulting continuum B_2 will not be simply-connected (Theorem 3). Three examples of non-simply-connected continua, B_3, B_4, and B_5, are constructed, each B_i being composed of two disks sewed together along a connected

curve K_i ($i = 1, \cdots, 5$). Here K_3, which consists of a triod and a spiral about it, is triod-like, K_4 is arc-like (also called "snake-like" or "chainable"), and K_5 is tree-like and arcwise connected.

The author says a map of a simple closed curve J into the union $X_1 \cup X_2$ is finitely oscillating if J can be represented as the union of a finite number of arcs such that the image of each of them lies in one set X_i ($i = 1, 2$). A set Y is said to have the extremal inverse property (e.i.p.) if, for every pair of its points p and q, there exists a continuum C in the disk D, with points p' and q' on the boundary of D, and a map of C into Y that takes p' and q' into p and q, respectively. Each arc-like continuum has the e.i.p. (Theorem 5). Let X_1 and X_2 be simply-connected LC^1 continua; all finitely oscillating maps of J into $X_1 \cup X_2$ are null-homotopic if and only if $X_1 \cap X_2$ has the e.i.p. (Theorem 9). It is left as an open question whether or not each triod-like plane continuum has the e.i.p. If the intersection of two disks is connected and locally-connected, their union is simply-connected (Theorem 12).

It turns out, moreover, that the topology of the intersection of two disks completely determines the simple connectivity or its lack in the union. Namely, let D_1, D_2 be disks, and X_1, X_2 be locally connected continua such that the intersections $D_1 \cap D_2$ and $X_1 \cap X_2$ are homeomorphic; if $X_1 \cup X_2$ is simply-connected, then $D_1 \cup D_2$ is also (Theorem 14). An example, built on a connected and locally connected curve K_6, shows that the inverse implication fails. More generally, if X_1, X_2, Y_1, Y_2 are simply-connected LC^1 continua such that $X_1 \cap X_2$ and $Y_1 \cap Y_2$ are homeomorphic, then the fundamental groups of $X_1 \cup X_2$ and of $Y_1 \cup Y_2$ are isomorphic (Theorem 15).

A. Lelek (Wrocław)

Bing, R. H.; Kister, J. M. 1626

Taming complexes in hyperplanes.

Duke Math. J. **31** (1964), 491–511.

The main result of this paper is Theorem 1.1: Let K be a finite k-dimensional complex and h an imbedding of K into an n-plane E^n in E^{n+k}, where $k+2 \leqq n$. Let ε be any positive number. Then there exists an isotopy g_t ($t \in I$) of E^{n+k} onto itself such that (1) g_0 is the identity; (2) $g_1 h$ is piecewise linear; (3) g_t is the identity outside an ε-neighborhood of $h(K)$ for each t in I; (4) each point of E^{n+k} moves along a polygonal path under g_t ($t \in I$) having length less than ε.

Also proved is the following theorem, which had previously been mistakenly assumed by several authors to be an immediate and direct consequence of Gugenheim's work [Proc. London Math. Soc. (3) **3** (1953), 29–53; MR 15, 336]. Theorem 5.5: Suppose K_1 and K_2 are two rectilinear k-complexes in E^n, $n \geqq 2k+2$, and h is a piecewise linear homeomorphism of K_1 onto K_2 that does not move any point as far as ε. Then there is an isotopy h_t ($t \in I$) of E^n onto itself such that (1) $h_0 =$ identity; (2) $h_1 = h$ on K_1; (3) each h_t is the identity outside an ε-neighborhood of K_1; (4) each point of E^n moves along a polygonal path of length less than ε.

The following results are also of interest. Corollary 6.3: If M is any 2-manifold, then any two imbeddings of M into an n-plane in E^{n+2} are equivalent. Corollary 6.4: Any two imbeddings of a k-sphere with handles into an n-plane in E^{n+k} are equivalent. Two imbeddings, f_1 and f_2, of M into E^{n+k} are "equivalent" if there is a homeomorphism g of E^{n+k} onto itself such that $gf_1 = f_2$.

Finally we mention Corollary 7.2: Every locally flat imbedding of a triangulated k-manifold in a combinatorial n-manifold, $n \geqq 2k+2$, is tame.

This extends results of the reviewer [Bull. Amer. Math. Soc. **69** (1963), 824–831; MR **27** #5241b] and Greathouse [ibid. **69** (1963), 820–823; MR **27** #5241a].

H. R. Gluck (Cambridge, Mass.)

Cantrell, J. C. 1627

n-frames in euclidean k-space.

Proc. Amer. Math. Soc. **15** (1964), 574–578.

An n-frame F_n is a union of n arcs, $F_n = \bigcup_{i=1}^n A_i$, with a distinguished point p such that p is an end point of each A_i and $A_i \cap A_j = p$ for $i \neq j$. Any n-frame in E^k for which each arc A_i is a straight line segment may be selected as the standard n-frame G_n. Then an n-frame F_n in E^k is said to be tame if there is a homeomorphism of E^k onto itself carrying F_n onto G_n; otherwise it is said to be wild. For $n > 1$, F_n is mildly wild if it is wild and if $F_n - (A_i - p)$ is tame for $i = 1, 2, \cdots, n$. DeBrunner and Fox [Duke Math. J. **27** (1960), 425–429; MR **22** #11359] have shown that for each $n > 1$ there are mildly wild n frames in E^3.

The author proves the following theorems. Theorem 1: There are no mildly wild n-frames in E^k for $k > 3$. Theorem 2: A wild arc or simple closed curve in E^k, $k > 3$, must fail to be locally flat at each point of some Cantor set.

H. R. Gluck (Cambridge, Mass.)

McMillan, D. R., Jr. 1628

Taming Cantor sets in E^n.

Bull. Amer. Math. Soc. **70** (1964), 706–708.

A Cantor set C in euclidean n-space E^n is called tame if there is a homeomorphism of E^n onto itself throwing C into a straight line segment. Homma [Yokohama Math. J. **7** (1959), 191–195; MR **23** #A1357] and Bing [Pacific J. Math. **11** (1961), 435–446; MR **24** #A539] have shown that a Cantor set C in E^3 is tame if and only if $E^3 - C$ is 1-ULC (in the sense of homotopy). The main result of the present paper is that the same is true of each Cantor set in E^n for any $n \neq 4$. *K. W. Kwun* (Princeton, N.J.)

Baxter, Glen; Joichi, J. T. 1629

On permutations induced by commuting functions, and an embedding question.

Math. Scand. **13** (1963), 140–150.

If f and g are commuting continuous self-transformations of a closed interval with only finitely many fixed points (precisely: fg has finitely many), there is an obvious implication from (1) f and g lie in a semigroup of continuous transformations isomorphic with the non-negative part of an additive subgroup of the real line, to (3) f and g have a common fixed point. The authors prove two theorems stronger than (1)⇒(2) and (2)⇒(3), where (2) is a strong and complicated set of restrictions on the behavior of f, g and fg between fixed points of fg. The work was, it seems, motivated particularly by the following application (of (1)⇒(2)): the Tchebycheff polynomials T_2, T_3 do not satisfy (1). *J. R. Isbell* (New Orleans, La.)

Wu, T. S. [Wu, Ta-Sun] 1630
Locally compact semigroups with dense maximal subgroups.
Trans. Amer. Math. Soc. **113** (1964), 151–168.
The author considers topological semigroups S which have the following two properties: (1) S is locally compact; (2) S has an identity 1 and the group H of units (elements having inverses relative to 1) is dense in S. Mostert showed that H need not be a topological group (i.e., inversion need not be continuous) [Math. Z. **82** (1963), 29–36; MR **27** #2964]; his construction is generalized so as to yield what the author calls a Mostert semigroup T: Let $\mathscr{F}$ be any family of semigroups S_i with an infinite cyclic subgroup G_i such that $B_i = S_i \setminus G_i$ is compact; it is shown that then B_i is a compact monothetic group and that all compact monothetic groups can occur in this fashion [Hofmann, Trans. Amer. Math. Soc. **106** (1963), 19–51, Theorem II; MR **26** #1865]; T is defined as the semigroup of all $(s_i) \in \prod_{\mathscr{F}} S_i$ such that s_i^n, $n = 1, 2, \cdots$, has a convergent subset for all but a finite number of indices i; the topology defined on T is the one first introduced into semigroups by Mostert [loc. cit.] and used by Braconnier [J. Math. Pures Appl. (9) **27** (1948), 1–85; MR **10**, 11] for locally compact abelian groups, and by various other authors for rings. The group of units in T then is not topological.

There are two main results in the paper of which the first requires a much longer proof. Suppose that S satisfies properties (1) and (2) above, property (3) H is not a topological group, and property (4) the topology of S has a countable basis. Then (I) if the component of 1 in S is trivial, S contains a Mostert semigroup with 1 as identity, and (II) if H is connected, there is a monomorphism of topological semigroups from the additive group of real numbers into H (which is not necessarily a monomorphism of topological groups). There are a variety of other smaller results which, together with Mostert's results in the paper mentioned above, constitute the present knowledge of the problem to find sufficient conditions under which H is a topological group when (1) and (2) are satisfied. *K. H. Hofmann* (New Orleans, La.)

Wu, Ta-Sun 1631
Continuous flows with closed orbits.
Duke Math. J. **31** (1964), 463–469.
The author characterizes topologically those subsets F of any given surface S such that F is the fixed-point set of some continuous flow in S whose orbits are all closed subsets of S; and likewise with "closed" replaced by "compact". The investigation relies heavily on work of A. Beck [Bull. Amer. Math. Soc. **66** (1960), 305–307; MR **22** #8489], who obtained characterizations of the sets F when S is the plane.
W. H. Gottschalk (Middletown, Conn.)

Auslander, Joseph 1632
On the proximal relation in topological dynamics.
Proc. Amer. Math. Soc. **11** (1960), 890–895.
Let (X, T) be a transformation group, where X is a compact Hausdorff space and T is an arbitrary topological group. Points x and y of X are said to be proximal provided that for each member α of the uniformity of X there exists $t \in T$ such that $(xt, yt) \in \alpha$. The author studies the proximal relation P of (X, T) by substantial use of the enveloping semigroup. Results include: (1) If $x \in X$ and if M is a nonempty closed invariant subset of the orbit-closure of x, then there exists $y \in M$ such that x and y are proximal. (2) If P is an equivalence relation in X, then X decomposes uniquely into invariant sets each consisting of those points whose orbit-closures contain a certain minimal subset of X. (3) If P is closed in $X \times X$, then P is an equivalence relation in X; the converse fails. (4) Proximal is an equivalence relation in a product transformation group if and only if proximal is an equivalence relation in each factor.
W. H. Gottschalk (Middletown, Conn.)

Vinokurov, V. R. 1633
On the definition of a dynamic limit point in general dynamical systems. (Russian)
Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 3 (40), 36–38.
Let R be a locally compact space with countable base and G a locally compact group of homeomorphic mappings of R into itself with countable base. The author gives an abstract definition of a dynamic limit point of a point of R. Theorem 1: If $y \in R$ is a dynamic limit point of $x \in R$, then y is also a dynamic limit point of $g(x)$ for any $g \in G$. Theorem 2: The dynamic limit points of a point x form an invariant closed set. *W. A. Coppel* (Canberra)

Clay, Jesse Paul 1634
Invariant attractors in transformation groups.
Illinois J. Math. **8** (1964), 473–479.
The author introduces and studies a notion of attraction which generalizes the one used in the theory of differential equations and which is applicable to general transformation groups. Thus, let (X, T) be a transformation group with uniformizable phase space X and phase group T, $\mathscr{B} \subset PT$, D an invariant subset of X, $x \in X$. Then x is $\mathscr{B}$-attracted to D if, given an index α of X, there exists $B \in \mathscr{B}$ with $xB \subset D\alpha$; x is regionally $\mathscr{B}$-attracted to D if, given an index α of X and U a neighborhood of x, then there exists $B \in \mathscr{B}$ and $y \in U$ with $yB \subset D\alpha$. Let $S(\mathscr{B})$ $[R(\mathscr{B})]$ denote the set of points which are attracted [regionally attracted] to D. Then $S(PT)$, $R(PT)$, $S(A)$, $R(A)$ and the relations among them are studied, where A is the collection of syndetic subsets of T.
R. Ellis (Philadelphia, Pa.)

ALGEBRAIC TOPOLOGY

See also 1286, 1603, 1653.

Goldman, A. J. 1635
Optimal matchings and degree-constrained subgraphs.
J. Res. Nat. Bur. Standards Sect. B **68B** (1964), 27–29.
From the author's summary: "The characterization of maximum-cardinality matchings in linear graphs by the nonexistence of augmenting paths has been extended by several authors to a similar characterization of maximum degree-constrained subgraphs. This paper contains a proof of the extended version by direct reduction to the case of matchings. Possible algorithmic implications of the reduction are suggested."
H. M. Trent (Washington, D.C.)

ıeke, Lowell W.; Harary, Frank 1636

n the thickness of the complete graph.

ull. Amer. Math. Soc. **70** (1964), 618–620.

thickness $t(G)$ of a graph G is the minimum number of ıar subgraphs whose union is G. The authors determine) (K_n = complete n-graph) for $n \equiv -1, 0, 1, 2 \pmod 6$. ir result, $t(K_n) = [(n+7)/6]$, is obtained by combining legant construction which shows that $t(K_{6m+2}) \leq m+1$ ı Euler's polyhedral formula from which $t(K_{6m-1}) \geq$ 1. It is perhaps worth pointing out that in the two aining cases $n \equiv 3, 4 \pmod 6$ the formula for $t(K_n)$; not generally hold, e.g., it is known [J. Battle, Iarary and Y. Kodama, same Bull. **68** (1962), 569–571; **27** #5248; W. T. Tutte, Canad. Math. Bull. **6** (1963), 330; MR **28** #2535] that $t(K_9) = t(K_{10}) = 3$. Moreover, ems that to obtain a formula for all n is a difficult ılem, as witness the considerable effort by several ıors that has gone into the determination of $t(K_9)$.

G. Sabidussi (Hamilton, Ont.)

ızsch, Herbert 1637

ur Theorie der diskreten Gebilde. 15. Mitteilung: usatzbemerkungen.

'iss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-'atur. Reihe **11** (1962), 733–736.

Note bringt zwei Ergänzungen zu früheren Arbeiten Theorie der diskreten Gebilde, die seit dem Jahre ; in der gleichen Zeitschrift (Band 5 ff.) erschienen [siehe z.B. **10** (1961/62), 843–850; MR **25** #2595, und dort zitierten Referate].

) In jedem Dreikantnetz $N_3(\vartheta, 0)$ auf der Kugel K_0 en sich, wie früher gezeigt wurde, die ganzzahligen ıaturen 1 bis ϑ derart auf die Ecken gleichverteilen, unter den $\mu\vartheta$ Signaturen der Randecken jedes ietes Γ ($\mu = 1, 2, \cdots$) jede der Signaturen 1 bis ϑ ch oft (μ-mal) vorkommt (Gleichverteilung schlechthin) außerdem eine gewisse Permutation dieser Eckenaturen sich bei monotoner Durchlaufung des Randes s Gebietes Γ periodisch ($(\mu-1)$-mal) wiederholt iodische Gleichverteilung). Das gilt, wie hier bemerkt l, nicht mehr für $\vartheta = 3$. Für eine Gleichverteilung echthin der Signaturen 1, 2, 3 auf die Ecken eines ikantnetzes $N_3(3, 0)$ auf der Kugel K_0 ist notwendige ingung, daß die Eckenanzahl e von $N_3(3, 0)$, die stets mod 4 ist, auch noch $\equiv 0 \bmod 3$ ist. Ob diese Beding auch hinreichend ist, bleibt offen, ebenso einige chließend aufgeworfenen Fragen.

) Es gibt auf der Kugel K_0 Dreikantnetze N_3, die durch Eckentransformation (Aufweiten von Ecken Dreiecken) in Kantennetze $N_3(2, 1)$ überführen lassen solche, bei denen das nicht möglich ist. Im letzten le entsteht die Minimumsaufgabe, durch Eckenısformation von N_3 die Anzahl der geraden Gebiete transformierten Netzes möglichst klein zu machen. wird ein Eckentransformationssatz hergeleitet, mit sen Hilfe man zeigen kann, daß diese Minimalzahl der den Gebiete stets 1 ist. *K. Strubecker* (Karlsruhe)

tzsch, Herbert 1638

ur Theorie der diskreten Gebilde. 16. Mitteilung: Ein ignierungssatz modulo 2 für Dreikantnetze auf der ugel.

Viss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-'atur. Reihe **11** (1962), 1207–1212.

Jeder Ecke E eines (endlichen, zusammenhängenden) Dreikantnetzes N_3 auf der Kugel K_0 werde eine Signatur $\sigma(E) = 0$ oder 1 zugeordnet; dieselbe Signatur $\sigma(E) = \sigma(WR)$ entspreche jedem der drei Winkelräume der Ecke E. Für alle Randecken E eines Gebietes G ist dann $s(G) = \sum \sigma(E) = \sum \sigma(WR)$. Die Note behandelt die in der vorhergehenden Arbeit aufgeworfene Frage, ob für ein beliebiges Dreikantnetz N_3 auf der Kugel K_0 und für beliebig zu seinen Gebieten G modulo 2 vorgegebene Werte $w(G) = 0$ oder 1 eine Signaturfunktion $\sigma(E)$ so bestimmt werden kann, daß für jedes Gebiet $s(G) \equiv w(G) \bmod 2$ ist und wie groß gegebenenfalls die Anzahl der Lösungen dieses Problems ist.

Wie sich zeigt, sind dabei drei Klassen zu unterscheiden, je nachdem N_3 auf K_0 (1) gerade, (2) halbgerade, (3) weder gerade noch halbgerade ist. Die Klassensignatur ist damit (1) $B = 2$, (2) $B = 1$, (3) $B = 0$. Das gestellte Problem wird durch die beiden folgenden Sätze beantwortet, deren Beweis auf einem Reduktionsweg geführt wird: (A) Existenzsatz: Wenn N_3 auf K_0 weder gerade noch halbgerade ist, dann existiert stets wenigstens eine Signaturfunktion $\sigma(E)$ der genannten Eigenschaft. Ist N_3 halbgerade aber nicht gerade, dann ist die Kongruenz $s \equiv 0 \bmod 2$, und wenn N_3 gerade ist, so sind für die $w(G) = 0$ oder 1 die Kongruenzen $s_1 \equiv s_2 \equiv s_3 \bmod 2$ notwendige und hinreichende Bedingungen dafür, daß eine geeignete Signaturfunktion $\sigma(E)$ existiert; im letzten Fall verteilen sich die Gebiete G von N_3 auf drei vollständige Systeme S_1, S_2, S_3 und s_m bedeutet die Summe aller jener $w(G)$, deren Gebiete G dem vollständigen System S_m angehören. (B) Anzahlsatz: Sind die Bedingungen (A) erfüllt, und ist f die Anzahl der Gebiete G von N_3 auf K_0, dann gibt es zu jedem vorgegebenen System der f Werte $w(G)$ genau 2^{f+B-4} verschiedene Lösungsfunktionen $\sigma(E)$. *K. Strubecker* (Karlsruhe)

Yue, Jing-zhong [Yueh, Ching-chung] 1639

Imbedding theory and cohomology of permutation groups. II.

Acta Math. Sinica **12** (1962), 341–351 (*Chinese*); *translated as Chinese Math.* **3** (1963), 372–381.

Let X be a topological space, p a prime, π a subgroup of the symmetric group S_p on p letters, G a π-module. In Part I [Acta Math. Sinica **12** (1962), 47–59; MR **27** #1951] the author introduced the imbedding groups $\Phi^q(X, \pi, G)$. Φ^q is a subgroup of $H_\pi{}^q(R(X), G)$, where $R(X)$ is the free part of the cartesian power X^p under the action of S_p, and $H_\pi{}^q$ is a π-equivariant cohomology. Let $I(X, \pi, G)$ be the largest q for which $\Phi^q \neq 0$. It was shown that if X is imbeddable in Y, then $I(X, \pi, G) \leq I(Y, \pi, G)$ for every π, G. In the present paper the author investigates the structure of Φ^* when π is Z_p or S_p and G is Z_p or Z. It is shown that Φ^* has the structure of a Hopf algebra, and the pth reduced powers in Φ^* are computed. In the second part of the paper, the theory is modified so as to apply to local imbeddings (immersions).

P. A. Smith (New York)

Uchida, Fuichi 1640

A note on the generalized homology theory.

Tôhoku Math. J. (2) **16** (1964), 81–89.

In this note, which may be considered as an addendum to G. W. Whitehead's paper [Trans. Amer. Math. Soc.

102 (1962), 227–283; MR **25** #573], the author shows that a spectrum defines a homology and a cohomology theory on the category of spaces with base points having the homotopy type of a *CW*-complex.

He also shows how, under suitable connectedness and co-connectedness assumptions on the spectrum, G. W. Whitehead's exact sequences involving the homology suspension have their analogues for generalized homology and cohomology theories. *J.-P. Meyer* (Baltimore, Md.)

tom Dieck, Tammo 1641

Die Hopfsche Invariante. (English and French summaries)

Ann. Univ. Sarav. Math.-Natur. Fak. No. 1 (1963), 56 *pp.*

Author's summary: "The first part of this paper deals with the intersection theory for triangulable homology manifolds and develops Hopf's original approach to his invariant. Further on we discuss functional cohomology operations and their relation to Hopf's original invariant. Finally, we give a proof of a theorem on Hopf's construction which (generalized) leads to a reduction of functional operations to ordinary ones." *H. Wada* (Sendai)

Samelson, Hans 1642

A note on the Bockstein operator.

Proc. Amer. Math. Soc. **15** (1964), 450–453.

For any pathwise connected space X, an element w in $H^1(X; Z_p)$ determines a unique homomorphism w: $H_1(X; Z)\to Z_p$. Let Z_p act on itself trivially and on Z_{p^2} by sending 1 into $p+1$. Composing w with the Hurewicz map, we have an action of $\Pi_1(X)$ on the sequence $0\to Z_p\to Z_{p^2}\to Z_p\to 0$ and thus a sequence of local coefficients $0\to Z_p\to \mathscr{Z}_{p^2}\to Z_p\to 0$. Let $\beta^w: H^*(X; Z_p)\to H^*(X; Z_p)$ be the Bockstein operator defined by this sequence and β_p: $H^*(X; Z_p)\to H^*(X; Z_p)$ the usual Bockstein operator. The main result of the paper is the following theorem: $\beta^w(v)=\beta_p(v)+w\cdot v$.

R. H. Szczarba (New Haven, Conn.)

Zhou, Xue-guang [Chow, Sho-kwan] 1643

A statement on my paper "Steenrod operations and homotopy groups. II".

Acta Math. Sinica **13** (1963), 544–547 (*Chinese*); *translated as Chinese Math.* **4** (1964), 593–597.

This statement is published to defend the author's original paper against certain remarks made in the present reviewer's original review [Sci. Record (N.S.) **2** (1958), 355–357, 358–363; MR **22** #7118a; Acta Math. Sinica **9** (1959), 227–263; MR **22** #7118b; Sci. Sinica **9** (1960), 155–171, 172–196; MR **22** #7118c]. It serves also to give extra details of his work, and to correct certain misprints in the Chinese versions. (Unfortunately, the English versions have different misprints.)

Briefly, the defence is successful. The fifth paragraph of the original review (concerning φ^{q+1}) is irrelevant, because it turns out that the operation the author has in mind is φ^{q+2}. As for the sixth paragraph of the original review (concerning the indeterminacy of the operation), it turns out that the reviewer was misled by a misprint in the English version [Sci. Record (N.S.) **2** (1958), 358–363, p. 360, line 11; MR **22** #7118a].

The reviewer would therefore like to withdraw the final part of the original review (from the words "Unless the details are misprinted" onwards) and express appropriate apologies to the author for misunderstanding his intentions. *J. F. Adams* (Manchester)

Green, Paul S. 1644

A cohomology theory based upon self-conjugacies of complex vector bundles.

Bull. Amer. Math. Soc. **70** (1964), 522–524.

Complex vector bundles over a space X (say a polyhedron) can be used to construct a Grothendieck group $K(X)$ [see the reviewer and Hirzebruch, Proc. Sympos. Pure Math., Vol. III, pp. 7–38, Amer. Math. Soc., Providence, R.I., 1961; MR **25** #2617]. The Bott periodicity theorem asserts that we have an isomorphism $K(X)\cong \tilde{K}(S^2(X))$, where $S^2(X)$ is the double suspension of X and $\tilde{K}$ denotes "K modulo the base point". Refinements of this theory arise when we consider vector bundles with non-degenerate bilinear forms in the fibres: if the forms are symmetric, we obtain groups $KO(X)$, while if they are skew-symmetric we obtain groups $KSP(X)$. The period for both these is 8. Moreover, $KO(X)$ can be interpreted alternatively (and more conventionally) as the Grothendieck group of real vector bundles and $KSP(X)$ as the Grothendieck group of quaternionic vector bundles. In the present note the author considers an intermediate theory in which the bilinear forms are not subjected to symmetry or skew-symmetry conditions. He obtains groups, denoted by $KSC(X)$, which have period 4.

This theory has also been developed, independently, by D. W. Anderson. *M. F. Atiyah* (Oxford)

Takeuchi, Masaru 1645

A remark on the theorem of Peterson-Adachi.

J. Fac. Sci. Univ. Tokyo Sect. I **10**, 124–128 (1964).

Using obstruction theory, the author proves the following theorem. Let X be a finite CW-complex with dim $X\leq n$, n even, and let f: $X\to BSO(n)$. Assume (1) the torsion of $H^{4r}(X; Z)$ is relatively prime to $a_r(2r-1)!$ for $r>0$, where $a_r=1$ if r is even and $a_r=2$ if r is odd, (2) $H^{8r+1}(X; Z_2)=0$ and $H^{8r+2}(X; Z_2)=0$ for $r>0$, and (3) if $n\equiv 6 \pmod 8$, then the torsion of $H^n(X; Z)$ is relatively prime to 2. Then the homotopy class of f is determined by the Pontrjagin classes, the second Stiefel-Whitney class, and the Euler characteristic of the bundle defined by f.

F. P. Peterson (Cambridge, Mass.)

O'Neill, Ronald C. 1646

On H-spaces that are CW-complexes. I.

Illinois J. Math. **8** (1964), 280–290.

H-spaces that are CW-complexes are examined from the point of view of the theory of loops. Homotopy classes of H-structure maps on an H-space X are shown to be in 1-1 correspondence with homotopy classes of maps $(X\times X,\ X\vee X)\to(X,\ e)$. Notions of homotopy power associativity and homotopy diassociativity are defined and related to the corresponding concepts for loops arising as homotopy sets. Results obtained here are to be applied to Postnikov systems of H-spaces in a sequel.

A. H. Clark (Providence, R.I.)

Hilton, P. J. 1647

On excision and principal fibrations.

Comment. Math. Helv. **35** (1961), 77–84.

In the category of based spaces with the homotopy type of CW-complex let Σ, Ω be suspension and loop-space, $\Pi(x, y)$ the based set of homotopy classes of maps, $\Pi_1(X, Y) = \Pi(\Sigma X, Y) \approx \Pi(X, \Omega Y)$. If $f: X \to Y$, let Cf be the mapping cone, i.e., the cofibre of a cofibration homotopy-equivalent to f, and C^*f the fibre of a fibration homotopy-equivalent to f. Set $\Pi_1(f, Z) = \Pi(Cf, Z)$, $\Pi_1(Z, f) = \Pi(Z, C^*f)$.

If $F \xrightarrow{j} X \xrightarrow{p} Y$ is a fibration, then ΣF may be compared with Cp and Y with Cj: if F and Y are acyclic in low dimensions and B aspherical in high dimensions, then $\Pi_1(p, B) \to \Pi_1(F, B)$ and $\Pi(Y, B) \to \Pi_1(j, B)$ are surjective monomorphisms. This is proved first when B is an Eilenberg-MacLane space, and extended inductively using a Postnikov system for B. It follows that a fibration $\Omega Q \to X \to Y$ is equivalent to one induced by a map $Y \to Q$ whenever Y is acyclic in low dimensions and the nonzero homotopy groups of Q lie in a narrow band.

Dually, if $Y \xrightarrow{i} X \xrightarrow{q} F$ is a cofibration, then ΩF may be compared with C^*i and Y with C^*q: if F and Y are acyclic in low dimensions and A is acocyclic in high dimensions, then $\Pi_1(A, i) \to \Pi_1(A, F)$ and $\Pi(A, Y) \to \Pi_1(A, q)$ are surjective monomorphisms. The proof is also dual, using a homology decomposition of A. It follows that a cofibration $Y \to X \to \Sigma Q$ is equivalent to one induced by a map $Q \to Y$ whenever Y is acyclic in low degrees and the nonzero homology groups of Q lie in a narrow band.

The arrows in diagram 1′ are reversed, a typographical error likely to lead to some confusion.

A. Heller (Urbana, Ill.)

TOPOLOGY AND GEOMETRY OF DIFFERENTIABLE MANIFOLDS

See also 1291, 1351, 1359, 1513, 1515, 1596, 1597, 1644, 1645.

Watabe, Tuyosi; Hirose, Tunehisa 1648

On the immersibility of almost parallelizable manifolds.

Proc. Japan Acad. **39** (1963), 715–716.

A closed manifold M is said to be almost parallelizable if $M - x$ is parallelizable, where x is a point of M. The authors prove the following: Any almost parallelizable C^∞ n-manifold can be immersed in euclidean $(n+1)$-space provided $n \not\equiv 0 \pmod 4$. The proof is a straightforward application of the results of J. F. Adams [Ann. of Math. (2) 75 (1962), 603–632; MR **25** #2614] and M. W. Hirsch [Trans. Amer. Math. Soc. **93** (1959), 242–276; MR **22** #9980].

R. H. Szczarba (New Haven, Conn.)

Mahowald, Mark 1649

On embedding manifolds which are bundles over spheres.

Proc. Amer. Math. Soc. **15** (1964), 579–583.

In this paper, the author investigates the embeddability of the total space of a fiber bundle over a sphere with fiber a π-manifold. The technique is to first exhibit a topological embedding and then apply the approximation theorem of Haefliger [Comment. Math. Helv. **36** (1961), 47–82; MR **26** #3069] to obtain a differentiable embedding. As an application of the main theorems, it is shown that RP^7 embeds topologically in R^{11}, differentiably in R^{12}, and that RP^{15} embeds differentiably in R^{24}.

R. H. Szczarba (New Haven, Conn.)

Kervaire, Michel 1650

La méthode de Pontryagin pour la classification des applications sur une sphère.

Topologia Differenziale (*Centro Internaz. Mat. Estivo*, 1° *Ciclo*, *Urbino*, 1962), *Lezione* 3, 13 *pp. Edizioni Cremonese, Rome.*

Expository article.

J. F. Adams (Manchester)

Roberts, Howard M. 1651

On the dual Pontrjagin classes of a manifold.

J. Math. Mech. **13** (1964), 511–520.

From the author's introduction: "This paper is devoted to proving that for manifolds of dimension n certain polynomials in the dual Pontrjagin classes must be zero modulo q, an odd prime. The methods are similar to those used by W. S. Massey [Amer. J. Math. **82** (1960), 92–102; MR **22** #1918] who showed that for certain values of n and k the $(n-k)$-dimensional dual Stiefel-Whitney class of an n-manifold is always zero. Later Massey and F. P. Peterson [Bol. Soc. Mat. Mexicana (2) **8** (1963), 1–13; MR **29** #628] showed that these results were, for non-orientable manifolds, the best possible. In 1962, E. H. Brown, Jr. and F. P. Peterson [Bull. Amer. Math. Soc. **69** (1963), 228–230; MR **26** #4369] determined all the relations that the Stiefel-Whitney classes of a manifold must satisfy.

"The results of this paper are a step toward solving the much more difficult problem of determining all the relations that the Pontrjagin classes of an n-manifold must satisfy."

R. H. Szczarba (New Haven, Conn.)

Edwards, H. M. 1652

A generalized Sturm theorem.

Ann. of Math. (2) **80** (1964), 22–57.

In his Colloquium volume [*The calculus of variations in the large*, Amer. Math. Soc., New York, 1934] M. Morse gave an analytical discussion of a generalization of the separation, comparison, and oscillation theorems of Sturm for positive definite, self-adjoint, second-order linear systems of ordinary differential equations with self-adjoint (and other) boundary conditions. R. Bott [Comm. Pure Appl. Math. **9** (1956), 171–206; MR **19**, 859] gave new geometric formulations and proofs of these results in terms of intersection numbers and deformations of cycles. In the paper under review a further generalization (in several directions) is made for systems of arbitrary order (necessarily even), following Bott's interpretation; furthermore, the exposition is self-contained and pleasant reading.

The two main ideas which enter here are the following. (1) An analysis of the index and nullity of certain quadratic forms in function spaces; this includes Morse's index theorem. (The author is aware of the close relation between Sturm theory and the spectral theorem in Hilbert space; however, it might be worthwhile to call attention to the work of M. Hestenes [Pacific J. Math. **1** (1951), 525–581, especially §§ 17–19; MR **13**, 759]. (2) A geometric

discussion of multiplicities of intersections of curves with certain subvarieties in a Grassmann manifold; in particular, a characterization is given of the desired intersection theory. Having established his general theorem in terms of quadratic forms, the author ends by a discussion of its relevance to classical formulation of Sturm's theory, and to the calculus of variations. *J. Eells* (Ithaca, N.Y.)

Conner, P. E.; Floyd, E. E. 1653

Periodic maps which preserve a complex structure.

Bull. Amer. Math. Soc. **70** (1964), 574–579.

In their informative Ergebnisse tract [*Differentiable periodic maps*, Academic Press, New York, 1964], the authors use an isomorphism between the set of bordism classes of free actions of a compact Lie group H on oriented manifolds and the bordism homology module $\Omega_*(B_H)$ to collect results on both sides. This paper announces the corresponding results in the complex case with some improvements.

Let H be a compact Lie group which acts differentiably on a compact manifold M and so on its stable tangent bundle also. A weakly complex action consists of such a pair (H, M) and a homotopy class of complex structures on the stable tangent bundle which are invariant under H. If H acts freely on M, then $M \to M/H$ is a principal H-bundle and determines a homotopy class $[M/H, B_H]$. Also M/H has a quotient weakly complex structure. This gives an isomorphism between the set of bordism classes of free weakly complex actions of H and $\mathscr{U}_*(B_H)$ (homology in the Milnor spectrum). The results are for the cases $H = \mathbf{Z}_p$, $(\mathbf{Z}_p)^k$, and $\mathbf{Z}_{p^k}$ with p prime: The reduced group $\tilde{\mathscr{U}}_*(B_{\mathbf{Z}_p})$ is generated as a $\mathscr{U}_*$-module by elements $[\mathbf{Z}_p, S^{2n-1}]$, one for each dimension, with $\mathbf{Z}_p \subset U(n)$ acting freely; the order of $[\mathbf{Z}_p, S^{2n-1}]$ is p^{k+1}, where $2k(p-1) < 2n-1 < 2(k+1)(p-1)$, and

$$p^k[\mathbf{Z}_p, S^{2k(p-1)+1}] = b[\mathbf{Z}_p, S^1][P_{p-1}(\mathbf{C})]^k$$

for $b \neq 0 \bmod p$. A consequence is that if M^{2n} is a closed almost-complex manifold with an involution respecting the structure, then the number of fixed points is of the form $k \cdot 2^n$ when they are isolated; moreover, $[M^{2n}] = k[P_1(\mathbf{C})]^n$ in $\mathscr{U}_*/(2\mathscr{U}_*)$. The elements of $\mathscr{U}_*$ which have representatives on which $(\mathbf{Z}_p)^k$ acts freely as a weakly complex action are precisely those whose Chern numbers are divisible by p. If a closed manifold M admits a weakly complex action of $\mathbf{Z}_{p^k}$, then its class in $\mathscr{U}_*$ is divisible by p, and there is a similar result for oriented bordism. Finally, a theorem on equivariant maps: The order of an element $\gamma \in \tilde{\mathscr{U}}_{2n-1}(B_{\mathbf{Z}_2})$ divides 2^n if and only if some representative $(\mathbf{Z}_2, M^{2n-1})$ has co-index $\leqq 2k$; it can be shown from this that there is no equivariant map of $P_{4n+1}(\mathbf{R})$ into S^{2n}. A fuller description of these results in the case $p=2$ is to be found in the authors' Seattle notes. *H. B. Shutrick* (Örebro)

PROBABILITY

See also 1222, 1689, 1721, 2110, 2111.

Freudenthal, Hans 1654

★Wahrscheinlichkeit und Statistik.

R. Oldenbourg, Munich, 1963. 143 *pp.* *DM* 18.00.

Eine deutsche, vom Verfasser ausgeführte Übersetzung der 1957 erschienenen holländischen Originalausgabe [Bohn, Haarlem, 1957]. Das Buch wendet sich an Leser mit einer normalen mathematischen Schulbildung. Es ist einerseits elementar und sehr anschaulich angelegt, andererseits ist es doch recht anspruchsvoll und führt verhältnismäßig weit: in der Wahrscheinlichkeitstheorie bis zu den klassischen Grenzwertsätzen und Markoffschen Ketten, in der Statistik über die Neyman-Pearsonsche Theorie einschließlich verteilungsfreier Methoden bis zu sequenten Verfahren und zur Spieltheorie. Als Ausgangspunkt dient die Beschreibung stochastischer Variabler und ihrer Verteilungen durch ein Urnenmodell, woran man sich allerdings nicht streng hält, und im letzten Kapitel, Philosophie der Wahrscheinlichkeit, erwähnt man die axiomatische Begründung. Viel wichtiger und nützlicher aber ist die Ausbildung eines Gefühls für wahrscheinlichkeitstheoretische und statistische Denkweisen anhand einer weit gestreuten Fülle gut gewählter Beispiele und Aufgaben. Manche möglichen Trugschlüsse werden in konkreter Form eingehend diskutiert. Der Stil ist lebhaft, oft witzig und schreckt auch vor Kalauern nicht zurück. *K. Krickeberg* (Heidelberg)

Fisz, Marek 1655

★Probability theory and mathematical statistics.

Third edition. Authorized translation from the Polish. Translated by R. Bartoszynski.

John Wiley & Sons, Inc., New York-London, 1963. xvi+677 *pp.* $15.75.

The second edition of this book was published simultaneously in Polish and in German in 1958 and was reviewed in these Reviews [Państwowe Wydawnictwo Naukowe, Warsaw, 1958; MR **22** #5995; VEB Deutscher Verlag Wiss., Berlin, 1958; MR **22** #5996]. The present third edition is more than just an English translation of the second edition. Although the table of contents has remained almost unchanged, substantial additions have been made in practically every chapter, containing some new material and bringing the presentation up-to-date. Each chapter is now followed by "Problems and Complements", which add up to about sixty pages, and contain exercises of different levels of difficulty and additional theory presented in sequences of problems. The Bibliography has grown from 11 pages in the second edition to 36 in the present edition. It is a saddening thought that the untimely death of the author has put an end to the development of this book which has been growing in stature with each consecutive edition.

Z. W. Birnbaum (Seattle, Wash.)

Darroch, J. N. 1656

On the distribution of the number of successes in independent trials.

Ann. Math. Statist. **35** (1964), 1317–1321.

Let $P_r = P_r(p_1, \cdots, p_n)$ denote the probability of r successes in n independent trials, where p_j is the probability of success in the jth trial. In Theorems 1 and 2, $p_j \neq 0, 1$ for all j. Theorem 1: $P_{r+1} - P_r \geqq 0$ implies $P_r - P_{r-1} > 0$, $r = 1, \cdots, n-1$. Theorem 2: $P_{r+1} - 2P_r + P_{r-1} \geqq 0$ and $P_r - P_{r-1} > 0$ imply $P_r - 2P_{r-1} + P_{r-2} > 0$, $r = 1, \cdots, n$. It follows that the sequence $\{P_r\}$ is "bell-shaped". The main content of Theorems 3 and 4 is that the

most probable number of successes is one of the two integers nearest to $\mu=\sum p_i$, the expected number of successes. (Theorem 1 also follows from

$$P_r^2 \geqq (1+r^{-1})(1+(n-r)^{-1})P_{r-1}P_{r+1}$$

[P. Lévy, *Théorie de l'addition des variables aléatoires*, p. 88, Gauthier-Villars, Paris, 1937].) The proof of Theorem 3 makes use of a theorem concerning the maximum of any linear function of $P_0, P_1, \cdots, P_n$, given μ, which is stated as Corollary 2.1 in the reviewer's paper [Ann. Math. Statist. **27** (1956), 713–721; MR **18**, 240]. {The reviewer wishes to point out that his Corollary 2.1 is an immediate extension of the second theorem in P. Tchebichef [J. Reine Angew. Math. **33** (1846), 259–267].}
W. Hoeffding (Chapel Hill, N.C.)

Kendall, David G. 1657
Extreme-point methods in stochastic analysis.
Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **1** (1962/63), 295–300.
The author shows how the Lévy-Khintchine integral representation for the Laplace-Stieltjes transform of an infinitely divisible distribution on [0, ∞] can be obtained from Choquet's theorem concerning the representation of the points of a non-empty compact convex subset K of a metrizable locally convex Hausdorf space as the centroid of a probability measure carried by the extreme points of K. *L. L. Helms* (Urbana, Ill.)

Newell, G. F. 1658
Asymptotic extremes for m-dependent random variables.
Ann. Math. Statist. **35** (1964), 1322–1325.
Let $X_1, X_2, \cdots$ be a stationary sequence of m-dependent random variables, and put $Y_n=\max(X_1, \cdots, X_n)$, $n\geqq 1$. The main result of the paper is that if $\{y_n : n\geqq 1\}$ is a real sequence such that $nP\{X_1>y_n\}$ is bounded for all $n\geqq 1$, then

$$\lim_{n\to\infty} (P\{Y_n \leqq y_n\} \times \exp[nP\{X_1 > y_n, \max(X_2, \cdots, X_m) \leqq y_n\}]) = 1.$$

This is applied to finding the limiting distribution of Y_n under conditions weaker than those given by Watson [same Ann. **25** (1954), 798–800; MR **16**, 385]. The author also gives an example showing that his condition is strictly weaker. *S. M. Berman* (New York)

Böge, Werner 1659
Zur Charakterisierung sukzessiv unendlich teilbarer Wahrscheinlichkeitsverteilungen auf lokalkompakten Gruppen.
Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2**, 380–394 (1964).
A probability measure b on a locally compact group G is successively infinitely (SI-) divisible if there are probability measures a_r, $r>0$ and rational, such that $b=a_1$ and $a_{r+s}=a_r * a_s$. This notion coincides with ordinary infinite divisibility for G finite or G the real line, but not in general. Here (Theorem 5.11) it is shown that coincidence holds for root-compact groups, those G for which to each compactum C and integer $n>0$ there corresponds a compactum C_n containing all members of sequences $x_1, x_2, \cdots, x_{n-1}, x_n=e$ such that Cx_iCx_k meets Cx_{i+k} whenever $i+k\leqq n$. There is also an extensive treatment of Poisson distributions on locally compact groups and of their relationship to SI-divisibility.
J. G. Wendel (Ann Arbor, Mich.)

Parthasarathy, K. R. 1660
The central limit theorem for the rotation group. (Russian summary)
Teor. Verojatnost. i Primenen. **9** (1964), 273–282.
Let G be a compact Lie group of dimension k, with T_g an irreducible representation and A_j, $j=1, 2, \cdots, k$, the corresponding infinitesimal operators along mutually orthogonal one-parameter subgroups. An infinitely divisible probability distribution μ on G that lacks idempotent factors is called Gaussian if for every T_g one has

$$\int T_g\, d\mu(g) = \exp\{\sum a_{ij}A_iA_j + \sum a_iA_i\},$$

$\|a_{ij}\|$ positive semidefinite, $a_i\geqq 0$. For $G=K_s$, the real orthogonal group in dimension s, the author obtains (Theorem 4.1) necessary and sufficient conditions in order that, for a sequence μ_n converging to unit mass at the identity, all limits of $\mu_n{}^{*n}$ shall be Gaussian. The necessary and sufficient conditions are analogous to those of Lindeberg and Lévy in the classical case.
J. G. Wendel (Ann Arbor, Mich.)

Parthasarathy, K. R.; Varadhan, S. R. S. 1661
Extension of stationary stochastic processes. (Russian summary)
Teor. Verojatnost. i Primenen. **9** (1964), 72–78.
The author proves that if x_t is a real-valued, stochastically continuous and relatively (strictly) stationary process defined in the interval $t\in[0, 1]$, then there exists a stochastically continuous stationary process X_t, $t\in R$ (real line), whose restriction to the unit interval is x_t. This implies the extension theorem of Kreĭn concerning positive definite functions.
T. Kawata (Washington, D.C.)

Fernique, Xavier 1662
Continuité des processus Gaussiens.
C. R. Acad. Sci. Paris **258** (1964), 6058–6060.
Let X be a centered Gaussian process with continuous covariance on $[0, 1]\times[0, 1]$ and φ an increasing positive function of a positive argument; φ is a weak module of X if

$$E\{[X(t)-X(s)]^2\} \leqq \varphi^2(|t-s|)$$

for any pair $(t, s)\in[0, 1]\times[0, 1]$. By means of this notion, a sufficient condition of almost sure continuity is given. Further, if φ is a positive function of a positive argument verifying certain conditions, there exist centered (stationary) Gaussian processes almost sure unbounded having φ as weak module. *R. Theodorescu* (Bucharest)

Maĭdanjuk, M. N.; Rubanik, V. P. 1663
Stochastic processes in the most elementary linear systems with retarded argument. (Russian)
Approximate methods of solving differential equations, pp. 60–63. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, **1963**.

Es seien a, b stetige über $[t_0, T]$ erklärte Funktionen und $\tau > 0$. $Y(t)$ sei eine über $[t_0-\tau, t_0]$ erklärte stetige zufällige Funktion. Es wird die stochastische Gleichung $dX(t)/dt + a(t)X(t)+b(t)X(t-\tau)=0$ mit der Anfangsbedingung $X(t)=Y(t)$, $t_0-\tau \leq t \leq t_0$ betrachtet. Daraus werden formal die Gleichung für den Mittelwert und die Korrelationsfunktion von $X(t)$ hergeleitet. *L. Schmetterer* (Vienna)

Šatašvili, A. D. 1664

On the transformation of Gaussian measures under linear transformations. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 437–440.

Author's summary: "Transformation of measures corresponding to Gaussian processes is considered in certain linear transformations. During these transformations measures corresponding to processes pass into measures that are absolutely continuous with respect to the initial measures. The measures obtained after transformation are characterized by calculating the density of these measures with respect to the initial ones."

Vasil'ev, P. I.; Kovalenko, I. N. 1665

A remark on stationary streams of homogeneous events. (Russian)

Ukrain. Mat. Ž. **16** (1964), 374–375.

The authors remark that if the parameter of a Poisson process $X(t)$ is chosen as a random variable with density proportional to $(x^2 \ln x)^{-1}$, $x \geq 2$, then $E[X(t)]=\infty$, $\lim_{t\to 0} t^{-1}$ Pr{at least one event occurs in $(0,t)$} $=\infty$, while $\lim_{t\to 0} t^{-1}$ Pr{at least two events occur in $(0,t)$} $=0$. The authors' desire to construct such an example was motivated by material in Hinčin's book [*Studies on mathematical queueing theory* (Russian), Fizmatgiz, Moscow, 1963]. [Since $X(t)$ does not have independent increments, the precise connection with Hinčin's work was not completely clear to the reviewer.] *E. Reich* (Stanford, Calif.)

Takács, L. [**Takács, Lajos**] 1666

★Processus stochastiques. Problèmes et solutions.

Traduction française d'après la traduction anglaise de P. Zádor. Monographies Dunod, 55.

Dunod, Paris, 1964. ix+190 *pp.* 18.00 *F.*

This is a translation into French from the English version [*Stochastic processes: Problems and solutions*, Methuen, London, 1960; MR **24** #A572], which was an outgrowth of *Theory of probability* (Hungarian) [Tankönyvkiadó, Budapest, 1957; MR **21** #3909] by P. Medgyessy and L. Takács.

Takács, Lajos 1667

A combinatorial method in the theory of Markov chains.

J. Math. Anal. Appl. **9** (1964), 153–161.

The combinatorial method of the title consists in the use of a generalization of the ballot theorem, given in earlier publications of the author [Operations Res. **9** (1961), 402–407; MR **23** #A2957; J. Amer. Statist. Assoc. **57** (1962), 327–337; MR **25** #1586], to the determination of two sequences of random variables, resembling the waiting time and queue size associated with imbedded Markov chains in queueing theory. More specifically, let $\{\nu_n\}$ be a sequence of mutually independent random variables with distribution $P\{\nu_n=j\}=q_j$, $j=0,1,\cdots$, and write $s_n=\nu_1+\cdots+\nu_n$; let $\xi_n=[\xi_{n-1}+1-\nu_n]^+$, $n=1,2,\cdots$ where ξ_0 is an arbitrary random variable assuming non-negative integer values and $[a]^+=\max(0,a)$. The sequence $\{\xi_n\}$ is a homogeneous Markov chain, and it is shown that for $k=1,2,\cdots$; $i=0,1,\cdots$,

$$P\{\xi_n \geq k, \xi_0=i\} = P\{s_n \leq n+i-k\} + \sum_{j=k}^{n-1} \frac{k}{j} P\{s_j=j-k\}P\{s_n-s_j > n+i-j\}.$$

A similar but more complicated result is obtained for the sequence $\{\zeta_n\}$ with ζ_0 a random variable of the same sort as ξ_0, and $\zeta_n=[\zeta_{n-1}-1]^++\nu_n$, $n=1,2,\cdots$. Finally, the limiting distributions of both sequences are found.

J. Riordan (Murray Hill, N.J.)

Rozenblat-Rot, M. [**Rosenblatt-Roth, Millu**] 1668

On the relative stability of sums of negative random variables arranged in a Markov chain. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 261–263.

Consider a Markov chain ξ_k, $1 \leq k < \infty$, in which every ξ_k takes non-negative values, put $S_n=\sum_{k=1}^n \xi_k$, $F_k(x)=P(\xi_k \leq x)$, and define

$$\alpha_k = 1-\sup_{x,y,E} |P(\xi_k \in E|\xi_{k-1}=x)-P(\xi_k\in E|\xi_{k-1}=y)|,$$

$$\alpha^{(n)} = \min_{1\leq i<n} \alpha_i,$$

where E is a Borel set on the line. S_n is said to be relatively stable if there exists a normalizing sequence $A_n>0$, $1\leq n<\infty$, such that $P(|A_n^{-1}S_n-1|\geq\delta)\to 0$ as $n\to\infty$ for any $\delta>0$. In case the ξ_k's are independent, conditions for the relative stability were studied by Khintchine and Bobrov [B. V. Gnedenko and A. N. Kolmogorov, *Limit distributions for sums of independent random variables* (Russian), GITTL, Moscow, 1949; MR **12**, 839; English transl. Addison-Wesley, Cambridge, Mass., 1954; MR **16**, 52]. The author announces various sufficient conditions for relative stability. A sample is as follows: For S_n to be relatively stable with a normalizing sequence $B_n>0$ subject to the condition (1) $\max_{1\leq k\leq n} P(B_n^{-1}\xi_k \geq \varepsilon)\to 0$ as $n\to\infty$ for any $\varepsilon>0$, it is sufficient that there exists A_n, $1\leq n<\infty$, such that

$$\sum_{k=1}^n P(\xi_k \geq \varepsilon A_n\alpha^{(n)}) \to 0,$$

$$\frac{1}{A_n}\sum_{k=1}^n \int_0^{\varepsilon A_n\alpha^{(n)}} x\,dF_k(x) \to 1$$

as $n\to\infty$ for any $\varepsilon>0$; in this case A_n, $1\leq n<\infty$, can be a normalizing sequence subject to (1).

G. Maruyama (Fukuoka)

Nagasawa, Masao; Sato, Keniti 1669

Some theorems on time change and killing of Markov processes.

Kōdai Math. Sem. Rep. **15** (1963), 195–219.

Consider a conservative Markovian motion x with invariant measure m:

$$\int m(da)P_a[x(t)\in db] = m(db);$$

let φ^{-1} be the inverse function of a positive additive

functional φ of x; let $\mathfrak{e}$ be an independent exponential holding time with law $P.[\mathfrak{e}>t]=e^{-t}$; and consider the derived motions: (a) $x(\varphi^{-1})$ defined by the time substitution $t\to\varphi^{-1}$, and (b) $x(t)$ annihilated at time $\varphi^{-1}(\mathfrak{e})$. Both motions are Markovian; the terminal measure of the killed motion with initial distribution e is defined as

$$\int e(da)P_a[x(\varphi^{-1}(\mathfrak{e})-)\in db].$$

A motion $\hat{x}$ is dual to x relative to m if

$$m(da)P_a[x(t)\in db] = m(db)\hat{P}_b[\hat{x}(t)\in da];$$

m is then an invariant measure for $\hat{x}$ also. Connections between additive functionals, time substitutions, killing, invariant measures, terminal measures, and duality are considered here under mild technical conditions on the basic motion x. A typical result is that if x and $\hat{x}$ are dual as above and if $\hat{\varphi}$ is an additive functional of $\hat{x}$, related to φ according to the (formal) rule:

$$E_a[d\varphi|x(t) = b]m(db) = \hat{E}_a[d\hat{\varphi}|\hat{x}(t) = b]m(db) \equiv dt\times n(db),$$

then $x(\varphi^{-1})$ and $\hat{x}(\hat{\varphi}^{-1})$ are dual relative to n, and x killed at time $\varphi^{-1}(\mathfrak{e})$ is dual to $\hat{x}$ killed at time $\hat{\varphi}^{-1}(\mathfrak{e})$ relative to m. Related results will be found in the papers of Blumenthal and Getoor [Trans. Amer. Math. Soc. **112** (1964), 131–163; MR **28** #3483; Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **3** (1964), 50–74; MR **29** #2849].

H. P. McKean, Jr. (Cambridge, Mass.)

Hennequin, Paul-Louis 1670

Processus de Markoff en cascade.

Ann. Inst. H. Poincaré **18**, 109–195 (1963).

Der Autor betrachtet eine spezielle Klasse Markovscher Prozesse, die mit Hilfe der "elementaren Übergänge" charakterisiert werden. Ein homogener Markovscher Prozeß, dessen Zustandsraum K_n aus den n-dimensionalen Vektoren mit ganzen Komponenten besteht, ist definiert durch die Übergangswahrscheinlichkeiten $P_I^J(t)$. Sei $\lim_{t\downarrow 0}(1-P_I^I(t))/t=Q_I\leqq\infty$ und $\lim_{t\downarrow 0}P_I^K(t)/t=Q_I^K<\infty$; $K\neq I$. Für $Q_I\neq 0$ wird gesetzt $Q_I^J=Q_I\Pi_I^J$, wenn $I\neq J$ und $\Pi_I^I=0$; fur $Q_I=0$ sei $\Pi_I^I=1$ und $\Pi_I^J=0$, wenn $J\neq I$. Ein Paar (I,J) mit $\Pi_I^J>0$ heißt ein "elementarer Übergang". Ein Markovscher Prozeß auf K_n wird Kaskadenprozeß genannt, wenn seine elementaren Übergänge durch $J-I\in\mathfrak{T}$ bestimmt sind, wobei $\mathfrak{T}$ eine endliche, von I unabhängige Teilmenge von K_n ist.

Bei den meisten Beispielen besteht der Zustandsraum nicht aus K_n, sondern aus der Menge E^n der Punkte aus R^n mit ganzen positiven Koordinaten, so daß eine Präzisierung der Definition der elementaren Übergänge an den Seitenflächen von E^n notwendig ist. Sodann werden die Eigenvektoren des Operators $\Pi=(\Pi_I^J)$ untersucht. Anschließend gibt der Autor einige Kriterien dafür an, daß eine Teilmenge von E^n eine Verweilmenge ist. Weiter werden die Trajektorien des Prozesses studiert, indem die Menge der Trajektorien auf eine Teilmenge A der positiven Halbgeraden abgebildet wird. Dies führt auf das Problem eines absorbierenden Randes eines homogenen Prozesses auf der Geraden, was sich mit Hilfe von erzeugenden Funktionen und einer Methode von A. Tortrat [C. R. Acad. Sci. Paris **245** (1957), 1872–1874; MR **19**, 1093] behandeln läßt. Insbesondere wird der Fall einer im Zustandsraum homogenen Kette betrachtet, d.h. $\Pi_I^{I+J}=A_J$ hängt nicht von I ab. Der Zustandsraum ist eine kommutative Gruppe G. Der Autor untersucht den Fall, wo alle Punkte von G und den Fall, wo die Elemente einer Halbgruppe von G, ausgehend von einem Element von G, erreicht werden können. Dabei interessiert er sich vor allem für die Eigenschaften des Fellerschen Randes.

Ein Ergebnis von K. Itô und H. P. McKean, Jr. [Illinois J. Math. **4** (1960), 119–132; MR **22** #12317] wird auf den Fall einer symmetrischen Irrfahrt verallgemeinert. Die Warte- und Verweilzeiten von G werden mit Hilfe von Differenzengleichungen, die auch bei der approximativen Lösung des Dirichletschen Problems auftreten, berechnet.

Schließlich werden diese Resultate auf inhomogene Ketten auf einer Gruppe G übertragen.

K. Dietz (Heidelberg)

Csiszár, Imre 1671

Eine informationstheoretische Ungleichung und ihre Anwendung auf den Beweis der Ergodizität von Markoffschen Ketten. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 85–108.

Let $f(u)$, $0<u<+\infty$, be a continuous convex function. Let P and Q be probability measures dominated by a σ-finite measure on the measurable space (X,S_X) with λ-densities or Radon-Nikodym derivatives $p(x)=P(dx)/\lambda(dx)$, $q(x)=Q(dx)/\lambda(dx)$. The f-divergence of the probability measures P and Q is defined by

$$\mathscr{I}_f(P,Q) = \int_X q(x)f(p(x)/q(x))\lambda(dx).$$

If P and Q are interpreted as distributions of signals arising from two different sources at the input to the channel $(X,Y,\nu(\cdot/x))$, then the distributions $\bar{P}$ and $\bar{Q}$ of the output signals are obtained by the formulas

$$\bar{P}(B) = \int_X \nu(B|x)P(dx), \qquad \bar{Q}(B) = \int_X \nu(B|x)Q(dx).$$

The author proves that

$$(*)\qquad \mathscr{I}_f(P,Q) \geqq \mathscr{I}_f(\bar{P},\bar{Q})$$

and gives the condition for equality in (*). The author defines the measure Q^* on the product space $(X\times Y, S_X\times S_Y)$ of the input and output spaces (X,S_X), (Y,S_Y), respectively, by $Q^*(A\times B)=\int_A\nu(B|x)Q(dx)$ for sets of the form $A\times B$, $A\in S_X$, $B\in S_Y$ and then by extension in the usual manner over $S_X\times S_Y$. The author proves a stability theorem: If the function $f(u)$, $0<u<+\infty$, is twice differentiable and $\inf_{0<u<b}f''(u)>0$ for every $0<b<+\infty$, then for every α, $0<\alpha<1$, a positive number δ, depending only on f and α, can be specified, such that for arbitrary $0<\varepsilon<1/\alpha$ and any two probability measures P, Q on (X,S_X) with finite $\mathscr{I}_f(\bar{P},\bar{Q})$ and

$$\mathscr{I}_f(P,Q)-\mathscr{I}_f(\bar{P},\bar{Q}) < \delta\varepsilon^2,$$

there holds the inequality

$$Q^*\left(\left\{(x,y):\left|\frac{p(x)}{q(x)}-\frac{\bar{p}(y)}{\bar{q}(y)}\right|\leqq\varepsilon\right\}\right)\geqq 1-\alpha.$$

The author applies results developed in the paper to give a pure information-theoretic proof of the ergodicity of certain Markov chains, without use of matrix theory.

S. Kullback (Washington, D.C.)

Blackwell, David; Freedman, David 1672

The tail σ-field of a Markov chain and a theorem of Orey. *Ann. Math. Statist.* **35** (1964), 1291–1295.

If $\{x_n : 0 \leqq n < \infty\}$ is a sequence of random variables on the probability triple $(\Omega, \mathfrak{F}, P)$, $\mathfrak{F}^{(n)}$ is the smallest σ-field over which x_i, $i \geqq n$, are measurable, and $\mathfrak{F}^{(\infty)} = \bigcap_{n=0}^{\infty} \mathfrak{F}^{(n)}$ is the tail σ-field. As usual, an event is called invariant if it is invariant under the shift operation, and an event is exchangeable if it depends measurably on the x_n, $n \geqq 0$, and is invariant under finite permutations of them.

The first main result is that when $\{x_n\}$ is a Markov chain with stationary transition probabilities, countable state space I, and all states recurrent, $P^{(\infty)}$ is atomic under P, each atom being of the form $[x_0 \in I_c]$ for some $c \in C$, where $\{I_c\}$, $c \in C$, is the partition of I into cyclically moving sub-classes. The proof uses the Hewitt-Savage zero-one law.

The second principal result, proved by means of a martingale convergence theorem, is that the sequence $\{x_n\}$ has a tail σ-field trivial under P if and only if

$$\lim_{n\to\infty} \left[\sup_{A \in \mathfrak{F}^{(n)}} |P(A \cap B) - P(A)P(B)| \right] = 0$$

for each $B \in \mathfrak{F}$.

It is pointed out that the results above imply that in case x_n is a Markov chain with stationary transition probability matrix p, and one ergodic class of recurrent aperiodic states, then $\lim_{n\to\infty} \sum_j |p^n(i_1, j) - p^n(i_2, j)| = 0$, which was shown by the reviewer [Z. Wahrscheinlichkeitstheorie Verw. Gebiete **1** (1962), 174–176; MR **26** #3117]. {Thus still another proof becomes available for the usual ergodic theorem for Markov chains first found by Kolmogorov.} There is a discussion of the tail σ-fields of a random walk on a countable Abelian group and the σ-field of exchangeable sets of recurrent Markov chains. The paper concludes with three examples of Markov chains with countable state space and stationary transition matrix p. In example 1 all states are recurrent, aperiodic and in the same ergodic class, but two independent particles starting in different states and moving according to p need never meet. In example 2 the chain, which is transient, has trivial invariant σ-field, but non-atomic tail σ-field. In the third example, in which the state space has only three states, the chain is aperiodic and ergodic but has a nontrivial σ-field of exchangeable events.

S. Orey (Minneapolis, Minn.)

Jadrenko, M. Ĭ. 1673

Isotropic random fields of Markov type in Euclidean space. (Ukrainian. Russian and English summaries) *Dopovidi Akad. Nauk Ukraïn. RSR* **1963**, 304–306.

Author's summary: "The spectral decomposition of an isotropic (but not necessarily homogeneous) random field is indicated in Euclidean space. Isotropic random fields possessing the following Markov properties are studied: at any $r_1 < r$ with probability equal to unity,

$$\mathrm{proj}_{H_\xi(r)}\, \xi(r_1, \theta_1, \cdots, \theta_{n-2}, \varphi) = \mathrm{proj}_{G_\xi(r)}\, \xi(r_1, \theta_1, \cdots, \theta_{n-2}, \varphi),$$

where $H_\xi(r)$ and $G_\xi(r)$ are closed linear manifolds generated by random values $\xi(u, \theta_1, \cdots, \theta_{n-2}, \varphi)$ with $u > r$ and $u = r$, respectively."

Zaslavskiĭ, A. H. 1674

On the isomorphism problem for stationary processes. (Russian. English summary) *Teor. Verojatnost. i Primenen.* **9** (1964), 318–326.

Author's summary: "The central problem in ergodic theory is that of isomorphism. In the paper the sufficient condition for isomorphism of the stationary process $\xi = (\cdots, \xi_{-1}, \xi_0, \xi_1, \cdots)$, $\xi_n = 0, 1, \cdots, l$, with some stationary process $\eta = (\cdots, \eta_{-1}, \eta_0, \eta_1, \cdots)$, $\eta_n = \alpha_1, \cdots, \alpha_m$, $m \leq l$, is found. This condition is expressed in terms of a one-dimensional distribution of the process ξ. Isomorphism is constructed with the aid of elementary codes $(i) = \eta_1{}^i \eta_2{}^i \cdots \eta_{\omega_i}{}^i$ $(i = 1, \cdots, l)$, which code the elementary words

$$(i) = i\underbrace{00\cdots0}_{\omega_i}.$$

One of the examples considered proves that it is possible to construct a system of elementary codes for any arbitrary l and m. This system possesses some properties which secure unique decoding of the sequence η."

C. W. Helstrom (Pittsburgh, Pa.)

Dvoretzky, Aryeh 1675

On the oscillation of the Brownian motion process. *Israel J. Math.* **1** (1963), 212–214.

Let $x(t)$, $-\infty < t < \infty$, be a separable Brownian motion process on the real line. It is proved that there is a positive constant c $(\leqq \infty)$ such that

$$\inf_{-\infty < t < \infty} \limsup_{h \to 0+} \frac{|x(t+h) - x(t)|}{h^{1/2}} \geqq c$$

with probability one. This improves a classical result of Paley, Wiener and Zygmund [Math. Z. **37** (1933), 647–668] that almost no Brownian paths satisfy a uniform Lipschitz condition of order greater than $\frac{1}{2}$.

T. Watanabe (Urbana, Ill.)

Bártfai, Pál 1676

Irrfahrtsprobleme mit einer spiegelnden Wand. (Russian summary) *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **8** (1963), 125–133.

Let $\xi_1, \xi_2, \cdots$, be a sequence of independent, identically distributed random variables on the non-negative integers. If the $(n-1)$st step of a random walk on the integers finds it at k, let the nth step be $|k - \xi_n|$. The author shows that if ξ_n has finite mean, then all states of the associated Markov chain are recurrent and have finite mean recurrence time. He finds the limit distribution for the chain and the mean recurrence time to the origin. If ξ_n has infinite mean, he shows that there are no recurrent states.

P. E. Ney (Ithaca, N.Y.)

Taylor, S. J. 1677

The exact Hausdorff measure of the sample path for planar Brownian motion. *Proc. Cambridge Philos. Soc.* **60** (1964), 253–258.

Let $\psi(a) = a^2 \log(1/a) \log\log\log(1/a)$, and let $C(t, \omega)$ be the segment of the planar Brownian motion path up to time t. Z. Ciesielski and the author [Trans. Amer. Math. Soc. **103** (1962), 434–450; MR **26** #816] conjectured that the

Hausdorff ψ-measure, denoted by ψ-m, will be exact for planar Brownian motion in the following sense: with probability one,

$$(*) \qquad c_1 \leqq \psi\text{-m}(C(1, \omega)) \leqq c_2$$

for suitable positive constants c_1, c_2. The first inequality was proved by D. Ray [ibid. **106** (1963), 436–444; MR **26** #3129]. In the present paper the author obtains an upper bound in (*), completing the proof of the conjecture. The method employed here is to find an economical covering by squares of different sizes, based on the computations in the paper of Ray. Finally, it is pointed out that (*) implies that there is a constant $\theta > 0$ satisfying

$$\psi\text{-m}(C(t, \omega)) = \theta t$$

for every t with probability one.
T. Watanabe (Urbana, Ill.)

Viskov, O. V.; Prohorov, Ju. V. 1678
Probability of loss of a call in heavy traffic. (Russian. English summary)
Teor. Verojatnost. i Primenen. **9** (1964), 99–104.
The authors study the loss probability p of a system of n fully available trunks with refusals under the assumptions of independent interarrival periods, exponential service times with parameter μ, and high arrival density θ, $\theta = \mu^{-1}$/mean interarrival time. Using the exact expression for p due to Palm and Takács [Takács, Acta Math. Acad. Sci. Hungar. 8 (1957), 319–324; MR **20** #2801], the following limit theorem for a sequence of systems indexed by N is obtained. Suppose (i) $\mu_N \equiv \mu$; (ii) $n_N = (1 - \alpha_N)\theta_N$, where $0 < \alpha_N < 1$, $\theta_N \alpha_N{}^2 \to \infty$; (iii) the variance of the interarrival periods is bounded uniformly with respect to N. Then $p_N/\alpha_N \to 1$. The significance of the result is that the asymptotic behavior of p_N is independent of the form of the interarrival distribution.
The authors make some remarks on the more complicated limiting case when $\theta\alpha^2$ is small.
E. Reich (Stanford, Calif.)

Weiss, George H. 1679
Effects of a distribution on gap acceptance functions on pedestrian queues.
J. Res. Nat. Bur. Standards Sect. B **68B** (1964), 31–33.
Author's summary: "The theory of pedestrian queueing has been developed by several authors under the assumption that all pedestrians have the same gap acceptance functions. In this paper the theory is modified to take into account a distribution of gap acceptance functions. Some calculations made with the new formulas seem to indicate that only if traffic is very heavy will the difference between the results of the two theories become noticeable."

Dall'Aglio, Giorgio 1680
Present value of a renewal process.
Ann. Math. Statist. **35** (1964), 1326–1331.
$X_1, X_2, \cdots$ are independent and identically distributed, non-negative random variables, which are the "inter-replacement" times in a renewal process. Suppose the cost of each replacement is the same and consider the present value of all items replaced between times 0 and ∞. If $C(\rho)$ is the present value and $\rho > 0$ is the rate of interest, under some circumstances, $C(\rho)$ is asymptotically normally distributed as $\rho \to 0$. *M. Dwass* (Evanston, Ill.)

Iglehart, Donald L. 1681
Multivariate competition processes.
Ann. Math. Statist. **35** (1964), 350–361.
A multivariate competition process (M.C.P.) is a stationary, continuous time Markov process whose state space is the lattice points of the positive orthant in N-dimensional space and whose transition probability matrix only allows jumps to certain nearest neighbors. It is the natural generalization of birth-and-death processes. Results of Reuter [Proc. Fourth Berkeley Sympos. Math. Statist. and Prob., Vol. II, pp. 421–430, Univ. California Press, Berkeley, Calif., 1961; MR **26** #812] are extended to obtain sufficient conditions for an M.C.P. to be regular, positive recurrent, absorbed with certainty, and to have finite mean absorption time. *S. Karlin* (Stanford, Calif.)

STATISTICS

See also 1509, 1654, 1655, 1681, 1723, 1758, 2098, 2113, 2120.

Wine, R. Lowell 1682
★Statistics for scientists and engineers.
Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964. xvi + 671 *pp.* \$12.00.
An introductory text. Table of Contents: (1) Introduction; (2) Frequency distributions; (3) Theoretical distributions; (4) Probability; (5) Sampling and sampling distributions—expectation and estimation; (6) Sampling from normal populations; (7) Sampling from normal populations—the chi-square distribution; (8) Sampling from normal populations—the Student t-distribution; (9) Sampling from normal populations—F-distribution with applications; (10) Analysis of variance—one-way classification; (11) More about the one-way classification; (12) Analysis of variance—multiway classifications; (13) An introduction to factorials; (14) Regression and related topics; (15) Analysis of counted data; (16) Distribution-free methods. Statistical tables.

Finch, P. D. 1683
The theory of information and statistical inference. I.
J. Appl. Probability **1** (1964), 121–140.
From the author's introduction: "The purpose of this paper is to construct a theory of the amount of information provided by an experiment which does not rely on what Good (1962) has termed the modern Bayesian principle that it is legitimate to use the axioms of probability even when this involves the use of probabilities of hypotheses. In this respect the theory of this paper differs from ... Lindley (1956), Mallows (1959) and Good (1960), each of which is written from a Bayesian viewpoint."

In order to achieve this purpose, the author replaces probabilities of hypotheses by their "acceptabilities", although a Bayesian would prefer to interpret this term as an expected utility, initial, intermediate, or final (prior, intermediate, or posterior).

Amongst other things, there is a desideratum-explicatum approach to conditional and unconditional amounts of information and to discrimination, which leads, for example, to a measure of discrimination between two hypotheses essentially identical with "weight of evidence".

A frequency definition is suggested for a (generalized) probability given a disjunction of hypotheses. This definition depends on an imaginary experiment in which the hypotheses occur with equal frequency.

I. J. Good (Ashford)

Johnson, N. L.; Nixon, Eric; Amos, D. E.; Pearson, E. S. 1684

Table of percentage points of Pearson curves, for given $\sqrt{\beta_1}$ and β_2, expressed in standard measure.

Biometrika **50** (1963), 459–498.

Tables are given for the median and the lower and upper 0.25, 0.5, 1.0, 2.5, 5.0, 10.0 and 25.0 percentage points, all in standard measure, of the Pearson frequency curves for which $\sqrt{\beta_1}=0.0(0.1)2.0$ and β_2 increases by steps of 0.2 within a region indicated by Fig. 1 of the paper. Interpolation procedures, illustrative examples, and methods of computation are described.

R. L. Plackett (Newcastle upon Tyne)

Chino, Sadako 1685

The relation between correlation ratio and success rate in the classification by the quantification method. (Japanese. English summary)

Proc. Inst. Statist. Math. **11** (1963/64), 7–24.

Bunke, Olaf 1686

Theorie der Bereichschätzung statistischer Parameter. (Russian summary)

Teor. Verojatnost. i Primenen. **8** (1963), 330–337.

Let X be an observable random variable with values $x\in\mathfrak{X}$ whose distribution is known to belong to a family indexed by a vector parameter $\omega\in\Omega$ with s components. The problem is to estimate a subset $\omega^0\subset\Omega^0$ of these components, the others being nuisance parameters. For each $x\in\mathfrak{X}$ the estimator of ω^0 is $B(x)$, a closed subset of Ω^0. Generalizing the ideas of E. Lehmann, the author treats this problem using a loss function Φ non-decreasing in the following three arguments: $\varphi_B(\omega^0)$ equal to zero or unity according to whether $\omega^0\in B$ or not; $\mu(B)$, the Lebesgue measure of B and $\rho(\omega^0, B)$, the position function of ω^0 with respect to B, perhaps $\rho(\omega^0, B)=\inf\|\omega_1{}^0B\|$ over $\omega_1{}^0\in B$. With certain specializations of Φ, some existence and completeness theorems are proved.

J. Neyman (Berkeley, Calif.)

Riedwil, Hans 1687

Approximation empirischer Verteilungen unter Einsatz elektronischer Rechengeräte. (French, English, and Italian summaries)

Mitt. Verein. Schweiz. Versich.-Math. **64** (1964), 59–117.

Author's summary: "The author deals with certain frequency curves, namely the Normal distribution, the Gram-Charlier distribution, the Edgeworth distribution, the Lognormal distribution, the Logarithmic Gram-Charlier distribution, and the Logarithmic Edgeworth distribution. He shows their relation between each other and also to the Normal standard distribution and explains how they are qualified for the approximation of empirical distributions with the help of electronic computers. For the estimation of parameters the methods of maximum likelihood and of moments are briefly dealt with. For the goodness of fit the author describes the most important tests at present in use, as the Chi-squared test, the Kolmogorov test, the Cramér-von Mises-Smirnov test, and the Anderson-Darling test. He explains especially the most adapted formulas for the solution of numerical problems derived from these tests."

Rahman, N. A. 1688

Some generalisations of the distributions of product statistics arising from rectangular populations.

J. Amer. Statist. Assoc. **59** (1964), 557–563.

Author's summary: "This paper presents a unified approach to the distributions of the product of $k\,(\geq 2)$ maximum values, and the product of k sample ranges from rectangular populations, both for equal and unequal sample sizes. The results obtained are generalisations based on some earlier work of Rider. It is proved that, in all cases, the distributions can be conveniently transformed to χ^2 by the use of a well-known logarithmic transformation. It is also suggested that these results might be of use in quality control procedures in high-precision production engineering with close specification limits."

P. S. Dwyer (Ann Arbor, Mich.)

Davis, A. W. 1689

A note on the characteristic functional for a replacement process of Gani.

J. Appl. Probability **1** (1964), 157–160.

Gani and Yeo [Austral. J. Statist. **4** (1962), 55–60] have considered the age-distribution of n ranked elements after several replacements. In a separate paper by the author [J. Austral. Math. Soc. **4** (1964), 233–243] the characteristic functional of the replacement process described by Gani and Yeo has been obtained explicitly. The purpose of the present note is to consider the product densities for this process. Explicit formulae are given for the first- and second-order product densities.

H. L. Harter (Dayton, Ohio)

Cochran, William G. 1690

On the performance of the linear discriminant function. (French summary)

Bull. Inst. Internat. Statist. **39** (1962), *livraison* 2, 435–447.

The problem of selecting variates to be included in a linear discriminant function is studied in the paper with the following chief objects: first, to derive rules for discarding variates which show little separation of the two populations in question, and second, to find rough estimates of the discriminating power of the linear discriminant function without computing it. The author discusses the contribution of individual variates to the discriminating power of the discriminant function and derives rules for the rejection of poorly separating variates. The results of the study are confronted with numerical applications reported in the literature.

J. Machek (Prague)

Papamichail, D. 1691
Sur un théorème de Fisher à l'estimation.
Bull. Soc. Math. Grèce (N.S.) **4** (1963), *no.* 1, 74–78.

The author proves the following theorem. Let $f(x, \theta)$ be a distribution law depending on an unknown parameter θ. Let $\psi(\theta)$ be a function of the parameter. If t_1 and t_2 are two unbiased estimators of $\psi(\theta)$ whose limiting distribution is a two-dimensional normal distribution and if t_1 is efficient (i.e., its variance is equal to the Cramér-Rao lower bound), then the correlation coefficient of t_1 and t_2 is equal to the second square root of Var t_1/Var t_2. For the case $\psi(\theta) \equiv \theta$ this is the well-known result due to R. A. Fisher.
J. Machek (Prague)

Fox, Martin; Rubin, Herman 1692
Admissibility of quantile estimates of a single location parameter.
Ann. Math. Statist. **35** (1964), 1019–1030.

C. Stein [same Ann. **30** (1959), 970–979; MR **22** #278] gave a proof of the admissibility of Pitman's estimator of a location parameter, loss measured by square error, under certain assumptions on moments. Preceding Stein's was a paper by D. Blackwell [ibid. **22** (1951), 393–399; MR **13**, 260]. Blackwell treated more general types of loss functions under very much more restrictive assumptions on the probabilities of the random variables. The present paper is one of several papers by various authors which are about to appear and which try to develop further ideas in the papers by Blackwell and Stein. Variations on the theme which have been considered by various authors include fixed sample size and sequential procedures, different types of loss functions, invariant and non-invariant estimators, and relationships between assumptions on moments and the goodness of the result obtained.

The present paper works with the fixed sample size problem for the loss function described below. $L(\theta, d) = a(\theta - d)$ if $d \leqq \theta$, $L(\theta, d) = b(d - \theta)$ if $d \geqq \theta$, where $a > 0$ and $b > 0$. The reviewer has shown that for loss functions like $L(\cdot)$ if there is more than one minimax invariant estimator, then every invariant estimator is inadmissible. The authors consider the admissibility of the minimax invariant estimator of θ on the hypothesis that the minimax invariant estimator is uniquely determined.

One observes a random variable (X, Y), where X is real-valued and the conditional probability measure of $X - \theta$, given Y, is $P(\cdot, Y) \cdot \theta$, assumed unknown, is the location parameter to be estimated. An estimator δ of the form $\delta(X, Y) = X + \delta_0(Y)$ is said to be an invariant estimator.

Most of the paper consists of a tremendous calculation to show, for the measure of loss specified above, that if the minimax invariant estimator is uniquely determined and square integrable, then it is an almost admissible estimator of θ. That is, if ρ is the risk function, "δ is almost admissible provided that, given any other procedure δ', if $\rho(\theta, \delta') \leqq \rho(\theta, \delta)$ for all θ, then $\rho(\theta, \delta') = \rho(\theta, \delta)$ a.e. (μ)". μ is Lebesgue measure on the line. In addition, the authors prove that if, moreover, $P(\cdot, y)$ is absolutely continuous for each y or has its points of increase in a fixed lattice for all y, then the minimax invariant estimator is admissible. Two examples are given to show that the conditions just stated for admissibility cannot be relaxed.
R. H. Farrell (Ithaca, N.Y.)

Ziaud-Din, M. [Ziaud Din, M.] 1693
Development of symmetric functional and k-statistics. (French summary)
Bull. Inst. Internat. Statist. **39** (1962), *livraison* 2, 501–508.

The author reviews the history and importance of the application of symmetric functions in statistics. Their application includes the calculation of moments, cumulants, k-statistics, and polykays. After describing the contributions of a number of men, he concludes by announcing his own construction of the k-statistic of order 13, expressed in terms of augmented symmetric functions and also in terms of power sums.
S. W. Nash (Vancouver, B.C.)

Speckman, Janace A. 1694
Determinations based on duplication of readings.
J. Res. Nat. Bur. Standards Sect. B **68B** (1964), 49–53.

Author's summary: "This paper is concerned with a statistical estimation procedure in which measurements of a quantity are taken until two identical readings are obtained; this duplicated value is then taken as the estimate of the magnitude of the quantity concerned. The properties of this estimation procedure have been investigated numerically, under the assumptions that the individual observations are rounded values of measurements which have a normal distribution, and this estimator is compared with the arithmetic mean of two observations. It is shown that an arithmetic mean of two observations from the rounded distribution is almost always superior to the estimator described above. The exception is where the rounding interval is so wide and the rounding lattice is so advantageously placed that the only real reason for taking repeat measurements would be as a protection against gross errors."

Bain, Lee J.; Weeks, David L. 1695
A note on the truncated exponential distribution.
Ann. Math. Statist. **35** (1964), 1366–1367.

The exact distribution is given for the minimal sufficient estimator of the parameter c in the truncated exponential distribution $ce^{-cx}(1 - e^{-cx_0})^{-1}$ $(0 < x < x_0)$.
J. Gani (E. Lansing, Mich.)

Li, Ši-lin' [Li, Shih-lin] 1696
An asymptotically most powerful test for testing composite hypotheses. (Ukrainian. Russian and English summaries)
Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 583–588.

Author's summary: "Neyman's result [*Probability and statistics*, pp. 213–234, Almqvist & Wiksell, Stockholm, 1959; MR **22** #3056] is extended in this note. It is proved that in some cases Neyman's criterion is locally asymptotically most powerful not only in class $C(\alpha)$, but among all criteria of class $K(\alpha)$ as well. Examples are given of the construction of such criteria."

Farlie, D. J. G. 1697
The asymptotic efficiency of Daniels's generalized correlation coefficient.
Biometrika **50** (1963), 499–504.

In previous papers [Biometrika **47** (1960), 307–323; MR **22** #10078; J. Roy. Statist. Soc. Ser. B **23** (1961), 128–142; MR **23** #A1425] the author proposed the form $H(x, y) = F(x)G(y)\{1 + \alpha \sum A_r(F)B_r(G)\}$ (summed over $r = 1$ to k) for a bivariate cumulative distribution function suitable for investigating tests of association. Independence corresponds to $\alpha = 0$. Using this type of distribution he investigated the asymptotic efficiency of rank correlation coefficients of Daniels's type (defined explicitly in this paper) as statistics for independence tests. Previously, the discussion was restricted to the case in which neither marginal distribution depended on α; in the present paper, the restriction is lifted to allow one marginal distribution to be a function of α, but k is taken to be 1. Expressions are found for the asymptotic efficiency, its maximum value and the corresponding test statistic. It is shown that for certain particular forms of distributions corresponding to simple trends in mean, standard deviation and scale parameter, the maximum asymptotic efficiency is $\frac{3}{4}$, for a Daniels rank correlation. *C. A. B. Smith* (London)

Linnik, Ju. V. 1698

Statistical problems with nuisance parameters. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 49–51.

Consider a family of probability densities (with respect to s-dimensional Lebesgue measure) of the form

$$C(\theta_1, \cdots, \theta_s) \exp\left(-\sum_{i=1}^{s} \theta_i T_i\right) p(T_1, \cdots, T_s),$$

where $\theta_1, \cdots, \theta_s$ are parameters. The hypothesis H_0 states that a homogeneous polynomial of $\theta_1, \cdots, \theta_s$ equals zero. The alternative H_1 is specified by a probability distribution on the parameter space. A theorem is stated to the effect that under certain regularity conditions for every $\alpha \in (0, 1)$ and every $\varepsilon > 0$ there is a randomized similar size α test of H_0 which is ε-optimal against H_1 and which is of a type introduced by Wijsman [Ann. Math. Statist. **29** (1958), 1028–1045; MR **21** #5256].
W. Hoeffding (Chapel Hill, N.C.)

Barton, D. E.; David, Florence N. 1699

Randomization bases for multivariate tests. I. The bivariate case. Randomness of N points in a plane. (French summary)

Bull. Inst. Internat. Statist. **39** (1962), *livraison* 2, 455–467.

Consider n sets of points on a plane area, r_t points in the tth set, N points altogether. The authors consider statistics to test for the random distribution of the points of the various sets. Such studies of randomness are of interest in plant ecology and in the study of the distribution of chromosome pairs. These statistics are chosen so as to be invariant under transformations of the coordinate system by translation, rotation, and reflection, so as to vary from 0 to 1, and so as to be analogous to tests under assumptions of normality. Essentially, they are ratios of the sum of squared distances of points within the same set, summed over all sets, to the sum of squared distances between all N points, or are closely related to such ratios. The authors consider the distributions of such statistics under the assumption that all distinguishable permutations of the N points are equally likely. The probability function (not given) was found for smaller N to have a saw-tooth profile. The first three moments are calculated. A beta density agreeing in the first two moments with the statistic appears to give a satisfactory fit. If the coordinate system is so chosen that the new X, Y of the points are uncorrelated and each has variance one, the moments are much simplified. An appendix lists moment functions up to order 8, which are invariant under choice of axes.
S. W. Nash (Vancouver, B.C.)

Bell, C. B. 1700

A characterization of multisample distribution-free statistics.

Ann. Math. Statist. **35** (1964), 735–738.

For one-sample statistics it is known that, under certain mild assumptions, a statistic is strongly distribution-free with respect to the class of continuous cumulative probability functions if and only if it is of the form $\psi[F(X_1), F(X_2), \cdots, F(X_n)]$, where $X_1, X_2, \cdots, X_n$ is a sample of a random variable with c.p.f. F. In the present paper this statement is extended to the k-sample case, i.e., the case when $X_{i1}, X_{i2}, \cdots, X_{in_i}$ is a sample of a random variable with c.p.f. F_i, for $i = 1, 2, \cdots, k$ and $k \geqq 2$. The precise formulation of this result involves concepts and assumptions which cannot be briefly summarized.
Z. W. Birnbaum (Seattle, Wash.)

Kshirsagar, A. M. 1701

Confidence intervals for discriminant function coefficients.

J. Indian Statist. Assoc. **1** (1963), 1–7.

Consider the discriminant analysis of $(q+1)$ p-variate populations. Suppose that the true means of all populations are collinear. There is then just one discriminant function, say $\alpha_1 x_1 + \cdots + \alpha_p x_p$. The paper derives two exact procedures using F statistics for getting confidence intervals for the ratios of coefficients, α_j/α_p, under the assumption that all populations are multivariate normal with the same non-singular covariance matrix. For the case of non-collinearity, another procedure is derived for getting confidence intervals for the ratios of coefficients of the ith discriminant function. This procedure is not exact, however, since certain parameters must be estimated. All these procedures and an earlier procedure due M. S. Bartlett [Ann. Eugenics **16** (1951), 199–214; MR **13**, 666] are compared numerically. While Bartlett's procedure gives the narrowest intervals, followed by the author's second procedure, the latter, in the reviewer's opinion, is to be preferred since it is exact, while Bartlett's procedure is only asymptotically exact.
S. W. Nash (Vancouver, B.C.)

Cooper, Paul W. 1702

Statistical classification with quadratic forms.

Biometrika **50** (1963), 439–448.

This paper describes some multivariate extensions of Pearson Type II and Type VII distributions. For a special class of classification problems (equal prior probabilities, equal costs of errors, equal costs of correct decisions) the appropriate statistic turns out to be a quadratic form.
H. E. Reinhardt (Missoula, Mont.)

Takács, Lajos 1703

An application of a ballot theorem in order statistics.

Ann. Math. Statist. **35** (1964), 1356–1358.

The "ballot theorem" is used to give another variation on the explicit determination of the one-sided, two-sample Kolmogorov-Smirnov probabilities.

M. Dwass (Evanston, Ill.)

Anderson, T. W.; Das Gupta, S. 1704

A monotonicity property of the power functions of some tests of the equality of two covariance matrices.

Ann. Math. Statist. **35** (1964), 1059–1063.

Let $\mathbf{\Sigma}_i, \mathbf{S}_i$ $(i=1,2)$ denote the population and sample covariance matrices, respectively, of two non-singular p-variate normal populations. More generally, the populations may have densities of the type

$$p(\mathbf{x};\mathbf{\Sigma}) = |\mathbf{\Sigma}|^{-1/2}g(\mathbf{x}'\mathbf{\Sigma}^{-1}\mathbf{x}),$$

where, without loss of generality, $E(\mathbf{x})=\mathbf{0}$. Tests of the hypothesis $\mathbf{\Sigma}_1=\mathbf{\Sigma}_2$, invariant under linear transformations, are based on the characteristic roots of $\mathbf{S}_1\mathbf{S}_2^{-1}$, say $c_1 \geqq c_2 \geqq \cdots \geqq c_p$. Let ω be a set in the space of the characteristic roots such that, if $(c_1,\cdots,c_p)\in\omega$ and if $\bar{c}_i \leqq c_i$ for $i=1$ to p, then $(\bar{c}_1,\cdots,\bar{c}_p)\in\omega$. Then the probability of ω depends on $\mathbf{\Sigma}_1$, $\mathbf{\Sigma}_2$ only through the characteristic roots γ_i of $(\mathbf{\Sigma}_1\mathbf{\Sigma}_2^{-1})$ and is a monotonically decreasing function of each of the characteristic roots of this matrix. If $g(c_1,\cdots,c_p)$ is monotonically increasing in each of its arguments, a test with acceptance region $g(c_1,\cdots,c_p)\leqq\mu$ has a power function which is monotonically increasing in each γ_i. Similar results are gotten for one sample, the hypothesis $\mathbf{\Sigma}=\mathbf{\Sigma}_0$, and tests based on the characteristic roots to S.

S. W. Nash (Vancouver, B.C.)

Anderson, T. W. 1705

On Bayes procedures for a problem with choice of observations.

Ann. Math. Statist. **35** (1964), 1128–1135.

The problem considered is that of determining, a priori, how many measurements to make in an experimental situation where a decision has to be made between two simple hypotheses, H_0 and H_1.

On the basis of known losses for wrong rejection of H_0 and H_1 and known costs of the measurements, the author considers the nature of the set, S, of values of g_0 (the prior probability of H_0) for which the larger of two possible sets of measurements gives the smaller minimum expected loss for the best Bayes procedure. If X and Y are sufficient statistics for the probability measures of the smaller set of measurements and the additional measurements (respectively), it is shown that S is an interval when X and Y are normal. Other examples are given where S is not an interval. When S is an interval, the admissible procedures are classified into five cases according as the Type I and Type II errors are relatively small, medium or large. For example, it is shown that when Type I and Type II errors are both "intermediate", the admissible procedures consist of admissible procedures based on both X and Y.

E. J. Hannan (Canberra)

Schwartz, Loraine [Schwartz, Lorraine] 1706

On consistency of Bayes procedures.

Proc. Nat. Acad. Sci. U.S.A. **52** (1964), 46–49.

The author announces rather weak conditions which guarantee consistency of Bayes estimators. The conditions are of a different nature than those given by Doob [*Le calcul des probabilités et ses applications*, pp. 23–27, Centre Nat. Rech. Sci., Paris, 1949; MR **11**, 444] and include, in an essential way, conditions which are dependent on the choice of prior distribution.

H. E. Reinhardt (Missoula, Mont.)

Cheng, Ping 1707

Bayes sequential procedures with K-decision actions for the exponential Pólya type III distributions.

Acta Math. Sinica **13** (1963), 119–135 (*Chinese*); *translated as Chinese Math.* **4** (1963), 131–147.

Work of Karlin [Proc. Third Berkeley Sympos. Math. Statist. and Probability, 1954–1955, Vol. I, pp. 115–128, Univ. California Press, Berkeley, Calif., 1956; MR **18**, 947] is exploited and extended to derive properties, particularly monotonicity, of the procedures described in the title under standard regularity assumptions (e.g., bounded loss function, parameter space an interval of real numbers). An approximation is suggested.

H. E. Reinhardt (Missoula, Mont.)

Lehmann, E. L. 1708

Asymptotically nonparametric inference: An alternative approach to linear models.

Ann. Math. Statist. **34** (1963), 1494–1506.

The author develops tests of linear hypotheses, confidence intervals for individual contrasts, and simultaneous intervals for all contrasts for linear models with several observations per cell which are asymptotically distribution-free. The asymptotic efficiency relative to standard procedures is the same as that of the Wilcoxon test relative to Student's t-test. The procedures are based on those in an earlier paper by the author [same Ann. **34** (1963), 957–966; MR **27** #2051]. The author provides a historical introduction, heuristic motivation for the techniques, and a worked example, making the techniques accessible to statistical practitioners.

H. E. Reinhardt (Missoula, Mont.)

Lehmann, E. L. 1709

Nonparametric confidence intervals for a shift parameter.

Ann. Math. Statist. **34** (1963), 1507–1512.

Let $X_1,\cdots,X_m$ and $Y_1,\cdots,Y_n$ be independent observations from distributions $P(X_i\leqq x)=F(x)=\int_{-\infty}^{x}f(t)\,dt$ and $P(Y_i\leqq y)=F(y-\Delta)$. The Wilcoxon two-sample test is used to produce non-parametric confidence intervals for Δ, which, in turn, are used to produce a consistent estimator of the quantity $1/\int f^2(t)\,dt$. Such an estimator is required for some other techniques developed by the author [#1708].

H. E. Reinhardt (Missoula, Mont.)

Lehmann, E. L. 1710

Asymptotically nonparametric inference in some linear models with one observation per cell.

Ann. Math. Statist. **35** (1964), 726–734.

The author considers, for observations $X_{i\alpha}$ $(i=1,\cdots,c;$ $\alpha=1,\cdots,N)$, the linear model $X_{i\alpha}=\nu+\xi_i+\mu_\alpha+U_{i\alpha}$ $(\sum\xi_i=\sum\mu_\alpha=0)$, where the U's are independently distributed according to a common continuous distribution.

Interest is in making inferences about the ξ's in the presence of nuisance parameters, the μ's. The author gives estimates of single contrasts, compatible estimates of all contrasts and tests of hypotheses concerning the ξ's, and derives asymptotic distributions and efficiency of the procedures as $N \to \infty$. The work is a continuation of that reported by the author [#1709].

H. E. Reinhardt (Missoula, Mont.)

Singh, Daljit 1711

A method of working out the sum of totals over blocks containing a specified variety in the analysis of a $n \times n$ balanced lattice design.

J. Indian Soc. Agric. Statist. **14** (1962), 151–156.

Author's summary: "A step in the analysis of incomplete block designs is the obtaining of the sum of block totals containing specific varieties. Based upon properties of Latin squares generated by the application of Galois fields for the general case of $n \times n$ designs and cyclic operations upon the columns for the particular case of $p \times p$ designs (where p is a prime number), methods are presented which enable such totals to be obtained with relative ease. This is especially useful when the number of characters to be analysed is quite large—the layouts being the same or when the size of the lattice employed is big."

H. B. Mann (Madison, Wis.)

Collier, Raymond O., Jr.; Baker, Frank B. 1712

The randomization distribution of F-ratios for the split-plot design—an empirical investigation.

Biometrika **50** (1963), 431–438.

From the authors' summary: "Randomization distributions were obtained for the F-ratios of a split-plot design with four replications, six whole plots and four split plots on each of four sets of 96 observations. Comparisons were made between the observed and theoretical frequencies for the various F-ratios. Results indicated that in the case of the actual psychological data none of the randomization distributions for the whole-plot, split-plot, and whole-plot $\times$ split-plot F-ratios departed significantly, at the 0.20 level, from the corresponding F-distributions suggested by normal theory. For both exponential and rectangular samples the randomization distributions for the whole-plot and split-plot F-ratios did not depart significantly from corresponding F-distributions at the 0.10 level. However, for the split-plot $\times$ whole-plot interaction F-ratios both types of data yielded distributions which departed from the F-distribution at the 0.02 level. In the log-normal case, the randomization distributions for each of the F-ratios in the split-plot design departed significantly from corresponding F-distributions at the 0.05 level. In general, the discrepancy between observed and theoretical distributions seems to be due largely to an excess of cases in the central portion of the distributions. The greatest departure from normal theory was observed for samples from the highly leptokurtic log-normal distribution.

"The proportions of observations falling beyond the upper 0.025, 0.05 and 0.10 points of the F-distribution were obtained for the randomization distributions of each type of F-ratio and the agreement with normal theory was found to be quite close. With few exceptions the empirical proportions were found to be less than corresponding F-distribution proportions."

S. Addelman (Durham, N.C.)

Yamamoto, Sumiyasu; Fujii, Yoshio 1713

Analysis of partially balanced incomplete block designs.

J. Sci. Hiroshima Univ. Ser. A-I Math. **27** (1963), 119–135.

Consider a partially balanced incomplete block design (PBIBD) with m associate classes and with parameters v, b, r, k, λ_i, $p_{jk}{}^i$ $(i, j, k = 1, 2, \cdots, m)$. Then the total number of plots is $n = vr = bk$. The algebra R generated by the relationship matrices I, G, B, T_i $(i = 1, 2, \cdots, m)$ over the real field is said to be the relationship algebra, where I is the identity matrix of order n, G is the $n \times n$ matrix with $+1$ everywhere, B is the $n \times n$ matrix having 1 or 0 in the (f, g)th position according as the fth and gth plots belong to the same block or not and T_i is the $n \times n$ matrix having 1 or 0 in the (f, g)th position according as the treatments applied to the fth and gth plots are ith associates or not. The main results of this paper are the following. (i) If and only if a PBIBD is connected, there occurs no component of the treatment sum of squares (SS), which is completely confounded with the block space; (ii) If and only if a PBIBD degenerates into a complete block design, all components of the treatment SS are orthogonal to the block space; (iii) In a PBIBD with three or more associate classes all of the following components of the treatment SS: (a) orthogonal to the block space, (b) completely confounded with the block space, and (c) partially confounded with the block space, may occur simultaneously; (iv) The dimension of the relationship algebra R is obtained, and the unique decomposition of its unit element into mutually orthogonal idempotents is given.

The result in (ii) was obtained earlier by B. V. Shah [Calcutta Statist. Assoc. Bull. **8** (1958), 73–80; MR **21** #1672]. However, it seems that the techniques applied here in proving the results are new. An inequality relationship between v and b is also obtained in this paper which is essentially the same as that due to W. S. Connor and W. H. Clatworthy [Ann. Math. Statist. **25** (1954), 100–112; MR **15**, 494].

S. S. Shrikhande (Bombay)

Hinkelmann, K.; Kempthorne, O. 1714

Two classes of group divisible partial diallel crosses.

Biometrika **50** (1963), 281–291.

The authors note that a partially balanced incomplete block design (PBIB) can be utilized to derive a partial diallel cross (PDC) by making a correspondence between line i of a PDC and treatment i of a PBIB and another correspondence between a cross $(i \times j)$ of the PDC and a block containing treatments i and j of the PBIB. A general method of analyzing PDC's derived from PBIB's is also described. It is shown that generalized group divisible PBIB designs with m associate classes (GGD/m-PBIB) and extended group divisible PBIB designs with $m = 2^v - 1$ associate classes (EGD/$(2^v - 1)$-PBIB) can be utilized to derive PDC's. Methods for analyzing GGD/m-PDC's and EGD/$(2^v - 1)$-PDC's are outlined. Examples of the derivations of a GGD-PDC and an EGD-PDC from corresponding PBIB's and a table indicating the possible PDC plans that can be derived from the generalized and extended group divisible designs are presented in two appendices to the paper.

S. Addelman (Durham, N.C.)

Ghosh, M. N. 1715

Uniform approximation of minimax point estimates.

Ann. Math. Statist. **35** (1964), 1031–1047.

Author's summary: "In this paper the problem of minimax point estimation of a function $g(\theta)$ of a parameter θ is considered when the loss function is of the form $W(u(x), g(\theta)) = |u(x) - g(\theta)|^p$ $(p > 1)$, and $u(x)$ is an estimate with bounded risk. When Conditions A and B stated later hold, it is shown that a unique minimax estimate $u_0(x)$ exists, and if $\{u_n(x)\}$ is any uniformly bounded minimax sequence, then the risk functions of $u_n(x)$ converge uniformly to the risk function of $u_0(x)$, so that no almost subminimax estimate can exist which, though not a minimax estimate, has for a wide range of values of the parameter θ a lower value of the risk than that of the minimax estimate. Under some additional conditions, it is shown that an approximation to the minimax estimate $u_0(x)$ in the space $F_\infty^{(p)}$ of functions with bounded risk may be obtained by the minimax estimate $\bar{u}_N(x)$ in the finite-dimensional linear space spanned by N basis vectors $v_1, \cdots, v_N$ of $F_\infty^{(p)}$, so that the maximum risk of $\bar{u}_N(x)$ converges to that of $u_0(x)$. This may help in finding an approximation to a minimax estimate in non-standard problems, where it is difficult to guess a minimax estimate from invariance or other considerations and specially when the problem is a perturbation of a standard problem." *J. Kiefer* (Ithaca, N.Y.)

Buckland, W. R. 1716

★Statistical assessment of the life characteristic: A bibliographic guide.

Griffin's Statistical Monographs & Courses, No. 13. *Hafner Publishing Co., New York.* 1964. 125 *pp.* $4.25.

This little book is divided into four chapters: 1. "Some General Problems"; 2. "Problems of Estimation"; 3. "Tests of Hypotheses"; 4. "Analysis of Fatigue Tests". Contrary to what the title on the cover implies, viz. "Statistical Assessment of the Life Characteristic", and what the titles of particular chapters might suggest, the contents do not offer any kind of systematic presentation of problems of life testing which could be helpful in gaining competence in the matter. Rather, as indicated on the title page and in the Preface, the purpose of the monograph is limited to providing a bibliographic guide to statisticians who become involved in life testing. The guide covers a substantial part of post World War II literature disseminated in many journals, both statistical and engineering. There are also quite a few references to earlier publications. The monograph has an unconventional but a very convenient form, each odd page being given to the author's comments with the facing, even page providing the references. The book will be found useful in those cases where a statistician is in need of a ready source of information as to the literature concerned with some particular aspect of the problem of life testing. For example, the reviewer was glad to find a list of papers dealing with the question of adequacy of the exponential distribution to represent the life time of a manufactured product. Completeness of references is difficult to attain and the reader is frequently referred to other sources. It is likely that many statisticians will find the author's monograph very useful. This usefulness might have been even greater if the book contained a little more substantive information, particularly on the engineering aspects of the problem. *J. Neyman* (Berkeley, Calif.)

Des Raj 1717

On forming strata of equal aggregate size.

J. Amer. Statist. Assoc. **59** (1964), 481–486.

The object of the paper is to test the efficiency of strata of equal aggregate size on four theoretical distributions, and the results of the paper supplement and reinforce the findings of Cochran in this direction. It is the view of the reviewer that the uses and the merits of the principle of equipartition in stratified random sampling which were advocated by Mahalanobis, Hansen, Hurwitz and Madow should be recommended on some restricted distributions under incomplete information. *T. Kitagawa* (Fukuoka)

Hannan, E. J. 1718

Regression for time series with errors of measurement.

Biometrika **50** (1963), 293–302.

From the author's introduction: "We consider the problem of estimating, from n consecutive observations, the regression relation

$$(1) \qquad y_t = \alpha \sum_{j=1}^{q} \beta_j x_{j,t} + e_t \qquad (t = 1, 2, \cdots),$$

where the $x_{j,t}$ are not directly observable but instead we record $\xi_{j,t} = x_{j,t} + \delta_{j,t}$. It is well known that if e_t and the $\delta_{j,t}$ are sequences of uncorrelated random variables, then the β_j cannot be identified by means of estimates based only on variances and covariances. Reiersøl [Econometrica **9** (1941), 1–24; MR **2**, 237] and Geary [ibid. **17** (1949), 30–58; MR **10**, 465] have, amongst others, considered the use of 'instrumental variables' to enable the relation (1) to be estimated. The main instrumental variables considered by them were lagged values of the $\xi_{j,t}$ themselves, it being assumed that the $\delta_{j,t}$ are not serially correlated while the $x_{j,t}$ are. We will discuss the asymptotic properties, including the asymptotic efficiency, of Geary's and Reiersøl's procedures using spectral methods. . . . The methods which we shall use to obtain estimates which have higher asymptotic efficiency than those of Reiersøl and Geary will be the optimal combination of estimates obtained from each of a large number of frequency bands. These methods will often be computationally arduous and would not be used unless n is large."

P. H. Diananda (Singapore)

Beran, M.; Molyneux, J. 1719

Statistical properties of the electric field in a medium with small random variations in permittivity. (Italian summary)

Nuovo Cimento (10) **30** (1963), 1406–1422.

Authors' summary: "We consider in this paper the problem of determining the two-point electric field correlation function in a random medium subjected to a constant electric field. The medium is assumed to have small random variations in permittivity that in an ensemble sense are statistically homogeneous and isotropic. For this case the effective permittivity is derived. Perturbation equations are also set up to allow the determination of higher-order approximations and the first-order equations governing a random medium in the presence of a nonconstant average electric field are derived. The paper concludes with a discussion of the mathematically trivial though nevertheless interesting one-dimensional problem."

S. Prager (Minneapolis, Minn.)

Dmitrieva, F. A. 1720

On the definition of wind velocity in a given time interval. (Russian. English summary)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **19** (1964), *no.* 1, 166–168.

Author's summary: "A new formula recently given by H. Cramér [Bull. Amer. Math. Soc. **68** (1962), 512–516; MR **25** #3562] in the theory of normal stationary stochastic processes is applied to predict the behaviour of the maximum wind velocity in a given time interval at the height of 32 meters above sea level. The observations agree with the predictions in a satisfactory way."

NUMERICAL METHODS

See also 1302, 1363, 1364, 1435, 1460, 1473, 1475, 1476, 1479, 1480, 1481, 1485, 1510, 1756, 1757, 1763, 1764, 1945, 2089, 2101.

Marsaglia, George 1721

Random variables and computers.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 499–512. *Publ. House Czech. Acad. Sci., Prague,* 1964.

Author's summary: "This paper is concerned with methods for generating, in a digital computer, sequences of numbers which will serve as realizations of a sequence of random variables. The problem has two parts—the problem of producing independent uniform random variables, and the problem of producing arbitrary random variables in terms of uniform random variables. Section 1 describes arithmetic procedures for generating uniform variables. It contains nothing new. Section 2 considers properties of linear transformations, modulo 1, of uniform variables. These are the theoretical analogues of the arithmetic procedures for generating uniform variables. The rest of the paper is devoted to the problem of producing arbitrary random variables in terms of uniform random variables."

Shibuya, Masaaki 1722

Generation of pseudo-random numbers. (Japanese)

Sûgaku **15** (1963), 68–71.

An expository discussion of generation of pseudo-random numbers with reference to, among others, P. Peach [J. Amer. Statist. Assoc. **56** (1961), 610–618; MR **24** #B505], M. Greenberger [Math. Comp. **15** (1961), 383–389; MR **26** #2033], and S. Yamada [J. Operations Res. Soc. Japan **3** (1960/61), 113–123; MR **24** #B2084]. *S. Ikehara* (Tokyo)

Amos, D. E. 1723

Additional percentage points for the incomplete beta distribution.

Biometrika **50** (1963), 449–457.

Tables are given for the solution of the equation $B_x(a,b)/B_1(a,b)=\alpha$, where

$$B_x(a,b) = \int_0^x u^{a-1}(1-u)^{b-1}\,du,$$

when $\alpha=0.0010, 0.0025$; $2a=1(1)30, 40, 60, 120, \infty$; $2b=1(1)10, 12, 15, 20, 24, 30, 40, 60, 120$. The method of evaluation is described. Corrections are listed to certain percentage points appearing in Table 16 of *Biometrika tables for statisticians*, Vol. I [E. S. Pearson and H. O. Hartley (Editors), Cambridge Univ. Press, London, 1954; MR **16**, 53]. *R. L. Plackett* (Newcastle upon Tyne)

Tsuda, Takao; Kiyono, Takeshi 1724

Application of the Monte Carlo method to systems of nonlinear algebraic equations.

Numer. Math. **6** (1964), 59–67.

The paper is an exposition of the interesting idea of investigating the roots of a system

$$F_i(x_1,\cdots,x_n)=0, \qquad 1\leq i\leq n,$$

of equations by tracing the behavior of a "gas particle" moving under the influence of the "potential"

$$U=\sum_{i=1}^{m} a_i^2F_i^2(x_1,\cdots,x_n)$$

and of classical collision laws, and then looking for maxima of the resulting spatial distribution. It is proposed to use these estimates of the roots as starting values for a more precise iterative procedure. No comparisons are made with other methods. *R. R. Coveyou* (Oak Ridge, Tenn.)

Ahlin, A. C. 1725

A bivariate generalization of Hermite's interpolation formula.

Math. Comp. **18** (1964), 264–273.

Author's summary: "One of the most commonly used methods for deriving formulas for bivariate interpolation is that of extending to two variables the formulas of Lagrange, Aitken, Newton, Gauss, Stirling, Everett, Bessel, etc., in which forward, backward and central differences are used. These formulas have the property that the resulting interpolation polynomial agrees with the interpolated function, $f(x,y)$, at each of the node points of a Cartesian grid. In this study, we shall investigate the existence of a wider class of interpolation formulas, together with their associated error terms, than those obtainable by the method just described. To this end, we develop a bivariate osculatory interpolation polynomial which not only agrees with $f(x,y)$ in function values at each of the node points of a Cartesian grid but which also enjoys the property that agreement in values of partial and mixed partial derivatives up to specified, arbitrary orders is obtainable at these points. The result is essentially a bivariate generalization of Hermite's interpolation formula."

T. Fort (Columbia, S.C.)

Bengtsson, Bengt-Erik; Nordbeck, Stig 1726

Construction of isarithms and isarithmic maps by computers.

Nordisk Tidskr. Informations-Behandling **4** (1964), 87–105.

Authors' summary: "The most important consideration in map construction is that the finished map shall give an objective picture of the actuality it represents. It is also important that the map can be produced quickly and efficiently with a minimum of manual labor. The coordinate mapping method fulfils these requirements to a

very high degree. This article deals with isarithmic mapping methods adapted to the coordinate mapping method and to computers and automatic data processing."

Steinberg, A. S. 1727
On a modification of the method of balanced descent for the problem of the generalized Chebyshev minimax. (Ukrainian. **Russian and English summaries**)
Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 318–321.

Carter, I. P. V. 1728
Optimization techniques.
Nordisk Tidskr. Informations-Behandling **3** (1963), 69–92.
A survey paper which contains little information which is not well known in the literature. It contains almost no mathematics, apart from a description of the Lagrange multiplier technique, and has no bibliography or acknowledgements. *A. D. Booth* (Saskatoon, Sask.)

Rich, Robert P.; Shaw, Harry 1729
A method for finding all the zeros of $f(z)$.
J. Assoc. Comput. Mach. **10** (1963), 545–549.
Let P be a convex polygon and let $f(z)=u(x,y)+iv(x,y)$ be a single-valued analytic function in and on P. The problem is to find the zeros of f which lie in the interior of P as well as their orders. A one-dimensional search technique is proposed which provides an original estimate. This is then improved by employing the following theorem: If z is a zero of f of order N, then on any sufficiently small circle with center z there are $2N$ zeros of u and $2N$ zeros of v. These $4N$ zeros are approximately evenly spaced about the circle, the zeros of u and of v alternating.

Computer results are displayed for $f(z)=(z-1)^N$ and $f(z)=\sin(2\pi/z)$, and high accuracy is claimed.
P. J. Davis (Providence, R.I.)

Krabs, Werner 1730
★Einige Methoden zur Lösung des diskreten linearen Tschebyscheff-Problems.
Dissertation zur Erlangung des Doktorgrades der Mathematisch-Naturwissenschaftlichen Fakultät der Universität Hamburg, Hamburg, 1963. 67 *pp.*
This thesis is concerned with connections between ways of calculating the solutions to linear Čebyšev approximation problems on finite point sets and the methods of linear and quadratic programming. The thesis is divided into two parts. The first part discusses "minimizing" methods. The author shows that the problem of finding a "restricted" best least squares approximation ("restricted" means that only vectors within a given distance, as measured by the maximum norm, of the vector being approximated are admitted in the least squares competition) is equivalent to a quadratic programming problem. He gives a method related to the simplex method for finding increasingly restricted best least squares approximations. In the second part the author presents a modified simplex method which is a "maximizing" method equivalent to Stiefel's exchange algorithm. Examples of the methods are given.
T. J. Rivlin (Yorktown Heights, N.Y.)

Dreyfus, Stuart 1731
The numerical solution of variational problems.
J. Math. Anal. Appl. **5** (1962), 30–45.
Consider the problem of determining values of a decision variable $z(t)$ that minimize $\phi(\mathrm{y},t)$ at $t=T$, where T is the first time that $\boldsymbol{\theta}(\mathrm{y},t)=\mathbf{0}$, and y is subject to the difference equation $\mathrm{y}(t+\Delta)=(\mathrm{y}t)+\mathrm{g}(\mathrm{y},t,z)\Delta$. In the paper, techniques are developed for solving this problem numerically using dynamic programming theory. The author first discusses the case $\boldsymbol{\theta}=\Psi$, a scalar, and goes on to consider the case of a vector $\boldsymbol{\theta}$. He then discusses a generalization of this model to one in which there are inequality constraints on $\mathrm{y}(t)$ for all t, and finally solves numerically a constrained brachistochrone problem in which the particle must remain above a specified (sloping) straight line.

The method is extendable from terminal to intermediate constraints, and from terminal criteria $\phi(\mathrm{y},T)$ to path criteria $\sum_1^n \phi(\mathrm{y}(r\Delta),r\Delta)$. Instead of using the less efficient method of "successive approximation to the problem", in which we zero in on the initial values of the Lagrange multipliers, at each stage finding the optimal solution for the approximate problem, the author uses a technique of "successive approximation to the solution", at each stage of which approximately optimal solutions (decisions) are found for the true problem, and we zero in on the optimal decisions $z(t)$. *B. Gluss* (Berkeley, Calif.)

Pinsker, I. Š. 1732
The method of alternants (solution of a non-linear programming problem). (Russian. English summary)
Avtomat. i Telemeh. **25** (1964), 302–311.
The following programming problem is formulated: Determine $x=(x_1,\cdots,x_n)$ which minimizes $u(s^*,x)$ subject to $u(s,x)\leqq b(s)$, where $(s,x)\in Q_{m+n}$. Here (s,x) stands for $(s_1,\cdots,s_m,x_1,\cdots,x_n)$, Q_{m+n} is a given subset of $(m+n)$-dimensional Euclidean space, $u(s,x)$ is continuous in Q_{m+n} and $b(s)$ is continuous in Q_m^s—the set of points in R_m whose coordinates are the first m coordinates of points in Q_{m+n}. It is supposed that s^* is an isolated point in Q_m^s and $b(s^*)=0$.

Taking advantage of the particular nature of the restrictions, an algorithm is proposed for finding a local minimum, which relies heavily on methods developed by the author in the field of Chebyshev approximations and reported in earlier papers [E. P. Novodvorskiĭ and the author, Uspehi Mat. Nauk **6** (1951), no. 6 (46), 174–181; MR **13**, 728; the author, Dokl. Akad. Nauk SSSR **95** (1954), 21–24; MR **15**, 858]. *B. Bereanu* (Bucharest)

Fox, L. 1733
★An introduction to numerical linear algebra.
Monographs on Numerical Analysis.
Clarendon Press, Oxford, 1964. xi+295 *pp.* $8.00.
A practical development of numerical techniques, with many numerical examples presented in illustration. There are, however, no "problems for the student". Chapter headings are as follows: Introduction; Matrix algebra; Elimination methods of Gauss, Jordan, and Aitken; Compact elimination methods of Doolittle, Crout, Banachiewicz and Cholesky; Orthogonalization methods; Condition, accuracy and precision; Comparison of methods, measure of work; Iterative and gradient methods; Iterative methods for latent roots and vectors; Notes on error

analysis for latent roots and vectors. Each chapter contains a brief commentary on references. The discussion of errors is much less complete and systematic than that given by Wilkinson in his recent book [*Rounding errors in algebraic processes*, Prentice-Hall, Englewood Cliffs, N.J., 1963; MR **28** #4661], and the two books in some measure supplement each other.

Altogether, this is a very good introduction that presupposes very little background in mathematics.

A. S. Householder (Oak Ridge, Tenn.)

Gautschi, Walter 1734

On inverses of Vandermonde and confluent Vandermonde matrices. II.

Numer. Math. **5** (1963), 425–430.

The confluent Vandermonde matrix U_{2n} with (i, j)th element

$$\begin{array}{lll} x_j^{i-1} & \text{for } 1 \leqq i \leqq 2n, & 1 \leqq j \leqq n, \\ 0 & \text{for } i = 1, & n+1 \leqq j \leqq 2n, \\ (i-1)x_j^{i-2} & \text{for } 2 \leqq i \leqq 2n, & n+1 \leqq j \leqq 2n, \end{array}$$

is of interest in the construction of Gaussian quadrature formulas and Hermite interpolation. The author sharpens slightly his previous upper bound for $\|U_{2n}^{-1}\|$ [Numer. Math. **4** (1962), 117–123; MR **25** #3059] and gives an algorithm for computing U_{2n}^{-1}.

W. W. Hooker (Palo Alto, Calif.)

Dufour, H. M. 1735

Résolution des systèmes linéaires par la méthode des résidus conjugués.

Rev. Française Traitement Information [*Chiffres*] **7** (1964), 125–134.

Author's summary: "L'auteur présente l'organigramme et les possibilités d'une méthode de résolution des systèmes linéaires dérivée de la méthode des gradients conjugués. Elle a été spécialement étudiée pour avoir une convergence pratique aussi rapide que possible, et pour être efficace contre les erreurs d'arrondis."

Kredell, Bengt 1736

On complex successive overrelaxation.

Nordisk Tidskr. Informations-Behandling **2** (1962), 143–152.

Author's summary: "A very natural generalization of the successive overrelaxation method for the solution of complex linear systems is studied. An optimum complex overrelaxation factor is determined and some numerical experiments verify the theory."

Eve, J. 1737

The evaluation of polynomials.

Numer. Math. **6** (1964), 17–21.

It is shown how to evaluate polynomials of degree n by $[\frac{1}{2}(n+4)]$ multiplications and n additions by a modification of a method of D. E. Knuth [Comm. ACM **5** (1962), 595–599; MR **27** #970]. This is best possible for odd n and requires at most one too many multiplications for n even. {See also T. S. Motzkin [Bull. Amer. Math. Soc. **61** (1955), 163], E. G. Belaga [Dokl. Akad. Nauk SSSR **123** (1958), 775–777; MR **21** #3935], and V. Ja. Pan [Problemy Kibernetiki **5** (1959), 17–29; Dokl. Akad. Nauk SSSR **127** (1959), 266–269; MR **23** #B560].} *John Todd* (Pasadena, Calif.)

Comtet, L. 1738

Calcul pratique des coefficients de Taylor d'une fonction algébrique.

Enseignement Math. (2) **10** (1964), 267–270.

By using the linear differential equation satisfied by algebraic functions, the author obtains recursions for the Taylor coefficients of the functions. Two examples are worked out. *P. J. Davis* (Providence, R.I.)

Freudenstein, Ferdinand; Roth, Bernard 1739

Numerical solution of systems of nonlinear equations.

J. Assoc. Comput. Mach. **10** (1963), 550–556.

Authors' summary: "For simultaneous nonlinear equations, the convergence of all functional iteration procedures is dependent upon a good initial approximation to the desired root. In this paper, the restriction on the choice of the initial approximation has been circumvented by dividing each problem into a number of subsidiary problems in accordance with a procedure which is essentially a numerical adaptation of the implicit function theorem. Additional procedures are developed for dealing with sets of equations with singularities in the domain of the iteration." *J. T. Day* (Zürich)

Bahvalov, N. S. 1740

On the conjecture of the independence of round-off errors in numerical integration. (**Russian**)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 399–404.

The author proves that the rounding error in a certain type of quadrature formula is a random variable. The quadrature formula is assumed to have the form

$$\int_0^1 f(x)\,dx \simeq \frac{1}{rN}\sum_{k=0}^{N} q_k f\left(\frac{k}{N}\right),$$

where the q_k and r are integers. It is also assumed that the computation is performed on a computer in fixed-point arithmetic and that the scale factor for f is chosen so that $|f(x)| < 1$ for $0 \leqq x \leqq 1$. *A. H. Stroud* (Lawrence, Kans.)

Haber, Seymour 1741

A note on some quadrature formulas for the interval $(-\infty, \infty)$.

Math. Comp. **18** (1964), 313–314.

Für das Integral $\int_{-\infty}^{+\infty} (1+x^2)^{-\alpha} g(x)\,dx$ mit $\alpha > \frac{1}{2}$ und beschränkter Funktion $g(x)$ wird eine Quadraturformel vom Gaußschen Typ hergeleitet. Die Formel soll exakt sein auf dem Raum der Funktionen

$$a_0 + \sum_{\nu=1}^{m} \frac{a_\nu + b_\nu x}{(1+x^2)^\nu}$$

mit möglichst großem m bei fester Stützstellenzahl. Durch eine einfache Substitution wird das Problem der Bestimmung der Stützstellen und Gewichte mit Hilfe des Orthogonalsystems der Jacobischen Polynome (bzw. der zugehörigen Quadraturformel für endliches Intervall) gelöst. *G. Meinardus* (Clausthal-Zellerfeld)

Lemaire, B.; Munteanu, Ion 1742
Évaluation des cycles limites de certains systèmes d'équations différentielles.
Rev. Française Traitement Information [*Chiffres*] **6** (1963), 255–274.
Authors' summary: "Dans une première partie du travail on démontre que, lorsque ε tend vers 0, la famille des cycles limites $\{\Gamma_\varepsilon\}$ de certains systèmes du type:

$$\varepsilon\frac{dx}{dt} = h(y)-F(x), \qquad \frac{dy}{dt} = -\varepsilon g(x)$$

(où ε est un paramètre positif) converge vers une courbe fermée Γ_0. Dans la deuxième partie, on donne une méthode pour le calcul à la machine des cycles limites. Cette méthode est appliquée à un exemple."

Reimer, Manfred 1743
Optimale Verfahren zur numerischen Integration von Anfangswertproblemen.
Math. Z. **84** (1964), 70–79.
The application of consistent linear multistep methods of order k,

$$y_{n+k} = \sum_{i=0}^{k-1} a_i y_{n+i} - h\sum_{i=0}^{k} b_i f(x_{n+i}, y_{n+i}),$$

to a single differential equation,

$$\frac{dy}{dx} = f(x,y) \quad \left(\frac{\partial f}{\partial y} \leqq g < 0, \quad g = \text{const}, \quad a \leqq x \leqq b\right),$$

is discussed. A method is absolutely stable if for any sufficiently small positive K, there exists an $h^*(K)$ such that for all h, $0<h\leq h^*(K)$, the error will never exceed K if the errors of the initial values do not exceed K. Theorem: $a_i>0$ for all i implies absolute stability implies $a_i \geqq 0$ for all i.

In order to choose an optimal method within the class considered, the author introduces a parameter D_k (called "Genauigkeit") that characterizes the computational situation and the problem at hand, e.g.,

$$D_3 = (1+\lambda)M_5K^{-1}|g|^{-5},$$

$M_5 = \text{u.b.}|y^{(5)}(x)|$, $\lambda = \text{u.b.}|r_n/t_n|$, r_n = local rounding error, t_n = upper bound of the local discretization error. For each multistep method a quantity $H^*(D_k)$ is defined that is a lower bound and probably a good substitute for the largest possible value of $|gh^*(K)|$ for the given problem and the given K. For $k=2$ and $k=3$ the methods which maximize $H^*(D_k)$ are determined explicitly. ($\partial f/\partial y$ is here a constant.) When $D_3\to\infty$, max $H^*(D_3)\sim(180/D_3)^{1/4}$, and the optimal method converges to Simpson's rule.
G. Dahlquist (Stockholm)

Melamed, V. G. 1744
A method of numerical solution of a boundary-value problem for the equation $y''+p(x)y'+q(x)y=0$ on a half-line. (Russian)
Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 156–160.
The numerical solution of the boundary-value problem

(1) $y''+p(x)y'+q(x)y = 0, \qquad y(x_0) = 1, \qquad y(\infty) = 0,$

where $x_0>0$, $q(x)<0$, is difficult if, as is often the case, the differential equation possesses solutions which are unbounded. Instead of (1) the author then considers the equivalent problem

(2) $z' = -[z^2+p(x)z+q(x)] = -[z-u_1(x)][z-u_2(x)],$

$$\int_{x_0}^{\infty} z(s)\,ds = -\infty,$$

obtained by the change of variables $z=y'/y$. For definiteness, let $u_1(x)<0$, $u_2(x)>0$. Assume, moreover, that $u_1(x)$ is monotone. Then the solution of (2) may be approximated by integrating the differential equation in (2) backwards from x_k to x_0, where x_k is large, using as initial value $z(x_k)=u_1(x_k)$. If $u_1(x)$ is strictly decreasing, e.g., and

(3) $$\lim_{x\to\infty}(u_1'(x)/u_1(x)) = 0,$$

the resulting error is less than ε provided x_k is such that $u_1'(x)/u_1(x)\leqq\varepsilon$ for $x\geqq x_k$. Similar statements are proved when (3) fails to hold, but $\lim_{x\to\infty}(u_1'(x)/u_1^2(x))=0$, and also in the case where $u_1(x)$ is increasing and $\lim_{x\to\infty}u_1(x)=a$, $a<0$. {The paper contains typographical errors.}
Walter Gautschi (Lafayette, Ind.)

Verlan', A. F. 1745
A method of simulating differential equations along with the boundary conditions on analog computers. (Ukrainian. Russian and English summaries)
Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 451–453.
Author's summary: "The author suggests finding the solution of a system of differential equations (1) $dX(t)/dt = f[t, X(t)]$ with boundary conditions $\alpha X(0)+\lambda X(t_i)+\beta X(T)=\gamma$ by means of the iterative formula $\alpha X_{k+1}(0)=\gamma-\lambda X_k(t_i)-\beta X_k(t)$, where k is the number of iterations. The vectors $X_k(t_i)$ and $X_k(T_i)$ are determined by solving system (1) on an analog computer with initial conditions equal to $X_k(0)$. Automation of the process of solution requires a memory cell and a model of the algebraic equations."

Sofronov, I. D. 1746
On the sweep method for solving difference boundary-value problems. (Russian)
Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 256–266.
The primary concern of this paper is a linear recursion formula of the form

(1) $\sum_{j=0}^{p} X_{n,j}U_{n+j} = Y_n, \quad n = 0,\cdots,N-p;\ X_{n,0} = I,$

where the unknowns U_n and inhomogeneous terms Y_n are s-vectors and the coefficients $X_{n,k}$ are $s\times s$ matrices together with $S_L=k_0s$ conditions at the left-hand boundary (these may be put in the form (2) $\sum_0^{p-k} L_j^{(k)}U_j=C^{(k)}$ $[k=1,\cdots,k_0]$) and $S_R=(p-k_0)s$ conditions on the right, (3) $\sum_0^{p-1} R_jU_{n-p+1+j}=C$, with C an S_R-vector and R_j an $S_R\times s$ matrix. The basic idea of the "sweep-method" for boundary-value problems is to sweep through the sequence of points $\{n\}$, transforming the S_L restrictions on the solution space $R^{(0)}$ of (1), originally given as relations between $U_0,\cdots,U_{p-1}$, into S_L linearly independent conditions on $U_{n-p+1},\cdots,U_n$ which, together with the original S_R right-hand conditions, suffice to determine uniquely $U_{n-p+1},\cdots,U_n$ so that (1) may now be used for a backward sweep computing successively the remaining unknowns, $U_{n-p},\cdots,U_1,U_0$.

This author describes an algorithm for the determination of the $\{U_n\}$ in (1), (2), (3) and discusses its computational stability under conditions which appear reasonable for a scheme obtained from a well-posed boundary-value problem for a differential equation with reasonably smooth coefficients. Some generalizations and modifications are also described. The main algorithm is defined by making k_0 forward sweeps (each carrying s of the S_L left-hand conditions across) defined by

$$H_n^{(k)} = (X_{n,1}^{(k)} - X_{n,1}^{(k-1)})^{-1}$$
$$[k = 1, \cdots, k_0;\ n = 0, \cdots, N-p];$$
$$X_{n,j}^{(k)} = H_{n-1}^{(k)}(X_{n-1,j+1}^{(k)} - X_{n-1,j+1}^{(k-1)}),$$
$$Y_n^{(k)} = H_{n-1}^{(k)}(Y_{n-1}^{(k)} - Y_{n-1}^{(k-1)})$$
$$(4_k)\qquad [k = 1, \cdots, k_0;\ n = 1, \cdots, j-p+k;$$
$$j = 1, \cdots, p-k;\ X_{n,p-k+1}^{(k)} = 0];$$
$$X_{n,j}^{(0)} = X_{n,j}, \qquad Y_n^{(0)} = Y_n$$
$$[n = 0, \cdots, N-p;\ j = 1, \cdots, p];$$
$$X_{0,j}^{(k)} = L_j^{(k)}, \qquad Y_0^{(k)} = C^{(k)}$$
$$[k = 1, \cdots, k_0;\ j = 1, \cdots, p-k]$$

to obtain, finally, the S_L relations

$$(5)\qquad \sum_{j=0}^{p-k} X_{N-p+1,j}^{(k)} U_{N-p+1+j} = Y_{N-p+1}^{(k)}, \quad k = 1, \cdots, k_0,$$

at the right end to go with (3). The kth sweep (4_k) uses the kth left-hand condition (2) together with the result of the preceding sweep, i.e., one computes $H_0^{(1)}$, $X_{1,1}^{(1)}, \cdots, X_{1,p-1}^{(1)}$, $Y_1^{(1)}$, $H_1^{(1)}$, $X_{2,1}^{(1)}, \cdots$, $Y_2^{(1)}, \cdots$, $H_{N-p}^{(1)}$, $X_{N-p+1,1}^{(1)}, \cdots, X_{N-p+1,p-1}^{(1)}$, $Y_{N-p+1}^{(1)}$, then $H_0^{(2)}$, $X_{1,1}^{(2)}, \cdots$, etc. *T. I. Seidman* (Seattle, Wash.)

Larkin, Bert K. 1747

Some stable explicit difference approximations to the diffusion equation.

Math. Comp. **18** (1964), 196–202.

This paper extends a difference scheme of Saul'ev to obtain several two-dimensional explicit alternating-direction schemes for solving the heat equation. Unlike most explicit schemes, these are shown to be stable. Instead of a restriction on the ratio $\Delta t/\Delta x^2$, these schemes require that $\Delta t/\Delta x \to 0$ as $\Delta x \to 0$ to guarantee consistency. The numerical comparison made by the author with the Peaceman-Rachford alternating-direction implicit scheme used to solve the heat equation in two dimensions indicates the superiority of the implicit scheme. As Δt increases from 0.0025 to 0.02, the error in the explicit scheme varies between 0 and 0.0255, while the error in the implicit scheme varies from 0.0019 to 0.0032. If, as the author asserts, the implicit scheme requires 20% more computing, this can only be due to poor programming. Equation (2.11) shows that 7 computations per point are required for one sweep of the explicit procedure. Each sweep of the implicit procedure uses the Gaussian elimination solution for tridiagonal matrices [Douglas, J. Assoc. Comput. Mach. **6** (1959), 48–58; MR **21** #4555]. A glance at the equations for this solution shows that one sweep of the implicit procedure requires 6 computations per point if we restrict storage, and only 4 computations per point if we store once and for all the β_i and the ratios c_i/β_i (excluding from our comparison the number of steps needed to compute d_i, the right-hand side, in both explicit and implicit cases). Further, the examples presented in the paper are not very interesting since they are computed for ratios $\Delta t/\Delta x^2$ of 2, 1, 0.5 and 0.25. In the last case the ordinary forward difference equation is stable and the other values are not much larger than the stable case. *A. O. Garder, Jr.* (St. Louis, Mo.)

Saul'ev, V. K. 1748

Solution of certain boundary-value problems on high-speed computers by the fictitious-domain method. (Russian)

Sibirsk. Mat. Ž. **4** (1963), 912–925.

In euclidean n-space let G be a connected n-dimensional region with piecewise smooth $(n-1)$-dimensional boundary S. Let $a(x)$ be continuous, bounded, non-negative, and let $a_{ij}(x)$ be continuously differentiable in G, with $\sum a_{ij}\xi_i\xi_j \geq \mu \sum \xi_i^2$ for some $\mu > 0$. Let u be a solution of the Dirichlet problem

$$Lu = \sum_{i,j=1}^{n} (a_{ij}u_{x_j})_{x_i} - au = -f \quad \text{in } G,$$
$$u = 0 \quad \text{on } S.$$

Let G' be another n-dimensional region contiguous to G, and let γ be the common boundary of G and G'. Let v satisfy $Lv = -f$ in G; $\Delta u = \sum v_{x_i x_i} = 0$ in G'; $v = 0$ on the "outer" boundary of $G \cup G'$; and on γ (i) let v be continuous, and (ii) let $dv/dN = D\partial v/\partial \nu$, where d/dN denotes differentiation in the conormal direction corresponding to a_{ij} on the G side of γ, and $\partial/\partial z$ denotes the ordinary normal derivative on the G' side; and $D > \mu$ is a sufficiently large positive constant. The author shows that, in terms of the L_2 norm in G, $\|v-u\| < cD^{-0.5}\|f\|$, where c is a constant independent of u, v, and D. He uses this result as the basis for a "universal" computer program to solve numerically the Dirichlet problem with vanishing boundary conditions for one of the equations (1) $L'U = D\Delta U - aU = -f$, $L'U = -\lambda U$, or $U_t = L'U + f$, where Δ is the plane or axisymmetric Laplacian operator. G is assumed to be bounded by polygons with sides parallel to the coordinate axes. If R is the smallest rectangle with such a boundary that encloses G, then $G' = R - G$. In G the coefficients D and f are assumed to be piecewise constant, with jumps only on segments parallel to the axes. In G' a very large positive constant value is assigned to D, and the value 0 to a and f. The numerical solution U of the Dirichlet problem on G for one of the equations (1) with conditions (i) and (ii) at the discontinuities of D is approximated by the solution V of the corresponding problem for the enlarged region R. If R is systematically divided into a set of subrectangles $\{R_{ij}\}$ by lines that include all the loci of discontinuities of D, the organization of the computation with a five-point approximation to Δ becomes very straightforward, so that by the specification of a few parameters a single program can be used to deal with a wide diversity of regions G.

J. H. Giese (Aberdeen, Md.)

Cannon, John R.; Douglas, Jim, Jr. 1749

Three-level alternating-direction iterative methods.

Contributions to Differential Equations **3** (1964), 189–198.

This is a preliminary study of the application of three-level alternating direction iterative methods for the solution of

elliptic difference equations. Three-level alternating direction methods are analyzed in some detail for the standard five- and nine-point Laplace difference equations on the unit square. *M. Lees* (Pasadena, Calif.)

Cannon, J. R. [**Cannon, John R.**] 1750
The numerical solution of the Dirichlet problem for Laplace's equation by linear programming.
J. Soc. Indust. Appl. Math. **12** (1964), 233–237.

The author describes a method for solving numerically the Dirichlet problem for Laplace's equation in a plane domain which contains the origin in its interior and is bounded by a simple closed rectifiable Jordan curve. The Dirichlet data are assumed to be known only approximately at a finite number of boundary points. With the aid of a theorem of Runge and Walsh on the uniform approximation of a harmonic function by a sequence of harmonic polynomials, the author shows how to construct a harmonic polynomial approximation to the solution of the Dirichlet problem; the coefficients of this approximation are determined as the solution of a linear programming problem. An example is given which shows that the method compares favorably with the Peaceman-Rachford alternating-direction method. *M. Lees* (Pasadena, Calif.)

Gatun, V. P. 1751
On the numerical solution of three-dimensional boundary-value problems in potential theory by the superposition of representations. (Russian)
Approximate methods of solving differential equations, pp. 19–26. *Izdat. Akad. Nauk Ukrain. SSR, Kiev,* 1963.

The author considers the finite-difference equations corresponding to the Dirichlet problem in a three-dimensional region. He gives a proof of the analogue of the Schwarz alternating procedure [see R. Courant and D. Hilbert, *Methods of mathematical physics*, Vol. II, p. 293, Interscience, New York, 1962; MR **25** #4216]. An example is given where the region consists of two parallelepipeds. In each of these he uses an explicit solution of the finite-difference equations which was derived by G. N. Položiĭ [Ukrain. Mat. Ž. **12** (1960), 308–323; MR **26** #905; *Numerical solution of two- and three-dimensional boundary-value problems of mathematical physics and functions of a discrete argument* (Russian), Izdat. Kiev. Univ., Kiev, 1962]. *H. J. Norton* (Teddington)

Leroy, Roger 1752
Méthodes variationnelles de résolution des équations fonctionnelles. Exemple des systèmes linéaires d'équations intégrales de Fredholm.
C. R. Acad. Sci. Paris **258** (1964), 2255–2258.

Es sei E ein linearer Raum, in welchem ein inneres Produkt definiert ist, und a ein linearer Operator, der E in E abbildet. Der Verfasser beschreibt einige Näherungsmethoden zur Lösung von Gleichungen $ay=b$ in E (Verfahren des steilsten Abstiegs, Verfahren von Ritz-Galerkin). *J. Schröder* (Cologne)

Medvedev, V. A. 1753
On the convergence of the Bubnov-Galerkin method.
Prikl. Mat. Meh. **27** (1963), 1148–1151 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1769–1774.

The author studies the convergence of the Bubnov-Galerkin method for linear equations in Hilbert spaces. Let a linear operator L be given in a separable Hilbert space H. The domain of definition and the range of values of L are each dense in H. The problem is to solve the equation $Lx=f$, $f\in H$. The approximate method of Bubnov and Galerkin consists in the following. Let $\{R_n\}$ be a sequence of finite-dimensional subspaces from the domain of definition of L and let P_n be orthogonal projection operators on these subspaces. The sequence $\{R_n\}$ is assumed to be projectionally complete in H [see N. I. Pol'skiĭ, Dokl. Akad. Nauk SSSR **143** (1962), 787–790; MR **26** #3173]. One seeks an approximate solution $x_n\in R_n$ of the equation $P_nLx_n=P_nf$. The author bases his research on Pol'skiĭ's paper and gives a new proof of the convergence of Bubnov-Galerkin method for equations with completely continuous operators. *D. D. Stancu* (Cluj)

Douglas, Jim, Jr. 1754
Mathematical programming and integral equations.
Sympos. on the Numerical Treatment of Ordinary Differential Equations, Integral and Integro-Differential Equations (Rome, 1960), pp. 269–274. *Birkhäuser, Basel,* 1960.

Let $\mathscr{B}$ be a linear space, and let T be an operator (possibly non-linear, with unbounded inverse) mapping $\mathscr{B}$ into $\mathscr{B}$. The author considers the functional equation (1) $Tf=g$, $g\in\mathscr{B}$. After observing that the usual discretization procedures (for $\mathscr{B}$ a function space) are often unstable, he suggests altering the meaning of (1) in the following manner: Let $\mathscr{F}\subset\mathscr{B}$ be a normal family of permissible input functions, and set $e(f)=\|T(f)-g\|$, $f\in\mathscr{F}$. Now, for given $g\in\mathscr{B}$, determine $f\in\mathscr{F}$ such that

$$(2)\qquad e(f)=\min_{f^*\in\mathscr{F}} e(f^*).$$

This amounts to solving a mathematical programming problem which, in general, is neither linear nor finite-dimensional. In order to find the solution of (2) in a constructive manner, it is suggested to consider a sequence of subfamilies $\mathscr{F}_n\subset\mathscr{F}$ such that $\bigcup\mathscr{F}_n=\mathscr{F}$ and to solve the problem corresponding to (2) for each $\mathscr{F}_n$. Under certain conditions the resulting mathematical programming problems are linear and finite-dimensional (for instance, if T is an integral operator of Volterra type), and the minima thus obtained converge to $e(f)$. If the solution f of (2) is unique, it is the limit of the solutions of the approximating problems. *P. Henrici* (Zürich)

Puhov, G. È.; Borkovskiĭ, B. A.; Stepanov, A. È. 1755
The method of simulation of continuous operators. (Ukrainian. Russian and English summaries)
Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 325–331.

Authors' summary: "A simulation method is proposed for mathematical operators of the form

$$(1)\qquad \sum_{i=1}^n a_ix_i+\sum_{i=1}^n b_i\frac{dx_i}{dt}+\sum_{i=1}^n c_if_i(x_1,\cdots,x_n)=0,$$

where t is the time; a_i, b_i, c_i are constant coefficients; x_i are constants or functions of time. The method permits one to obtain the voltage x_k corresponding to (1) at any signal terminal when the voltages x_i are applied to $n-1$ others. The method may be used for the simulation of

mathematical multidimensional operators. Simulation of systems of linear algebraic equations with an arbitrary coefficient matrix is discussed in detail."

COMPUTING MACHINES

See also 1148, 1248, 1573, 1726, 1745, 2138.

Bazilevskii, Yu. Ya. [Bazilevskiĭ, Ju. Ja.] (Editor) 1756

★The theory of mathematical machines.

Translated from the Russian by C. A. R. Hoare; translation edited by J. M. Jackson. A Pergamon Press Book.

The Macmillan Co., New York, 1963. xii + 264 *pp.* $10.00.

If the present volume, originally published in Russian six years ago [Fizmatgiz, Moscow, 1958], is indicative of current Russian research in mathematical machines, it only serves to emphasize the relatively large gap which exists in this country between the computer specialists and the theoretical mathematicians and logicians. For this book contains primarily a sound mathematical treatment of mathematical machines which are reasonably close to actual existing computers in their properties. One gathers from the preface that this collection of eight papers is intended as the first of a series.

We shall comment briefly on the eight papers. "The theory of sequential logical functions" (Bazilevskiĭ) presents a fairly standard treatment of the sequential logical functions as Boolean functions with delay and feedback added, and then discusses the use of periodic functions to represent given sequential functions.

"The structure of memory or storage systems" (Bazilevskiĭ) gives representations for various computer functions in terms of the functions developed in the preceding paper and extensions of these. The reader is cautioned that Bazilevskiĭ includes as functions of memory the various operations which in this country are assigned to the arithmetic or control units of a computer.

In the longest and most interesting paper, "Some general questions in programming", I. Ja. Akushskiĭ develops a matrix algebra of operators to handle standard programming operations. He includes a discussion of the question of "programmability", the choice of appropriate programming operators, and the simultaneous solution of several problems. This paper would provide a firm basis for a "theory of programming".

"Programming and recursive functions" (Ju. A. Shreider) discusses the choice of recursive functions to form a basis for programming. The basic list Shreider presents is admittedly redundant, but much closer to the usual basic computer functions than most lists of recursive function generators.

"Methods of estimating the efficiency of universal digital computers with programme control" (Bazilevskiĭ and Shreider) presents formulas for estimating efficiency in terms of the amount of machine time involved in the solution of a problem. The authors state that the formulas should be extended to account for the difficulty of programming for various machines, but they do not attempt this.

"The solution of systems of linear algebraic equations by a Monte Carlo method" (Shreider) seems out of place in the present collection since it is strictly numerical analysis, as the title indicates. The two methods discussed are more appropriate for special-purpose computers with built-in statistical operations.

In "The most economic address system for a digital computer", V. S. Linskiĭ discusses the relative merits of one-, two-, three-, and four-address computers, and rules in favor of the single-address machines.

Finally, Akushskiĭ presents some algorithms for parallel computation of more complex arithmetic functions in "Multi-register schemes for arithmetical operations".

In the rather extensive bibliography included with the eight papers, one notes that at least one paper in the original Russian volume has not been included in the translation. *R. R. Korfhage* (W. Lafayette, Ind.)

Harris, L. Dale 1757

★Numerical methods using Fortran.

Charles E. Merrill Books, Inc., Columbus, Ohio, 1964. xi + 244 *pp.* $9.00.

This author has not written an introduction to FORTRAN and a text on numerical methods and bound them together, as so many have done. He has made a real effort toward integrating the two goals of the book, and he has done a fine job. As the name implies, he is concerned with numerical methods, rather than numerical analysis, so one can search in vain for error analyses and developments of the mathematics behind some of the methods. For example, a single example of a Runge-Kutta method is given, without any hint as to where it came from or that there are other Runge-Kutta methods. Moreover, it is suggested that because computers are fast, the step size in the "marching" (i.e., Euler's) methods may be made arbitrarily small to reduce the truncation error. No mention is made of the round-off difficulties that are thereby introduced. It is also unfortunate that the author chose to confuse the traditional role of the symbol Q in computer methods associated with Runge-Kutta solutions. The numerical topics covered are interpolation, "differentiating and integration" (it is not clear why this combination of words was chosen), ordinary differential equations, roots of an equation and simultaneous linear equations, the last involving a little bit of matrix manipulation. FORTRAN is covered rather thoroughly, but occasionally one finds such comments as: "... the computer is not bright enough to obtain the product of 2 and 2.0." In section 8.1 a subroutine is introduced called ZERO which computes the value zero for several variables. No programmer would do this in such a situation, and no pedagogical use is made of ZERO. Another minor criticism is the poor format used in printing arrays. Although the author distinguishes between arrays and matrices and uses brackets only for the latter, it is unfortunate that he omitted the brackets entirely for arrays. It is easy to find things to criticize in any book. On balance, this book looks quite good for what it tries to do.

B. A. Galler (Ann Arbor, Mich.)

Aliev, T. A. 1758

The introduction of graphical data into universal computers for solving statistical problems. (Russian. Azerbaijani summary)

Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Mat. Tehn. Nauk **1963**, *no.* 6, 47–51.

Bacchus, Pierre; Pouzet, Pierre 1759
Autoprogrammation pour calculateur BULL (Gamma E.T.) APB.
Rev. Française Traitement Information [*Chiffres*] **7** (1964), 3–14.
Authors' summary: "One donne une description du langage APB par référence au langage ALGOL. L'emploi des variables indicées à progression automatique est caractéristique de ce langage. Créé en vue des travaux pratiques faits aux étudiants débutants, donc d'utilisation simple, il est enseigné dans les universités françaises dotées de ce matériel. Un compilateur APB existe aussi sur IBM 1620."

Fabian, Václav 1760
A recursive procedure for compiling expressions.
Rev. Française Traitement Information [*Chiffres*] **6** (1963), 275–281.
Author's summary: "A simple ALGOL 60 procedure is given for compiling expressions, and the properties claimed for the procedure are proved. Essential use is made of the recursion facility of ALGOL 60."

Karlgren, Hans 1761
Representation of text strings in binary computers.
Nordisk Tidskr. Informations-Behandling **3** (1963), 52–59.

Dahlstrand, Ingemar 1762
A half year's experience with the Facit-Algol 1 compiler.
Nordisk Tidskr. Informations-Behandling **2** (1962), 137–142.
Author's summary: "The Facit-Algol 1 compiler is described briefly. Restrictions compared to Algol 60 and compatibility with other compilers is mentioned. The paper then discusses the experience with the compiler in the areas of teaching, programming, punching, debugging, and program exchanging. The conclusion is that Algol has done well and represents a great step forward."

Danielsson, Per-Erik 1763
Converting a curve to right-angled increments.
Nordisk Tidskr. Informations-Behandling **3** (1963), 213–221.
Author's summary: "Servo systems used in numerical control are to be fed by incremental commands. For the two-dimensional case the Taylor expansion of $f(x, y)$ is a useful way to compute these increments. The proper increment in each moment can simply be deduced from the sign of the terms involved in the Taylor expansion. Three special curves are analyzed: the straight line, the circle or the circular arc and the general curve of second order. The computation and the execution of the commands can be more or less synchronous. Simplified flow diagrams for the computer program under two different assumptions are shown."

Babuška, Ivo; Prager, Václav; Vitásek, Emil 1764
Numerische Stabilität von Rechenprozessen.
Wiss. Z. Techn. Univ. Dresden **12** (1963), 101–110
Authors' summary: "Die Verfasser weisen auf wesentliche Fragestellungen der numerischen Mathematik hin, die insbesondere beim Einsatz von digitalen Rechenautomaten auftreten. An Beispielen aus verschiedenen Gebieten der praktischen Analysis werden numerische Stabilitätseigenschaften ausführlich diskutiert. Theoretische Überlegungen führen dabei zur Definition neuer Stabilitätsbegriffe, deren Brauchbarkeit an verschiedenen Aufgaben gezeigt wird."

Bubenko, Janis A., Jr. 1765
Multiprogramming—An introduction. (Swedish. English summary)
Nordisk Tidskr. Informations-Behandling **3** (1963), 1–26.
Author's summary: "The need for effective utilization of computer facilities has resulted in a great interest in parallel processing of several independent programs on a computer. The introductory parts of this paper deal with multiprogramming both for batch and real-time computer systems. Also some important hardware features such as automatic program interrupts, memory lock-outs and autonomous facilities are discussed. In the following chapter elementary program-mechanics are introduced and attention is paid to priority assignment to different programs. Some main functions of a Master Control Program are then discussed and finally some aspects are given on multiprogram scheduling of computer runs. Multiprogramming is still in an early stage of its development and therefore this paper makes no claim to completeness—it rather attempts to render some important principles of these new techniques."

Kulikowski, T. 1766
On a programming of simple arithmetic expressions.
Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 51–52.
The author establishes the conjecture of Z. Pawlak [Bull. Acad. Polon. Sci. Sér. Sci. Tech. **9** (1961), 487–490] that (1) a programme can be written which will compute an n-operation arithmetic expression using a number of storage locations for partial or final results, S_n, so that $S_n \leq E[\log_2 (n+1)]$, and that (2) a similar programme exists for which $S_n \geq E[\log_2 (n+1)]$. $E[x]$ is the integral part of x.
A. D. Booth (Saskatoon, Sask.)

Takahashi, Hidetoshi 1767
Electronic computers and number theory. (Japanese)
Sûgaku **15** (1963), 1–6.
An expository paper on certain applications of electronic computers to number theory, and those of number theory to electronic computers. *S. Ikehara* (Tokyo)

Mizuno, Yukio; Kato, Ryuji 1768
Comparison between radix and merge sorting methods on their processing efficiencies.
Information Processing in Japan **2** (1962), 24–27.
Authors' summary: "The purpose of this paper is to describe and discuss the processing efficiencies in sorting operations. The typical methods adapted to large-volume sorting are radix sorting and merge sorting. The following note is to compare the processing efficiencies of these methods and to decide the most efficient sorting condition."

Flores, Ivan 1769
Derivation of a waiting-time factor for a multiple-bank memory.
J. Assoc. Comput. Mach. **11** (1964). 265–282.
Author's summary: "The limiting effect of the number of banks of memory upon the response of the computer is studied. It is shown that this effect is diminished and almost removed by increasing the number of memory banks. A quantitative equation relating a waiting-time factor, $\mathfrak{F}$, to the memory cycle time, the input/output time and worst-case calculation time for different numbers of banks of memory is derived. This derivation is based upon a number of simplifying assumptions, all of which are explained and justified. Two graphs convey these relations and provide a means for estimating an effective number of banks for a given computational ability."

Puhov, G. È.; Stepanov, A. È. 1770
Electronic computer synthesis of algebraic objects. (Ukrainian. Russian and English summaries)
Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 447–451.
Authors' summary: "A simulation method is proposed for algebraical objects. This method uses both voltages and the currents and permits reducing the number of electronic amplifiers."

Morimoto, Hiroshi 1771
On an attempt at reappearance of human thinking by an automatic calculator.
Math. Japon. 8 (1963), 59–79.
This paper describes a conventional interpretive programme for a phrase structure language designed for mathematical problem solving. The programme has been used on the HIPAC 101-B and takes of the order of 5 minutes to implement an input sentence. Typical sentences are:
(1) "Find two numbers such that 2 times the smaller added to 3 times the larger is 37, while the difference between numbers is $\frac{3}{5}$ their sum."
(2) "I subtract 5 from a certain number, multiply the results by 7, and then add 11. The result is 67. Find the number." *A. D. Booth* (Saskatoon, Sask.)

Shimizu, Tatsujiro; Sugibayashi, Masutaro 1772
Computer-programming for problem-solving of verbal problems in arithmetic and algebra.
Math. Japon. **8** (1963), 47–57.
This paper describes the transformation grammar which is the basis of the programme described by Morimoto [#1771]. *A. D. Booth* (Saskatoon, Sask.)

GENERAL APPLIED MATHEMATICS

See also 1563.

Madelung, Erwin 1773
★Die mathematischen Hilfsmittel des Physikers.
Siebente Auflage. Die Grundlehren der mathematischen Wissenschaften, Band 4.
Springer-Verlag, Berlin, 1964. xx + 536 *pp*.
DM 49.70.
The fourth edition of this well-known and much-used book appeared in 1950 and its review will be found in MR **12**, 484. There is no foreword to the seventh edition, but a casual inspection indicates that it is identical to the fourth edition with the exception that the index has been enlarged, and an addendum to p. 279 (on linear differential equations) has been inserted.
It is, of course, important to have compendia in the mathematical literature. The difficulty with them is that one cannot remember what they contain, but then there are the pleasures of browsing.
P. J. Davis (Providence, R.I.)

Palacios, J. 1774
★Dimensional analysis.
Translated by P. Lee in conjunction with L. Roth.
Macmillan & Co., Ltd., London; St. Martin's Press, New York, 1964. xviii + 236 *pp*. 42 *s*.
This work is a slightly extended version and translation of the first Spanish edition [Espasa-Calpe, Madrid, 1956]. As is the case with many other publications by the author, definitely unorthodox views are expressed. This is evident also from the Preface: "... the novelty is of such a kind, and is in such contradiction to the ideas professed by all specialists in questions related to Dimensional Analysis and even with the philosophy accepted by the majority of contemporary physicists, that I have found it necessary to justify my statements to the utmost and to demonstrate that the new method, when applied to all chapters of physics, surpasses those held by the most highly regarded specialists".
Despite a serious attempt, the reviewer is unable to grasp the essence of the author's approach or to follow the logical development of the book.
The following is a brief summary of the contents. Part I: Homogeneous functions, Physical entities, Fundamental postulates, Concept of dimension, π-theorem. Part II (Applications): Formulation of problems, Rigid bodies, Deformable solids, Fluids, Aerodynamics, Temperature, Thermal conductivity, Thermodynamics, Statistical mechanics, Quantum mechanics, Electromagnetism, Relativity, Corpuscular physics. *H. Rund* (Pretoria)

Brillouin, Léon 1775
★Les tenseurs en mécanique et en élasticité.
Deuxième édition (deuxième tirage). Cours de Physique Théorique.
Masson et C^ie, Éditeurs, Libraires de l'Académie de Médecine, Paris, 1960. iii + 370 *pp*. 32 *F*.
The book, whose original edition appeared in 1938, consists of the following chapters: 1. Remarques générales; 2. Géométrie vectorielle. Définition des tenseurs; 3. Les pseudo-tenseurs en géométrie vectorielle. Densités et capacités tensorielles; 4. Les principaux opérateurs différentiels utilisables en géométrie vectorielle; 5. Postulat de transport parallèle; dérivée covariante en géométrie affine; 6. Géométrie métrique, espace de Riemann; 7. Les opérateurs différentiels et la dérivée covariante en géométrie métrique; 8. La mécanique rationnelle et l'emploi des géométries de Riemann; 9. Le passage à la mécanique ondulatoire; 10. Élasticité; 11. Ondes élastiques dans les solides; 12. La théorie des solides et les quanta.

Ravstin, J.; Mesnard, G. 1776
Formalisme portant sur des tableaux de fonctions, susceptible d'applications à la physique mathématique. (Italian summary)
Nuovo Cimento (10) **32** (1964), 1015–1022.
Operations on triangular tables of functions are defined; applications to field theory are anticipated.
K. Forster (Los Angeles, Calif.)

MECHANICS OF PARTICLES AND SYSTEMS

See also 1403, 1775.

Forbat, N. 1777
Trennung der Veränderlichen in mechanischen Systemen. (Russian, English, and French summaries)
Wiss. Z. Humboldt-Univ. Berlin Math.-Natur. Reihe **12** (1962/63), 533–541.
Nach einem zusammenfassenden Referat über die klassischen Ergebnisse der Hamilton-Jacobi'schen Theorie der Integration allgemeiner physikalischer Systeme werden sogenannte verallgemeinerte Stäckel'sche Systeme untersucht. Es sind dies i.a. nichtkonservative Systeme, die gekennzeichnet sind durch bestimmte Bedingungen, denen die Elemente der in den Ausdruck für die kinetische Energie eingehenden Matrix genügen müssen. Der Verfasser zeigt, daß die Variablen derartiger Systeme getrennt werden können, so dass eine allgemeine Integration der Bewegungsgleichungen durch Quadraturen möglich wird. Ist n die Zahl der Freiheitsgrade des Systems, dann existieren n quadratische Integrale dieser Gleichungen, von denen eines die Hamilton'sche Funktion ist. Die Variablen des Systems sind i.a. nicht-periodische Funktionen der Zeit, aber sie können im Falle konservativer Systeme als Summen periodischer Funktionen dargestellt werden. Die Untersuchungen des Verfassers interessieren im Hinblick auf mögliche Anwendungen in der Theorie nichtlinearer Systeme. *K. Magnus* (Stuttgart)

Wehrli, Christoph 1778
Dynamisches Verhalten einer einfach besetzten rotierenden Welle mit Kardangelenken. (English summary)
Z. Angew. Math. Phys. **15** (1964), 154–166.
Author's summary: "If a rotating, massless, elastic shaft carrying a disk is supported at the ends by Cardan links, the motion of the disk depends on the angles at the joints and the torques transmitted by the joints. The system is considered for constant angular velocity and constant torques of the driving shafts. The investigation of this nonstationary system leads to two second-order differential equations with periodic coefficients. In order to establish conditions for instability the characteristic exponents are calculated by means of generalized Hill's determinants. It is found that there exist critical intervals for the angular velocity."

Harding, C. F. 1779
Solution to Euler's gyrodynamics. I.
Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 325–328.
Given the angular velocity of a rigid body as a function of time, one may determine the "attitude" of the body as a function of time by integrating Euler's kinematical equations, which are non-linear and lead to a description of the attitude in terms of Eulerian angles. In this paper it is shown that, alternatively, the attitude of the body can be found by integrating a certain set of linear differential equations, and an infinite series solution of these equations is described. *T. R. Kane* (Stanford, Calif.)

Galeev, A. U.; Peršic, Ju. I. 1780
The distribution of accelerations in a rigid body rotating about a fixed point. (Russian)
Vsesojuz. Zaočn. Inst. Železn. Transport. Učen. Zap. Trudy Kafedr Vysš. Mat. i Teoret. Meh. No. 7 (1961), 241–250.
En utilisant les formules classiques, les auteurs trouvent que l'accélération peut être décomposée en deux composantes, l'une est l'accélération de rotation et la deuxième est une accélération dirigée du point M vers l'axe instantanné de rotation communément appelée accélération centrifuge.

Les auteurs étudient en détail ces deux composantes et donnent deux exemples pour illustrer leurs calculs. Le premier exemple concerne le cas où la vitesse angulaire est de la forme:

$$\omega_x = 5 \sin \tfrac{1}{2}\pi t/\text{sec.},$$
$$\omega_y = 5 \cos \tfrac{1}{2}\pi t/\text{sec.},$$
$$\omega_z = 1/\text{sec.}$$

Il s'agit de déterminer les composantes de l'accélération et le rayon de courbure au point M.

Dans le deuxième exemple, on donne les coordonnées de M (3, 4, 0) pour déterminer la vitesse de rotation ω et les composantes de l'accélération.
M. Kiveliovitch (Paris)

Tabueva, V. A. 1781
Existence conditions for circular motions of the Froude pendulum. (Russian)
Izv. Vysš. Učebn. Zaved. Matematika **1961**, *no.* 5 (24), 61–68.
L'auteur étudie l'équation du mouvement du pendule de Froude $I\ddot{\varphi}+b\dot{\varphi}+Maq\sin\varphi=F\cdot(\Omega-\dot{\varphi})$, où φ est l'angle de déviation par rapport à la verticale, I le moment d'inertie du pendule, $b\dot{\varphi}$ le moment du frottement ordinaire, $Maq\sin\varphi$ le moment de la force de la pesanteur du pendule, $F\cdot(\Omega-\dot{\varphi})$ le moment de la force de frottement du manchon sur la barre, Ω la constante de rotation de la barre du pendule. *M. Kiveliovitch* (Paris)

Troitskii, V. A. [Troickiĭ, V. A.] 1782
On the optimization of a vibration transporter process.
Prikl. Mat. Meh. **27** (1963), 1117–1123 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1715–1726.
The motion of a particle with friction on a conveyer is considered with conveyer plane performing (a) a vibration at an angle of, say, 30° to the plane, (b) independent vibrations in the plane and perpendicular to it. The

problem posed is to find among the continuous functions representing the displacement of the plane, having piecewise continuous accelerations, those which impart maximum displacement to the particle within one period of oscillation. Optimal functions are obtained showing that the addition of a perpendicular component of acceleration affords a significant increase in particle displacement. Regimes of motion in which the particle is "tossed up" are not considered. *K. Forster* (Los Angeles, Calif.)

Molerus, Otto 1783

Zur Instabilität von Schwingungsformen. (French summary)

Z. Angew. Math. Phys. **15** (1964), 262–272.

When the point of support of an elastic double pendulum (consisting of two particles attached to linear springs) is forced to oscillate on a vertical line, the particles can move on the same vertical line. Stability of such motions is investigated by means of approximate determinations of characteristic exponents. *T. R. Kane* (Stanford, Calif.)

Liashenko, V. F. [Ljašenko, V. F.] 1784

On the sufficient conditions of stability in the theory of a horizontal gyrocompass.

Prikl. Mat. Meh. **27** (1963), 1106–1107 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1696–1698.

This brief note consists of comments related to results obtained previously [Prikl. Mat. Meh. **27** (1963), 373–376]. *T. R. Kane* (Stanford, Calif.)

ELASTICITY, PLASTICITY

See also 1512, 1774, 1775.

★Progress in Solid Mechanics. Vol. III. 1785

Dynamical problems in elasticity by V. D. Kupradze. Edited by I. N. Sneddon and R. Hill.

North-Holland Publishing Co., Amsterdam; Interscience Publishers [John Wiley & Sons, Inc.], New York, 1963. vii + 259 *pp.* $10.50.

This book is concerned with those boundary-value problems of linearized elasticity theory related to steady-state oscillations and to equilibrium of isotropic bodies. It contains much of the author's former work on existence and uniqueness previously available only in Russian. Existence theorems for both the displacement and traction problems are established; the proofs are based on the theory of multi-dimensional singular integral equations. The book closes with a chapter on approximate solutions.

M. E. Gurtin (Providence, R.I.)

Hayart, René 1786

Extension des formules de Murnaghan relatives au solide en phase d'élasticité finie, au cas de couples superficiels.

C. R. Acad. Sci. Paris **258** (1964), 1390–1391.

En tenant compte de M^0 (position initiale du point M), de M, des voisinages des deux premiers ordres et de la température absolue T en M, on calcule la variation de la fonction φ, rapportée à un repère orthonormé (sans utiliser les trièdres des Cosserat). Différents cas particuliers sont pris en considération.

P. P. Teodorescu (Bucharest)

Truesdell, C. 1787

Second-order effects in the mechanics of materials.

Second-order Effects in Elasticity, Plasticity and Fluid Dynamics (Internat. Sympos., Haifa, 1962), *pp.* 1–47. *Jerusalem Academic Press, Jerusalem; Pergamon, Oxford,* 1964.

This paper is a reprint of a general lecture giving an exposition of the theory of non-linear effects in the behaviour of continuous media. The author summarises the general theories of Reiner for fluids, of Rivlin for nonlinear fluids and highly elastic solids, and of Noll in the classification of different kinds of non-linear materials. He then proceeds to consider second-order effects based upon the general theories, but using second-order approximations to the corresponding linear theories. Topics covered include hypoelastic materials, memory effects, and anisotropic fluids. The paper includes an extensive list of references. *J. E. Adkins* (Nottingham)

Ericksen, J. L. 1788

Non-uniqueness and non-existence in linearized elasticity theory.

Contributions to Differential Equations **3** (1964), 295–300.

This is an extension of the author's previous work on nonexistence theorems in classical linear elastostatics [Arch. Rational Mech. Anal. **14** (1963), 180–183; MR **27** #6423]. Here the author deals with the dynamical equations of the theory of small deformations superposed on large. Some definitions: A stress wave is a smooth moving surface on which the displacement vector u_α of the superposed small deformation is to be continuous, but its first partial derivatives may have finite discontinuities. A null displacement u_α satisfies the equations of motion in a region R for $0 \leqq t \leqq t_0$, with $u_\alpha = \partial u_\alpha/\partial t = 0$ in R at $t=0$, $u_\alpha = 0$ on ∂R for $0 \leqq t \leqq t_0$, $u_\alpha \not\equiv 0$ in R for $t = t_0$. The main result is the theorem that if the governing equations admit a null displacement in some region R for any non-zero interval of time t_0, then for this space-time region, there will be smooth boundary and initial data for which no solution exists, not even if the possibility of stress-waves is admitted. Stated otherwise, in general, non-uniqueness implies non-existence. The paper is a contribution of fundamental importance to the whole area of uniqueness and existence questions in linearised elasticity theory.

M. Hayes (Newcastle upon Tyne)

Kawatate, Kazuo 1789

A note on the stress function.

Rep. Res. Inst. Appl. Mech. Kyushu Univ. **11** (1963), 21–27.

The method of Dorn and Schild [Quart. Appl. Math. **14** (1956), 209–213; MR **18**, 83] for representing the stress tensor in terms of stress functions is derived anew in general tensor notation. No mention of the limited validity of this type of representation, as observed by Gurtin [Arch. Rational Mech. Anal. **13** (1963), 321–329; MR **27** #3123], is offered. *R. L. Fosdick* (Chicago, Ill.)

Seeger, Alfred 1790

The application of second-order effects in elasticity to problems of crystal physics.

Second-order Effects in Elasticity, Plasticity and Fluid Dynamics (*Internat. Sympos., Haifa*, 1962), *pp.* 129–144. *Jerusalem Academic Press, Jerusalem; Pergamon, Oxford*, 1964.

An expository paper. Non-linear effects occur in the theory of crystal defects within the cores of defects, where details of atomic structure make continuum theory inapplicable, and in the surrounding strain field, where second-order effects may be treated as perturbations of the linear theory. The two available methods are those of displacement gradients (Murnaghan, Green and Adkins, Rivlin) and of stress functions (Kröner and Seeger). Illustrative examples and the experimental determination of the third-order elastic constants are discussed.

F. R. N. Nabarro (Johannesburg)

Guo, Zhong-heng 1791

Homographic representation of the theory of finite thermoelastic deformations. (Polish and Russian summaries)

Arch. Mech. Stos. **15** (1963), 475–505.

This paper contains no new information in continuum mechanics and deals strictly with the notational aspects of the calculus of vectorial homography as applied to the theory of finite deformations. The basis of the paper appears to be an outgrowth of a study of the Italian literature on finite deformations and contains many references to this work. The paper aims at introducing the reader to the mathematical formalism of the method of homographic representation.

R. L. Fosdick (Chicago, Ill.)

Guo, Zhong-heng 1792

Some notes on hypoelasticity. (Polish and Russian summaries)

Arch. Mech. Stos. **15** (1963), 683–690.

The author examines the theory of hypo-elasticity, but starting with a definition which assumes one stress rate to be unsymmetrical. He then defines two kinds of hypo-elasticity based upon two different definitions of the stress rate. He concludes that one definition includes the other as a special case. Taking the broader definition he examines the problem of simple acceleration here. Not all of the equations in the paper appear to be tensorially invariant.

J. E. Adkins (Nottingham)

Bhargava, R. D.; Kapoor, O. P. 1793

Circular inclusion in an infinite elastic medium with a circular hole.

Proc. Cambridge Philos. Soc. **60** (1964), 675–682.

Authors' summary: "This paper deals with the two-dimensional problem of a circular inclusion undergoing spontaneous dimensional changes in an infinite elastic medium with a circular hole. This provides a more realistic model for some physical phenomena where inclusions have found applications. The effect of a concentrated force acting at an arbitrary point of an infinite medium with a circular hole has been found as an auxiliary result. After the evaluation of the effect of a concentrated force, Eshelby's point force approach has been used to get an exact solution to the inclusion problem."

Fox, N. 1794

On asymptotic expansions in plate theory.

Proc. Roy. Soc. Ser. A **278** (1964), 228–233.

A three-dimensional harmonic function $\phi(x, y, z, h)$ is sought in the plate region bounded by the faces $|z| \leq h$ and a cylindrical surface S (the edge) perpendicular to the (x, y)-plane. The closed curve of intersection with the plane and its arc length are denoted by C and A. On the boundaries ϕ is prescribed: $\phi(x, y, \pm h, h) = \phi_f(x, y)$; $\phi = \phi_f(x, y) + \phi_C(s, z/h)$ on S, where ϕ_f and ϕ_C are smooth functions, $\phi_C(s, \pm 1) = 0$ and ϕ_C is an even function of z. The solution, $\Phi(x, y, \zeta, h) \equiv \phi(x, y, z, h)$, where $\zeta = z/h$, is assumed to have a power series expansion in h. The coefficients are explicitly determined by substituting the expansion into the differential equation and the face boundary conditions and equating coefficients of like powers of h. The expansions do not, in general, satisfy the conditions on S.

To obtain an expansion that is valid near the edge, a "stretched" normal coordinate $\nu \equiv n/h$ is introduced, where n is the distance measured along an inwardly drawn normal to C. In the region $B: 0 \leq s \leq l$, $0 \leq \nu \leq d/h$, $|\zeta| \leq 1$, the solution $\Psi(s, \nu, \zeta, h) \equiv \phi(s, n, z, h)$ is assumed to possess a power series in h. Here the length d is selected so that every point of B in the (x, y)-plane lies on one, and only one, normal to C. The coefficients in the expansion, which satisfy two-dimensional Poisson equations in ν and ζ with s as a parameter, are required to vanish outside of B. Boundary conditions for the coefficients are obtained by specifying that $\Phi + \Psi$ satisfy the original boundary conditions.

The first terms Φ_0 and Ψ_0 in each expansion are explicitly determined. It is then shown that

$$(\phi - \Phi_0 - \Psi_0)/(\Phi_0 + \Psi_0) \to 0$$

as $h \to 0$ uniformly in the plate region.

E. L. Reiss (New York)

Nelson, Frederick C. 1795

Frequency cross-over of simply-supported circular ring segments.

J. Franklin Inst. **278** (1964), 20–27.

Author's summary: "As the opening angle of a simply-supported circular ring segment is increased the frequency curves associated with the symmetric and antisymmetric mode shapes cross each other. This paper establishes the initial cross-over angles for the fundamental and first harmonic as a function of a center line extension parameter. A comparison is made to Reissner's result for a shallow sinusoidal arch."

Oravas, G. Æ. 1796

Direct kinematic theory of flexure of beams.

Österreich. Ing.-Arch. **18** (1964/65), 131–152.

An expository paper on the classical small-deflection theory of elastic beams without torsion.

R. P. Nordgren (Houston, Tex.)

Özbek, Tekin 1797

Die analytische Lösung der gekrümmten elastischen ebenen Stäbe. (Turkish summary)

İstanbul Tek. Üniv. Böl. **16** (1963), 39–47.

Author's summary: "In der vorliegenden Arbeit wurden die Schnittgrössen und Formänderungen ebener gekrümmter elastischer Stäbe in Abhängigkeit von der gegebenen äusseren Belastung analytisch ermittelt. Man hat gezeigt, dass die tangentiale Verschiebung aus der gegebenen Belastung mit Hilfe einer Differentialgleichung berechnet werden kann und die andere Formänderungskomponente und Schnittgrössen als Funktionen der tangentialen Verschiebung ausgedrückt werden können, wenn die Belastung in der Stabebene ist. Die Ergebnisse sind zu den parabolischen Stäben angewendet und ein Beispiel angegeben."

Sarkisjan, V. S. 1798

On the solution of the problem of the bending of anisotropic (non-orthotropic) plates on an elastic base. (Russian. Armenian summary)

Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. **17** (1964), *no.* 2, 65–79.

By a formal perturbation procedure, the author reduces the nonseparable problem of the title to a recursive series of problems for an associated orthotropic plate. The leading two terms are obtained in Fourier series for a simply supported rectangular plate under a sinusoidally distributed load. *R. P. Nordgren* (Houston, Tex.)

Yu, Yi-Yuan 1799

Generalized Hamilton's principle and variational equation of motion in nonlinear elasticity theory, with application to plate theory.

J. Acoust. Soc. Amer. **36** (1964), 111–120.

Author's summary: "A generalized Hamilton's principle and the associated variational equation of motion are presented for finite elastic deformations. By means of the generalized variational equation of motion, it is possible to deduce immediately the usual nonlinear equations of elasticity, such as were discussed by Novozhilov. The inconsistence in Novozhilov's work for a simplified nonlinear case of technical importance is pointed out. In its linearized version, the generalized variational equation of motion is next employed in the derivation of plate equations for both flexural and extensional motions, which are finally compared with those given by previous authors." *W. D. Kroll* (Washington, D.C.)

Sur, S. P. 1800

Vibration of inhomogeneous spherical shell of aeolotropic materials.

Indian J. Mech. Math. **2** (1964), 6–9.

Author's summary: "In a recent paper, Chakravorty [Bull. Calcutta Math. Soc. **47** (1955), 235–238; MR **18**, 434] studied the problems of radial and rotatory vibration of a spherical shell of aeolotropic elastic material. In this paper, the problems of radial and rotatory vibration of aeolotropic spherical shell have been considered when the materials of the shell are nonhomogeneous."

Vaĭnberg, D. V.; Itenberg, B. Z. 1801

Cylindrical ribbed shell under the action of end-face discrete forces. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 457–462.

Authors' summary: "The authors consider the problem of a cylindrical ribbed shell, the end face of which is reinforced with a rigid ring, to which discrete forces and moments are applied, or loads distributed along various areas of the end face of the shell. A system of basic resolving differential equations was obtained for the displacement problem on the basis of a model of a constructively orthotropic shell. A numerical investigation of certain cases was carried out."

Naubereit, Harry 1802

Einflussflächen der Kreiszylinderschale, belastet durch Einzelmomente.

Wiss. Z. Techn. Univ. Dresden **12** (1963), 147–156.

Author's summary: "Die drei gekoppelten partiellen Differentialgleichungen der Kreiszylinderschale lassen sich unter Verwendung einer Spannungsfunktion (für den jeweiligen Belastungsfall) in eine gewöhnliche Differentialgleichung 8. Ordnung mit konstanten Koeffizienten überführen. Das Problem wurde an der geschlossenen unendlich langen Schale, bei Verwendung von im Gleichgewicht stehenden Einzelmomenten betrachtet. Die Belastung wurde mittels Fourier-Reihe in Ringrichtung und Fourier-Integral in Richtung der Erzeugenden dargestellt. Das in der Spannungsfunktion auftretende Fourier-Integral ist geschlossen lösbar und führt auf Greensche Funktionen. Die Verschiebungen und Schnittlasten lassen sich aus der Spannungsfunktion ermitteln."

Ghosh, Basudev 1803

On the stresses and displacement in a twisted spherically isotropic medium containing a spherical cavity having rigidity varying periodically with distance.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **97** (1962/63), 1001–1013.

Author's summary: "Stress and displacement components have been calculated in a non-homogeneous spherically isotropic material containing a spherical cavity when a twist is applied on the inner surface of the cavity. The non-homogeneity has been assumed to be periodic with respect to radial distance. Expressions for the same have been derived for a thick spherical shell made of the same material possessing same non-homogeneity when a twist is applied on the outer surface."

Deutsch, E. 1804

Torsion of a transversely isotropic semi-infinite body. (Polish and Russian summaries)

Arch. Mech. Stos. **15** (1963), 517–533.

The author generalises, for transverse isotropy, previous work on the twisting of an isotropic elastic semi-space (full references are given in the paper). The partial differential equation governing the solution of this axisymmetric problem is reduced by means of a Hankel transform, and formal solutions for the displacement and the shearing stresses are obtained in the following cases: (i) when shearing forces are prescribed over a circular patch

on the plane base of the semi-infinite solid, and (ii) when a torsional displacement is prescribed over a circular patch, the surface exterior to the patch being stress-free. Some numerical work has been carried out, and the results are displayed graphically. *J. Hubert Wilkinson* (Battersea)

Harris, Paul; Greene, Richard F. 1805
Decay of elastic waves in solids with dislocations.
J. Appl. Phys. **35** (1964), 2170–2173.
For one-dimensional visco-elastic strain the Lagrangian strain is assumed to be given by a power series in the stress, each term σ^n having an independent relaxation kernel $A_n(t-\tau)$. For linear relaxation, A_1 is determined to represent the Granato-Lücke dislocation relaxation. An attempt is made to treat the non-linear "break-away" terms of the G-L theory by considering a single harmonic term $n \approx 15$. A complex representation $\sigma = \exp i\omega t$ is used, and σ^n is written as $\exp ni\omega t$, implying the real part $\cos n\omega t$. The reviewer suggests that the original equation implies the use of $\cos^n \omega t$.
F. R. N. Nabarro (Johannesburg)

England, A. H. 1806
A note on cracks under longitudinal shear.
Mathematika **10** (1963), 107–113.
Considering anti-plane strain with displacements $u=0$, $v=0$, $w=w(x,y)$, Barenblatt and Čerepanov [Prikl. Mat. Meh. 25 (1961), 1110–1119] treated the effect of homogeneous shear on a body containing a crack running from $x=-a$ to $x=+a$, or containing an infinite row of such cracks along either the x- or the y-axis. The results are here generalised to the case of a stress distribution which takes arbitrary values far from the cracks.
F. R. N. Nabarro (Johannesburg)

Vodička, Václav 1807
Free vibrations of a composite circular plate.
Acta Phys. Austriaca **17** (1963/64), 319–332.
This paper outlines a process for finding the normal modes of oscillation of a thin plate made up of concentric annuli of different materials. The solutions are expressed in matrix form. *J. W. Craggs* (Melbourne)

Nowinski, J. L. 1808
Large-amplitude oscillations of oblique panels with an initial curvature.
AIAA J. **2** (1964), 1025–1031.
Author's summary: "von Kármán field equations for flexible oblique plates with an initial curvature are extended to a dynamical case. Using series representation of initial and additional deflections and Galerkin's procedure, the governing equation for an admissible mode time function is established. Using this single assumed modal deflection, and assuming built-in edge free to move in the inplane directions, the following particular cases are discussed: buckling of an oblique plate under uniaxial compressive load, free linear vibrations of a square plate, large deflections of a uniformly loaded square plate, snap-through phenomena of a curved oblique plate under uniform transverse load, and free nonlinear vibrations. A numerical example concerning a rhombic plate is discussed in more detail. The well-known fact of a decrease of the period of nonlinear vibrations with an increasing amplitude is corroborated, this relation being less pronounced for larger sweep angles."

Pipes, Louis A. 1809
The dynamic stability of a uniform straight column excited by a pulsating load.
J. Franklin Inst. **277** (1964), 534–551.
Author's summary: "The parametrically excited transverse oscillations of a uniform straight column whose ends are pinned are investigated. The ends of the column are considered to be subjected to a pulsating load whose time graph is a square wave. By the use of matrix algebra, an expression giving the regions of instability is obtained. The effects of more general boundary conditions are studied and a nonlinear differential equation to account for the limitation of the amplitude of the oscillations is developed."

Hoff, Nicholas J.; Chao, Chi-Chang; Madsen, Wayne A. 1810
Buckling of a thin-walled circular cylindrical shell heated along an axial strip.
Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 253–258.
The authors investigate by small-deflection theory the stability of a thin cylinder heated uniformly over an axial band, the remainder of the cylinder being unheated. The cylinder is restrained from axial elongation, so that the band is uniformly stressed in the axial direction and the remainder unstressed. The authors' statement that this condition would apply to the middle of a very long cylinder seems incorrect unless the same restraint were applied to it, and in this case Euler buckling would also be a factor.

Like a number of recent similar studies of the stability of cylinders under nonuniform axial compression, the authors find a buckling stress only slightly greater than for uniform compression unless the width of the stressed strip is small, say less than $2.5(ah)^{1/2}$, where a and h are the radius and thickness. Since the corresponding theory for uniform compression yields buckling stresses 2 to 10 times those found in experiments and it has been found necessary to consider large deflection effects and defects to explain all the discrepancies, it is not clear what physical significance, if any, such studies have; this is not discussed. The accuracy of the simplified Donnell equations used by the authors for this case, where there are only one or two waves in the circumference, is also questionable. *L. H. Donnell* (Chesterton, Ind.)

Reismann, Herbert 1811
Response of a prestressed elastic plate strip to a moving pressure load.
J. Franklin Inst. **278** (1964), 8–19.
Author's summary: "The dynamic response of an infinite, prestressed plate strip under a moving, distributed pressure load is determined. The transverse pressure load is assumed to propagate, with constant speed, in the direction which is parallel to the infinite edges of the plate strip. The constant, uniform prestress acts in the plane of the plate in a direction which is parallel to the infinite edges of the plate strip. The problem is formulated as a boundary-value problem within the framework of the

classical, small-deflection theory of thin plates, and solutions are obtained in terms of trigonometric series. It is shown that the shape of the resulting deflection profile of the plate depends strongly on the speed of propagation of the load as well as on the magnitude (and sense) of the prestress. Because damping is neglected, there exist combinations of load speed and prestress for which deflections become unbounded (critical speeds). The concepts of group and phase velocity are used to determine the steady-state response for supercritical speeds."

Jones, J. P. 1812
Wave propagation in a two-layered medium.
Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 213–222.
This paper treats the problem of elastic wave propagation in a plate consisting of two layers bonded together at the interface. The layers are of finite thicknesses h_1 and h_2, but of infinite extent. The main part of the paper deals with the propagation of a Rayleigh type train of waves, treated as a two-dimensional problem. The equations of motion together with the boundary conditions lead to an eighth-order determinant whose vanishing determines the phase velocity for a given wave number. This determinant involves six parameters, essentially two elastic constants and one density for each medium, and its complexity makes it difficult to obtain analytic information from it. The paper presents in graphical form the results of digital computer solution of the determinantal equation for one particular set of values of the six parameters, and for $h_1/(h_1+h_2)=0.1(0.1)0.9$. The graphs show a plot of dimensionless wave number against dimensionless phase velocity.

The analysis for *SH* waves (transverse vibrations parallel to the surfaces and perpendicular to the direction of the propagation) is also considered, but no calculations are presented. *I. M. Longman* (Rehovot)

Yeh, Gordon C. K. 1813
The diffraction of a plane compressional wave by a spherical cavity in an elastic medium. (German summary)
Z. Angew. Math. Phys. **15** (1964), 237–252.
The title problem is treated by use of a modified form of Laplace transform, and expansion in zonal harmonics. The transforms of the stresses and displacements are quoted but not inverted. *J. W. Craggs* (Melbourne)

Lubliner, Jacob 1814
A generalized theory of strain-rate-dependent plastic wave propagation in bars.
J. Mech. Phys. Solids **12** (1964), 59–65.
The general incremental quasi-linear stress-strain relation

$$\dot{\varepsilon} = f(\sigma, \varepsilon)\dot{\sigma} + g(\sigma, \varepsilon)$$

is discussed in relation to previously proposed constitutive laws and to experimental observations. It is shown that the general law offers some greater flexibility in representing diverse aspects of material behavior. For example, it is shown that the law leads asymptotically to a general stress-strain relation at very high strain rates rather than to only elastic response. Also, as with other rate laws, it predicts that superimposed stress increments propagate at the elastic velocity. *S. R. Bodner* (Providence, R.I.)

Ventura, Gerardo L.; Cicala, Placido 1815
Stati di tensione nei pannelli paraboloidici sottili poco svergolati.
Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **97** (1962/63), 1205–1222.
Authors' summary: "Per le strutture a guscio con superficie media appartenente ad un paraboloide iperbolico si determinano gli stati di tensione e deformazione atti a rappresentare gli effetti di distribuzioni continue di carichi applicate lungo i lati di pannelli quadrilateri con piccolo svergolamento."

Kühn, G. J. 1816
Symmetry of energy-transfer ratios for elastic waves at a boundary between two media.
J. Acoust. Soc. Amer. **36** (1964), 423–427.
From the author's summary: "A proof is presented that the energy-transfer ratios of the various elastic-wave modes reflected and transmitted at a plane boundary between two media are symmetrical and equal to the product of the forward- and backward-reflection coefficients. This relationship is shown to hold for several types of lossless boundary conditions at a solid-solid interface."
L. S. D. Morley (Farnborough)

Sinha, S. B. 1817
A study of the transverse waves in layered hypo-elastic media.
J. Sci. Engrg. Res. **7** (1963), 293–306.
Author's summary: "The propagation of transverse waves in hypo-elastic plates of grade one has been investigated. It is found that the transverse wave velocity retains the same character as in the classical linear elastic body. The fundamental difference comes through the fact that none of the six components of stress tensor vanishes, while in classical linear theory only two shear components have non-zero values." *M. Hayes* (Newcastle upon Tyne)

Sawczuk, Antoni 1818
On initiation of the membrane action in rigid-plastic plates.
J. Mécanique **3** (1964), 15–23.
The aim of the paper is to make some assessment of the effect of membrane stress on the collapse loads of flat plates under normal pressure. A virtual velocity method is used, and is such that a first-order estimate of the effect can be made without detailed knowledge of the flow rule, provided that the collapse load in the absence of membrane stresses is known. *J. W. Craggs* (Melbourne)

Lomakin, V. A. 1819
On the theory of plasticity of anisotropic media. (Russian. English summary)
Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, *no.* 4, 49–53.
Author's summary: "Stress-strain relations are proposed for work-hardening anisotropic media. The theory is based on von Mises' yield condition for anisotropic media, the plastic potential hypothesis and the work-hardening hypothesis according to which the work-hardening parameter is a function of the work of stresses over plastic deformations. In the isotropic case the proposed relations coincide with Reiss' theory."

Conroy, M. F. 1820
The behavior of rigid-ideally plastic beams due to dynamic loading.
J. Mécanique **2** (1963), 455–473.
Author's summary: "This paper is concerned with the behavior of a thin, infinite or semi-infinite, uniform beam under the action of a dynamic load. In the case of the semi-infinite beam, the load consists of a transverse shear force, $F(t)$, and a bending moment, $M(t)$, applied at the free finite end. In the case of the infinite beam, the load consists of a concentrated transverse force, $P(t)$. The beam is assumed to be made of a rigid-ideally plastic material. The effects of transverse shearing deformations, strain rate, and rotary inertia are taken to be negligible, and the yield condition is assumed to depend only upon the bending moment. Two possible types of initial deformation phase are found. The later phases of deformation are then considered. The application of the results of the paper to finite beams is discussed." *T. W. Ting* (Raleigh, N.C.)

Rychlewski, J.; Ostrowska, J. 1821
On the initial plastic flow of a body with arbitrarily small non-homogeneity. (Polish and Russian summaries)
Arch. Mech. Stos. **15** (1963), 697–710.
The authors give the equations of plane strain of an inhomogeneous rigid-plastic body and consider the case in which the shear yield stress K has the form $K = K^0 + \delta K$, where K^0 is a constant and δK is a function of position which is everywhere small compared to K^0. The theory is applied to the problem of indentation of a half-plane by a smooth rigid punch, it being assumed that the slip-line field is a perturbation of a known slip-line field for the homogeneous body. This basic field is taken to be the well-known field for this problem but modified so as to be asymmetrical about the centre line of the punch. A parameter λ defines the asymmetry of the field. By taking δK to be various functions of position, a number of different cases of inhomogeneity are treated. For each case, the indentation pressure is obtained as a function of λ, and λ is chosen so as to minimise this indentation pressure. The related problem of the tension of a deeply notched strip is treated in a similar manner.

The reviewer notes that the authors do not take into account the limits on the intensity of stress singularities in plastic rigid bodies derived by Hill [J. Mech. Phys. Solids **2** (1954), 278–285; MR **16**, 94]. The application of Hill's inequalities seems in every case to determine λ to have the same value as is obtained by the authors' method, but in a rather more direct manner.
A. J. M. Spencer (Nottingham)

Hult, Jan 1822
On the stationarity of stress and strain distributions in creep.
Second-order Effects in Elasticity, Plasticity and Fluid Dynamics (Internat. Sympos., Haifa, 1962), pp. 352–361. *Jerusalem Academic Press, Jerusalem; Pergamon, Oxford,* 1964.
The expression "stationarity of stress" is used by the author to indicate that the ratio of any two components of the stress tensor is independent of time. It is shown that if the constitutive equation for stress and strain rate is of a type attributed to Odqvist (where the strain rate tensor is proportional to the stress deviator tensor, with the proportionality factor involving unspecified functions of stress and strain invariants) certain restrictions must be satisfied by the unspecified functions which enter into the constitutive equation in order to ensure that the stress and strain states remain stationary. These restrictions are suggested by two simple one-dimensional problems involving a truss and a beam. It is shown in the simpler examples that one-dimensional creep laws of the types attributed to Norton and Nadai, will lead to stationarity of stress, whereas other laws, such as those proposed by Soderberg and Prandtl, will not lead to stationarity of stress. It is suggested that this result may offer the possibility of determining experimentally which of various suggested creep laws give the best description of creep due to time-dependent loading. *B. Paul* (Bedminster, N.J.)

FLUID MECHANICS, ACOUSTICS

See also 1313, 1787.

Trustrum, Kathleen 1823
Rotating and stratified fluid flow.
J. Fluid Mech. **19** (1964), 415–432.
The problem of fluid systems with basic density or vorticity fields is greatly complicated by the non-linearity of the governing equations. In recent years some progress has been made in gaining an understanding of such systems when the flow patterns are steady; under these circumstances a first integral of the equations of motion may be found and in some cases one can deal with a linear differential equation. No such trick is possible when the flow is non-steady. In the present paper, the author overcomes this difficulty to some extent by using an Oseen approximation to the primitive equations. Her paper serves to explain a number of phenomena which have been observed experimentally, in particular, the well-known blocking effects. This approach may be useful in attacking more practical problems, such as airflow over mountains in which it may be shown that the inertial terms are of fundamental importance. *R. R. Long* (Baltimore, Md.)

Lahiri, Sukumar 1824
On the uniform shearing motion of a fluid past a projection.
Indian J. Mech. Math. **2** (1964), 14–16.
Author's summary: "The problem of two-dimensional flow with constant shear past a projection in the form of an arc of a circle has been considered in this paper. The positions of the stagnation points have been determined for different values of a non-dimensional parameter."

Majumder, S. R. 1825
A circular disc in the presence of a point source on its axis in perfect fluid.
Indian J. Mech. Math. **2** (1964), 2–5.
Author's summary: "In this paper 'the problem of a circular disc of finite radius in perfect fluid having a source at a point on its axis' has been considered. The solution of this problem has been deduced from that of an oblate

spheroid by making the eccentricity of the meridian section tend to unity. The potential function for the fluid motion, the velocity of the fluid on the disc and the pressure on the disc all have been found out in terms of infinite series. *J. W. Craggs* (Melbourne)

MacCamy, R. C. 1826

The motion of cylinders of shallow draft.

J. Ship Res. **7** (1963/64), *no.* 3, 1-11.

Author's summary: "The earlier work of the author [same J. **5** (1961/62), no. 3, 34-43] is extended. A perturbation procedure is developed for the determination of the two-dimensional fluid motion produced by a long cylinder, of shallow draft, in forced oscillation. The limiting case is that of a raft of zero draft. The leading term for heave was previously calculated numerically [loc. cit.]. Here the corresponding first term for roll is computed. Also the first correction term for heave is determined. This requires some alteration of the numerical technique. A discussion is given of the validity of the perturbation procedure." *F. Ursell* (Manchester)

Burke, J. E. 1827

Scattering of surface waves on an infinitely deep fluid.

J. Mathematical Phys. **5** (1964), 805-819.

From the author's summary: "The reflection and transmission of small amplitude waves incident on a plane barrier submerged in an infinitely deep fluid are investigated; the barrier has a finite width and is parallel to the undisturbed free surface of the fluid. Green's function techniques are used to represent the velocity potential in terms of its discontinuity (pressure difference) across the barrier. The application of the boundary condition at the barrier leads to a one-dimensional integral equation for the pressure difference. This equation is extended by introducing the fluid velocity normal to the plane of the barrier, and then it is analyzed by the complex Fourier transform methods of Wiener and Hopf. The velocity transforms are found to be characterized by a pair of dual inhomogeneous integral equations which allow systematic approximation by iteration. A convolution provides representations for the velocity potential in the different regions of the fluid, and these are the basis for investigating the reflection and transmission of waves. Results for some related problems (e.g., scattering by a semi-infinite barrier, by a finite dock, etc.) are obtained as limiting cases." *A. E. Heins* (Ann Arbor, Mich.)

Erošin, V. A. 1828

Motion of a slender body in a liquid of variable depth. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, *no.* 3, 66-69.

This paper presents the solution of the boundary-value problem corresponding to the axial motion of a slender cone, floating in an inverted position with its base in the plane of the free surface. The fluid is bounded by a sloping plane bottom of angle $\pi/2n$, with n an integer, and the high-frequency (i.e., vanishing velocity potential) boundary condition is applied on the free surface. The problem is solved in a straightforward manner by the method of images. Detailed results are given for the velocity potential and axial force in the case $n=1$, corresponding to a vertical wall. *J. N. Newman* (Washington, D.C.)

Saint-Guily, B. 1829

Ondes liquides de gravité en milieu inhomogène et dans un bassin tournant.

J. Mécanique **2** (1963), 425-435.

The author finds the nature of the solutions to the title problem to depend on the relative value of the period of inertia with respect to the Väisälä periods. For example, when the period of inertia is greater than the Väisälä periods, there is a progressive surface wave only for periods less than that of the greater Väisälä period, while there is a series of internal waves for all positive periods: these internal waves are progressive for periods between the lesser Väisälä period and the period of inertia but are stationary for positive periods outside this range. Phase velocities and group velocities are obtained. An interesting feature is that the group velocity for internal waves is negative when the period of inertia is less than the Väisälä periods. *L. R. Mack* (Austin, Tex.)

Concus, Paul 1830

"Standing capillary-gravity waves of finite amplitude": Corrigendum.

J. Fluid Mech. **19** (1964), 264-266.

The uniqueness condition previously utilized by the author [same J. **14** (1962), 568-576; MR **26** #3301] is shown to be unacceptable because the set of excluded depths is everywhere dense. The presence of viscosity is invoked to permit retention of the previously obtained solutions both of the author and of others.
L. R. Mack (Austin, Tex.)

Ball, F. K. 1831

Energy transfer between external and internal gravity waves.

J. Fluid Mech. **19** (1964), 465-478.

Solutions are obtained for resonant gravity wave interactions in a two-fluid model. The nonlinear coupling between internal and external waves is examined in detail for the case of long waves.
D. J. Benney (Cambridge, Mass.)

Moffatt, H. K. 1832

Magnetic eddies in an incompressible viscous fluid of high electrical conductivity.

J. Fluid Mech. **17** (1963), 225-239.

Author's summary: "It is shown that in an incompressible fluid in which the magnetic diffusivity λ is much less than the kinematic viscosity ν, certain magnetic field distributions of limited spatial extent (conveniently described as magnetic eddies) can exist on a length scale such that the associated Reynolds number and magnetic Reynolds number are respectively small and large compared with unity. The Lorentz forces are in equilibrium with the dynamic forces associated with the fluid motion. The boundary condition imposed on this motion is that at a large distance from a magnetic eddy the velocity field should be a uniform axisymmetric irrotational straining

motion. The eddies are steady in the limit $\lambda \to 0$, but decay slowly in a fluid of finite conductivity. Two particular eddies are considered in detail: a disk-shaped eddy in a compressive straining motion, and a spherical eddy in an extensive straining motion. Possible applications to turbulance in interstellar gas clouds are qualitatively considered."

Berker, Ratip 1833
Contrainte sur une paroi en contact avec un fluide visqueux classique, un fluide de Stokes, un fluide de Coleman-Noll.
C. R. Acad. Sci. Paris **258** (1964), 5144–5147.
Author's summary: "Nous nous proposons de donner une expression simple de la contrainte qui s'exerce sur une paroi en contact avec un fluide. Nous supposerons successivement qu'il s'agit: (a) d'un fluide visqueux classique; (b) d'un fluide de Stokes; (c) d'un fluide du deuxième ordre au sens de Coleman et Noll. La paroi sera supposée d'abord fixe, ensuite en mouvement."
D. R. Breach (Toronto, Ont.)

Okabe, Jun-ichi 1834
Similarity in boundary layer separations of two-dimensional flows.
Rep. Res. Inst. Appl. Mech. Kyushu Univ. **11** (1963), 33–37.
If the external velocity is given by $u_1 = \alpha\phi(x)$ or $x = \beta\psi(u_1)$, then the author shows from Howarth's or Pohlhausen's approximate forms for the momentum-integral equation that in incompressible laminar boundary-layer flow the position of separation is independent of α in the first case and that the value of u_1 at separation is independent of β in the second case. Actually, these results may easily be derived from the full boundary-layer equations.

Using an approximate form for the momentum integral equation, the author proves the same results for turbulent incompressible boundary layers.
J. C. Cooke (Farnborough)

Barcilon, Victor 1835
Role of the Ekman layers in the stability of the symmetric regime obtained in a rotating annulus.
J. Atmospheric Sci. **21** (1964), 291–299.
Author's summary: "The stability of the symmetric regime observed in the rotating heated annulus experiments is investigated. It is shown that the Ekman layers, which have been neglected in previous investigations, are important. In addition to their role in the dissipation of the kinetic energy of the disturbances, they induce vertical velocities which can affect the release of the available potential energy necessary for the occurrence of baroclinic instability. The transition curve obtained for marginal stability is in qualitative agreement with the experimental data."

Dolapčiev, Blagovest; Čobanov, Ivan 1836
Geschwindigkeitsregime einer Flüssigkeit bei Anwesenheit von Kármánschen Wirbelstrassen. (Bulgarian. German summary)
Annuaire Univ. Sofia Fac. Sci. Phys. Math. Livre 1 Math. **56** (1961/62), 35–72 (1963).

Yu, Yun-Sheng; Sun, Dah-Chen 1837
Stability of a viscous flow between two rotating coaxial cylinders.
J. Franklin Inst. **277** (1964), 140–149.
Authors' summary: "The stability of a viscous flow between two rotating coaxial cylinders with large ratio of gap-width to cylinder-radius is investigated. The mathematical characteristic-value problem is solved approximately. Numerical calculations for the radius-ratio equal to 1.71 are carried out to determine the critical Taylor numbers at the onset of instability for different values of the ratio of the rotational speed of the outer cylinder to that of the inner cylinder. The calculated results are compared to Lewis's experiments."

To integrate the differential equation for the radial velocity the authors replace r^{-1} by $a+br^2+cr^4$, where a, b and c are rational functions of η, the ratio of the radii of the two cylinders, and an undefined variable ξ. This last variable, according to the authors, must be selected to lie between 1 and η. Then a, b and c are treated as constants. It is not explained how this approximation was derived nor are the conditions under which it is valid mentioned.
D. R. Breach (Toronto, Ont.)

Verma, Y. K.; Verma, Pratibha 1838
The Rayleigh-Taylor instability of a compressible conducting fluid in the presence of a magnetic field.
Proc. Nat. Inst. Sci. India Part A **29** (1963), 309–315.
Authors' summary: "The Rayleigh-Taylor instability for an interface separating two compressible media has been investigated in the presence of horizontal magnetic field. In incompressible fluids for the disturbances perpendicular to the direction of H, the magnetic field does not affect the development of Rayleigh-Taylor instability, but in the case of compressible fluids, the magnetic field affects the development of Rayleigh-Taylor instability for such disturbances also. The critical wave number below which Rayleigh modes do not exist is found, which is less than that obtained by Vandervoort. Thus magnetic field has a stabilizing effect on Rayleigh-Taylor disturbances."

Verma, Y. K.; Verma, Pratibha 1839
The Rayleigh-Taylor instability of a rotating compressible inviscid fluid.
Proc. Nat. Inst. Sci. India Part A **29** (1963), 316–318.
Authors' summary: "Rayleigh-Taylor instability of a rotating compressible inviscid fluid has been investigated. The Coriolis force for unstable configuration decreases the critical wave-length of perturbations which give rise to Rayleigh modes, while for stable configuration it increases this critical wave-length."

Verma, Y. K.; Verma, Pratibha 1840
The Rayleigh-Taylor instability of a rotating compressible conducting fluid in the presence of a magnetic field.
Proc. Nat. Inst. Sci. India Part A **29** (1963), 319–324.
Authors' summary: "The Rayleigh-Taylor instability of a rotating compressible conducting fluid in the presence of a magnetic field has been investigated. It is found that for stable configuration magnetic field and rotation both enhance the critical wave-length of the perturbation which

gives rise to Rayleigh modes, while for unstable configuration a magnetic field enhances the critical wave-length whereas rotation decreases this critical wave-length."

Meksyn, D. 1841

Stability of laminar flow between parallel planes for two- and three-dimensional finite disturbances.

Z. Physik **178** (1964), 159–172.

The author considers the stability of laminar flow between parallel planes for two- and three-dimensional disturbances. He makes tentative calculations in several cases to arrive at an observed critical Reynolds number $R \sim 1000$. He further concludes that three-dimensional disturbances that could generate turbulence have a more destabilising effect than two-dimensional disturbances, especially with increasing Reynolds number for the flow.

K. M. Ghosh (Belgharia)

Scriven, L. E.; Sternling, C. V. 1842

On cellular convection driven by surface-tension gradients: Effects of mean surface tension and surface viscosity.

J. Fluid Mech. **19** (1964), 321–340.

In a series of now classic experiments on the cellular circulation patterns occurring in a shallow pool of liquid heated from below, Bénard observed hexagonal cells with a spacing of 3.27 or more times the liquid depth. For many years, convective instability was the accepted explanation of this effect; indeed, Lord Rayleigh's analysis of the problem indicated that if hexagonal cells are formed, their spacing would be about 3.28 times the liquid depth. Over the past 25 years, however, this explanation has grown more and more unsatisfactory: upwelling of hot liquid has been found always to occur below the centers of depression of the liquid surface; Bénard cells have been observed in liquids cooled from below; drying paint films—on the ceiling as well as the floor—often display Bénard cells; cellular convection can occur when the temperature gradient is tenfold less than required by Rayleigh's theory.

In 1952, Block drew the conclusion that Bénard cells are caused by variations in the surface tension coefficient, which are in turn due to non-uniformities of temperature over the face surface. A theoretical treatment by Pearson, based upon this explanation, agrees in many essentials with the various experimental observations.

In the present paper, the authors relax certain restrictions of Pearson's treatment. By accounting for the possibility of shape-deformations of the free surface, they show that there is no critical Marangoni number for the onset of stationary instability, that the limiting case of zero wave-number is always unstable, and that surface viscosity inhibits stationary instability.

W. Langlois (San Jose, Calif.)

Pillow, A. F. 1843

Diffusion of heat and circulation in potential flow convection.

J. Math. Mech. **13** (1964), 521–545.

Author's summary: "A study is made of the diffusion of a scalar quantity, such as heat or circulation, in a standardized convection field which describes a potential flow past any two-dimensional, finite or semi-infinite body. The problem is of some interest in that it provides the start of an iterative description of the vorticity field in the outer region of a real incompressible viscous flow. The potential flow field outside the streamline which divides at the body is first mapped onto a stagnation point flow in a half-plane. The temperature field of a unit point source of heat arbitrarily placed in this potential flow is then found by means of a double Fourier transform. Properties of the incomplete Bessel functions which arise in this solution are investigated and series representations of the source field are obtained in terms of a suitable collection of multipole fields. Separable solutions in cartesian coordinates provide a representation in terms of fields produced by a complete set of source or dipole distributions along a line parallel to the real axis. These separable solutions permit the construction of the vorticity field outside of the dividing streamline, if the vorticity distribution on it is known downstream of its point of attachment."

R. Gerber (Grenoble)

Kumar, Prem 1844

Decay of temperature fluctuations in homogeneous turbulence.

Tensor (*N.S.*) **15** (1964), 54–60.

The author considers the decay of turbulent temperature correlations between quantities taken at various points in the fluid and at various times. By neglecting the triple correlations in two-point, two-time correlation equations, or the quadruple correlations in three-point, three-time equations, as was done by the reviewer for velocity correlations [NASA TR R-96 (1961)], closed-form solutions are obtained. The result obtained for two-time temperature correlation reduces to that of Loeffler and Deissler [Internat. J. Heat and Mass Transfer **1** (1961), 312–324] when the time delay is zero, and to that of Corrsin [J. Aeronaut. Sci. **18** (1951), 417–423; MR **14**, 328] when, in addition, the decay time is large. {The quantity $\overline{T^2}$ in equation (38) should evidently be written as $\overline{TT'}$ to indicate that the temperatures are considered at two times.}

R. G. Deissler (Cleveland, Ohio)

Korjavov, P. P. 1845

Numerical calculation of the turbulent mixing of two homogeneous gas flows. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 495–511.

The turbulent mixing of two uniform gas flows is considered in the boundary-layer approximation. Temperature and velocity differences are not required to be small. A generalized Prandtl hypothesis, in which three mixing lengths appear, is used to characterize heat and momentum transfer by the turbulent fluctuations. A transformation of independent variables is made to obtain a boundary-value problem for mean velocity and temperature which does not contain the mixing lengths explicitly. The turbulent Prandtl number which does appear is assumed to be independent of downstream distance, and is assigned constant values (0.5 or 1.0) for the numerical computations. The boundary-value problem just mentioned is shown to have a similar solution determined by a set of ordinary differential equations and boundary conditions of precisely the same form as those found for the case of laminar mixing. The boundary-value problem for the similar solution is transformed to a set of integral equations, for which

numerical solutions are obtained by successive approximations. The author reports that improved convergence is obtained when each new cycle is launched using appropriately weighted mean values of the last two approximations, rather than the last approximation itself.

D. A. Lee (Dayton, Ohio)

v. Krzywoblocki, M. Z. 1846

On the existence proof of the flow in the boundary layer in a compressible fluid. III.

J. Reine Angew. Math. **208** (1961), 145–163.

The author claims to have completed here the existence proof stated in Part I [same J. **206** (1961), 175–191; MR **26** #981] by showing that "the results of the two-point boundary layer theory can be embedded into the viscous compressible fluid flow theory derived in Part II [ibid. **207** (1961), 113–128; MR **26** #981]". *G. Capriz* (Pisa)

Uehara, Takeyuki; Iri, Yumi; Iri, Masao 1847

The geometry of the Cartan space associated with general compressible fluid flows. (French, German, Italian, and Russian summaries)

Internat. J. Engrg. Sci. **1** (1963), 497–508.

In Section I, the authors consider a geometrization of the motion of a compressible fluid, assuming that the viscosity can be neglected and the flow is irrotational and isentropic. In order to achieve such geometrization they use Cartan spaces.

If Φ is the 4-dimensional velocity potential defined by $\mathbf{q}=\nabla\Phi$, $\phi=\partial\Phi/\partial t$, where $\mathbf{q}$ is the velocity of the fluid and ϕ the negative of the total energy per unit mass, the equation for Φ is given by

$$(1)\quad \frac{\partial^2\Phi}{\partial t^2}+\frac{\partial}{\partial t}(\nabla\Phi)^2+[(\nabla\Phi)\cdot(\nabla\Phi)-a^2I]:(\nabla\nabla\Phi)+\frac{\partial\Omega}{\partial t}+(\nabla\Phi)\cdot(\nabla\Omega) = 0.$$

It is found in the paper that this equation can be derived from a variational principle by setting $\delta\iiiint p(w)\,dxdydzdt=0$, where p depends only on

$$w = -\frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial x}\right)^2+\left(\frac{\partial\Phi}{\partial y}\right)^2+\left(\frac{\partial\Phi}{\partial z}\right)^2\right]-\frac{\partial\Phi}{\partial t}-\Omega(x,y,z,t).$$

In connection with this a 5-dimensional Cartan space is introduced, the contact element of which is given by $(x^1, x^2, x^3, t, \Phi; q_1, q_2, q_3, \phi, 1)$, as the underlying space of the considered motion, where x^1, x^2, x^3 are the ordinary cartesian coordinates and q_1, q_2, q_3 the velocity components. The contravariant and covariant components of the metric tensor of this space are given in a complete table. Some interesting properties of the metric tensor are investigated, corresponding to physical properties of the flow. Shock wave and Mach cone and Earnshaw-type relations are treated.

In Section II, the general case of compressible flow, without the restrictions of Section I, is investigated; a more general equation corresponding to (1) is found. Physical quantities such as entropy, vorticity, etc., are expressed in terms of geometrical entities in the Cartan space. Finally, the case of anisentropy is connected with the non-vanishing of the mean curvature of suitable hypersurfaces of the space. *H. A. Eliopoulos* (Windsor, Ont.)

Holder, D. W. 1848

The transonic flow past two-dimensional aerofoils.

J. Roy. Aero. Soc. **68** (1964), 501–516.

This article contains the text of the Second Reynolds-Prandtl Lecture which was given in Cologne on 23 March 1964.

Zolver, René 1849

Écoulement hypersonique asymptotique d'un gaz parfait non visqueux, non conducteur de la chaleur sur un obstacle en loi de puissance.

C. R. Acad. Sci. Paris **258** (1964), 5152–5155.

The author tries to improve the calculations in the problem of the hypersonic flow past a power-law obstacle by combining the method of asymptotic expansions with that of integral equations. In the lowest order approximation this leads to the same results as obtained by blast wave theory. These expansions break down for particular values of the power index; in this event they must be replaced by modified forms. *P. Mandl* (Ottawa, Ont.)

Guiraud, Jean-Pierre 1850

Bruit balistique et focalisation.

C. R. Acad. Sci. Paris **258** (1964), 5148–5151.

This paper is concerned with the propagation of sonic bangs in a quiet homogeneous atmosphere. Using results of a previous paper based on principles of geometrical acoustics [same C. R. **258** (1964), 4425–4428], the author derives an asymptotic formula for the pressure rise across the cusped wave fronts. *P. Mandl* (Ottawa, Ont.)

Gear, C. W. 1851

Singular solutions at boundary intersections in two dimensional quasi-linear homogeneous partial differential equations.

J. Math. and Phys. **43** (1964), 100–110.

Si l'on considère l'écoulement stationnaire bidimensionnel d'un fluide parfait et compressible autour d'un dièdre, on sait que, lorsque les conditions aux limites sont données sur le dièdre et l'onde de choc, la solution ne s'exprime pas au moyen de fonctions analytiques; à moins que l'angle du dièdre satisfasse à des conditions bien déterminées. L'auteur reprend ce problème, en étudiant plus généralement le cas d'un système elliptique quasi linéaire de n équations homogènes à 2 inconnues avec des conditions aux limites sur deux courbes concourantes.

Après avoir montré que la solution présente une singularité au point d'intersection (pointe du dièdre dans le cas particulier de l'écoulement), l'auteur effectue une transformation de coordonnées introduisant un paramètre supplémentaire t caractérisant la nature de la singularité et qui sera déterminé par les conditions aux limites. La solution est cherchée sous forme d'une série de puissances de l'une des nouvelles variables indépendantes avec des coefficients dépendant de l'autre variable. Dans le cas de l'écoulement autour d'un dièdre, cas étudié ici en détail, le paramètre t est obtenu explicitement, alors qu'il n'en est pas ainsi pour un système de plus de 2 équations. Enfin, l'auteur considère le cas d'un système d'équations non homogènes. *R. Peyret* (Montrouge)

Soubbaramayer 1852
Structure des chocs ionisants lents.
C. R. Acad. Sci. Paris **258** (1964), 436–439.
The author's study of ionizing-shock structure [same C. R. **254** (1962), 2135–2137] is extended to so-called slow shocks. It is assumed that the gas is un-ionized below and fully ionized above a certain temperature. Integral curves have been computed by machine for a numerical case, including a range of values of the upstream conductivity (wherein, of course, the ionizing shock described above corresponds to the value zero). The conclusion is that the ionizing slow shock behaves like a slow magnetogas-dynamic shock; notably, the conditions for upstream and downstream are the same. Also, the beginning of the slow ionizing shock is the same as that of the magneto-gasdynamic sub-shock [see Germain, O.N.E.R.A. Publ. No. 97 (1959); MR **22** #8971]; thus supersonic upstream flow is a requirement. *W. R. Sears* (Ithaca, N.Y.)

Oberai, M. M.; Peyret, R. 1853
Écoulement à symétrie sphérique d'un gaz visqueux dans un champ de gravitation.
J. Mécanique **3** (1964), 101–117.
On peut étudier complètement l'écoulement stationnaire à symétrie sphérique d'un fluide parfait, soumis à un champ de forces radial, lorsque les lignes de courant sont les rayons issus du centre de symétrie. Les auteurs considèrent le même problème mais dans le cas d'un fluide faiblement dissipatif et déterminent complètement le premier terme des développements asymptotiques exprimant la solution. Le cas traité correspond à un gaz parfait à chaleurs spécifiques constantes; le nombre de Prandtl est égal à $\frac{3}{4}$ et les coefficients de viscosité vérifient la relation de Stokes. Le champ de forces est celui dû à l'attraction newtonienne du centre. On sait alors que suivant les conditions aux limites (au centre et à l'infini) l'écoulement de fluide parfait peut être continu ou non. L'intérêt du travail des auteurs est précisément d'étudier complètement le cas de l'écoulement avec onde de choc. Il faut alors, de façon essentielle, tenir compte des modifications à apporter aux conditions de choc de Rankine-Hugoniot tenant compte des effets dissipatifs du fluide [P. Germain et J.-P. Guiraud, O.N.E.R.A. Publ. No. 105 (1962)]. *P. Germain* (Paris)

Peyret, Roger 1854
Sur la structure du choc lent dans un schéma à deux fluides non dissipatif.
C. R. Acad. Sci. Paris **258** (1964), 2973–2976.
Author's summary: "On étudie la structure du choc lent dans le cas d'un schéma à deux fluides *sans dissipation*. Il est possible d'avoir, lorsque l'écoulement est supersonique avant le choc, une solution discontinue (sous-choc) périodique autour de l'état amont, puis autour de l'état aval, la position du sous-choc restant arbitraire."

Kouzov, D. P. 1855
Diffraction of a plane hydro-acoustic wave at a crack in an elastic plate.
Prikl. Mat. Meh. **27** (1963), 1037–1043 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1593–1601.
An infinite horizontal elastic plate is divided into two identical parts by an infinitesimally thin rectilinear crack. The half-space below the plate is filled by an ideal compressible fluid and a plane monochromatic acoustic wave originating in the depths of the fluid propagates normally to the crack. The solution for the diffraction field is found as a special case of a more general problem solved by the author [Prikl. Mat. Meh. **27** (1963), 541–546].
For small wave numbers the diffracted field is shown to split into a cylindrical wave and a surface wave in the fluid accompanied by a corresponding wave in the plate. The energy transport by the elastic wave in the plate is four times the energy transport by the surface wave in the fluid. *H. C. Levey* (Perth)

Graebel, W. P. 1856
The hydrodynamic stability of a Bingham fluid in Couette flow.
Second-order Effects in Elasticity, Plasticity and Fluid Dynamics (Internat. Sympos., Haifa, 1962), pp. 636–649. *Jerusalem Academic Press, Jerusalem; Pergamon, Oxford*, 1964.
The possibility of a specific type of unstable motion occurring in the title problem is investigated. The original formulation of the constitutive equation used here appeared first in the paper by K. Hohenemser and W. Prager [Z. Angew. Math. Mech. **12** (1932), 216–226].
Small rotationally symmetric motions are superposed on the well-known Couette flow solution for this material. The restriction on the superposed motions is not discussed, but conditions under which the chosen motions can occur are determined. Predictions for several specific cases are given in graphical form.
The stabilizing effect of the presence of a yield stress is discussed with the aid of physically motivated arguments. *P. R. Paslay* (Houston, Tex.)

Erdoğan, M. Emin 1857
Transmission de la chaleur à un fluide visqueux se mouvant entre deux sphères concentriques tournantes. (English summary)
Z. Angew. Math. Phys. **15** (1964), 66–67.
Author's summary: "The problem investigated is the temperature distribution in an incompressible viscous fluid moving between two concentric rotating spheres whose temperatures are maintained constant, i.e., the inner sphere temperature is T_a and that of the outer sphere T_b."

Erdoğan, M. Emin 1858
Transmission de la chaleur à un fluide visqueux se mouvant autour d'une sphère tournante. (English summary)
Z. Angew. Math. Phys. **15** (1964), 311–312.
Author's summary: "The problem investigated is the temperature distribution in an incompressible viscous fluid past a rotating sphere. The cases in which the thermal conductivity of the sphere is large and small are considered separately."

Erdoğan, M. E. [Erdoğan, M. Emin] 1859
An exact solution in magnetohydrodynamics.
J. Mécanique **2** (1963), 505–512.

Steady flow of an electrically conducting incompressible viscous fluid between two concentric rotating spheres in the presence of a magnetic field produced by a source placed at the common centre of the spheres is discussed. It is claimed that an exact solution of the equations of magnetohydrodynamics has been found. The reviewer does not agree with this claim. In this problem, v_r, v_θ cannot be taken to be zero everywhere in the fluid. This will render the equations (21) and (22) inconsistent even when the magnetic field is zero. The reviewer also fails to understand the physical significance of the magnetic source *S. D. Nigam* (Madras)

Gundersen, Roy M. 1860

Quasi-one-dimensional magnetohydrodynamic flow with heat addition. [II] Oblique field. (German summary)

Z. Angew. Math. Phys. **15** (1964), 210–213.

The first part appeared in same Z. **14** (1963), 294–297 [MR 28 #2771]. This is a very short discussion of formal properties of a certain hyperbolic system of differential equations in two independent variables governing a special type of magnetohydrodynamic flow. The main result is a "general solution" of an approximating linear system with constant coefficients. The linear system governs flows nearly uniform. This result is incorrect because of important errors made in deriving it.

C. S. Gardner (New York)

Tkalič, V. S. 1861

A stationary problem in magnetohydrodynamics when two coordinates are interdependent; the transformation of Chaplygin. (Russian)

Inž. Ž. **2** (1962), *no.* 3, 43–53.

In curvilinear orthogonal coordinates (q_1, q_2, q_3), three-dimensional magnetohydrodynamic equations for a perfectly conducting gas are integrated to be reduced to a system of two equations for the density and a quantity uniquely related to stream-functions in the (q_1, q_2)-plane under the conditions: (1) every quantity, as well as Lamé's constants, is independent of q_3; (2) the q_3-component of the electric field is zero.

Further simplification is possible in the case of Cartesian coordinates by use of effective total pressure, yielding linear equations in a kind of hodograph plane.

Briefly sketched examples are (1) a head-on collision of two-dimensional jets in the presence of a magnetic field perpendicular to the plane on the assumption that the adiabatic index is two, (2) generalized Alfvén waves, and (3) special magnetohydrostatic problems which are solved exactly and may approximate real situations, e.g., in cylindrical geometry. *H. Hasimoto* (Baltimore, Md.)

Grandori Guagenti, Elisa 1862

Superficie di Lamb e di Bernoulli nella magnetofluidodinamica.

Boll. Un. Mat. Ital. (3) **19** (1964), 40–52.

Conditions for the existence of magnetic "Lamb surfaces" in a conducting fluid, i.e., surfaces which everywhere contain both the magnetic vector **H** and the velocity vector **v**, are discussed.

H. K. Moffatt (Cambridge, England)

Miles, John W. 1863

Stability of a continuum plasma.

Phys. Fluids **7** (1964), 223–228.

Author's summary: "Weibel has considered the stability of a cylindrical column of plasma, regarded as a viscous compressible fluid, that is confined by a combination of dc and rf magnetic fields. The integral equation in which his analysis culminates is considered here. The normal modes are first determined for a dc magnetic field and found to be definitely stable for all finite wavelengths and neutrally stable for infinite wavelengths. It then is shown that the addition of an rf magnetic field renders a finite number of these modes unstable in certain spectral bands. This last result is based on the hypothesis that the strength of the rf magnetic field is small compared with that of the dc field (or small in some other appropriate sense), in consequence of which the modes of the plasma in the presence of the rf magnetic field may be regarded as approximately independent."

J. K. Thurber (New York)

Pratelli, Aldo M. 1864

Su alcune discontinuità trasversali in magnetofluidodinamica. (English summary)

Boll. Un. Mat. Ital. (3) **18** (1963), 382–397.

Author's summary: "The author proves that, when the electrical conductivity is infinite across the Alfvén wave front, which propagates as well in incompressible fluids and in ideal gases, as in gases which have the 1st (shear) viscosity coefficient equal to zero and the 2nd (compressional) viscosity coefficient positive, the gradient of the 'total pressure' (sum of the mean pressure and of the 'magnetic' pressure) remains continuous. The author proves afterwards that, still under the assumption that the 1st viscosity coefficient is null and not the 2nd, there may exist wave fronts which propagate with any velocity relative to the fluid; across them the second derivatives of the velocity are discontinuous and, under convenient assumptions, also the second derivatives of the magnetic induction." *H. K. Moffatt* (Cambridge, England)

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 1368, 1369, 1432, 1463, 1719, 1755, 2073.

★Progress in Optics. Vol. III. 1865

Edited by E. Wolf.

North-Holland Publishing Co., Amsterdam; Interscience Publishers [John Wiley & Sons, Inc.], New York, 1964. xii + 340 *pp.* (4 *plates*) $12.50.

This volume contains three papers, "The elements of radiative transfer", by F. Kottler; "Apodisation", by P. Jacquinot and B. Roizen-Dossier; "Matrix treatment of partial coherence", by H. Gamo; these papers will be reviewed individually.

The second volume (1963) in this series has already appeared.

Broer, L. J. F. 1866

Non-linear optics. (Dutch)

Nederl. Tijdschr. Natuurk. **30** (1964), 159–168.

An expository lecture.

Drougard, Renee 1867
Optical transfer properties of fiber bundles.
J. Opt. Soc. Amer. 54 (1964), 907–914.
Author's summary: "The point spread function of a static fiber bundle is not spatially invariant, and the system cannot therefore be strictly characterized by an optical transfer function. It is shown both theoretically and experimentally that, in the case of circular fibers of diameter d, the modulation of the image of an object varies with the position of the fiber bundle and lies between two extreme values which become closer together as the spatial frequency decreases. Thus, for sufficiently low frequencies (of the order $1/8d$), an optical transfer function can be used. The experimental results are in good agreement with the theory. A randomly vibrating bundle is found to have a spatially invariant point spread function and can therefore be characterized by an optical transfer function."

Galitsky, V. M. [**Galickiĭ, V. M.**]; 1868
Gurevich, I. I. [**Gurevič, I. I.**]
Coherence effects in ultra-relativistic electron bremsstrahlung. (Italian summary)
Nuovo Cimento (10) **32** (1964), 396–407.
The authors derive formulas for ultra-relativistic Bremsstrahlung using classical and dimensional arguments. They take into account coherence polarization, multiple scattering, and absorption effects. They compare the relative importance of these effects in different energy regimes. *F. R. Halpern* (La Jolla, Calif.)

Mehta, C. L.; Wolf, E. 1869
Coherence properties of blackbody radiation. I. Correlation tensors of the classical field.
Phys. Rev. (2) **134** (1964), A1143–A1149.
The electromagnetic field of the black-body radiation is spatially and temporally correlated. The quantity

$$\mathscr{E}_{ij}^{(r)}(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2) \equiv \langle E_i^{(r)}(\mathbf{r}_1, t_1) E_j^{(r)}(\mathbf{r}_2, t_2)\rangle$$

is evaluated and discussed. The evaluation is based on a previous semiclassical derivation by Kano and Wolf [Proc. Phys. Soc. **80** (1962), 1273–1276].
J. M. Jauch (Geneva)

Mehta, C. L.; Wolf, E. 1870
Coherence properties of blackbody radiation. II. Correlation tensors of the quantized field.
Phys. Rev. (2) **134** (1964), A1149–A1153.
An improved derivation of the electromagnetic correlation tensor is given for black-body radiation based on the quantum theory of radiation. The result for the second-order correlation is found to be identical with that obtained by the semiclassical method of Part I [#1869 above].
J. M. Jauch (Geneva)

Fer, Francis 1871
Dynamique et rayonnement électromagnétique d'un fluide chargé et cohérent.
C. R. Acad. Sci. Paris **258** (1964), 2983–2986.
The author discusses in a classical framework the problem of the motion of, and radiation from, an accelerated charged particle and attempts to avoid the well-known difficulties in such an approach by invoking, in a modified form, Madelung's hypothesis (i.e., the classical electron is pictured not as a rigid particle but as a fluid).
T. J. M. Boyd (Culham)

Ward, G. N. 1872
On the integration of Maxwell's equations, and charge and dipole fields.
Proc. Roy. Soc. Ser. A **279** (1964), 562–571.
This paper presents the solution of Maxwell's equations in a homogeneous medium for the case of the initial-value problem for a space-like initial surface. The solution, which is essentially a specialized version of the standard one for a symmetric hyperbolic system based on Riesz's method, is applied to the special cases of a moving point charge and a moving dipole. *I. Kay* (Ann Arbor, Mich.)

Fischer, Gaston 1873
Microwave surface impedance of a periodic medium.
J. Mathematical Phys. 5 (1964), 1158–1167.
The surface impedance of a medium is calculated, the complex scalar conductivity of which is a periodic function in a direction parallel to the surface. Radiation is assumed to impinge vertically onto the surface. It is shown that there are two independent field configurations according to whether the induced microwave current flows in a direction parallel to the conductivity pattern or perpendicular to it. These two configurations lead to different surface impedances, the one arising in the perpendicular configuration being generally larger.

The calculation is carried through when the conductivity is of the form $a+b \cos (ky)$ with a local relation assumed between the current and the field. Equations are also derived for the more general problem when the periodic conductivity is an arbitrary Fourier series and a nonlocal relation is assumed between the current and the field. These problems are of interest in measurements of the microwave surface impedance of hard superconductors
J. H. Richmond (Columbus, Ohio)

Bacry, Henri 1874
★Les moments multipolaires en relativité restreinte. Application à la diffusion des rayons X par une particule douée d'un moment magnétique quelconque.
Thèses présentées à la Faculté des Sciences de l'Université d'Aix-Marseille pour obtenir le grade de Docteur ès Sciences Physiques.
Masson & Cie, Éditeurs, Paris, 1963. vii + 43 *pp.*
In this paper the author carries out a rigorous and systematic study of the motion of a system of charged particles (or of a particle possessing a charge structure) in an electromagnetic field. It is the first attempt to derive a general formulation covering several particular cases already discussed by different authors. By maintaining a general, relativistically covariant procedure as long as possible, the author shows that the derivation of many results previously obtained on this subject implicitly involves the use of certain approximations, and he can determine to what extent such approximations are justified.

This work contains two sections, together with some

appendices devoted to technical details. Part 1 is a review of some basic notions of relativistic electrodynamics. Part 2 contains the discussion of the chosen subject. Such a discussion is developed along the following lines: (i) the general relativistic definition of the (static and dynamic) multipole moments of a system of charges, and of their properties; (ii) the general relativistic formulation of Maxwell's equations in the presence of matter; (iii) the equations of motion of a particle with a charge structure in an electromagnetic field.

There follows an application of the results obtained to consider the scattering of electromagnetic radiation by a polarized particle. The most significant conclusions from a physical point of view are summarized at the end of the work. The most interesting one is the emphasis on the different roles played by the "normal" and the "anomalous" magnetic moments of a particle in the equations of motion.

The interest of this work consists in the fact that it unites many apparently unconnected physical problems in a unique general frame. This approach can be very useful for a thorough understanding of these phenomena and also for applications in the future to more complex problems. However, the general character of the formulation entails the use of a complicated and massive notation, which may discourage a reader not closely acquainted with problems of this kind. *E. Ferrari* (Rome)

Leite Lopes, J. 1875

★Fundamentos da eletrodinâmica clássica [Fundamentals of classical electrodynamics].

Série Científica, No. 3.

Universidade do Brasil, Rio de Janeiro, 1960. 108 *pp.*

Maslennikov, M. V.; Suškevič, T. A. 1876

Asymptotic properties of the solution of the characteristic equation of the transport theory of radiation in strongly absorbent media. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 23–34.

In the characteristic integral equation

$$(1) \qquad M(\lambda)(1+\lambda\mu)\varphi_\lambda(\mu) = \int_{-1}^{1} g(\mu,\mu')\varphi_\lambda(\mu')\,d\mu'$$

arising in the asymptotic theory of the transfer of radiation through thick media, the kernel is assumed to be of the form

$$(2) \qquad g(\mu,\mu') = \sum_{s=0}^{N} \frac{2s+1}{2} g_s P_s(\mu)P_s(\mu'),$$

where $P_s(\mu)$ is the Legendre polynomial of order s. For λ close to unity (strong absorption), a convergent series solution is obtained for (1), and its leading terms are given explicitly. The given terms determine the asymptotic behavior of the solution as λ approaches one.

L. B. Rall (Madison, Wis.)

Thomas, D. P. 1877

Electromagnetic diffraction by two coaxial discs.

Proc. Cambridge Philos. Soc. **60** (1964), 621–634.

The author studies the solution of a problem of low-frequency electromagnetic diffraction by two perfectly conducting coaxial circular disks upon which is incident an electromagnetic plane wave travelling along the axis of symmetry of the obstacle. A method of Lebedev and Skal'skaja [Ž. Tehn. Fiz. **29** (1959), 700–710; MR **24** #B1507] enables the author to reduce this problem to two scalar problems. Following W. E. Williams' work [Proc. Roy. Soc. Ser. A **267** (1962), 77–87; MR **25** #4772], the boundary-value problems for these independent scalar problems may be reduced to a pair of simultaneous Fredholm integral equations of the second kind. The calculation of the scattering coefficient is considered in detail. *A. E. Heins* (Ann Arbor, Mich.)

Islam, Mohammed Azizul 1878

Scattering of electromagnetic waves by a composite cylinder.

J. Mathematical Phys. **5** (1964), 550–557.

Using the approach outlined by the reviewer in his monograph [*Electromagnetic radiation from cylindrical structures*, Pergamon, New York, 1959; MR **21** #4742], the author carries out a formal solution for electromagnetic scattering by an obliquely incident plane wave on a cylindrical obstacle of infinite length. In this particular instance, the obstacle consists of a circular homogeneous dielectric core with a concentric homogeneous dielectric sleeve. No numerical results are given, although certain special cases are considered.

J. R. Wait (Boulder, Colo.)

Pavageau, James 1879

Diffusion par des surfaces imparfaitement planes.

C. R. Acad. Sci. Paris **258** (1964), 5828–5831.

A study is made of the diffraction of a plane electromagnetic wave impinging on a dielectric surface of sinusoidal profile and with amplitude small relative to the wavelength. This leads to the corresponding analysis for a general, nearly plane surface, which may be regarded as a superposition of sinusoidal surfaces.

E. W. Marchand (Rochester, N.Y.)

Bolotovskiĭ, B. M.; Sedrakjan, D. M. 1880

Particle emission from the open end of a waveguide. (Russian. Armenian summary)

Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk **17** (1964), *no.* 2, 119–126.

Consider a semi-infinite electromagnetic waveguide, of cylindrical shape with infinitely thin and perfectly conducting walls. A charged particle of specified (high) velocity enters the open end of the guide, moving along a line parallel to the axis of the latter. The authors calculate the radiation emitted by the particle on entering the waveguide, owing to eddy currents generated over the conducting surface, under the approximation that the change produced in the motion of the particle can be neglected.

The procedure used is entirely that of classical boundary-value theory for the electromagnetic field. The vector potential of the incoming particle is first expressed in terms of Bessel functions, adapted to the cylindrical form of the waveguide. The boundary conditions on the conducting surface are then satisfied by formal series expansions involving Bessel functions. A formula is obtained giving the radiation emitted by the particle at small angles with its direction of motion. {It is not clear

to the reviewer whether this gives the radiation emitted into the waveguide or exterior to it.} A general formula is given for the spectral and angular distribution of the radiation. *E. L. Hill* (Minneapolis, Minn.)

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See also 1446, 1517.

Alekseev, B. V. 1881
Laminar boundary layer of a sublimating surface. (Russian)
Ž. Vyčisl. Mat. i Mat. Fiz. 4 (1964), 512–524.

Il s'agit de la couche limite au voisinage du point d'arrêt d'un véhicule hypersonique dont la paroi est en graphite qui se sublime. Le gaz est de l'air dissocié qui réagit avec la vapeur de graphite. On considère successivement les cas où il y a et où il n'y a pas de réaction hétérogène à la paroi. La couche limite elle même a été considérée soit en équilibre, soit gelée. Des résultats sont présentés donnant les concentrations de O_2, N_2, O, N, CO, CO_2, C sous forme de courbes, et aussi la répartition de température. *J. P. Guiraud* (Paris)

QUANTUM MECHANICS

See also 1518, 1774, 1776, 1868, 1874, 1979.

Landau, L. D. [Ландау, Л. Д.]; 1882
Lifšic, E. M. [Лифшиц, Е. М.]
★Quantum mechanics. Non-relativistic theory [Квантовая механика. Нерелятивистская теория].
Second revised and augmented edition. Theoretical Physics, Vol. III.
Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1963. 702 *pp.* 1.47 *r.*

The first edition [OGIZ, Moscow, 1948; MR **10**, 582; English transl., Pergamon, London, 1958; MR **19**, 1230] has been reviewed. The second edition is unchanged in general scope and methodology. Improvements appear throughout the text. The most substantial additions are chapters on the vector coupling model (Racah-Wigner coefficients) and on nuclear theory. The chapters on scattering theory have been considerably amplified.

{As in the first edition, the strength of this text for physics students lies primarily in the second half of the book dealing with applications to physical problems. Class use of the English edition by the reviewer has shown the discussion of the physical background of wave mechanics and the treatment of the general theory to be distinctly inadequate. The second edition seems to be a little smoother in places but not materially better. The first chapter subjects the reader to one of the least appealing and most indefensible forms of the Heisenberg-Bohr theory of physical measurement. This is capped by a whole-hearted adoption of the Dirac symbolism for function spaces and the spectral theory of linear operators (both bounded and unbounded). In the present state of the art the student of theoretical physics still must struggle with these handicaps, but let the mathematical reader be forewarned.} *E. L. Hill* (Minneapolis, Minn.)

Chako, Nicholas 1883
Some remarks on the foundations of quantum mechanics.
Acta Phys. Polon. **24** (1963), 629–640.

There has been a continuing thread of interest in exploring the formal correspondence between classical Hamilton-Jacobi theory and quantum mechanics. The present paper summarizes the current status of the author's work in this direction—following generally the ideas of G. D. Birkhoff [Proc. Nat. Acad. Sci. U.S.A. **19** (1933), 339–344; ibid. **19** (1933), 475]. It is regrettable that publications in this interesting area have tended to be fragmented and poorly coordinated: Certain results and speculations are rediscovered over and over again, and the pace of genuine advance is exceedingly slow. This is reflected in the severely parochial selection of literature citations appended to the present paper. A sampler of some additional pertinent references might include I. Fényes [Z. Physik **132** (1952), 81–106; MR **15**, 78], E. Guth [Phys. Rev. (2) **126** (1962), 1213–1215; MR **25** #2794], L. Motz [ibid. (2) **126** (1962), 378–382; MR **25** #2793], L. Motz and A. Selzer [ibid. (2) **133** (1964), B1622–B1624; MR **29** #918]. *T. Erber* (Chicago, Ill.)

Rosen, Nathan 1884
The relation between classical and quantum mechanics.
Amer. J. Phys. **32** (1964), 597–600.

Author's summary: "It is pointed out that, if classical mechanics is to be regarded as a limiting case of quantum mechanics, then it must admit superpositions of states, which are generally regarded as unacceptable. To avoid this, it is proposed that classical mechanics be described by a nonlinear equation which is equivalent to the Hamilton-Jacobi equation and is not always the limit of the Schrödinger equation. It is conjectured that the transition from the wave-mechanical equation to the classical equation is characterized by a mass $m_0 = \hbar c/\gamma^{1/2} \approx 10^{-5} g$. This nonlinear equation may help one to understand the process of measurement of a quantum-mechanical system by means of a classical measuring instrument."

Marcus, R. A. 1885
Separation of sets of variables in quantum mechanics.
J. Chem. Phys. **41** (1964), 603–609.

Author's summary: "Separation of the Schrödinger equation for molecular dynamics into sets of variables can sometimes be performed when separation into individual variables is neither possible nor for certain purposes necessary. Sufficient conditions for such a separation are derived. They are the same as those found by Stäckel for the corresponding Hamilton-Jacobi problem, with an additional one which is the analogue of the Robertson condition for one-dimensional sets. Expressions are also derived for operators whose eigenvalues are the separation constants. They provide a variational property for these constants. For use in aperiodic problems an expression is obtained for the probability current in curvilinear coordinates in an invariant form. Application of these results to reaction rate theory is made elsewhere."

Marcus, R. A. 1886
Local approximation of potential-energy surfaces by surfaces permitting separation of variables.
J. Chem. Phys. **41** (1964), 610–616.

Author's summary: "In the immediate vicinity of a potential-energy minimum or of a saddle point, it is shown that major topographical features of a 'non-separable' potential-energy surface can be imitated by those of a surface permitting separation of variables. For each extremal path of descent or ascent to the cited critical point of the surface, there is an exact match of the tangent, the first curvature vector in configuration space, and the force constant along that path provided that the known curvature vector satisfies an equation containing the metric tensor of the selected coordinate system and known force constants. Because of the wide choice of coordinate systems available for selection, it is anticipated that this relation may be fulfilled for each extremal path, partly by choice of the coordinate system and partly by subsequent choice of the curvilinear coordinates of the critical point. There are several possible applications of this local approximation, including those to problems involving anharmonic coupling of normal modes and those involving n-dimensional tunneling and other calculations in reaction-rate theory. Use will be made of the formalism to extend the activated complex theory in chemical kinetics. As a preliminary test of the local-approximation concept, literature data on n- and one-dimensional tunneling rates are compared. They are found to be fairly similar when proper cognizance is taken of zero-point energies."

Birman, M. Š.; Èntina, S. B. 1887

A stationary approach in the abstract theory of scattering. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 506–508.

Es wird skizziert mathematisch strenge Behandlung der sogenannten quantentheoretischen Streungstheorie: es werden gegeben hinreichende Bedingungen für Existenz der Wellenoperatoren und des S-Operators und ihre Darstellung als Grenzwerte von Resolventen (Formel von Lippmann-Schwinger). Wir können hier nur den Satz 3 dieser interessanten Note anführen. Es seien A und A_0 selbst-adjungierte Operatoren mit gemeinsamen Definitionsgebiet: $D(A)=D(A_0)$, $V=A-A_0$. Falls $|V|^{1/2}(A_0-iI)^{-n}$ und $|V|^{1/2}(A-iI)^{-n}$, $n=1, 2, \cdots$, vom Hilbert-Schmidtschen Typus sind, dann existieren die Wellenoperatoren $W_\pm(A, A_0)$, $W_\pm(A_0, A)$.

K. Maurin (Warsaw)

Nicholson, A. F. 1888

Two-body bound state and scattering with non-central forces.

Austral. J. Phys. **17** (1964), 1–7.

The equations representing two-particle scattering in the presence of noncentral forces are coupled, and, for a given J, have the form

$$(1)\qquad [T_1+V_{11}]u = V_{12}w,$$

$$[T_2+V_{22}]w = V_{12}u.$$

These equations reduce to a single equation if we impose the condition

$$(2)\qquad w = \beta u.$$

Condition (2) imposes restrictions on the potential V; a class of potentials satisfying (2) is considered.

L. M. Delves (Kensington)

Reed, Robert D.; Nodvik, John S. 1889

Small-angle multiple scattering of fast charged particles using Molière screening.

Phys. Rev. (2) **135** (1964), B199–B202.

The formula for the cross-section of elastic scattering of an electron by a screened Coulomb potential obtained by Dalitz in the second Born approximation has been generalized to a potential represented by the sum of three such terms. Following the work of Bethe, Nigam, Sundaresan and Wu, the distribution function for multiple scattering is obtained. The experimental results of Hanson, Lanzl, Lyman and Scott for the $1/e$-widths are compared and found to agree with the results obtained by Nigam et al. (as recalculated by Scott).

B. P. Nigam (Buffalo, N.Y.)

Desai, Bipin R. 1890

Diffraction width in terms of f^0 and the residue of the Pomeranchuk pole.

Phys. Rev. (2) **135** (1964), B180–B182.

If the amplitude is dominated at high energies by a single Pomeranchuk pole, the phase of the residue of this pole should be a multiple of 2π. This is consistent with experiments if the particle f^0 lies on this trajectory. On the other hand, from Yukawa potentials one obtains the result that this phase should be small. Thus, either the potential theory does not apply very well to the relativistic Pomeranchuk trajectory or the dominance of a single pole is incorrect. *A. O. Barut* (Boulder, Colo.)

Choudhury, M. H. 1891

On the analytic properties and asymptotic behaviour for large l of partial-wave amplitudes in many-channel potential scattering. (Italian summary)

Nuovo Cimento (10) **32** (1964), 943–954.

For the many-channel scattering of spinless particles by a superposition of Yukawa potentials it is shown that the amplitude has correct asymptotic properties for large angular momentum l along the imaginary axis to permit a Watson-Sommerfeld transformation. This point was not complete in previous publications. The method used is based on Fredholm theory. *A. O. Barut* (Boulder, Colo.)

Greider, K. R. 1892

Scattering approximation for long-range forces.

Phys. Rev. Lett. **9** (1962), 392–396.

There have been several approximation methods in scattering theory which treat many-body scattering processes in terms of the physical two-body scattering amplitudes. They have, in general, been useful for short-range potentials and for large energies and momentum transfers. This letter describes a method which appears to be valid for all energies and momentum transfers. Its restriction is that it requires at least one of the interactions to be long-range. An expression is derived which formally obtains the many-body scattering matrix in terms of the two-body amplitudes. For elastic and inelastic scattering, a suitable expression can be obtained with a slight modification of existing formulations. Rearrangement collisions may be readily treated by the new method. The method is applied to two problems: the

neutron transfer reaction in low energy ion-ion scattering and the high energy (p.d.) pickup process. Good agreement is obtained with experiments. *H. Lustig* (New York)

Maslov, V. 1893
The scattering problem in the quasi-classical approximation.
Dokl. Akad. Nauk SSSR **151** (1963), 306–309 (*Russian*); *translated as Soviet Physics Dokl.* 8 (1964), 666–668.

Rectangular coordinates and angular coordinates in n-dimensional space are denoted by x_i and ϕ_i, respectively. The asymptotic properties of $\psi(x)$ and $f(\phi)$ as $h\to 0$ are sought, where

$$-h^2\Delta\psi+v(x)\psi = E\psi,$$

$$\psi = \exp(iE^{1/2}x_1/h)+|x|^{(1-n)/2}f(\phi)\exp(iE^{1/2}|x|/h) + o(|x|^{(1-n)/2}),$$

the results for $f(\phi)$ correcting previously obtained formulae.

Various preliminary definitions are made concerning trajectories, Jacobians, foci, phase of paths, action, homotopic paths, determinate paths and resonance points, illustrated by two examples when $n=1$. The asymptotic form for $f(\phi)$ is then stated and applied to the case of the one-dimensional barrier and to the case $n=3$ with $v(x)=v(r)$, $r=|x|$. *J. Heading* (Southampton)

Eden, R. J. 1894
Legendre transforms of scattering amplitudes and a crossing-symmetric representation. (Italian summary)
Nuovo Cimento (10) **31** (1964), 998–1012.

The scattering amplitude is expanded in a double Legendre series of the form

$$A(x,y) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}(2l+1)(2m+1)C(l,m)P_l(x)P_m(y).$$

The variables l, m do not correspond to angular momenta in the two channels, and as such, the unitarity will be very complicated for the amplitudes $C(l,m)$. On the other hand, crossing symmetry can be expressed in a convenient way. A similar approach was given by Khuri [Phys. Rev. (2) **130** (1963), 429–435; MR **29** #3173].
A. O. Barut (Boulder, Colo.)

Scadron, M.; Weinberg, S. 1895
Potential theory calculations by the quasiparticle method.
Phys. Rev. (2) **133** (1964), B1589–B1596.

Authors' summary: "The quasiparticle method [S. Weinberg, same Rev. (2) **130** (1963), 776–783; MR **27** #6499; ibid. (2) **131** (1963), 440–460] is used to find binding energies, scattering lengths, and cross-sections for one particle in a strong Yukawa, Hulthen or exponential potential. The results are excellent in the lowest approximation." *W. M. Frank* (Silver Spring, Md.)

Balachandran, A. P. 1896
Boundary conditions for a partial-wave amplitude.
J. Mathematical Phys. **5** (1964), 614–621.

Author's summary: "A partial-wave amplitude is frequently subject to constraints which specify its values at a given set of points. This happens, for instance, when one insists on its correct threshold behavior. We investigate such constraints in this paper and derive a class of inequalities which are necessary conditions for the existence of such an amplitude." *E. Ferrari* (Rome)

Gnedin, Yu. N. [Gnedin, Ju. N.]; Dolginov, A. Z. 1897
Theory of multiple scattering.
Ž. Èksper. Teoret. Fiz. **45** (1963), 1136–1149 (*Russian. English summary*); *translated as Soviet Physics JETP* **18** (1964), 784–792.

Multiple elastic scattering of a particle (or a set of mutually non-interacting particles) with a set of identical, immovable scattering centers is described in T-matrix formalism, assuming that inelastic scattering may be treated as absorption. A statistical theory of the scattering from a homogeneous macroscopic body is then obtained by independently averaging the position of each scattering center over the volume of the body, by considering only cases where the single-particle scattering amplitude a is much less than the distance $N_0^{-1/3}$ between force centers and where the incident particle wavelength λ is much less than the dimension of the body, and by dropping terms in the perturbation expansion of the scattering matrix which are of order $N_0a^2\lambda$ or smaller compared with unity.

Coherent and incoherent scattering are treated separately. Both treatments depend heavily on assuming that the momentum dependence of the single-particle scattering amplitude is much weaker than that of the free-particle propagator or that of the average Fourier transform of the density of a scattering center. For coherent scattering an integral equation is obtained for the average transition amplitude. For incoherent scattering an extensive analysis based on diagram techniques permits the scattering intensity to be expressed in terms of a quantity satisfying a specific integral equation. This equation is an analogue of the usual transport equation, but the correspondence is not traced in detail. For the case of scattering from a spherical target approximate solutions of the relevant integral equations are given and a number of quantities of physical interest are given. An approximation procedure is suggested for solving the equations in more general cases. *R. C. Mjolsness* (Los Alamos, N.M.)

Brysk, Henry 1898
Some non-Born phase shifts.
Phys. Rev. (2) **133** (1964), B1625–B1632.

In the paper under review the author derives an expression for the tangent of the phase shift by what he calls the Drukarev transformation. The first-order approximation of this expression is identical with the formula the author derived in an earlier paper (same Rev. (2) **126** (1962), 1589–1595 [MR **25** #4791]; in the review of this paper the reviewer pointed out that the phase shift expression is derivable from a Volterra-type integral equation derived by Newton which is the same as the Drukarev transformation). In another paper [J. Mathematical Phys. 4 (1963), 1536–1538; MR **28** #1456] the author showed that the new expression for the phase shift is identical with the one derived by the Fredholm determinantal method. To first order this has been shown

also by Falk [Phys. Rev. (2) **129** (1963), 2340–2341; MR 28 #3679]. Another approximation is due to Swan [Nuclear Phys. **18** (1960), 245–270; MR **22** #6489]. In the paper under review it is shown that Swan's and the author's approximations are quite similar. In the second part of the paper the author shifts to numerical calculations. He computes the scattering length for the neutron-proton triplet state by successive approximations based on the Born, the Brysk and the Swan series. In addition, he performs two variational calculations based on the Brysk and Swan wave functions. The Born series gives a completely wrong result. The other four methods are very good and converge surprisingly fast to the exact result. He furthermore has calculated the s-wave phase shifts for square wells and barriers as a function of the strength of the potential and for different ranges. Again the Born approximation gives very poor results. The first-order Brysk approximation is far superior even for very small phase shifts. The same is essentially true for the Swan approximation and the variational methods. However, in the resonance region and for larger ranges none of the mentioned methods is very satisfactory. Only the variational method based on the Brysk wave function is superior in the resonance region. Some extensions to non-central potentials are also given. *S. Rosendorff* (Haifa)

Belinfante, Johan G. 1899

Existence of scattering solutions for the Schrödinger equation.

J. Mathematical Phys. **5** (1964), 1070–1074.

A proof is given, using Banach space methods [W. Hunziker, Helv. Phys. Acta **34** (1961), 593–620; MR **25** #1856], that the Schrödinger (Lippmann-Schwinger) equation

$$\langle \mathbf{x}|\mathbf{k}\rangle = \exp(i\mathbf{k}\cdot\mathbf{x}) - \frac{1}{4\pi}\int d^3\mathbf{y}\,\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\,V(y)\langle \mathbf{y}|\mathbf{k}\rangle$$

with $\int_0^\infty y\,dy|V(y)| < \infty$, satisfies the Fredholm alternative, that is, either it has for each real $\mathbf{k}$ a bounded and continuous solution, or else the corresponding homogeneous equation has a nontrivial bounded and continuous solution. In fact, in the present case, the homogeneous equation cannot have a solution for real $\mathbf{k}$, except for zero energy. *A. O. Barut* (Boulder, Colo.)

Banerjee, Haridas 1900

Remarks on the analytic continuation of the partial-wave amplitude.

Phys. Rev. (2) **131** (1963), 2810–2817.

The author considers the elastic scattering of neutral pseudoscalar mesons under the conditions (a) the Mandelstam representation is valid, and (b) the absorptive part of the amplitude vanishes suitably as the channel variables become infinite. It is shown that the function with the same left-hand discontinuity in the energy plane as the partial wave amplitude, but not right-hand discontinuity, is meromorphic in the complex angular momentum plane, and vanishes for large angular momentum. The same is only true for the complete partial wave amplitude under further rather strong conditions, which are difficult to reconcile with crossing symmetry.
John G. Taylor (Cambridge, England)

Stapp, Henry P. 1901

Derivation of the CPT theorem and the connection between spin and statistics from postulates of the S-matrix theory.

Phys. Rev. (2) **125** (1962), 2139–2162,

Author's summary: "The CPT theorem and the normal connection between spin and statistics are shown to be consequences of postulates of the S-matrix approach to elementary particle physics. The postulates are much weaker than those of field theory. Neither local fields nor any reference to space-time points are used. Quantum commutation relations and properties of the vacuum play no role. Completeness of the asymptotic states and positive definiteness of the metric are not required, though certain weaker asymptotic conditions prevail. The proofs depend on unitarity, macroscopic relativistic invariance, and a very weak analyticity requirement on the mass-shell scattering functions. The proofs are in the framework of the new S-matrix approach to elementary particle physics, which is established on a formal basis."

Leutwyler, H. 1902

Gravitational field: Equivalence of Feynman quantization and canonical quantization.

Phys. Rev. (2) **134** (1964), B1155–B1182.

The Feynman function integral quantization method is applied to the gravitational field. Owing to the peculiarities of this and other gauge fields, a number of ambiguities exist in the path quantization method. The author's prescription involves the use of a lattice to perform the sum and leads to a choice of the measure differing from those found by previous investigators (the Feynman method has been applied to this problem by C. W. Misner [Rev. Modern Phys. **29** (1957), 497–509; MR **19**, 1238], B. Laurent [Ark. Fys. **16** (1960), 279–383; MR **22** #2425], J. R. Klauder [Nuovo Cimento (10) **19** (1961), 1059–1061; MR **23** #B417], and B. S. DeWitt [J. Mathematical Phys. **3** (1962), 1073–1093; MR **26** #5880].

For the measure he obtains $Dg = |\det g|^{-3/2} N \prod dg_{\mu\nu}$, where $\det g$ is the determinant of the full $g_{\mu\nu}$ and $N = (-g^{00})^{-1/2}$ with integration between two $X^0 = \text{const}$ hypersurfaces. The process of determining the measure involves appropriate averaging over contributions which differ only by gauge but correspond to the same physical configuration. Electromagnetism is reviewed as a simple example of how to average over gauges (subsidiary conditions) but the differences between it and relativity are also noted, especially the fact that the lattice itself (or the coordinates) is affected by change of gauge. The derivation proceeds by first introducing Gaussian coordinates as a specific gauge choice. Aside from questions of singularity of such frames, this (or any) choice removes the degeneracy of summing over gauges. The gauge group is then brought back by summing over all coordinate transformations between arbitrary and the Gaussian frames. This brings back the initial-value constraint equation of the theory, in a way analogous to Dirac's Hamiltonian treatment. Once the measure is derived, a comparison is made with Dirac's Hamiltonian approach to quantization and equivalence is found, with a particular factor ordering of the operators in the $G_\mu{}^0$ constraint equations. Since consistency of quantization of the gravitational field by any approach is still an open problem, the equivalence obtained must be formal in nature.

No attempt is made in this paper to prove consistency of path integral quantization within its own framework, e.g., by setting up the Lorentz group generator relations.

S. Deser (Waltham, Mass.)

Yamagata, Hideo 1903

On unified representation of state vectors in quantum field theory. II.

Proc. Japan Acad. **39** (1963), 569–574.

A suitable linear subspace of the sequence space of Part I [the author and Ishihara, same Proc. **38** (1962), 339–344; MR **26** #1060] is shown to be the direct sum of the space of square-integrable functions and a space which contains a non-separable Hilbert space with base composed of "square roots" of S-functions introduced in earlier papers [see, e.g., ibid. **38** (1962), 715–720; MR **27** #6517]. This decomposition is claimed to correspond to the continuous and discrete representation of the commutation relations for free quantitised fields, respectively, though this is not made very clear. The paper is written in a very condensed style, with certain apparently crucial symbols not explicitly defined. *John G. Taylor* (Cambridge, England)

Sudarshan, E. C. G. 1904

A mechanism of induction of symmetries among the strong interactions.

Phys. Lett. **9** (1964), 286–288.

Author's introduction: "Recently Cutkosky [Phys. Rev. (2) **131** (1963), 1888–1890] has presented a model of N vector mesons interacting amongst themselves in such a manner that each vector meson may be considered as a bound state of pairs of vector mesons. He then proceeds to show that N must be equal to the dimension of some Lie algebra and that the (renormalized) coupling constants must be proportional to the structure constants of the group. Amongst the postulates involved is the conservation of electric charge (or, more generally, any additive quantum number which is not the same for all members of the multiplet). Cutkosky raised the possibility that this postulate may be redundant, but left the question open. It is the purpose of this note to present a discussion of the model, very similar to the work of Cutkosky, in which the charge conservation does not need to be explicitly imposed, but may be derived as a consequence."

S. L. Glashow (Berkeley, Calif.)

Nguyen Van-Hieu [Nguyen Van Hieu]; Faustov, R. N. 1905

Quasi-optical potential in quantum field theory.

Nuclear Phys. **53** (1964), 337–344.

Authors' summary: "The construction of potentials dependent on energy in the model of a quantum field theory with scalar particles is considered according to perturbation theory. The method of the two-time Green's functions and that of the scattering amplitude on the mass-shell are used. The calculated corrections to the Coulomb energy levels coincide in the two methods. However, the second method is more effective due to the locality and the analyticity of the potential."

J. W. Moffat (Baltimore, Md.)

Spitzer, Richard 1906

The parity of the neutral K meson.

Nuclear Phys. **47** (1963), 367–384.

An experiment is suggested to test the conjecture that the intrinsic parity of the neutral K-meson is imaginary, so that the fields K and $\bar{K}$ have opposite parities. This will manifest itself in a dependence of the effective mass-difference $K_1 - K_2$ on the component of an external magnetic field parallel to the K-momentum.

The paper also contains a long and substantially correct discussion of the physical meaning of super-selection rules.

{Reviewer's opinion: While the theoretical analysis here concerning the consequences of assuming an imaginary parity for the K-meson is correct, it is unlikely, to say the least, that it will provide a complete explanation of the conservation of strangeness, as hoped by the author. In particular, the selection rules the theory gives are always multiplicative, and not additive like strangeness. For example, the existence of the Ω^- (discovered since the paper was submitted) rules out the hypothesis that strangeness is conserved only mod 4, since if this were the case, $\Omega^- \to n + K^0 + \pi^-$ would occur strongly, and so the Ω^- would not exist as a particle.}

In the theoretical analysis, the author has done a service by pointing out that, contrary to popular belief, the absolute K parity might well be meaningful.

The incorrect contrary view, namely, that the conservation of strangeness prohibits the absolute determination of the K parity, has appeared in a number of papers, and the author chooses to call this the "current view-point". The author is not in disagreement with the viewpoint of Wigner, Wightman and Wick [Wick, Wightman and Wigner, Phys. Rev. (2) **88** (1952), 101–105; MR **14**, 827]. His analysis goes further than that given there, but is included in the theory worked out by Wightman since 1952 [Nuovo Cimento (10) **14** (1959), suppl., 81–94; MR **22** #4444; Streater and Wightman, *PCT, spin and statistics and all that*, Benjamin, New York, 1964; MR **28** #4807].

R. F. Streater (London)

Høgaasen, H. [Høgåsen, H.] 1907

A covariant polarization operator for $s = 1$ particles.

Z. Physik **179** (1964), 221–227.

A covariant polarization operator for a Dirac particle has been given by Michel and Wightman. The present author gives a corresponding operator expression for a particle of unit spin by using the Duffin-Kemmer-Petiau equation.

K. Nishijima (Champaign, Ill.)

Bertocchi, L.; Fubini, S.; Furlan, G. 1908

Bound states and renormalization properties. (Italian summary)

Nuovo Cimento (10) **32** (1964), 745–756.

The connection between bound states and renormalizability is discussed in potential scattering and quantum field theory. In potential theory the bound state condition is that the vertex renormalization constant Z vanishes. The terms in the perturbation expansion for Z are cut-off dependent, but the zeros of the bound state condition are cut-off independent. In the field theory discussion, the approach is taken that some particles in nature are elementary and the rest are considered as bound states. The composite particles (bound states) are treated as external

unquantized fields and the bound state condition is applied only to them. The quantitative discussion is restricted mainly to the case when the theory is renormalizable and the renormalization constants are logarithmically divergent. *M. Schwartz* (Bronx, N.Y.)

Rose, M. E.; Morita, M. 1909

Formal consequences of conservation of angular momentum projections.

Progr. Theoret. Phys. **31** (1964), 103–106.

It is a well-known fact that a particle of intrinsic spin s can have total angular momentum projections with respect to its direction of propagation lying only between $\pm s$. This result is proved from the existence of formal sum rules involving $3j$ and $6j$ symbols. The general sum rule is, in particular, applied to the emission of particles of spin less than 1. *H. Lustig* (New York)

Dombey, N. 1910

On massive and massless photons. (Italian summary)

Nuovo Cimento (10) **32** (1964), 1696–1706.

Author's summary: "The vanishing of the longitudinal mode of a massive photon as the photon mass goes to zero in gauge-invariant electrodynamics is investigated with the help of examples. A distinction between two forms of gauge-invariant amplitudes is made in order to clarify this limit; conditions are then obtained so that the limiting process is smooth. From these conditions results on the equality of particle charge, which are usually derived by means of Ward's identity, can be obtained."

J. Franklin (Livermore, Calif.)

Mugibayashi, Nobumichi; Kato, Yusuke 1911

Regular perturbation and asymptotic limits of operators in fixed-source theory.

Progr. Theoret. Phys. **31** (1964), 300–310.

Authors' summary: "Within the theory of regular perturbation the restriction on a source function is slightly relaxed so that the theory may include cases in which the interaction part of the Hamiltonian not only has no dense domain in common with the free part, but also it is never defined in a Hilbert space. A smallness condition of the interaction part expressed in terms of symmetric quadratic forms ensures the total Hamiltonian to be uniquely defined as a self-adjoint operator. All the results concerning strong asymptotic limits of operators which have been established previously for more restrictive source functions are retained in these cases."

W. M. Frank (Silver Spring, Md.)

Critchfield, Charles L. 1912

The WKB method in three dimensions.

Amer. J. Phys. **32** (1964), 542–545.

Author's summary: "The WKB method of solution to Schrödinger's equation is usually applied individually to partial waves of definite angular momentum, i.e., in one dimension. However, the partial differential equation may be solved directly, by this method, and the first two terms are derived for the solution to scattering by a central potential. The resulting cross-section is the same as in classical theory."

Tassie, L. J. 1913

Canonical quantization of the Schrödinger equation.

Amer. J. Phys. **32** (1964), 609–611.

Author's summary: "In the usual treatment of the field theory of the nonrelativistic Schrödinger equation, there are more dynamical variables than is warranted by the number of degrees of freedom. This causes difficulties in the canonical quantization. Working in momentum space it is possible to eliminate the extra dynamical variables, leaving a number of independent dynamical variables appropriate to the number of degrees of freedom. The canonical commutation relations can then be applied in a straightforward manner."

Tani, S.; Uhlenbrock, D. A. 1914

Hard core produced by orthogonality constraints.

J. Mathematical Phys. **3** (1962), 1161–1170.

The authors write: "It is shown that the effect of a hard core as an external force can be reproduced by requirements of orthogonality alone. A fictitious orthonormal set of functions is introduced, which form a complete set in the domain occupied by the hard core. The actual wave function is constructed as the eigenfunction of a modified kinetic energy operator, and is required to be orthogonal to all members of the fictitious set. The limit of infinitely many constraints is carefully discussed. It turns out to be impossible to regard the hard core effect as equivalent to some kind of 'potential' acting on free waves (for which the Hamiltonian would be well defined). However, the eigenfunction is well defined in the limit of infinitely many constraints, and is independent of the way in which the fictitious set is chosen." *S. Rosendorff* (Haifa)

Boulware, D. G.; Deser, S. 1915

External sources in gauge theories. (Italian summary)

Nuovo Cimento (10) **30** (1963), 1009–1019.

Authors' summary: "The notion of prescribed external sources is discussed for gauge fields. Such sources necessarily obey conservation laws which generally involve the fields themselves. The implications of these conservation requirements are analysed for classical and quantized vector multiplets (such as the Yang-Mills field) and for (classical) general relativity. A number of the vector source components may be prescribed independently of the field. The remaining components are those coupled to the gauge parts of the field, and may be eliminated by a suitable choice of gauge. On the other hand, although the gravitational conservation laws also leave a number of source components unconstrained, the rest depend nonlocally in time on the gravitational field variables. It is shown that these non-local components cannot be eliminated from the theory by a choice of coordinate frame (gauge), and therefore concluded that external gravitational sources are not useful in analysing the field's properties." [See also J. Schwinger, Theoretical Physics (Lectures Seminar Theoretical Phys., I.A.E.A., Trieste, 1962), Internat. Atomic Energy Agency, Vienna, 1963; MR **27** #3339]. *P. W. Higgs* (Edinburgh)

Pluhař, Z.; Tolar, J. 1916

Transformation matrix for the isotropic harmonic oscillator eigenvectors in $\{n_1n_2n_3\}$ and $\{nlm\}$ representations.

Czechoslovak J. Phys. **14** (1964), 287–293.

Authors' summary: "In this paper the properties of the elements $(n_1n_2n_3 | nlm\rangle$ of the transformation matrix connecting the eigenvectors $|n_1n_2n_3)$ and $|nlm\rangle$ of the isotropic harmonic oscillator are investigated. An explicit expression and recursion relations for these elements are derived." *J. N. Islam* (College Park, Md.)

Bargmann, V. 1917

Note on Wigner's theorem on symmetry operations.

J. Mathematical Phys. **5** (1964), 862–868.

This paper contains a proof of Wigner's theorem on symmetry operations: Every symmetry transformation of a coherent quantum mechanical system may be represented by a unitary or anti-unitary vector transformation of Hilbert space. The proof employs a completion of a construction indicated by Wigner. The author criticizes other demonstrations of this theorem and reaffirms the simplicity and transparency of the original construction by Wigner. *J. M. Jauch* (Geneva)

Mitter, H. 1918

Quantization of nonlinear field theories and scale transformation. (Italian summary)

Nuovo Cimento (10) **32** (1964), 1789–1808.

It has been occasionally suggested [see, in particular, H. A. Kastrup, Ann. Physik (7) **9** (1961/62), 388–428; MR **26** #2231] that in the asymptotic region of energies, i.e., for energies which are very large compared to the rest mass of a proton, physics could be invariant under a scale transformation (dilatation) of space-time. The author interprets and analyzes this possibility in the following manner. A quantum field theory is determined by a set of "Green's functions", for instance, by the set of vacuum expectation values of the products of field operators

(1) $$W^{(m)}(x_1\cdots x_n) = \langle\psi_0|\Phi(x_1)\cdots\Phi(x_m)|\psi_0\rangle.$$

Here ψ_0 is the vacuum state, $\Phi(x)$ the field operator. Consider a field equation of the form

(2) $$\Box\Phi(x) = g\Phi^n(x),$$

which is invariant under the transformation

(3) $$\Phi(x)\to\Phi'(x) = \lambda^\alpha\Phi(\lambda x), \qquad \alpha = \frac{2}{n-1},$$

where λ is an arbitrary scale parameter. It is then conjectured that the Green's functions (1) can be split into an invariant part W_s and a remainder W_R, where W_s dominates at high energies (or short distances) and is invariant if Φ' is inserted instead of Φ in the right-hand side of (1) and the vacuum state ψ_0 is left unchanged, i.e.,

(4) $$W_s^{(m)}(x_1\cdots x_m) = \lambda^{m\alpha}W_s(\lambda x_1, \cdots, \lambda x_m).$$

The author points out that if (4) is applied to the 2-point function $m=2$, then the resulting singularity at short distances and on the light cone in x-space is stronger than in perturbation theory for the two renormalizable theories $n=2$, 3, but it is weaker for the non-renormalizable theories $n\geqq 4$. A weaker singularity than that given by perturbation theory is in conflict with the customary postulates of quantum field theory, in particular, with the assumption of a positive definite metric in Hilbert space. Thus the author concludes that these non-renormalizable theories bear out Heisenberg's conjecture that an indefinite metric in Hilbert space has to be used for strongly nonlinear theories.

Incidentally, the author discusses also the case in which the continuous transformation group (3) is replaced by a discrete one in which λ can have only a denumerable set of values. *R. Haag* (Urbana, Ill.)

Kwiecinski, J.; Suranyi, P. 1919

On the analytic properties of the pion pion scattering amplitude in the complex angular momentum plane.

Phys. Lett. **9** (1964), 283–286.

The approximate amplitude for the scattering of pseudoscalar particles, obtained as a solution of a Bethe-Salpeter equation whose kernel corresponds to the Feynman graph with the exchange of one bubble, has an energy-independent branch point as its rightmost singularity in the angular momentum plane [R. F. Sawyer, Phys. Rev. (2) **131** (1963), 1384–1391; MR **28** #884; P. Suranyi, Phys. Lett. **6** (1963), 59–62; MR **27** #6523; G. Domokos and P. Suranyi, Nuclear Phys. **54** (1964), 529–548; MR **29** #3172]. In this letter it is shown that if one uses instead, as the kernel of the Bethe-Salpeter equation, the sum of all iterated bubble graphs, the rightmost singularity is a Regge pole. *F. Calogero* (Rome)

Ciulli, S.; Ghika, Gr.; Stihi, M. 1920

Regge poles in the presence of a hard core. (Italian summary)

Nuovo Cimento (10) **32** (1964), 955–970.

Authors' summary: "A hard-core potential is introduced in addition to the usual Yukawian potential in order to obtain the analytic continuation of the S-matrix in the left half-plane. The Watson transform cannot be simply performed because of the existence of an essential singularity at infinity. However, using the relation between the two kinds of Legendre functions this difficulty is avoided and besides the 'pole-sum' of the Regge terms one obtains an integral which can be evaluated by the 'steepest descent method'. The nature of the singularity if $R=0$ ($R=$ the radius of the hard core) is studied. One obtains the position of the first poles and the corresponding residues in the case of weak potentials." *J. W. Moffat* (Baltimore, Md.)

Kretzschmar, M. 1921

A new Regge representation for potential scattering. (Italian summary)

Nuovo Cimento (10) **32** (1964), 1405–1421.

The main result of this paper is the following: For a superposition of Yukawa potentials one can separate for each Regge pole term a part which represents the effect of the trajectory in the right-hand angular momentum plane ($\mathrm{Re}\, l > -\frac{1}{2}$) only, the other part being included in the background term. The separated part satisfies a Mandelstam representation with a spectral function which is non-vanishing in a "strip" determined by the energy interval over which the trajectory is in the right-hand angular

momentum plane. Whether it is advantageous to lump a known part of the amplitude into the unknown background term remains to be seen.

A. O. Barut (Boulder, Colo.)

Cunningham, J. 1922

Complex singularities in production amplitudes.

Austral. J. Phys. **17** (1964), 161–164.

Author's summary: "Complex singularities in the $\pi+N\rightarrow\pi+\pi+N$ amplitude corresponding to vertex contractions of the single-loop Feynman diagram exist on the physical sheet of the amplitude irrespective of whether the theory is formulated in terms of time-ordered or retarded products."

Quinn, John J. 1923

Conductivity tensor of an anisotropic quantum plasma in a uniform magnetic field.

Phys. Rev. (2) **135** (1964), A181–A185.

Author's summary: "The quantum magnetoconductivity tensor for a system in which the constant energy surfaces are ellipsoids tilted away from the direction of the dc magnetic field is derived. The results are applied to a system with cubic symmetry in which the Fermi surface consists of a set of ellipsoids. The dispersion relation for low-frequency magnetoplasma oscillations (helicons) is investigated, and some qualitative discussion of the attenuation of helicon waves in anisotropic materials is presented."

D. B. Lichtenberg (Bloomington, Ind.)

Rozics, J. D.; Johnson, W. R. 1924

Exact calculation of bremsstrahlung from polarized electrons.

Phys. Rev. (2) **135** (1964), B56–B60.

The authors give an unwieldy but exact series solution for the angular distribution of first-order bremsstrahlung from polarised electrons in the Coulomb field of a nucleus. In the case computed, the numerical results show that the usual Born approximation is rather poor.

L. M. Delves (Kensington)

Sakmar, Ismail A. 1925

Possible connections between the fermion trajectories.

Phys. Rev. (2) **135** (1964), B249–B252.

Author's summary: "Regge poles appearing in the odd and even continuations of the meson-baryon scattering amplitudes as a function of the energy are investigated from the point of view of relating different fermion trajectories. Many of the existing strange-particle resonances with odd and even parity can be related whereas there are no pion-nucleon resonances with the proper quantum numbers which can be related within this scheme. The expected form of the trajectories is given in the complex J-plane and also as Re $\alpha(W)$ and Re $\alpha(W^2)$, the last one corresponding to the Chew-Frautschi diagram. With the discovery of new resonances, further possibilities arise for the behavior of the fermion Regge trajectories, one of which is suggested for the 1765-MeV resonance."

J. W. Moffat (Baltimore, Md.)

Wolf, J. 1926

On the calculation of the ρ-trajectory.

Ann. Physik (7) **12** (1963/64), 396–397.

Author's summary: "The calculation of the ρ-trajectory is considered in more detail. For given experimental values of the trajectory at $t=0$, the value of the trajectory at threshold is varied. The right half-width of the ρ-resonance is obtained if this value at threshold is greater than one-half. Comment is made on the behaviour at the first threshold and on a possible oscillatory behaviour at higher thresholds."

Dowker, J. S. 1927

Cosmology and weak interactions.

Nuovo Cimento (10) **32** (1964), 1816–1818.

On the basis of the Salam-Ward conjecture (that the vacuum expectation value of the K^0 field does not vanish) the energy momentum tensor of the coupled K^0 and baryon field is derived, using the Einstein cosmological equation and the general Robertson cosmological metric. If $\langle K^0\rangle$ is computed from hyperon decay rates, an absurd value is found for the trace of the energy momentum tensor.

P. Roman (Boston, Mass.)

Baur, H. 1928

Zur Algebra ladungsartiger Eigenschaften.

Z. Physik **178** (1964), 390–408.

From the author's summary: "Using the formalism of the predicate calculus, possible types of interaction between charged systems are studied. A special algebra is developed applicable to systems additive in the change parameter. The possibility is raised that the resulting combinatorial scheme could be useful in classifying elementary particles."

A. J. Coleman (Kingston, Ont.)

Gross, Leonard 1929

Norm invariance of mass-zero equations under the conformal group.

J. Mathematical Phys. **5** (1964), 687–695.

From the author's introduction: "It has been known for a long time that Maxwell's equations are invariant not only under the ten-dimensional inhomogeneous Lorentz group, but also under the conformal group of Minkowski space.... One might expect that a unitary representation of the conformal group corresponding to the photon arises from Maxwell's equations in the same way that a unitary representation of the Lorentz group does, i.e., that one could extend the Maxwell representation of the Lorentz group to a unitary representation of the conformal group. ... In this paper we show that this representation of the inhomogeneous Lorentz group does in fact extend to a unitary representation of the conformal group, and that the same is true of other mass-zero, discrete spin representations of the Lorentz group."

J. Feldman (Berkeley, Calif.)

Coester, F.; Hamermesh, M.; McGlinn, W. D. 1930

Internal symmetry and Lorentz invariance.

Phys. Rev. (2) **135** (1964), B451–B452.

This is a continuation of the paper by McGlinn [Phys. Rev. Lett. **12** (1964), 467–469; MR **29** #976]. Given a semi-simple group of rank l. If the mutually commuting

generators H_i, $i=1, \cdots, l$, of this group commute with all the generators of the proper inhomogeneous Lorentz group (Poincaré group), then the group G which consists only of the generators of the semisimple group and the Poincaré group is a direct product of the two. In the opinion of the reviewer, the assumptions of the theorem are too restrictive for physical purposes. The internal symmetry group, such as SU_3, must emerge as a limiting case from a larger group which contains additional generators, because the internal symmetry group of the type SU_3 is necessarily approximative. *A. O. Barut* (Boulder, Colo.)

Sesma, J.; Biel, J.; Garrido, L. M. 1931
Relation between generalized Foldy-Wouthuysen and Lorentz transformations.
Amer. J. Phys. **32** (1964), 559–562.
The generalized Foldy-Wouthuysen transformations leave invariant the Hamiltonian form of the field equations. A relation is shown between such transformations and the Lorentz transformation from the laboratory system to another reference system. *P. Franklin* (Cambridge, Mass.)

Yoon Suk Koh; Goldhammer, Paul 1932
Variational treatment of hard-core interactions.
Nuclear Phys. **53** (1964), 287–296.
Authors' summary: "The problem of constructing variational trial wave functions for hard-core interactions is investigated using simple radial functions whose form is suggested by perturbation theory. Application is made to the harmonic oscillator with a hard core replacing the centre of attraction and to the deuteron using interactions of the Gammel-Thaler type. Results are compared with functions of the type used by Blatt, Derrick and Lyness in calculating the triton and indicate that some improvement in these calculations might be obtained by a simple modification in the radial functions."
J. L. Gammel (College Station, Tex.)

Gourdin, M.; Piketty, C. A. 1933
Remarks about polarization in elastic electron-deuteron scattering. (French and Italian summaries)
Nuovo Cimento (10) **32** (1964), 1137–1143.
Authors' summary: "La section efficace élastique électron-deutéron est calculée dans le cas d'une cible polarisée ou du deutéron de recul polarisé. Il est alors possible de séparer les facteurs de forme de charge et quadrupolaire du deutéron et d'en déduire un test de l'approximation non relativiste." *S. Azuma* (Fukusima)

DeBenedetti, Sergio 1934
★Nuclear interactions.
John Wiley & Sons, Inc., New York-London-Sydney, 1964. xxiii + 636 *pp.*
This monumental volume is truly unusual—both in the organization of its material and its way of exposition. As the author jocularly points out in the preface, his purpose for writing the book was that he might have a book from which he personally could benefit after reading. More seriously, the book seems to be aimed to benefit experimental physicists working in low- and medium-high energy physics and serves equally well as a vivid introduction on these topics for the young theoretician. It is certainly a book which every graduate student in these fields should carefully read at some stage of his career.

The first chapter covers the general properties of nuclear forces. A short survey of space-time symmetries is followed by the phenomenology of the neutron-proton system and the general characterization of nucleon-nucleon forces. Chapter II deals with nuclear models, giving a fair treatment—not only to the conventional models, but also encompassing the more modern aspects of many-body theory. The following chapter is entitled "Analysis of Scattering Experiments", and deals with S-wave neutron scattering, behavior of neutrons in matter as well as low- and high-energy scattering of nucleons. Chapter IV deals with the topic of interaction between nucleons and electromagnetic radiation, and the treatment here is based on the elementary quantization of the radiation field. The next chapter is on nuclear reactions and besides giving the formalism of complex collision theory, discusses nuclear reactions in relation to nuclear models. Chapter VI is an exposition of relativistic quantum electrodynamics. Chapter VII is on meson physics and contains a wealth of facts and modern theoretical developments. The last chapter is on weak interactions, and presents a systematic and almost up-to-date review.

It should be mentioned that really high-energy phenomena, such as strange particles, higher symmetries, and extreme high-energy asymptotics, are not covered. Finally, while a conscientious theoretician may find the treatment occasionally superficial, it is beyond doubt that this book will be valuable reading material for everyone.
P. Roman (Boston, Mass.)

Kwo, W. T. 1935
Alternant molecular orbital method.
Phys. Rev. Lett. **12** (1964), 563.
The author claims that Löwdin's formulation of the alternant molecular orbital method is incomplete and that there is an additional overlap integral. According to the reviewer's opinion, however, the argument is incorrect since the author counts erroneously the same function twice.

More precisely, Löwdin's method proposes to build up two sets of basic functions starting from a set of $4n$ atomic orbitals. By taking special combinations of those functions, he constructs $4n$ linearly independent alternant molecular orbitals. The author automatically took all combinations and thus obtained $8n$ functions, of which the extra $4n$ functions are obtainable from the original $4n$ functions and should be discarded. *T. Arai* (Argonne, Ill.)

Hagen, C. R. 1936
Lower bounds on the Lehmann weights in spin-zero meson theory.
J. Mathematical Phys. **5** (1964), 900–902.
The author studies the problem of the magnitude of the renormalization constants in spin-zero pseudoscalar meson theory. If the renormalization constants are assumed to be finite, the Lehmann spectral functions cannot decrease arbitrarily rapidly as functions of the energy. This problem has previously been investigated in detail by the reviewer in a series of papers [Nuclear Phys. **13** (1959), 150–155; MR **22** #2411; ibid. **14** (1959/60), 682–684; ibid. **16** (1960),

304–312; MR **23** #B853]. The paper under review presents an obvious extension of the reviewer's results to the fermion Green's function, and it contains the remark, based on the work of G. Baym [Phys. Rev. (2) **117** (1960), 886–888; MR **22** #3513], that the existence of a ground state requires that the pion-pion coupling constant λ is less than zero. It has already been shown in detail by the reviewer that $\lambda < 0$ leads to a finite mass renormalization constant δm^2 provided the asymptotic behavior of the spectral function is suitably restricted. In the present paper it is pointed out that an infinite mass renormalization in the exact Lagrangian leads to somewhat less stringent conditions on the spectral function.
J. W. Moffat (Baltimore, Md.)

Nutt, Gerald L. 1937
Fredholm determinants applied to electron-hydrogen scattering.
Phys. Rev. (2) **135** (1964), A345–A352.
Author's summary: "The properties of the Fredholm determinant for the electron-hydrogen atom system are discussed. These properties are used to derive formulas for the elastic-scattering phase shifts. S- and P-wave phase shifts are calculated for low energies and the results are compared with other recent calculations. Exchange effects are fully accounted for in the formalism, and a formula is given for the scattering amplitude when two exit channels are present." *J. W. Moffat* (Baltimore, Md.)

Nagarajan, M. A. 1938
Separability of center-of-mass motion in the nuclear shell model.
Phys. Rev. (2) **135** (1964), B34–B36.
Author's summary: "It is shown that one can find an orthogonal transformation that will enable one to split the motion of a many-body system to a center-of-mass motion and an internal motion. A particular orthogonal transformation has been chosen which retains the independent-particle aspect of a harmonic-oscillator shell-model Hamiltonian. It is suggested that one could easily study and eliminate the states with spurious motions of the center of mass by a direct transformation of the shell-model wave function into the new coordinate system."
A. C. Hurley (Melbourne)

Belinfante, J. G. 1939
Harmonic-oscillator model of the spin of the ω-meson. (Italian summary)
Nuovo Cimento (10) **32** (1964), 1384–1391.
A model of the ω-meson is made in which the constituent π-mesons interact via a harmonic oscillator potential. The spin and isotopic spin and symmetry are treated exactly. The lowest allowed state is an antisymmetric spin-one state. An attempt at relativistic generalization is considered. *F. R. Halpern* (La Jolla, Calif.)

Sperber, D. 1940
Recursion formula for the interaction energy of a configuration of n equivalent particles. (Italian summary)
Nuovo Cimento (10) **32** (1964), 1772–1778.
Author's summary: "Explicit simple formulas for the coefficients of fractional parentage are developed. The interaction energy in an n-particle configuration is expressed as a linear combination of the interaction energy of the allowed terms in an $(n-1)$-particle configuration."
K. K. Gupta (Bombay)

Nikiforov, A. F. [Никифоров, А. Ф.]; Uvarov, V. B. [Уваров, В. Б.]; Levitan, Ju. L. [Левитан, Ю. Л.] 1941
★Tables of Racah coefficients [Таблицы коэффициентов Рака].
Vyčisl. Centr Akad. Nauk SSSR, Moscow, 1962. 320 *pp.* 2.41 *r.*
Racah coefficients occur in the quantum theory of angular momentum and may be briefly characterized as the matrix elements of invariant operators formed in the coupling of three tensor operators. These coefficients are of great importance in atomic and nuclear spectroscopy, in angular correlation theory, and in the quantum theory of angular momentum itself. The importance of the Racah coefficient may be judged from the fact that some 15 more or less extensive tabulations have appeared since 1952. The two most extensive recent tabulations (besides the compilation under review) are: *The 3−j and 6−j Symbols*, Rotenberg et al. [Tech. Press of M.I.T., Cambridge, Mass., 1959] and *Tables of the Racah coefficients*, edited by Ishidzu et al. [Pan-Pacific Press, Tokyo, 1960].

The present tabulation gives the Racah coefficient $W(abcd; ef)$ as 8-place decimal fractions and has as its chief merit the extensive range of variables. The tables are divided into three sections: (I) a, b, c, d half-integral $(\frac{1}{2}(1)17/2)$; $e, f = 0(1)17$ (153 tables). (II) a, b, c, d integral $1(1)9$; $e, f = 1(1)18$ (162 tables). (III) a, c, e half-integral: $a, c = \frac{1}{2}(1)17/2$; e half-integral $\frac{1}{2}(1)35/2$; b, d, f integral $1(1)9$ (153 tables).

The present tabulation suffers in comparison to earlier work (in particular, those mentioned above) in that it lacks both a discussion of the properties of the Racah coefficients, as well as algebraic tables of the Racah coefficients (these are often more useful than numerical values in theoretical applications). It should be mentioned, too, that the Racah coefficients are square roots of rational numbers; tabulation as decimal fractions involves some loss of information (only the Ishidzu et al. tabulation—of the recent work—is expressed in exact (non-decimal) form). *L. C. Biedenharn* (Durham, N.C.)

Hillion, Pierre; Vigier, Jean-Pierre 1942
Un test possible de la symétrie d'hypercharge.
C. R. Acad. Sci. Paris **258** (1964), 5591–5593.
This is an attempt to incorporate the recently discovered Ω-hyperon and a conjectural Z^+ into the symmetry-scheme of relativistic rotator theories.
Y. Ne'eman (Pasadena, Calif.)

Kalos, M. H. 1943
Monte Carlo integration of the Schrödinger equation.
Trans. New York Acad. Sci. (2) **26** (1963/64), 497–504.
The author concludes that the Monte Carlo technique is very difficult to apply to problems with realistic nucleon-nucleon potentials. The hard core is the difficulty. Some

encouraging work aimed at circumventing the difficulty is reported, but no results are given. Some speculations on application of the method to scattering problems are made.
J. L. Gammel (College Station, Tex.)

Koehler, D. R.; Mann, R. A. 1944
Phenomenological potentials and the D(n, p)2n reaction.
Phys. Rev. (2) **135** (1964), B91–B98.
Authors' summary: The D(n, p)2n reaction is treated in a direct interaction framework wherein the phenomenological nucleon-nucleon potentials of Gammel and Thaler are used to describe the perturbation interactions. Tensor forces as well as central forces are included and final-state interactions among the particles is considered. The final state is pictured as a continuum deuteron plus free neutron or as a continuum dineutron plus free proton. To the extent that this picture is accurate it is found that the main contributions to the cross section are supplied by the doublet to doublet transitions. The calculations furthermore suggest a 'best choice' for the triplet—even parity—central potential. A comparison with the experimental results of Ilakovac and co-workers at 14.4 MeV is presented." *J. L. Gammel* (College Station, Tex.)

Qvist, B.; Törn, A. 1945
Calculation of some energy integrals with the Monte Carlo method.
Acta Acad. Abo. Math. Phys. **23**, *no.* 8, 12 *pp.* (1963).
Monte Carlo techniques for evaluating diagrams arising in the many-body problem are investigated. Such techniques have been extensively employed by Baker, Gammel, and Hill [Phys. Rev. (2) **132** (1963), 1373–1387] for such work. The results reported may be of some interest, but are too fragmentary to compare to the work cited. Vaguely expressed, the conclusion drawn is that Monte Carlo techniques work.
J. L. Gammel (College Station, Tex.)

Dahlblom, T. 1946
The pure long-range energy of a four-body cluster.
Acta Acad. Abo. Math. Phys. **23**, *no.* 12, 10 *pp.* (1963).
The work in this paper is somewhat similar to, but far less complete than, the work of Baker, Gammel, and Hill [Phys. Rev. (2) **132** (1963), 1373–1387]. The author claims to compute the four-body cluster corrections to the Brueckner theory. Indeed, Fig. 1 of the paper fails to include a certain type of diagram (cf. diagram II.A.5 of Baker, Gammel, and Hill [loc. cit.]). We shall not attempt to give a complete list of omitted diagrams. The three diagrams in Fig. 1 are what Baker, Gammel, and Hill called I.A.1, I.A.3, and I.A.2, respectively. However, it is not very easy to compare the results of the two papers because the potential shapes are different. The results shown in Table I for $E^{(4)}/E^{(3)}$ and $E^{(3)}/E^{(2)}$ agree with what one finds in Table IV of Baker, Gammel, and Hill for $k_F c \approx 1.0$, but the cluster terms become more important for $k_F c \approx 2.0$ than Table I of the present paper suggests. For $k_F c \approx 0.25$, the three diagrams are completely negligible according to Table IV of Baker, Gammel, and Hill.
J. L. Gammel (College Station, Tex.)

Dahlblom, T.; Fogel, K.-G. 1947
Calculation of the three-body-cluster energy for a long-range tensor force.
Acta Acad. Abo. Math. Phys. **23**, *no.* 5, 8 *pp.* (1963).
The one-pion exchange potential is used with a cut-off for investigating the direct second-order diagram and the direct three-body cluster diagram occurring in the Goldstone-Hugenholtz perturbation expansion of the ground state energy of nuclear matter.
J. W. Moffat (Baltimore, Md.)

Kihlberg, A. 1948
Separation of the rotational degrees of freedom from the N-particle Schrödinger equation. (Italian summary)
Nuovo Cimento (10) **32** (1964), 1295–1300.
By introducing the interparticle distances, together with Eulerian angles and the barycentric co-ordinates, into the N-particle Hamiltonian, a separation of the rotational motion to zeroth order in the vibrational amplitudes is obtained in the particular case when the particles possess relative equilibrium positions. The author derives an expression for the first-order coupling term and the Hamiltonian for the quantal rigid body.
G. F. Dell'Antonio (Naples)

Vail, J. 1949
The self-consistent Mathieu problem.
Canad. J. Phys. **42** (1964), 329–348.
In this paper the author discusses the Hartree solution for a one-dimensional many-fermion system with weak attractive interactions. The case of a periodic self-consistent potential is examined in detail. It is shown that the periodic state has lower energy than the plane-wave solutions when the Fermi energy lies near the edge of the first Brillouin zone. *K. K. Gupta* (Bombay)

Byers, N.; Yang, C. N. 1950
Physical regions in invariant variables for n particles and the phase-space volume element.
Rev. Modern Phys. **36** (1964), 595–609.
In considering the processes of decay ($A \to 1+2+\cdots+n$) or production ($A+\bar{1} \to 2+\cdots+n$) of particles, the authors state as the purpose of their paper the investigation of the following problems: (a) To find the kinematically allowed values of the scalar products of the 4-momenta of the particles i and j; (b) To find for a given assignment of values for these products which is kinematically allowed, how many inequivalent kinematic configurations are realizable (two configurations are inequivalent if they cannot be related by an orthochronous proper Lorentz transformation); (c) To obtain the relativistic phase-space element for the reactions under discussion in terms of the scalar products $p_i \cdot p_j$. The first two topics are discussed in Sections I and II and the results are summarized at the end of the second section. The volume elements for $n=3$ and 4 are given in Equations (34) and (45), while for the case $n \geq 5$ a symmetrical expression (in the indices i, j) is obtained in Equations (79) and (82). The paper contains also an interesting analysis of the geometry of the physical regions for the processes under study.
P. Singer (New York)

STATISTICAL PHYSICS, STRUCTURE OF MATTER

See also 1386.

de Boer, J.; Uhlenbeck, G. E. (Editors) 1951
★Studies in statistical mechanics. Vol. II.
North-Holland Publishing Co., Amsterdam; Interscience Publishers [*John Wiley & Sons, Inc.*], *New York*, 1964. xiii + 272 *pp*. $11.50.

Volume I (1962) was reviewed earlier [MR **24** #B2418]. The present volume contains three papers, the first by Kerson Huang, the second by J. M. H. Levelt and E. G. D. Cohen, and the third by C. S. Wang Chang, G. E. Uhlenbeck and J. de Boer; each of these three papers will be reviewed separately.

Ree, Francis H.; Hoover, William G. 1952
On the signs of the hard sphere virial coefficients.
J. Chem. Phys. **40** (1964), 2048–2049.

In this letter the authors tabulate the results of their evaluation of the fourth virial coefficient of a gas of d-dimensional hard spheres. The coefficient is known analytically for $d=3$; for $d=2$ and $4 \leqq d \leqq 9$ the authors calculated it by a Monte Carlo method. For $d=8$ and 9, they found it to be negative.

The authors conclude that for $d=3$ some of the higher virial coefficients for hard spheres may well be negative also. In any case, even if the popular conjecture that they are all positive were true, it seems now that the result could not be readily attributed to a single general property of a system of spheres, since it would involve dimensionality as well as spherical symmetry of the potential. (It has already been shown that parallel hard cubes in 3 dimensions have some negative virial coefficients [Hoover and De Rocco, same J. **36** (1962), 3141–3162].) *G. R. Stell* (New York)

Ree, Francis H.; Hoover, William G. 1953
Fifth and sixth virial coefficients for hard spheres and hard disks.
J. Chem. Phys. **40** (1964), 939–950.

The fifth and sixth virial coefficients, B_5 and B_6, are evaluated for systems of hard disks and hard spheres. The values are

$$\text{spheres}: B_5/b^4 = 0.1103 \pm 0.0003;$$
$$B_6/b^5 = 0.0386 \pm 0.0004,$$
$$\text{disks}: B_5/b^4 = 0.3338 \pm 0.0005;$$
$$B_6/b^5 = 0.1992 \pm 0.0008,$$

where b is the second virial coefficient.

The authors use a Monte Carlo method and facilitate their computations by expressing the coefficients as a sum of modified cluster integrals written in terms of both the Mayer f-function and $(f+1)$; their expressions contain fewer integrals than the original Mayer expressions.

The authors' method will probably prove easier to extend to the computation of B_7 than the somewhat different approaches of Katsura and Abe [same J. **39** (1963), 2068–2080] and Rowlinson [Proc. Roy. Soc. Ser. A **279** (1964), 147–160]. *G. R. Stell* (New York)

Snider, R. F. 1954
Variational methods for solving the Boltzmann equation.
J. Chem. Phys. **41** (1964), 591–595.

Author's summary: "Known variational methods are reviewed and their applicability to the Boltzmann equation when detailed balance does not hold is discussed. The relation between these variational methods and time-reversal and space-inversion symmetry is indicated."

Fröhlich, H.; Taylor, A. W. B. 1955
The Boltzmann equations in electron-phonon systems.
Proc. Phys. Soc. **83** (1964), 739–748.

In this very instructive paper the authors carefully discuss conditions under which the Boltzmann equations for an electron-phonon system (two coupled equations) can be derived without the use of the objectionable random phase approximations. Formally exact expressions for the rate of change of coarse-grained one-particle distribution functions are derived. They become identical with the collision terms in the Boltzmann equations if (i) the influence of an external field is negligible, (iia) particle correlation is absent at an initial time, and (iib) the relevant series in terms of the interaction parameter converges and is at all times dominated by a certain term. (iib) excludes the possibility of extensive correlations as they occur in superconductivity. In the derivation the importance of the coarse-graining and of the convergence of the series (iib) is emphasized.

{In the reviewer's opinion the condition (iia) is too strong and could be replaced by a weaker one such as the assumption of the localization of the interaction processes which ensure (iib) at the same time.}
S. Fujita (University Park, Pa.)

Stecki, J. 1956
Weak-coupling version of Bogoliubov's kinetic theory of gases.
Phys. Fluids **7** (1964), 33–37.

Bogoliubov's theory of irreversible statistical mechanics is developed by using an expansion in powers of the coupling constant λ. The explicit formal expressions for the kinetic equations and for the correlation functions to arbitrary order in λ are obtained. The link with Prigogine's theory is discussed. The application of this method to the kinetic equation for plasmas is briefly indicated.
R. Balescu (Brussels)

Resibois, P. 1957
On the asymptotic form of the transport equation in dense homogeneous gases.
Phys. Lett. **9** (1964), 139–141.

An apparent contradiction between the author's [J. Mathematical Phys. **4** (1963), 166–173; MR **27** #1252] and J. Stecki's [#1956 above] results on the master equation for the velocity distribution of a dense gas, including three-body collisions, is clarified. It is shown that the contradiction can be suppressed if a proper prescription for the passage to the limit of long times is used. This discussion substantiates Stecki's result, showing that there is complete equivalence between Bogoliubov's and Prigogine's formalisms of non-equilibrium statistical mechanics. *R. Balescu* (Brussels)

Nelkin, Mark; Ghatak, Ajoy 1958
Simple binary collision model for Van Hove's $G_s(r, t)$.
Phys. Rev. (2) **135** (1964), A4–A9.

Authors' summary: "It is noted that the Van Hove self-correlation function $G_s(r, t)$ for a dilute fluid is determined by a linearized Boltzmann equation identical to that occurring in the theory of neutron diffusion. A simple model for the collision integral, in which a molecule emerges from a collision with a Maxwellian distribution, allows some interesting analytic results to be obtained. The double Fourier transform $S_s(\kappa, \omega)$, of $G_s(r, t)$ is expressible in terms of the probability integral for complex arguments. Since $S_s(\kappa, \omega)$ is directly proportional to the Mössbauer line shape, the transition of the line shape from Doppler broadening to simple diffusion broadening is explicitly exhibited as a function of momentum transfer. The spatial moments of $G_s(r, t)$ are calculated as a function of time. The model used gives the same mean-square displacement as does the Langevin equation. The resulting $G_s(r, t)$ is not, however, Gaussian, as is shown by the mean fourth power of the displacement which is 25% greater at intermediate times than would be predicted by the Langevin equation. The non-Gaussian effects lead to an appreciable narrowing of $S_s(\kappa, \omega)$ for intermediate values of κ." *S. Ueno* (Kyoto)

Lee, T. D.; Nauenberg, M. 1959
Degenerate systems and mass singularities.
Phys. Rev. (2) **133** (1964), B1549–B1562.

Authors' summary: "For a system with degenerate energies, the power series expansions of the S-matrix elements may become singular. An elementary theorem in quantum mechanics is proved which shows that under certain general conditions such singularities do not appear in the power series expansion of the transition probabilities, provided these are averaged over an appropriate ensemble of degenerate states. Application of this theorem leads to the cancellations of mass singularities and infrared divergences in quantum electrodynamics. The question of whether a charged particle can have zero mass is studied." *J. N. Islam* (College Park, Md.)

Bochvar, D. A. [Bočvar, D. A.]; 1960
Stankevich, I. V. [Stankevič, I. V.];
Chistyakov, A. L. [Čistjakov, A. L.]
Concerning the entropy expression of the uncertainty principle.
Dokl. Akad. Nauk SSSR **149** (1963), 68–71 (*Russian*); *translated as Soviet Physics Dokl.* **8** (1963), 262–264.

The authors discuss the properties and advantages of the entropy functional

$$H_s = -\int_{-\infty}^{\infty} |\langle s|\psi\rangle|^2 \log|\langle s|\psi\rangle|^2\, ds$$

[cf. Hirschman, Amer. J. Math. **79** (1957), 152–156; MR **19**, 622] as a measure of the uncertainty of the variable s (either position or momentum) in the state $|\psi\rangle$, as compared with the traditional measure of uncertainty, the dispersion D_s. The uncertainty principle takes respectively the forms

(1) $$H_x + H_p \geqq C,$$

(2) $$D_x D_p \geqq C_1.$$

It is shown that (a) for certain $|\psi\rangle$, the entropy functional is a much more accurate measure of the uncertainty of one of the variables than the dispersion, whence the relation (1) in such a case yields a more enlightening estimate of the uncertainty of the other variable; (b) C is not smaller than 2.100, which represents an improvement over Hirschman's lower bound $\log 2\pi = 1.838$ and a step toward realizing his conjecture that $C = \log \pi + 1 = 2.145$. *A. Siegel* (Boston, Mass.)

Leff, Harvey S. 1961
Statistical thermodynamics of incompletely specified systems.
J. Chem. Phys. **41** (1964), 596–599.

Author's summary: "A formal theory of incompletely specified systems based on reasonable physical assumptions is presented. It is shown that the free energy for such systems involves the average of the logarithm of the partition function, rather than the average of the partition function itself. It is further shown that one cannot always replace the logarithm of the partition function by its average in all of the equations of statistical thermodynamics. The method differs from that of Mazo. We do not obtain a term corresponding to an entropy of mixing due to random elements. Finally, fluctuations as calculated here differ from those implied by Mazo's method. Due to the formal nature of the problem, a strict right-wrong distinction between the two methods cannot be made."

Živopiscev, F. A. 1962
Some properties of irreversible quantum-mechanical processes. (Russian)
Vestnik Moskov. Univ. Ser. III Fiz. Astronom. **1964**, *no.* 2, 8–15.

An irreversible equation for the density matrix is derived under the assumption of weak coupling. This equation differs from the usual Bloch equation by the presence of an additional dissipative term. The equation is applied to the problem of radiation damping of an electron moving through an electromagnetic field. *R. Balescu* (Brussels)

van Kampen, N. G. 1963
Condensation of a classical gas with long-range attraction.
Phys. Rev. (2) **135** (1964), A362–A369.

The statistical mechanics of a classical gas of hard spheres with long-range attractive interaction is developed. It is shown that condensation occurs, and the thermodynamical properties of the system are derived. It is proved that the supercritical isotherms for this system are given by the van der Waals equation of state. At lower temperatures the high and low density portions of the van der Waals isotherms are to be connected with a horizontal tie-line, the position of which is fixed by the Gibbs equal-area construction. Density fluctuations are also calculated. This is a paper of great fundamental significance. *J. S. Dahler* (Minneapolis, Minn.)

Rostoker, Norman 1964
Superposition of dressed test particles.
Phys. Fluids **7** (1964), 479–490.

Author's summary: "Kinetic theory for a spatially homogeneous plasma with Coulomb interactions is considered within the framework of the adiabatic approximation or secular perturbation theory. Time-dependent solutions are obtained for the pair correlation function, multiple time distribution functions, etc. Auto-correlation functions and spectral densities for fluctuating quantities are defined and obtained for a stable or unstable plasma. The results show that the plasma behaves as though it consisted of statistically independent quasiparticles. The relationship between test-particle problems and the more general kinetic theory is clarified. Physical interpretation of each term in the Fokker-Planck equation is provided in terms of spontaneous emission, induced emission and absorption."
R. Balescu (Brussels)

Rostoker, Norman 1965
Test particle method in kinetic theory of a plasma.
Phys. Fluids **7** (1964), 491–498.
Author's summary: "A test particle of coordinates $X=(x, v)$ is surrounded by a shield cloud of field particles of coordinates X' characterized by a conditional probability function $P(X|X't)$. A relationship is found between this function, the one-particle function $f(X, t)$, and the two-particle correlation function $G(X, X'; t)$. It is

$$G(X, X'; t) = f(Xt)P(X|X't)+f(X't)P(X'|Xt) + n\int dX''f(X''t)P(X''|Xt)P(X''|X't).$$

The first two terms indicate that each of the two particles involved is a test particle as well as part of the shield cloud of the other particle. This relation has been established without solving explicitly for anything and has none of the usual restrictions such as spatial homogeneity, adiabatic time behavior, etc., usually necessary for obtaining explicit solutions. It is useful because the problem of kinetic theory is reduced to determining P, which involves only the Vlasov equation. In addition, superposition principles for fluctuations, etc., are apparent at the outset."
R. Balescu (Brussels)

Lamborn, B. N. A.; Lafferty, D. L. 1966
Electrostatic waves in beam-plasma systems.
Phys. Fluids **7** (1964), 292–305.
Systems composed of a plasma and a set of moving nonrelativistic charged particle streams are described by means of collective coordinates. The physical quantities are expanded in a Fourier series in the space variables with a cutoff determined by the Debye screening distance. In terms of this expansion the collective interactions of the components of the system are discussed. The authors apply this method to the interaction of a beam of particles with a finite plasma where the time taken by the particles to traverse the plasma is short. Two growing plasma waves are obtained. One due to the coupling of the beam and plasma modes and the second when the beam oscillations match the background frequency. The relation between their results and some recent experiments of M. J. Kofoid are discussed.
{Misprints: In relations (7) and (8), $\mathbf{p}_{tk}$ should be ρ_{tk}.}
J. K. Thurber (New York)

Vacca, Jacopa 1967
Sull'equilibrio radiativo magnetodinamico di una massa gassosa sferica uniformemente rotante e gravitante.
Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **97** (1962/63), 1014–1026.
Author's summary: "Si determinano le equazioni differenziali che reggono l'equilibrio radiativo magnetodinamico di una massa gassosa sferica in rotazione uniforme, altamente conduttrice dell'elettricità, soggetta ad azione gravitazionale ed immersa inizialmente in un campo magnetico assiale costante."

Oberman, C.; Dawson, J. 1968
Plasma stability criteria from conservation laws.
Phys. Fluids **7** (1964), 773–777.
Authors' summary: "It is shown that the Penrose [Phys. Fluids **3** (1960), 258–265] criteria for the existence of unstable electrostatic oscillations of a Vlasov plasma can be derived simply from the energy-momentum conservation laws applied to marginally unstable modes. These criteria also are obtained variationally as necessary conditions for instability by extremizing the possible motions which can convert kinetic energy into electric field energy."
S. Gartenhaus (Lafayette, Ind.)

Quemada, Daniel 1969
Une condition de fermeture "thermodynamique" pour le système des équations hydrodynamiques dans un plasma sans collision, en équilibre maxwellien.
C. R. Acad. Sci. Paris **259** (1964), 315–318.
Author's summary: "Dans le cas d'un plasma sans collision, en équilibre maxwellien, on déduit une relation de type thermodynamique entre la conductivité $\boldsymbol{\sigma}$ et le coefficient thermoélectrique $\boldsymbol{\chi}$ pour chaque espèce de particules, de charge électrique q

$$\chi = \frac{1}{q}\frac{\partial}{\partial\beta}(\sigma), \qquad \beta = \frac{1}{\mathrm{KT}}.$$

En l'absence de champ magnétique extérieur, ou dans le cas de perturbations se propageant le long du champ magnétique, cette relation est une condition de fermeture exacte du système d'équations hydrodynamiques déduites de l'équation de Vlasov."

Rand, S. 1970
Transfer of energy in a quantized plasma.
Phys. Fluids **7** (1964), 807–813.
Author's summary: "The exchange of energy between a plasma, and a test particle in a time-independent potential, has been studied. The motion of the test particle and the plasma interactions are treated quantum mechanically. A general formula, obtained for the energy transfer, is applied to the problems of the free particle and to the linear harmonic oscillator in the dipole approximation. Contributions to the energy transfer are separated into parts associated with friction and with thermal agitation or velocity diffusion. The results are compared with the classical limits. For the free particle, the corrections to the classical results are logarithmic, whereas for the harmonic oscillator the corrections are very important, when the oscillator transition energies exceed the random

electron energies in the plasma. The frictional contribution to the energy loss by the oscillator is related to the electrical conductivity."

Baldwin, David E. 1971
Perturbation method for waves in a slowly varying plasma.
Phys. Fluids 7 (1964), 782–791.
Author's summary: "The method of expansion in instantaneous eigenfunctions is used to obtain approximate solutions to the linearized Vlasov equation when the unperturbed function is time-dependent. The method is useful when the time-dependence of the unperturbed function is slow compared with the plasma frequency. The general results are used to obtain the growth of wave energies in the quasilinear limitation of weak two-stream instabilities. Deviations from previous results for this problem are found."

Shapiro, V. D. [Šapiro, V. D.] 1972
Nonlinear theory of the interaction of a monoenergetic beam with a plasma.
Ž. Èksper. Teoret. Fiz. **44** (1963), 613–625 (*Russian. English summary*); *translated as Soviet Physics JETP* **17** (1963), 416–423.
Author's summary: "The nonlinear theory of oscillations excited by the interaction of a monoenergetic beam with a plasma has been investigated. The oscillation amplitude at saturation and the growth of the thermal energy of the beam and plasma resulting from the interaction are determined." *R. A. Gerwin* (Seattle, Wash.)

Fisher, Michael E. 1973
Correlation functions and the critical region of simple fluids.
J. Mathematical Phys. 5 (1964), 944–962.
Author's summary: "The 'classical' (e.g. van der Waals) theories of the gas-liquid critical point are reviewed briefly and the predictions concerning the nature of the singularities of the coexistence curve, the specific heat, and the compressibilities are compared critically with experiment and with the analytical and numerical results for lattice gas models. The critical singularities are related to the behavior of the pair correlation function $G(r)=g(r)-1$ and the Ornstein-Zernike theory of critical scattering is reviewed. Alternative derivations of the theory are discussed and its validity is assessed in relation to experiment and to more detailed theoretical calculations. The nature and magnitude of the expected deviations from the 'classical' theory are described. The analogies with critical magnetic phenomena are mentioned briefly."

Lebowitz, J. L.; Rubin, E. 1974
Dynamical study of Brownian motion.
Phys. Rev. (2) **131** (1963), 2381–2396.
Authors' summary: "We study the motion of a Brownian particle in a fluid from a dynamical point of view, i.e., without the a priori introduction of purely stochastic elements. The Brownian particle is distinguished primarily by having a mass M which is much greater than the mass of the fluid particles m. Our method consists of rewriting the Liouville equation for μ, the joint distribution of fluid and Brownian particle, as a pair of coupled equations for the distribution function of the Brownian particle f and the conditional distribution of the fluid $P=(\mu/f)$. The equation for P is then solved formally in a perturbation series in the square root of the mass ratio (m/M), which is then substituted in the equation for f to obtain a collision term δf representing the effect of the fluid on f. We consider two situations: (1) A constant external force acts on the Brownian particle and f is stationary, the external force being balanced by δf, and (2) a general time-dependent f. We find in both cases, as expected, that to lowest order δf has the form of a Fokker-Planck type collision term, though in the second case this only holds for times much larger than the fluid relaxation time after an initial time at which μ is arbitrary. The next order terms in δf differ for the two cases. Furthermore, because of the limitations on the times at which δf is valid in the second case, $f(t)$ does not really obey a Markoffian equation to this order when the initial state is arbitrary. In the Appendices we consider the formal structure of δf, the form of f in the stationary case, a 'quasistochastic' model of Brownian motion, the motion of a composite Brownian particle, and the motion of a Brownian particle in a crystal. The latter makes contact with the work of Hemmer and Rubin."
A. Siegel (Boston, Mass.)

Takahasi, Hidetosi 1975
Tables of Lorentz field for cubic lattice.
J. Phys. Soc. Japan **19** (1964), 499–503.
Author's summary: "The tables of numerical values of the Lorentz field are given for the simple cubic lattice. Ewald's method, suitable for machine calculation, was used."

Calais, Jean-Louis; Appel, Klaus 1976
Inversion of cyclic matrices.
J. Mathematical Phys. 5 (1964), 1001–1008.
Authors' summary: "The inversion of the overlap matrix in LCAO calculations of crystals is discussed for the general case, when an arbitrary number of neighbors is taken into account. Of the several alternative methods that are possible in one dimension, one is singled out—which can be used as well in three dimensions—to calculate the inverse to any desired accuracy. Particular attention is paid to the numerical aspects of the problem."
A. C. Hurley (Melbourne)

Sykes, M. F.; Essam, J. W. 1977
Exact critical percolation probabilities for site and bond problems in two dimensions.
J. Mathematical Phys. **5** (1964), 1117–1127.
Authors' summary: "An exact method for determining the critical percolation probability, p_c, for a number of two-dimensional site and bond problems is described. For the site problem on the plane triangular lattice, $p_c=\frac{1}{2}$. For the bond problem on the triangle, simple quadratic, and honeycomb lattices, $p_c=2\sin(\pi/18)$, $\frac{1}{2}$, $1-2\sin(\pi/18)$, respectively. A matching theorem for the mean number of finite clusters on certain two-dimensional lattices,

somewhat analogous to the duality transformation for the partition function of the Ising model, is described."
S. Sherman (Detroit, Mich.)

Suhubi, S. E. 1978
On a simple solution of the general multigroup neutron diffusion equations. (Turkish summary)
Istanbul Tek. Üniv. Bül. **16** (1963), 67–75.
Author's summary: "In this paper the solution of the most general integro-differential equations for multigroup neutron diffusion is given for one-group neutrons in finite multiplicative media taking delayed neutrons into consideration, the criticality condition in this case is obtained and an explicit expression is derived for a flux distribution initiated by sending given neutron fluxes to certain points of the medium."

RELATIVITY

See also 1774, 1874, 1902, 1942, 2092, 2093.

DeWitt, Bryce S. 1979
Gravity: A universal regulator?
Phys. Rev. Lett. **13** (1964), 114–118.
The linearized gravitational interaction of two scalar particles in the ladder approximation is considered, and the leading terms summed. The corresponding approximation to the gravitational self-energy is also found. The calculation is performed in De Donder gauge. In analogy with the "peratization" procedure recently introduced to deal with the singular intermediate vector meson theory, the high degree of divergence in the spin-two exchange is used to damp the high-energy parts of the amplitude. More precisely, appropriate summation for high-momentum transfer leads to an effective spin-two kernel displaced from its usual x^{-2} form to $(x^2-\lambda^2)^{-1}$, where the singularity is at a space-like hyperboloid defined by λ, the gravitational length, $\lambda \sim (\hbar G/c^3)^{1/2} \sim 10^{-33}$ cm. The corresponding Fourier transform is thus highly damped at high-momentum transfer. The two-particle ladder amplitude may be turned into a one-particle propagator with "nested" linearized gravitation bubbles by identifying the particle lines at one end of the ladder. This yields the self-energy, $\delta m = m^2 - m_0{}^2$ and wave-function renormalization, Z_2, contributions of the "nested" diagrams. Both are finite (as in peratization) with $m_0{}^2 \sim 2m^2$, $Z_2 \sim \frac{1}{2}$ to within negligible terms in an expansion in m/μ, where $\mu \sim \lambda^{-1} \sim 10^{19}$ Bev is the energy corresponding to λ. The m_0/m relation is reminiscent of the vector theory peratization, where the coupling constant renormalization (the analogue of mass renormalization here) is also finite, and of order unity.

Similar damping by linearized theory peratization of other self-energies, e.g., electrodynamic ones, is also discussed. The numerical values are very large, as the leading terms are characteristically $\delta m^2 \sim \alpha\mu^2$ due to the quadratic nature of δm^2 for scalar electrodynamics. It is also noted that the modified scalar propagator in the pure linearized case discussed above takes on its bare form for high momenta, in conformity with the Lehmann theorem.

While the author correctly states that this theorem really requires the existence of certain local properties of energy-momentum, their fulfillment or lack of it in general relativity is irrelevant to his work, which is purely within the framework of spin-two linearized theory, and not manifestly indicative of what the full Einstein theory would imply.

Finally, as noted in a footnote, gauge invariance is very delicate in quantum gravitation, and the ladder approximation is not a gauge-invariant subset of graphs. Further, as in vector theory peratization, the dominance of ladder graphs for exchange of massless particles is an entirely open question in the high-energy domain which is of interest here. *S. Deser* (Waltham, Mass.)

Hillion, Pierre 1980
Harmoniques et tenseurs hypersphériques.
Cahiers de Phys. **17** (1962/63), 455–463.
Author's summary: "Les harmoniques hypersphériques constituent une classe de solutions de l'équation d'onde $\Box W = 0$ dans un système de coordonnées valable seulement à l'extérieur du cône de lumière. Les tenseurs hypersphériques généralisent à l'espace-temps la notion de champ tensoriel qui s'introduit dans certains problèmes non relativistes en coordonnées sphériques."
P. Roman (Boston, Mass.)

Petrov, A. Z. 1981
On the gravitational fields with spherical symmetry.
Cahiers de Phys. **18** (1964), 1–8.
This is substantially identical with the author's paper in Ž. Èksper. Teoret. Fiz. **44** (1963), 1525–1533 [MR **28** #1965]. *F. A. E. Pirani* (Waltham, Mass.)

Burcev, P. 1982
On the mechanics of photon rockets in the general theory of relativity.
Czechoslovak J. Phys. **14** (1964), 294–301.
Author's summary: "Generally covariant equations of motion of a photon rocket comprising all the basic dynamic and thermodynamic processes are derived and are written for the case of a weak gravitational field. The equations lead to results which agree with those obtained by other means."

Havas, Peter 1983
The connection between conservation laws and laws of motion in affine spaces.
J. Mathematical Phys. **5** (1964), 373–378.
Generalizing earlier work [the author and J. N. Goldberg, Phys. Rev. (2) **128** (1962), 398–414; MR **26** #2290] valid in metric spaces, the author shows that for affine spaces for a collection of mass points subject to a covariant conservation law for a symmetric tensor density $p^{\mu\nu}$, both the geodesic law of motion and the form of $p^{\mu\nu}$ follow from the conservation law alone. Additional conditions on the affine space are required if the covariant conservation law applies instead to a symmetric tensor, since in an affine space one law does not necessarily imply the other as is true in a metric space. Among several applications is one to a bona fide affine space: the space-time of Newtonian mechanics. The constancy of mass and rectilinear motion are shown to be consequences of the conservation law. An analogous discussion is given for a

collection of point charges whose vector density J^ν satisfies a covariant conservation law in an affine space. From this alone follows the constancy of charge and the form of J^ν. *J. R. Klauder* (Murray Hill, N.J.)

Bialas, E.; Bialas, A. 1984

On the relation between the Penrose classification of gravitational fields and the classification according to group of motions.

Acta Phys. Polon. **24** (1963), 515–517.

It is shown that an algebraically special, empty space-time with non-degenerate Weyl tensor admits at most a two-parameter group of motions. *W. M. Tulczyjew* (Warsaw)

Papapetrou, Achille 1985

Ondes de choc et symétrie sphérique en relativité générale.

C. R. Acad. Sci. Paris **257** (1963), 2616–2619.

Let the metric $g_{\mu\nu}$ of a vacuum space-time V_4 be C^1 and (*) piecewise at least C^2. It is shown in Finkelstein coordinates [D. Finkelstein, Phys. Rev. (2) **110** (1958), 965–967] that if V_4 is Schwarzschild up to a spherically symmetric null surface S and $g_{\mu\nu,\rho\sigma}$ is discontinuous across S, then the discontinuity may be removed by a coordinate transformation preserving (*); this accords with Birkhoff's theorem. *F. A. E. Pirani* (Waltham, Mass.)

Latrémolière, Claude 1986

Sur les équations d'ordre supérieur du champ gravitationnel.

C. R. Acad. Sci. Paris **258** (1964), 1170–1172.

A tensor field $H_{\alpha\beta\lambda\mu}$ in a 4-dimensional Riemannian space is considered, it being supposed that this field satisfies the algebraic conditions $H_{\alpha\beta\lambda\mu}=-H_{\beta\alpha\lambda\mu}=-H_{\alpha\beta\mu\lambda}=H_{\lambda\mu\alpha\beta}$, $g^{\alpha\lambda}H_{\alpha\beta\lambda\mu}=0$, $\eta^{\alpha\beta\lambda\rho}H_{\alpha\beta\lambda\mu}=0$, together with the Bianchi identities. The integrability conditions of this system are obtained. These are discussed in terms of various Petrov types in terms of the spinor formalism of R. Penrose [Ann. Physics **10** (1960), 171–201; MR **22** #6563]. *H. Rund* (Pretoria)

Bichteler, Klaus 1987

Äusserer Differentialkalkül für Spinorformen und Anwendung auf das allgemeine reine Gravitationsstrahlungsfeld. (English summary)

Z. Physik **178** (1964), 488–500.

Author's summary: "An exterior differential calculus for spinor forms quite analogous to the well-known tensorial Cartan calculus is developed which renders possible the direct spinorial calculation of the Weyl- and Ricci-spinors. As an application, the metric and curvature spinors of the most general space-time admitting a normal shear-free null congruence are derived; the rather modest amount of calculation illustrates the advantages of the new method as compared to the techniques used thus far." *S. Ide* (Sapporo)

Chandrashekhar, K. 1988

Einstein's connections in nonholonomic frame. (Italian summary)

Nuovo Cimento (10) **32** (1964), 1166–1181.

The reviewer found a simple formula for the non-holonomic components $\Gamma_{jk}{}^i$ $(i,j,k=1,\cdots,4)$ of the Einstein unified connection [*Geometry of Einstein's unified field theory*, Chapter II, Theorem (13.1), Noordhoff, Groningen, 1957; MR **20** #5067]. The author writes out all these 64 equations. According to his explanation of the symbol ∇_i (his equation (4.4)) all these equations are faulty. This is due to the author's wrong application of ∇_i in his equation (4.3), where ∇_i (and Δ_i) should be replaced in the first three terms by the symbol ∂_i of the ordinary non-holonomic derivative.

{Reviewer's remark: The author's note at the end is not justified as far as the reviewer is concerned.}

V. Hlavatý (Bloomington, Ind.)

Droz-Vincent, Philippe 1989

Quantification du champ pentadimensionnel linéarisé.

Cahiers de Phys. **18** (1964), 25–37.

Extrait de l'introduction: "Dans cet article, nous calculerons les relations de commutation du champ unitaire de la théorie de Thiry. Nous nous limiterons évidemment à l'approximation linéaire, ce qui nous permettra de faire usage de transformations de Fourier pour établir des relations de commutation compatibles avec les équations du champ. La théorie pentadimensionnelle ayant le mérite de s'interpréter sans ambiguïté en termes de gravitation et d'électromagnétisme, les résultats seront immédiatement comparables à ceux qu'on peut obtenir séparément pour les deux champs."

A. Peres (Haifa)

Braunss, G. 1990

Zur einheitlichen Feldtheorie. (English summary)

Z. Naturforsch. **19a** (1964), 401–405.

Es werden die gekoppelten Feldgleichungen für das Einsteinsche Gravitationsfeld und ein skalares Materiefeld betrachtet, wobei beim Materiefeld an die Stelle des Massenterms ein in dem Potential φ nichtlineares Glied tritt, das nach steigenden geraden Potenzen von φ entwickelt gedacht wird. Unter Verwendung von Regularitätsforderungen am Koordinatenursprung und von Grenzbedingungen werden die statischen Kugelsymmetrischen Lösungen des Systems in der Nähe des Ursprungs und im Unendlichen näherungsweise diskutiert.

{Dem Referenten erscheinen die Ansätze ziemlich willkürlich. Der § 3 über konformebene Metriken ist dem Referenten bereits im Ansatz nicht verständlich, da der Verfasser statt einer vier- eine zweidimensionale Metrik für den kosmischen Raum einführt.} *H. Treder* (Berlin)

★Recent developments in general relativity. 1991

Pergamon Press, Oxford-New York-Paris; PWN —Polish Scientific Publishers, Warsaw, 1962. 472 pp. $8.00.

This collection of 45 articles on general relativity and related topics was commissioned in 1959 to celebrate the 60th birthday of Leopold Infeld (and is consequently often referred to as the "Infeld Festschrift"). Manuscripts were delivered by mid-1960, but various mishaps, not the responsibility of the editorial committee, delayed the appearance of the volume until April 1962. Contributors had been allowed to make minor corrections in text and additions in footnotes up to mid-1961.

Nevertheless, the book ranges widely over its field, and gives a fair indication of the state of the subject at the beginning of this decade. It contains a number of substantial papers which have not been published elsewhere and is still a valuable source of information. It is certainly essential to a serious library of the subject. The standard of printing and proof reading is high.

Part I comprises 7 review articles and Part II, 38 shorter original contributions. In each part the papers are arranged alphabetically by the editor. The subject matter overlaps to a considerable extent that of the Royaumont reports [see #1992 below]; they were prepared at about the same time, and 27 out of the 67 authors involved contribute to both. The separate papers from the two volumes are reviewed individually below [for arrangement see #1993]. *F. A. E. Pirani* (London)

★Les théories relativistes de la gravitation 1992
(Royaumont, 21–27 juin 1959).
Colloques Internationaux du Centre National de la Recherche Scientifique, XCI.
Éditions du Centre National de la Recherche Scientifique, Paris, 1962. 475 *pp.* 49 *NF.*

This is a collection of 46 reports, and an "allocution de clôture", presented at the Royaumont conference on gravitation theory in June 1959. It appears the manuscripts were delivered by mid-1960, but the volume did not appear until mid-1962. Contributors were able to make revisions in proof in 1961 {but no explanation of the delay in publication was ever offered to them, which is in striking contrast to the treatment accorded contributors to the *Infeld Festschrift* [see preceding review #1991]}.

The delay between conference and publication much reduced the value of the book. Much of the material was published elsewhere in the three-year interval, often in more extensive form.

The papers are printed in order of presentation, with a table of contents but no index, so that to find a particular paper one must search through the table. The reports of discussions include, as is often the case, many questions without answers and answers without questions. Most of the discussion has been translated into French, although it was conducted mainly in English {but the editors do not mention this}. There are many misprints.

The book has some value as a record of the conference proceedings, which cover much of the work in the field between 1957 and 1959. The separate papers are reviewed individually below, together with those from the *Infeld Festschrift* (see the preceding review [#1991] and for arrangement, the following review [#1993]). *F. A. E. Pirani* (London)

Bergmann, Peter G. 1993
Allocution de clôture.
Les théories relativistes de la gravitation (*Royaumont*, 1959), *pp.* 451–461. (*English version, pp.* 463–472). *Éditions du Centre National de la Recherche Scientifique, Paris*, 1962.

This is a bold and generally successful attempt to summarize the proceedings of the Royaumont conference (see preceding review [#1992]).

The main subjects distinguished in this allocution were unified field theory, programmes of quantization, non-quantum general relativity, and experimental work. Unity of physical theory, in particular of field theory, is the usual motivation in the first two subjects, and in the reviewer's opinion, understanding of the classical gravitational field as a preliminary to quantization, rather than for its own sake, should be a primary motive in non-quantum general relativity also. Otherwise one faces the prospect of engaging merely in a rather botanical kind of classical differential geometry. The weakness of the gravitational interaction places great obstacles in the way of direct experimental tests [but compare Weber (#2072 below)], but the prospect of indirect achievements, in generally covariant quantum theory, is a bright one, despite the difficulties which have been discovered in the past 30 years, and especially in the past decade, by attempts to quantize the gravitational field.

These difficulties, like most others in general relativity theory, stem from the general covariance of the theory, sometimes directly, more often because of the non-linearity which general covariance engenders. Many of the papers under review below attempt to defeat non-linearity, either in the formulation of the initial-value problem, or in constructing a canonical formalism, or in relating fields to their sources, or in redefining concepts, like energy, familiar in linear Lorentz-invariant theories.

In the following reviews, non-quantum general relativity is treated first. This includes classification theory, classical radiation theory, classical theory of internal and external solutions, exact solutions, measurement theory, construction of Green's functions, the problem of motion, energy and conservation laws. Next come classical preliminaries to quantization and quantum theory, and then theories of spinning particles, unified field theory, and other (non-Einstein) relativity theories. Experiments and observations, cosmology and Mach's principle are treated last. *F. A. E. Pirani* (London)

Penrose, R. 1994
General relativity in spinor form.
Les théories relativistes de la gravitation (*Royaumont*, 1959), *pp.* 429–432. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1962.

This is a brief outline of results which have been published in detail elsewhere [Ann. Physics **10** (1960), 171–201; MR **22** #6563]. It includes the "Penrose diagram", which does much to illuminate the Petrov classification (see Petrov, #1998a; #1998b) and indicates some other applications of the spinor formalism {whose exploitation in all sorts of directions, by the author and others, has been a major development since 1959 [see, for example, E. Newman and the author, J. Mathematical Phys. **3** (1962), 566–578; MR **25** #4904; P. Jordan, J. Ehlers and R. K. Sachs, Akad. Wiss. Lit. Mainz Abh. Math.-Natur. Kl. **1961**, 1–62; MR **27** #6601]}. *F. A. E. Pirani* (London)

Kundt, W. 1995
Note on the equivalence problem.
Recent developments in general relativity, pp. 307–311. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw*, 1962.

The problem of classifying gravitational fields has come much into prominence in recent years. It is, of course, a special case of the wider problem of classifying Riemannian spaces. This and the following four reviews are all of papers connected with this problem, although several of them have wider applications.

This paper is concerned with the local equivalence of two Riemannian spaces V_n, $\bar{V}_n$, that is to say, with the existence of a locally isometric map $V_n \to \bar{V}_n$. It is known [cf. É. Cartan, *Leçons sur la géométrie des espaces de Riemann*, 2ième éd., Chap. 13, Gauthier-Villars, Paris, 1951; MR **13**, 491] that two spaces are locally equivalent if and only if there exists a simultaneous transformation from one to the other of the set S comprising the metric tensor, the Riemann tensor, and the first p covariant derivatives of the Riemann tensor, with some $p \leq \frac{1}{2}n(n+1)$. However, for practical computational purposes it is almost essential to be able to compare sets of scalars (i.e., absolute scalar invariants) rather than sets of tensors.

The author shows by example that it is not in general sufficient to compare sets of scalars which are rational integral functions of the tensors of the set S. The example is the pp-wave space-time of Robinson [unpublished], with metric

$$ds^2 = 2dudv + dy^2 + dz^2 + 2H(u, y, z)\, du^2,$$

where (conditions for vacuum) $H_{yy} + H_{zz} = 0$; for this space-time, all such functions vanish [for many other properties see W. Kundt, Z. Physik **163** (1961), 77–86; MR **23** #B983]. He shows also that in a special case the scalars, rational integral and otherwise, of second and third order do not form a maximal functionally independent set. *F. A. E. Pirani* (London)

Kundt, Wolfgang 1996

Note on the symmetries of plane-fronted gravitational waves. (French summary)

Les théories relativistes de la gravitation (*Royaumont*, 1959), *pp.* 155–156. *Éditions du Centre National de la Recherche Scientifique*, *Paris*, 1962.

The author complements his Festschrift paper [#1995 above] by giving an explicit construction of three independent scalars (no more possible) for a pp-wave space-time. However, he adds in proof that he prefers a later {and much more complete} presentation by J. Ehlers and himself [in Chapter 2 of *Gravitation*, pp. 49–101, Wiley, New York, 1962; MR **26** #1177; this chapter describes also other methods and results of classification]. *F. A. E. Pirani* (London)

Petrov, A. Z. 1997

Gravitational field geometry as the geometry of automorphisms.

Recent developments in general relativity, pp. 379–386. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw*, 1962.

A very brief summary of the author's work on the classification of vacuum space-times by their isometry group structure. He treats a number of examples explicitly. For many more details see his book [*Einstein spaces* (Russian), Fizmatgiz, Moscow, 1961; MR **25** #4897]. *F. A. E. Pirani* (London)

Petrov, A. Z. 1998a

Invariant classification of gravitational fields.

Recent developments in general relativity, pp. 371–378. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw*, 1962.

Petrov, A. Z. 1998b

Classification invariante des champs de gravitation.

Les théories relativistes de la gravitation (*Royaumont*, 1959), *pp.* 107–112. *Éditions du Centre National de la Recherche Scientifique*, *Paris*, 1962.

These two papers are nearly identical. In each, the author reviews his now well-known local algebraic classification of vacuum Riemann tensors (or Weyl conformal curvature tensors) in space-time [Kazan. Gos. Univ. Učen. Zap. **114** (1954), no. 8, 55–69; MR **17**, 892]. He gives concrete examples, of type III and type N, and he discusses the question of boundary conditions in space-times of given type [see also Pirani, #2000; Lichnerowicz, #2002; Sachs, #2003]. *F. A. E. Pirani* (London)

Lanczos, C. 1999

Some properties of the Riemann-Christoffel curvature tensor.

Recent developments in general relativity, pp. 313–321. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw*, 1962.

This is in effect a study of the representation of a conformal curvature tensor in terms of two second-rank skew tensors. *F. A. E. Pirani* (London)

Pirani, F. A. E. 2000

Survey of gravitational radiation theory.

Recent developments in general relativity, pp. 89–105. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw*, 1962.

This paper and the following seven deal with classical gravitational radiation theory. From the summary: "A survey, to early 1960, of the covariant part of gravitational radiation theory, aimed at those with a working knowledge of general relativity. Approaches to gravitational radiation problems through weak field and other approximation methods and through the use of special coordinate conditions are not discussed. Most attention is given to (1) the theory of gravitational wave-fronts; (2) the algebraic and differential properties of the Riemann tensor and their implications for wave propagation; (3) exact vacuum metrics representing gravitational waves." *F. A. E. Pirani* (London)

Papapetrou, A.; Treder, H. 2001

Shock waves in general relativity.

Recent developments in general relativity, pp. 351–360. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw*, 1962.

A summary of work by Stellmacher [Math. Ann. **115** (1938), 740–783] and the authors on the theory of wave fronts, that is to say, of algebraic and differential properties of discontinuities in the metric tensor and its derivatives, in the presence (for example) of the vacuum

field equations. For a systematic study, see Treder's monograph [*Gravitative Stosswellen*, Akademie-Verlag, Berlin, 1962; MR **25** #1920]. *F. A. E. Pirani* (London)

Lichnerowicz, André 2002

Radiations en relativité générale.

Les théories relativistes de la gravitation (*Royaumont*, 1959), *pp.* 93–106. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1962.

A very clear introduction to the idea of a (pure) gravitational radiation field (type N field); the analogy with the pure electromagnetic radiation field is worked out in detail. (For a much more extended discussion see his survey article [Ann. Mat. Pura Appl. (4) **50** (1960), 1–95; MR **22** #6554], and for his more recent work on propagators, mentioned briefly in this report, see his paper [Inst. Hautes Études Sci. Publ. Math. No. 10 (1961); MR **28** #967]. On this subject compare also Synge [#2022]; De Witt [#2033a].) *F. A. E. Pirani* (London)

Sachs, R. 2003

Distance and the asymptotic behaviour of waves in general relativity.

Recent developments in general relativity, pp. 395–407. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw*, 1962.

This is an important contribution to the theory of gravitational wave propagation, based on a study of the differential properties of the Riemann tensor. From the author's summary: "It is shown that in electromagnetic null, gravitational null, or vacuum gravitational type III waves that spread out, there are universal algebraic relations between the parallax distance, the electromagnetic luminosity distance, the preferred parameter distance along the rays, and the rotation of the rays; when the rotation vanishes, all three kinds of distance are equal. The dependence of the electromagnetic field or Riemannian tensor on the distance is calculated explicitly and the waves interpreted as pure retarded fields seen at large distances from some source." {The author has subsequently carried these ideas much further [see his papers in Proc. Roy. Soc. Ser. A **264** (1961), 309–338; MR **27** #6598; ibid. **270** (1962), 103–126; MR **26** #7393; see also E. Newman and R. Penrose, J. Mathematical Phys. **3** (1962), 566–578; MR **25** #4904; I. Robinson and A. Trautman, Proc. Roy. Soc. Ser. A **265** (1961/62), 463–473; MR **24** #B1970].} *F. A. E. Pirani* (London)

Trautman, A. 2004

On the propagation of information by waves.

Recent developments in general relativity, pp. 459–463. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw*, 1962.

Here is an ingenious development of a physically essential property of gravitational waves, that they be able to propagate information. From this the author infers that wave like solutions of the equations of physical fields should depend on arbitrary functions, whose profiles convey the information. Assuming in each case a simple form for the field quantities, he carries out trials, first on a scalar field, then on the Maxwell field, and finally on the gravitational field, recovering in the last case Robinson's line element for pp-waves [see Kundt, #1995 above]. *F. A. E. Pirani* (London)

Brill, Dieter R. 2005

Time-symmetric gravitational waves. (French summary)

Les théories relativistes de la gravitation (*Royaumont*, 1959), *pp.* 147–153. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1962.

A survey of his earlier work [Ann. Physics **7** (1959), 466–483; MR **21** #7056], which he extends by considering certain spacewise-closed examples. *F. A. E. Pirani* (London)

Bonnor, W. B. 2006

Spherical gravitational waves. (French summary)

Les théories relativistes de la gravitation (*Royaumont*, 1959), *pp.* 141–146. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1962.

Brief summary of the methods and results of a much longer paper of the same title, in which approximations by double series are employed [Philos. Trans. Roy. Soc. London Ser. A **251** (1959), 233–271; MR **20** #7568]. *F. A. E. Pirani* (London)

Fock, V. [Fok, V.] 2007

Sur les ondes de gravitation émises par un système de masses en mouvement.

Les théories relativistes de la gravitation (*Royaumont*, 1959), *pp.* 137–140. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1962.

Reproduces the essence of paragraph 87 of his book [*Theory of space, time and gravitation* (Russian), GITTL, Moscow, 1955; MR **18**, 445; English transl., Pergamon, New York, 1959; MR **21** #7042]. {In the reviewer's opinion, little significance can be attached to the results found by the methods of Bonnor [#2006 above] and of Fock until there is some evidence for the convergence of the approximation methods which they use.} *F. A. E. Pirani* (London)

Papapetrou, A. 2008

Non-existence of periodically varying non-singular gravitational fields. (French summary)

Les théories relativistes de la gravitation (*Royaumont*, 1959), *pp.* 193–198. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1962.

This is a very readable summary of the results of a paper published elsewhere [the author and Treder, Ann. Physik (7) **3** (1959), 360–372; MR **21** #7054; see also the author and Treder, ibid. (7) **6** (1960), 311–327; MR **22** #7825; C. R. Acad. Sci. Paris **254** (1962), 4254–4256; MR **26** #5948; ibid. **254** (1962), 4431–4433; MR **26** #5949; see further, the following reviews: Peres and Rosen, #2009a; Taub, #2019b]. *F. A. E. Pirani* (London)

Peres, A.; Rosen, N. 2009a

Boundary conditions in general relativity theory.

Recent developments in general relativity, pp. 367–370. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw*, 1962.

Peres, A.; Rosen, N. 2009b

Some investigations of the gravitational field equations. (French summary)

Les théories relativistes de la gravitation (Royaumont, 1959), pp. 415–419. Éditions du Centre National de la Recherche Scientifique, Paris, 1962.

These two papers are more or less identical. Also, there is little in them which cannot be found in two other papers of the authors [Phys. Rev. (2) **115** (1959), 1085–1086; MR **21** #7061; Nuovo Cimento (10) **13** (1959), 430–438; MR **22** #605]. By taking time-averages in a weak field approximation, they reach the conclusion that the presence of non-linear terms in Einstein's vacuum field equations excludes the possibility of small stable oscillations of the field about an asymptotically Minkowskian equilibrium state. Similar conclusions have been reached independently by Papapetrou [see also preceding review #2008, and Taub, #2019b]. *F. A. E. Pirani* (London)

Finkelstein, David; Misner, Charles W. 2010

Further results in topological relativity. (French summary)

Les théories relativistes de la gravitation (Royaumont, 1959), pp. 409–413. Éditions du Centre National de la Recherche Scientifique, Paris, 1962.

The authors pursue their study [Ann. Physics **6** (1959), 230–243; MR **22** #2369] of the question: "Can a purely gravitic structure conceivably have half-integer spin?" It leads them to the conjecture, "without torsion, there is no half-spin". They discuss relations between the topology of space-time, regarded as a locally Euclidean topological space without metric structure, on the one hand, and the possible arrangements of the field of tangent light-cones, on the other. No derivations are given. *F. A. E. Pirani* (London)

Marder, L. 2011

Locally isometric space-times.

Recent developments in general relativity, pp. 333–338. Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw, 1962.

The author exhibits a class of solutions of Einstein's field equations having the properties: (1) The metric is axisymmetric and static; (2) The non-empty region consists of an ovaloidal shell S of finite thickness in whose interior cavity is fitted a cylindrical rod R, so that the empty region E between R and S has the topology of a toroid; (3) The material of R and S has everywhere positive density, and stresses small compared to the density; (4) The empty region F outside S is asymptotically flat; (5) The empty regions E of the different solutions in the class are locally, but not globally, isometric {in effect, a segment of variable angular size may be removed from the toroidal region E}.

{The word "complete" in the author's summary is not used in the sense of completeness with respect to geodesics—although these metrics probably are complete in this sense, this question is not examined. It means that the metric is defined explicitly for all values of the coordinates in assigned ranges.}

F. A. E. Pirani (London)

Hoffmann, Banesh 2012

Static, axially symmetric gravitational fields in general relativity involving mass singularities of both signs. (French summary)

Les théories relativistes de la gravitation (Royaumont, 1959), pp. 237–247. Éditions du Centre National de la Recherche Scientifique, Paris, 1962.

The author finds conditions for, and exhibits explicitly, solutions of Newton's gravitational equations representing mass points arranged in equilibrium on a straight line; negative masses are (necessarily) allowed and repel all other masses, while positive masses attract all other masses. He then finds corresponding solutions of Einstein's gravitational equations (his mass points are no longer monopoles); the requirement of regularity on the axis of symmetry yields a condition identical in form with the Newtonian equilibrium condition.

F. A. E. Pirani (London)

Fock, V. [Fok, V.] 2013

Einsteinian statics in conformal space.

Recent developments in general relativity, pp. 209–223. Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw, 1962.

Translation of Ž. Èksper. Teoret. Fiz. **38** (1960), 1476–1485; previously translated as Soviet Physics JETP **11** (1960), 1067–1072 [MR **22** #7822].

F. A. E. Pirani (London)

Ehlers, Jürgen 2014

Transformations of static exterior solutions of Einstein's gravitational field equations into different solutions by means of conformal mappings. (French summary)

Les théories relativistes de la gravitation (Royaumont, 1959), pp. 275–284. Éditions du Centre National de la Recherche Scientifique, Paris, 1962.

It is shown that the determination of static vacuum solutions of Einstein's field equations amounts to the determination of Riemannian 3-spaces with Ricci tensors of the form (*) $R_{ab} = -2U_{,a}U_{,b}$ (where U is some scalar). Buchdahl's transformation of static vacuum fields [Quart. J. Math. Oxford Ser. (2) **5** (1954), 116–119; MR **16**, 531], and a construction of stationary vacuum fields from static fields are deduced, and it is shown that the determination of interior solutions in which incoherent matter moves rigidly is also equivalent to (*) {for a comprehensive treatment see Ehlers and Kundt [*Gravitation*, pp. 49–101, Wiley, New York, 1962; MR **26** #1177]}.

F. A. E. Pirani (London)

Ozsváth, I.; Schücking, E. 2015

An anti-Mach metric.

Recent developments in general relativity, pp. 339–350. Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw, 1962.

The authors exhibit a solution of Einstein's vacuum field equations which is non-singular for all finite values of the coordinates, which is complete with respect to its geodesics, which admits a transitive group of motions, and which has non-vanishing Riemann tensor (of Petrov type N). This solution contradicts Mach's principle in the form "In the absence of matter, space-time should

necessarily be Minkowskian" [the reviewer, Jubilee of Relativity Theory (Bern, 1955), Proc., pp. 198–203, Helv. Phys. Acta, Suppl. IV, Birkhäuser, Basel, 1956; MR **19**, 506]. *F. A. E. Pirani* (London)

Hoffmann, B. 2016a

On the extension of Birkhoff's theorem to the case in which an electromagnetic field is present.

Recent developments in general relativity, pp. 279–282. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw*, 1962.

Bonnor, W. B. 2016b

On Birkhoff's theorem.

Recent developments in general relativity, pp. 167–169. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw*, 1962.

Bertotti, B. 2016c

The uniform electromagnetic field in the theory of general relativity. (French summary)

Les théories relativistes de la gravitation (Royaumont, 1959), *pp.* 249–251. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1962.

Birkhoff's theorem states that a spherically symmetric vacuum field must be static [G. D. Birkhoff, *Relativity and modern physics*, p. 253, Harvard Univ. Press, Cambridge, Mass., 1923; J. Eiesland, Trans. Amer. Math. Soc. **27** (1925), 213–245]. Hoffmann generalised this to Einstein-Maxwell fields [Quart. J. Math. Oxford Ser. **3** (1932), 226–237; ibid. **4** (1933), 179–183]. In the paper under review he corrects and completes his proof, and incidentally recovers as a singular case the uniform electromagnetic field of Bertotti.

Bertotti's paper is a summary of his previously published work on this uniform field [Phys. Rev. (2) **116** (1959), 1331–1333; MR **22** #1401].

Bonnor in his paper gives another proof of Birkhoff's theorem in a slightly different form; he excludes as unphysical the vacuum limit of Bertotti's field.

F. A. E. Pirani (London)

Rosen, G. 2017

Interpretation of Rainich geometry.

Recent developments in general relativity, pp. 387–393. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw*, 1962.

Discussion of the local geometrical significance, in terms of blades formed from Ricci principal directions, of the algebraic and differential conditions which must be satisfied by a solution of the Einstein-Maxwell equations. {Results extracted from the author's paper [Phys. Rev. (2) **114** (1959), 1179–1181; MR **21** #5482].}

F. A. E. Pirani (London)

Ehlers, J. 2018

Relativistic hydrodynamics and its relation to interior solutions of the gravitational field equations.

Recent developments in general relativity, pp. 201–207. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw*, 1962.

Outline of the kinematics of (generally) relativistic fluids, and brief consideration of a dust fluid, with examples. {For details, and many more results, see Jordan, Ehlers, Kundt and Sachs [Akad. Wiss. Lit. Mainz Abh. Math.-Natur. Kl. **1961**, 791–837; MR **27** #6602]. Compare also the following paper by Taub, #2019a].}

F. A. E. Pirani (London)

Taub, A. H. 2019a

On spherically symmetric distributions of incompressible fluids.

Recent developments in general relativity, pp. 449–457. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw*, 1962.

Taub, A. H. 2019b

Small motions of a spherically symmetric distribution of matter. (French summary)

Les théories relativistes de la gravitation (Royaumont, 1959), *pp.* 173–191. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1962.

In his Festschrift paper the author studies small spherically symmetric isentropic motions of a perfect fluid in general relativity. A perfect fluid motion is determined by Einstein's field equations $G^{ab}=-\kappa T^{ab}$, with energy tensor $T^{ab}=(\rho+p+\varepsilon)u^a u^b-pg^{ab}$, where u^a is the 4-velocity, ρ is the proper density, p the pressure, and $\varepsilon=\varepsilon(p,\rho)$ is the proper internal energy density. For isentropic motion, proper mass is conserved: $(\rho u^a)_{\|a}=0$. The spherical symmetry is introduced by assuming an appropriate form of metric, and the small motions by a perturbation method. The results include the recovery, in the non-relativistic limit, of a classical equation for adiabatic oscillations given by Rosseland [*The pulsation theory of variable stars*, Oxford Univ. Press, New York, 1949].

In his Royaumont report, he shows (without the restriction to small motions) that if, in addition, the fluid is incompressible ($\varepsilon=0 \Leftrightarrow \rho=$ const), then the solution must be static (Schwarzschild interior solution), as might be expected. *F. A. E. Pirani* (London)

Pham Mau Quan 2020

Le principe de Fermat en relativité générale.

Les théories relativistes de la gravitation (Royaumont, 1959), *pp.* 165–171. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1962.

Author's summary: "Dans un milieu de pouvoir diélectrique ε et de perméabilité magnétique μ, les rayons électromagnétiques sont géodésiques de longeur nulle de la variété riemannienne V_4 définie par la variété espace-temps munie de la métrique associée

$$d\bar{s}^2 = \left[g_{\alpha\beta}-\left(1-\frac{1}{\varepsilon\mu}\right)u_\alpha u_\beta\right] dx^\alpha dx^\beta$$

où $g_{\alpha\beta}$ est le tenseur métrique de l'espace-temps V_4 et u_α le vecteur vitesse unitaire.

"Si l'espace-temps V_4 admet un groupe connexe d'isométries globales à trajectoires orientées dans le temps et ne laissant invariant aucun point de V_4 et si le mouvement du milieu est permanent {le mouvement est permanent si le groupe d'isométries laisse invariant le

vecteur vitesse unitaire} la variété riemannienne associée $\bar{V}_4$ admet aussi un groupe d'isométries induit par celui de V_4. En projetant les géodésiques isotropes de $\bar{V}_4$ sur la variété quotient $\bar{V}_3$ de $\bar{V}_4$ par la relation d'équivalence définie par le groupe d'isométries, on obtient un théorème qui généralise le principe de Fermat en relativité générale.

"Dans le cas d'un espace-temps de Minkowski, ce théorème entraîne comme corollaire la formule relativiste de la composition des vitesses." [Cf. Pham Mau Quan, Arch. Rational. Mech. Anal. **1** (1957), 54–80; MR **19**, 926; Enseignement Math. (2) **4** (1958), 41–70; MR **20** #5064.] *F. A. E. Pirani* (London)

Fourès-Bruhat, Y. 2021

Les fluides chargés en relativité générale.

Les théories relativistes de la gravitation (*Royaumont*, 1959), *pp.* 157–163. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1962.

Summary of results which have been published elsewhere by the author [C. R. Acad. Sci. Paris **246** (1958), 1809–1812; MR **21** #4035a; ibid. **246** (1958), 3319–3322; MR **21** #4035b; ibid. **248** (1959), 1782–1784; MR **21** #3251; ibid. **248** (1959), 2558–2560; MR **21** #1811; Bull. Soc. Math. France **86** (1958), 155–175; MR **21** #4036].

F. A. E. Pirani (London)

Synge, J. L. 2022

Relativity based on chronometry.

Recent developments in general relativity, pp. 441–448. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw*, 1962.

This and the next paper reviewed deal with the general relativity theory of measurement. Here the author gives a brief and lucid outline (without proofs) of some of the techniques used in his well-known exposition of general relativity [*Relativity*, North-Holland, Amsterdam, 1960; MR **22** #9230]. The essential idea is to describe space-time in terms of observations by an observer carrying a standard clock to measure $\int ds$ along his world-line; this observer refers optical signals and dynamical measurements to a reference triad which is propagated along the world-line by an operationally determined rule. The author remarks that calculations of this type may be much simplified by the use of the world-function $\frac{1}{2}(\int ds)^2$ (which is discussed in great detail in his book [compare C. De Witt, #2033a; Lichnerowicz and Fourès-Bruhat, #2042]).

F. A. E. Pirani (London)

Kundt, W.; Hoffmann, B. 2023

Determination of gravitational standard time.

Recent developments in general relativity, pp. 303–306. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw*, 1962.

The authors distinguish between "non-gravitational standard time", determined (in principle) by experiments with rigid rods and standard clocks, and "gravitational standard time", determined (in principle) by experiments with geodetically moving test particles. The question of correspondence between these two time scales can, in their view, be answered only by experiment. They describe an experiment with test particles to measure the metric tensor and certain Christoffel symbols in the neighbourhood of a world-line. *F. A. E. Pirani* (London)

Bażański, S. 2024

The problem of motion.

Recent developments in general relativity, pp. 13–29. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw*, 1962.

This and the next eight reviews are of papers dealing with various aspects of the problem of motion. Stated generally, the problem is to deduce from the field equations limitations on the motion of particles of given structure interacting with a gravitational field. Difficulties arise from the non-linearity of the field equations and (as in Lorentz-invariant theories) from the treatment of particles as singularities. Because of the non-linearity it seems impossible to find Green's functions for the field in closed form, and so attacks on the problem generally either confine themselves to test particles or make use of approximation methods.

In this review article, the author recounts the development of approximation methods since the paper of Einstein, Infeld and Hoffmann [Ann. of Math. (2) **39** (1938), 65–100], dwelling mainly on the slow-motion method of Infeld and his school. The article forms a useful brief introduction to the book on the same subject by Infeld and Plebański [*Motion and relativity*, Pergamon, New York, 1960; MR **23** #B980] and to a few papers published after the book went to press.

F. A. E. Pirani (London)

Bażański, S. 2025

The equations of motion and the action principle in general relativity.

Recent developments in general relativity, pp. 137–150. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw*, 1962.

The author first derives a condition which must be satisfied by the Maxwell-Einstein field quantities if it is to be possible to deduce from the field Lagrangian an action principle of the Fokker type (depending only on particle coordinates and velocities and not involving field quantities explicitly). He then employs the condition to construct, in a very concise manner, the Lagrangian for motion of charged gravitating pole particles in the post-Newtonian approximation [cf. Plebański and the author, Acta Phys. Polon. **18** (1959), 307–345; MR **22** #3546]. *F. A. E. Pirani* (London)

Goldberg, J. N. 2026

Conservation laws and equations of motion. (French summary)

Les théories relativistes de la gravitation (*Royaumont*, 1959), *pp.* 31–43. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1962.

The author has concerned himself for some time with solving the problem of motion without recourse to approximation methods. Here he outlines the construction of conservation laws in the canonical formalism [for which see also Geheniau, #2038] and shows how this formalism may be used to construct surface integral restrictions on the motion of sources. He gives a careful discussion of the difficulties of this approach [see also the author, #2027; *Gravitation*, pp. 102–129, Wiley, New York, 1962; MR **26** #1178; A. Trautman, ibid., pp. 169–198; MR **26** #1180; and also #2034]. *F. A. E. Pirani* (London)

Goldberg, J. N. 2027
Dynamical variables and surface integrals.
Recent developments in general relativity, pp. 241–250. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw,* 1962.

By imposing suitable coordinate conditions, the author shows how in the linearized Einstein theory one may distinguish dynamical variables, i.e., those whose time development is uniquely determined by the vacuum field equations. He constructs surface integrals which lead to the equations of motion, but these "impose no limitation on the quadrupole moment of the matter distribution". He concludes that such surface integrals describe essentially unchanging features of the field, and cannot adequately describe situations involving interactions or radiation.
F. A. E. Pirani (London)

Fock, V. [Fok, V.] 2028
Quelques remarques sur les équations du mouvement et les conditions pour les coordonnées.
Les théories relativistes de la gravitation (Royaumont, 1959), *pp.* 67–74. *Éditions du Centre National de la Recherche Scientifique, Paris,* 1962.

The author shows that, in the first two orders of approximation, the coordinate conditions employed by Einstein, Infeld and Hoffman are identical with the harmonic conditions introduced by himself [cf. *Theory of space, time and gravitation,* GITTL, Moscow, 1955; MR **18**, 445; English transl., Pergamon, New York, 1959; MR **21** #7042]. The author and others, in the discussion, deliver further polemics on the significance of the harmonic conditions, which are an old bone of contention between Fock and the Infeld school. *F. A. E. Pirani* (London)

Tulczyjew, B.; Tulczyjew, W. 2029
On multipole formalism in general relativity.
Recent developments in general relativity, pp. 465–472. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw,* 1962.

A. Papapetrou [Proc. Roy. Soc. London Ser. A **209** (1951), 248–258; MR **13**, 695] gave a method for describing bodies by their multipole moments, and investigated by this method the motion of a particle endowed with mass and internal spin [see also M. Mathisson, Acta Phys. Polon. **6** (1937), 163–200]. In their paper, the authors put Papapetrou's method into a comparatively covariant form by relating all calculations to a set of axes which is Fermi-propagated along an arbitrarily chosen world-line (these methods are suitable for studying the motion of test bodies which can be described by their first few multipole moments only). *F. A. E. Pirani* (London)

Weyssenhoff, Jean 2030
Modèles relativistes de particules à spin et lignes d'univers isotropes.
Les théories relativistes de la gravitation (Royaumont, 1959), *pp.* 57–66. *Éditions du Centre National de la Recherche Scientifique, Paris,* 1962.

The author modifies the Mathisson theory [see review of Tulczyjew and Tulczyjew, #2029] so that the singular world-line becomes a null line. In this way he seeks to avoid difficulties which arise with the appearance of equations of motion of order higher than the second.
F. A. E. Pirani (London)

Havas, P. 2031
General relativity and the special relativistic equations of motion of point particles.
Recent developments in general relativity, pp. 259–277. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw,* 1962.

Collecting together a variety of results of his own and others in this useful paper, the author "describes a general method of establishing the equations of motion of special relativity for multipole singularities of arbitrary fields on the basis of the linearized equations of general relativity. . . . It is shown first that the differential identities of the linearized equations . . . imply the overall law of conservation of energy-momentum of the special theory. From this law the general form of the equations of motion of multipole singularities of arbitrary fields is derived by use of a modified form of Mathisson's method . . ." {from the introduction}. *F. A. E. Pirani* (London)

Peres, A. 2032
Motion and radiation of pole particles.
Recent developments in general relativity, pp. 361–366. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw,* 1962.

The author describes the results of an approximation procedure in which the contravariant metric tensor density is expanded in a series subject to Fock's harmonic condition, while the energy tensor is a sum of monopole terms. The radiated energy, calculated in the 7th approximation, has the same value as in the linearized theory [for details see the author, Nuovo Cimento (10) **11** (1959), 617–627; MR **21** #3252; ibid. (10) **11** (1959), 644–655; MR **21** #3253; ibid. (10) **13** (1959), 439–441; MR **21** #7067; ibid. (10) **13** (1959), 670; ibid. (10) **15** (1960), 351–369; MR **22** #3550]. *F. A. E. Pirani* (London)

De Witt, Cécile 2033a
Grandeurs relatives à plusieurs points, tenseurs généralisés.
Les théories relativistes de la gravitation (Royaumont, 1959), *pp.* 327–333. *Éditions du Centre National de la Recherche Scientifique, Paris,* 1962.

De Witt, Bryce S. 2033b
Freinage dû à la radiation d'une particule dans un champ de gravitation.
Les théories relativistes de la gravitation (Royaumont, 1959), *pp.* 335–343. *Éditions du Centre National de la Recherche Scientifique, Paris,* 1962.

These two papers are concerned with the radiation reaction problem. The first paper outlines the calculus of two-point tensors ("bi-tensors") in Riemannian space-time. This calculus was developed primarily with the object of constructing covariant Green's functions for wave equations in arbitrary Riemannian space-times. The second paper summarises the author's work with Brehme, making use of bi-tensor techniques on the determination

of the motion of a charged particle in gravitational and electromagnetic fields, and discusses the principle of equivalence in the light of their results [B. S. De Witt and R. W. Brehme, Ann. Physics **9** (1960), 220–259; MR **23** #B1528]. This work is mathematically very close to Lichnerowicz's study of propagators [for which see Lichnerowicz, Inst. Hautes Études Sci. Publ. Math. No. 10 (1961); MR **28** #967; Lichnerowicz and Fourès-Bruhat, #2042] and to Synge's theory of the world-function [see Synge, #2022]. *F. A. E. Pirani* (London)

Trautman, A. 2034

Sur les lois de conservation dans les espaces de Riemann.

Les théories relativistes de la gravitation (*Royaumont*, 1959), *pp.* 113–177. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1962.

The lack of suitable preferred coordinate systems, the non-existence of isometries in general Riemannian space-times, and the identity of mass with "gravitational charge", among other reasons, have led to difficulties in the definition of energy and in the construction of conservation laws for energy and momentum of the gravitational field. This and the following seven papers under review describe a variety of approaches to the resolution of these difficulties [see also Goldberg, #2027].

The author gives a brief review of his work on covariant conservation laws, particularly in space-times which do admit groups of motions [compare the author, Bull. Acad. Polon. Sci. Cl. III **4** (1956), 679–682; MR **19**, 219; ibid. **5** (1957), 721–727; MR **19**, 815; *Gravitation*, pp. 169–198, Wiley, New York, 1962; MR **26** #1180]. The paper concludes with a short critique of the interpretation of Einstein's and Møller's energy expressions, to which Møller replied in the discussion.

F. A. E. Pirani (London)

Møller, C. 2035

The energy-momentum complex in general relativity and related problems. (French summary)

Les théories relativistes de la gravitation (*Royaumont*, 1959), *pp.* 15–29. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1962.

The author surveys attempts by himself [Ann. Physics **4** (1958), 347–371; MR **20** #732; Mat.-Fys. Medd. Danske Vid. Selsk. **31** (1959), no. 14; MR **21** #4811] and Magnusson [ibid. **32** (1960), no. 6; MR **22** #4509] to construct a satisfactory definition of energy density. However, he states in a summary (added later) that he has now revised his views on this subject [compare Møller, Mat.-Fys. Skr. Danske Vid. Selsk. **1** (1961), no. 10; MR **25** #3772].

He then discusses quantization of the gravitational field, and proposes that experimental requirements may be met by regarding the components of the metric tensor as c-numbers and writing Einstein's gravitational field equations in the form

$$G_a{}^b = -\kappa\langle T_a{}^b\rangle,$$

where the crocheted expression on the right is the expectation value of the matter tensor. {In the discussion following the presentation of his paper, the author agreed that if there exist energy-carrying gravitational waves, then objections of the Bohr-Rosenfeld type could be raised against this procedure, which might be expected to come into conflict with the uncertainty principle.}

F. A. E. Pirani (London)

Pirani, F. A. E. 2036

Gauss's theorem and gravitational energy. (French summary)

Les théories relativistes de la gravitation (*Royaumont*, 1959), *pp.* 85–91. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1962.

This paper is along similar lines to Møller's [see preceding review, #2035]. From the author's summary: "Komar's expression for the gravitational energy flux density [A. Komar, Phys. Rev. (2) **113** (1959), 934–936; MR **21** #1196] is deduced from Gauss's theorem for gravitational mass in an arbitrary system of reference. A new condition is obtained which makes it possible to avoid certain difficulties in Komar's method" {but not all of them. It now seems to the reviewer rather doubtful whether the expression for gravitational energy dirived in this paper is of any use}. *F. A. E. Pirani* (London)

Synge, J. L. 2037

Tensorial integral conservation laws in general relativity.

Les théories relativistes de la gravitation (*Royaumont*, 1959), *pp.* 75–83. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1962.

The approach of this paper is related to that of the preceding two [Møller, #2035; Pirani, #2036]. The author also works with a covariant form of Gauss's theorem in a Riemannian space-time, with integrands constructed from the Riemann tensor and the world-function [see Synge, #2022]. By appealing to the weak-field approximation he identifies two of the resulting (tensorial) integrals as the flux of 4-momentum and of angular momentum of a matter distribution (both integrals vanish in empty space-time). *F. A. E. Pirani* (London)

Geheniau, J. [Géhéniau, J.] 2038

Remarques sur les "pseudo-tenseurs" et les identités satisfaites par un lagrangien.

Les théories relativistes de la gravitation (*Royaumont*, 1959), *pp.* 231–236. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1962.

The author's approach is related to that of Møller [#2035] and also to that of Goldberg [#2027]. He rederives identities satisfied by the derivatives of a scalar density [the author, Acad. Roy. Belg. Bull. Cl. Sci. (5) **28** (1942), 118–129; MR **7**, 183; ibid. (5) **45** (1959), 447–450; MR **21** #7783] and retrieves results of Goldberg, Møller, and others. *F. A. E. Pirani* (London)

Dirac, P. A. M. 2039

The energy of the gravitational field. (French summary)

Les théories relativistes de la gravitation (*Royaumont*, 1959), *pp.* 385–394. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1962.

The author's definition of energy density, within the framework of the Hamiltonian formalism [see Bergmann

and Komar, #2043, and the following 8 papers], is selected by the criterion of utility, and modified by a coordinate condition. He discusses some difficulties of his procedure, and gives several examples, including a calculation of the energy density for unidirectional waves. (A good deal of the same ground is covered in his American Physical Society Lecture [Phys. Rev. Lett. **2** (1959), 368–371; see also Proc. Roy. Soc. London Ser. A **246** (1958), 333–343 MR **20** #725].) *F. A. E. Pirani* (London)

Bel, L. 2040

La radiation gravitationnelle.

Les théories relativistes de la gravitation (Royaumont, 1959), *pp.* 119–126. *Éditions du Centre National de la Recherche Scientifique, Paris,* 1962.

This paper belongs to a different genre from those reviewed just above. Depending on the analogies between Riemann tensor and Maxwell tensor, rather than on identification of some kind of connection as the gravitational field, the author constructs from the vacuum Riemann tensor R^{abcd} and its (left) dual ${}^*R^{abcd}=\frac{1}{2}\eta^{ab}{}_{rs}R^{rscd}$ a trace-free totally symmetric tensor ("Bel-Robinson tensor", or "super-energy tensor")

$$T^{abcd} = \tfrac{1}{2}(R^{arcs}R^{b}{}_{r}{}^{d}{}_{s}+{}^*R^{arcs}{}^*R^{b}{}_{r}{}^{d}{}_{s})$$

with vanishing covariant divergence: $\nabla_a T^{abcd}=0$. The "gravitational super-energy" $T^{abcd}u_a u_b u_c u_d$ defined by any timelike vector u_a is shown to be non-negative, vanishing only when $R^{abcd}=0$. Certain algebraic properties of T^{abcd} distinguishing the various Petrov types are noted, and three tentative definitions of gravitational radiation put forward. {For a more extensive treatment by the same author, see, for example, his paper [Cahiers de Phys. **16** (1962), 59–81; MR **25** #1927; see also Lichnerowicz and Fourès-Bruhat, #2042].} *F. A. E. Pirani* (London)

Bondi, H. 2041

On the physical characteristics of gravitational waves. (French summary)

Les théories relativistes de la gravitation (Royaumont, 1959), *pp.* 129–135. *Éditions du Centre National de la Recherche Scientifique, Paris,* 1962.

The author approaches the problem of defining gravitational energy via Newtonian analogues. He directs his attention to the process of energy transfer, rather than the mere definition of energy density. First he reviews his work with McCrea [Proc. Cambridge Philos. Soc. **56** (1960), 410–413; MR **22** #10730] on transfer of energy in Newtonian gravitation theory, and then he discusses in general terms{but very entertainingly} transfer by classical radiative processes. *F. A. E. Pirani* (London)

Lichnerowicz, A.; Fourès-Bruhat, Y. 2042

Problèmes mathématiques en relativité.

Recent developments in general relativity, pp. 73–87. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw,* 1962.

A brief, lucid survey, with a good deal of motivation, of problems studied by the authors, indicating progress to date and outstanding questions. The principal problems are the problem of initial conditions and the problem of evolution for Einstein's field equations, for the cases energy-tensor$=0$ (cas extérieur) and energy-tensor$\neq 0$, the latter usually restricted to electromagnetic fields and perfect fluids {the problem of evolution is of course related to the problem of motion [see Bażański, #2024], and the problem of initial conditions is at the root of the difficulties of devising a satisfactory canonical formalism [see Bergmann and Komar, #2043]}.

They mention also certain global problems, including the question "Si un modèle d'univers est extérieur et regulier partout (donc vide) est-il localement euclidien?", which has been answered in the negative, at least to the reviewer's satisfaction, by Ozsváth and Schücking [see #2015]. *F. A. E. Pirani* (London)

Bergmann, Peter G.; Komar, Arthur B. 2043

Status report on the quantization of the gravitational field.

Recent developments in general relativity, pp. 31–46. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw,* 1962.

This and the next eight papers reviewed are mainly concerned with construction of a satisfactory canonical formalism for the classical gravitational field as a natural preliminary to quantization. Various difficulties arise from the general covariance of the theory (and its consequent non-linearity): for example, it is difficult to select physically significant variables which may be prescribed freely in the setting of initial data; generally, some partial differential equations have to be satisfied on an initial spacelike surface.

This review article is primarily an account of the search for observables. It constitutes a useful introduction to the work of Bergmann's group and some others, including discussions of the concept of constants of the motion, of the imposition of coordinate conditions, of intrinsic coordinates, and of the restoration of time-dependence, which may be frozen out of the equations by imposing sufficiently severe demands on the notion of "observable".

F. A. E. Pirani (London)

Bergmann, Peter G.; Komar, Arthur B. 2044

Observables and commutation relations. (French summary)

Les théories relativistes de la gravitation (Royaumont, 1959), *pp.* 309–325. *Éditions du Centre National de la Recherche Scientifique, Paris,* 1962.

The authors outline their programme of canonical formulation, and give some details of a procedure, not then complete, for the construction of locally defined observables from the Weyl tensor [compare, for a related approach, Bergmann and Komar, Phys. Rev. Lett. **4** (1960), 432–433]. {However, Bergmann later (in Aeronaut. Res. Lab. Rep. ARL 63-56 (1963)) wrote that the resulting expressions for Poisson brackets "though in closed form, are so involved as to appear useless".}

F. A. E. Pirani (London)

Anderson, James L. 2045

Generation of coordinate conditions and the construction of invariants in covariant theories. (French summary)

Les théories relativistes de la gravitation (Royaumont, 1959), *pp.* 373–384. *Éditions du Centre National de la Recherche Scientifique, Paris,* 1962.

The author describes his attempts to construct observables

by introducing conditions strong enough to make the coordinate system unique. He is able in the case of electrodynamics to carry out the analogous procedure exactly, but in the gravitational case approximation methods seem essential. *F. A. E. Pirani* (London)

Anderson, James L. 2046
Absolute change in general relativity.
Recent developments in general relativity, pp. 121–126. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw,* 1962.

The author attempts unfreezing of a canonical formalism frozen by the demand that observables be absolute invariants (i.e., numbers which are functionals of the Riemannian manifold, for example, the maximum value of a given scalar differential invariant in a given domain). He describes a procedure of Dirac's [Phys. Rev. (2) **114** (1959), 924–930; MR **21** #5483] for obtaining an unfrozen dynamics, and gives a summary of the method of his Royaumont paper [see #2045].

He defines "free variables" to be the (four) variables left over after the elimination of the variables for which the equations of constraint have been solved, and their canonical conjugates. Then absolute change is said to take place if there exists no coordinate system in which the free variables are constants of the motion. He discusses whether this formalism imposes limitations on the coordinate system. *F. A. E. Pirani* (London)

Dirac, P. A. M. 2047
Interacting gravitational and spinor fields.
Recent developments in general relativity, pp. 191–200. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw,* 1962.

Using the Vierbein ("four-leg") formalism, the author derives a Lagrangian for spinor field equations of the form given by him previously [*Max-Planck-Festschrift, 1958*, pp. 339–344, VEB. Deutscher Verlag Wiss., Berlin, 1959], and then by previously developed techniques [the author, Proc. Roy. Soc. London Ser. A **246** (1958), 333–343; MR **20** #725] he obtains a Hamiltonian for the interacting fields of the title, as well as the corresponding constraint equations.
F. A. E. Pirani (London)

Goldberg, I. 2048
On the quantum theory of gravitation.
Recent developments in general relativity, pp. 235–240. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw,* 1962.

The author derives the primary constraint equations after making Fourier transforms of the field variables and the Lagrangian. The secondary constraint calculations have also apparently been carried out, but are not reproduced.
F. A. E. Pirani (London)

DeWitt, B. S. 2049
Invariant commutators for the quantized gravitational field.
Recent developments in general relativity, pp. 175–189. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw,* 1962.

Formal development of a generally covariant dynamics, based on the theorem of classical transformation theory of which the author has previously [Rev. Modern Phys. **29** (1957), 377–397; MR **20** #2193] given a proof: "The variation in the functional form of the generator of a finite canonical transformation, due to independent infinitesimal canonical transformations of its arguments, is equal to the difference of the associated independent infinitesimal generators". The variations are expressed in terms of "bi-tensor" Green's functions [see C. De Witt, #2033a; B. S. De Witt, #2033b], and Poisson brackets between absolute invariants are constructed. Problems of the transition to quantum theory, including the problem of factor sequences, are also discussed.
F. A. E. Pirani (London)

Arnowitt, R. L.; Deser, S.; Misner, C. W. 2050
Canonical analysis of general relativity.
Recent developments in general relativity, pp. 127–136. *Pergamon, Oxford; PWN—Polish Scientific Publishers. Warsaw,* 1962.

Brief survey of the authors' programme of canonical formulation. (It is surveyed by them at greater length in Chapter 7 of *Gravitation* (edited by Witten), pp. 227–265, Wiley, New York, 1962 [MR **26** #1182], where reference may be found to the long series of original papers.) By analogy with the parametrized form of particle mechanics, the gravitational field Lagrangian is written in first-order form. The constraint equations are solved in principle by decomposing the variables into transverse and longitudinal parts (of which the latter may be discarded) and imposing coordinate conditions. In the end the action integral is written in terms of four canonical variables. Different choices of coordinate conditions lead to different canonical variables. The definition of energy and the self-energy problem are discussed.
F. A. E. Pirani (London)

Deser, S. 2051
Dynamical structure and definition of energy in general relativity. (French summary)
Les théories relativistes de la gravitation (Royaumont, 1959), pp. 395–407. *Éditions du Centre National de la Recherche Scientifique, Paris,* 1962.

Shortened version of a paper published jointly with Arnowitt and Misner [Phys. Rev. (2) **116** (1959), 1322–1330; MR **22** #4505a] in their series on canonical formalism [see #2050 above]. *F. A. E. Pirani* (London)

Belinfante, F. J. 2052
Two kinds of Schrödinger equations in general relativity theory.
Recent developments in general relativity, pp. 151–166. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw,* 1962.

Here the author discusses the dependence of generally covariant Schrödinger equations on the choice of spacelike surfaces and coordinate frames in both the Schrödinger picture and the interaction picture. For the Lorentz-invariant theory compare F. J. Belinfante and J. S. Lomont [Phys. Rev. (2) **84** (1951), 541–546; MR **13**,

1010]; F. J. Belinfante, D. I. Caplan, and W. L. Kennedy [Rev. Modern Phys. **29** (1957), 518–546; MR **19**, 810].
F. A. E. Pirani (London)

Belinfante, F. J. 2053
Work at Purdue University on the interaction of gravitation and Fermions. (French summary)
Les théories relativistes de la gravitation (*Royaumont*, 1959), *pp.* 345–357. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1962.
The author thinks that to a great extent he can by-pass the problem of constraints by the "muddification" process [F. J. Belinfante, D. I. Caplan, and W. L. Kennedy, Rev. Modern Phys. **29** (1957), 518–546; MR **19**, 810], although the problem of finding observables would remain. He discusses the alternative energy quasi-tensors which have been proposed for the gravitational field, the Schrödinger operators which can be formed from them, and the possibility of constructing an interaction picture with a unitary operator which depends on the choice of spacelike surface, but not on its parametrization. He relates his work to that of A. Komar [Phys. Rev. (2) **113** (1959), 934–936; MR **21** #1196].
F. A. E. Pirani (London)

Laurent, B. E. 2054
On a generally covariant quantum theory.
Recent developments in general relativity, pp. 323–332. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw*, 1962.
Identical with the author's earlier paper of the same title [Ark. Fys. **16** (1960), 237–245; MR **22** #2367].
F. A. E. Pirani (London)

Gupta, Suraj N. 2055
Quantum theory of gravitation.
Recent developments in general relativity, pp. 251–258. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw*, 1962.
This paper is a report of an invited address to the American Physical Society (on 29 January 1960). The author first summarises, without derivations, his earlier paper on the construction of gravitation theories, in particular, on the choice of Lagrangian [Rev. Modern Phys. **29** (1957), 334–336; MR **19**, 1022]. Then he outlines his work on the quantization of Einstein's field equations by expansion of the Lagrangian in powers of the gravitation constant (in flat space-time) [Proc. Phys. Soc. Sect. A **65** (1952), 161–169; MR **13**, 804; ibid. **65** (1952), 608–619; MR **14**, 417]. In his view, the success of his method, which he contrasts with the difficulties facing curved-space approaches, "strongly suggests that the space is flat and not curved" {the reviewer finds it difficult to understand this inference}.
F. A. E. Pirani (London)

Ivanenko, D. 2056
On the possible transmutations of ordinary matter in gravitation. (French summary)
Les théories relativistes de la gravitation (*Royaumont*, 1959), *pp.* 433–439. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1962.
The author writes down commutation relations for the quantization of the transverse part of the linearized gravitational field in interaction with a scalar field, and computes the cross-section for conversion of scalar particles (and also photons) into gravitons. He discusses several other possible procedures.
F. A. E. Pirani (London)

Scarf, Frederick L. 2057
A soluble quantum field theory in curved space. (French summary)
Les théories relativistes de la gravitation (*Royaumont*, 1959), *pp.* 421–428. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1962.
The author studies the equation $i\gamma^\mu\nabla_\mu\psi+2\lambda(\bar{\psi}\psi)\psi=0$ (a generalisation of Thirring's model [Ann. Physics **3** (1958), 91–112; MR **19**, 1016]) in a given 2-dimensional (therefore conformally flat) hyperbolic space. He finds that "(a) The definition of a state vector contains quantities which have meaning in all coordinate systems, but which must be evaluated in particular frames of reference" and "(b) For a large class of $g_{\mu\nu}$, matter is created by the metric".
F. A. E. Pirani (London)

Sciama, D. W. 2058
On the analogy between charge and spin in general relativity.
Recent developments in general relativity, pp. 415–439. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw*, 1962.
From the author's summary: "We . . . suggest that the spin tensor of a material system should be defined as the quantity coupled to the (Vierbein) affine connection $o_p(\alpha\beta)$. It follows from this definition that (a) spin gives rise to a skew part of the affine connection $\Gamma_{qr}{}^p$, (b) a particle with spin is acted on by a gravitational force which depends on the spin and on the curvature tensor. . . . A weak form of the {equivalence} principle is favoured, namely, that inertial forces are gravitational in origin. . . . We need in addition the principle of minimal gravitational coupling, which states that the entire gravitational coupling is obtained from the material Lagrangian of special relativity by replacing ordinary by covariant derivatives. This principle . . . accounts for the Galileo-Eötvös experiments.

"The main purpose of this paper is to point out that the theory of spin in general relativity is closely analogous to the theory of charge in electromagnetism . . . there is a minimal coupling principle {in electromagnetic theory} which states that the Lagrangian should contain no direct coupling of matter to electromagnetic field (no Pauli terms) . . . this principle . . . predicts that strangeness should be conserved in electromagnetic transitions of strange particles—a prediction which is in agreement with experiment. This experimental result is thus the experimental analogue of the Galileo-Eötvös result.

"In view of the close analogy between spin and the Vierbein connection on the one hand, and charge and the phase connection on the other, we believe that the phase connection should be made part of the geometrical structure of space" [see D. W. Sciama, Nuovo Cimento (10) **8** (1958), 417–431; MR **20** #744; see also T. W. B. Kibble, J. Mathematical Phys. **2** (1961), 212–221; MR **23** #B997].
F. A. E. Pirani (London)

Tonnelat, M.-A. [Tonnelat, Marie-Antoinette] 2059
Étude critique de la représentation de la matière dans la théorie asymétrique du champ unifié.
Les théories relativistes de la gravitation (Royaumont, 1959), pp. 199–223. Éditions du Centre National de la Recherche Scientifique, Paris, 1962.

Translated extract from the author's summary, with explanatory additions: The author describes in some detail the unified field theories of Einstein and Schrödinger based on a non-symmetric fundamental tensor [compare M.-A. Tonnelat, *La théorie du champ unifié d'Einstein et quelques-uns de ses développements*, Gauthier-Villars, Paris, 1955; MR **17**, 907]. She discusses the difficulties which arise in these theories when one tries to deduce the Lorentz force on a charged particle. This discussion leads her to develop a very general non-symmetric theory from which the Lorentz force may be correctly deduced, but this is achieved at the cost of partial renunciation of the use of a variational principle. *F. A. E. Pirani* (London)

Tonnelat, M. A. [Tonnelat, Marie-Antoinette] 2060
Les espoirs et les difficultés de la théorie du champ unifié d'Einstein.
Recent developments in general relativity, pp. 107–118. Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw, 1962.

The author reproduces her solution [C. R. Acad. Sci. Paris **231** (1950), 470–472; MR **12**, 291; ibid. **231** (1950), 487–489; MR **12**, 291; ibid. **231** (1950), 512–514; MR **12**, 292; J. Phys. Radium (8) **12** (1951), 81–88; MR **13**, 79; ibid. (8) **16** (1955), 21–38; MR **16**, 755] of the equations for the connection in Einstein's unified field theory [A. Einstein, Ann. of Math. (2) **46** (1945), 578–584; MR **7**, 266; A. Einstein and E. G. Straus, ibid. (2) **47** (1946), 731–741; MR **8**, 412] and indicates some {serious} difficulties connected with the physical interpretation of the field equations [compare Tonnelat, Nuovo Cimento (10) **3** (1956), 902–920; MR **18**, 177]. *F. A. E. Pirani* (London)

Thiry, Yves 2061
Sur les théories pentadimensionnelles.
Les théories relativistes de la gravitation (Royaumont, 1959), pp. 285–291. Éditions du Centre National de la Recherche Scientifique, Paris, 1962.

The author briefly summarizes the 5-dimensional Jordan-Thiry and Kaluza-Klein theories [cf. A. Lichnerowicz, *Théories relativistes de la gravitation et de l'électromagnétisme*, Masson, Paris, 1955; MR **17**, 199, for a detailed exposition] and discusses the definition of radiation in these theories in terms of surfaces of discontinuity and characteristic vectors. In the discussion he remarked "A mon avis aucune observation ne peut a ce jour infirmer ou confirmer la théorie que je viens d'exposer" {which may go a long way to explain the decline of interest in unified field theory—to which the still-flourishing school in Paris is almost the only exception—since about the time of Einstein's death in 1955}.
F. A. E. Pirani (London)

Droz-Vincent, M. 2062
Quantification de la théorie de Jordan-Thiry.
Les théories relativistes de la gravitation (Royaumont, 1959), pp. 127–128. Éditions du Centre National de la Recherche Scientifique, Paris, 1962.

Thiry's theory [see the preceding review, #2061] is linearized, under certain coordinate conditions, and commutation relations are written down for the resulting field quantities, which, according to the author, variously represent photons and gravitons.
F. A. E. Pirani (London)

Souriau, Jean-Marie 2063
Relativité multidimensionnelle non stationnaire.
Les théories relativistes de la gravitation (Royaumont, 1959), pp. 293–297. Éditions du Centre National de la Recherche Scientifique, Paris, 1962.

In order to describe the invariance properties of physical fields, the author considers, instead of space-time V_4, a manifold V_n, $n>4$. Usually all the fields are supposed invariant under a group of transformations $\mathfrak{G}$ of V_n; then $V_4=V_n|\mathfrak{G}$. The author finds this principle of "stationarity" unsatisfactory because it does not permit a sufficiently intimate unity of fields. He proposes to replace it by the assumption that $V_n=R^4\otimes W$, where W is a compact manifold of dimension $n-4$. He discusses, as an example, a "quasi-stationary" case, in which the field equations are linearized in the neighbourhood of a stationary solution. *F. A. E. Pirani* (London)

Kursunoglu, Behram [Kurşunoğlu, Behram] 2064
Fields, particles and quantum theory. (French summary)
Les théories relativistes de la gravitation (Royaumont, 1959), pp. 359–372. Éditions du Centre National de la Recherche Scientifique, Paris, 1962.

"The contents of this paper consist of some further developments of the author's version of Einstein's generalized theory of gravitation" [Phys. Rev. (2) 88 (1952), 1369–1379; MR **14**, 805]. The theory depends on a dimensional parameter, and reduces to the Einstein-Maxwell theory if the parameter is set equal to zero (eigenvalues of the parameter are to be interpreted as corresponding to masses).

Working by analogy with Lorentz-invariant theory, the author derives two action functions for his theory and discusses several examples. Publication of further results is promised. *F. A. E. Pirani* (London)

Costa de Beauregard, O. 2065
Quelques remarques d'analyse dimensionnelle pouvant intéresser les futures théories unitaires.
Les théories relativistes de la gravitation (Royaumont, 1959), pp. 303–308. Éditions du Centre National de la Recherche Scientifique, Paris, 1962.

Motivated by the Machian suggestion of Sciama [Monthly Notices Roy. Astronom. Soc. **113** (1953), 34–42; MR **15**, 566] and Park [J. Phys. Radium (8) **18** (1957), 11–16] that the strength of the gravitational interaction should be determined by the general distribution of matter in the universe, the author makes a formal comparison of the Lagrangian for interacting spinor and (linearized) gravitational fields, on the one hand, with that for interacting spinor and electromagnetic fields, on the other, whence, with the aid of some dimensional considerations, he proposes to identify $\hbar c$ as the square of the "gravitational

charge". He makes various remarks about the mass spectrum of the elementary particles and the definition of a fundamental length. *F. A. E. Pirani* (London)

Stephenson, G. 2066

Generally covariant variational principles. (French summary)

Les théories relativistes de la gravitation (*Royaumont*, 1959), *pp.* 225–229. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1962.

The metric tensor g_{pq} and affine connection Γ^r_{pq}, both assumed symmetric, are varied independently. Previous results [G. Stephenson, Nuovo Cimento (10) **9** (1958), 263–269; MR **20** #7571; P. W. Higgs, ibid. (10) **11** (1959), 816–820; MR **21** #4033] are summarized, and the variational principles $\delta\int F(R)(-g)^{1/2}\,d\tau=0$, $\delta\int F(S)(-g)^{1/2}\,d\tau=0$ are considered for various forms of the functions $F(R)$, $F(S)$, where $R=g^{km}R^p{}_{kmp}$, $S=g_{pq}g^{kl}g^{mn}g^{rs}R^p{}_{kmr}R^q{}_{lns}$, and $R^p{}_{kmr}=2(\partial_{[m}\Gamma_{r]}{}^p{}_k+\Gamma_s{}^p{}_{[m}\Gamma_{r]}{}^s{}_k)$ [cf. G. Stephenson, Proc. Cambridge Philos. Soc. **56** (1960), 247–251; MR **22** #11952; H. A. Buchdahl, ibid. **56** (1960), 396–400; MR **22** #11951].

F. A. E. Pirani (London)

Schild, A. 2067

Conservative gravitational theories of Whitehead's type.

Recent developments in general relativity, pp. 409–413. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw*, 1962.

The author writes down an action principle for an action-at-a-distance theory of gravitation in flat (Minkowski) space-time, which resembles Whitehead's theory [A. N. Whitehead, *The principle of relativity with applications to physical science*, Cambridge Univ. Press, London, 1922; J. L. Synge, Proc. Roy. Soc. London Ser. A **211** (1952), 303–319; MR **14**, 99], but which incorporates conservation laws for energy, linear momentum and angular momentum. The (tensor) interaction between particles is half-advanced, half-retarded (whereas in Whitehead's theory it is purely retarded). Two arbitrary parameters are present; assuming the Newtonian limit fixes one relation between them, and assuming also Einstein's value for the perihelion motion fixes them both uniquely. *F. A. E. Pirani* (London)

Jordan, P. 2068

Remarks about Ambarzumian's conception of pre-stellar matter.

Recent developments in general relativity, pp. 289–292. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw*, 1962.

Recognizing the substance of criticism by Pauli and by Fierz, the author now wishes his theory with a variable constant of gravitation [*Schwerkraft und Weltall*, zweite Aufl., Vieweg, Braunschweig, 1955; MR **17**, 1014] to be understood no longer as a theory of the creation of matter, but as a theory of "pre-stellar" matter (that is, of the material which subsequently condenses into stars and interstellar gas) which is supposed to exist in geometrical pockets in space-time, and to be converted into normal matter as time passes. *F. A. E. Pirani* (London)

Jordan, P. 2069

Empirical confirmation of Dirac's hypothesis of diminishing gravitation.

Recent developments in general relativity, pp. 283–288. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw*, 1962.

The author reviews some evidence, mainly geophysical, for the hypothesis, advanced tentatively by Dirac [Proc. Roy. Soc. London Ser. A **165** (1938), 199–208] that the gravitational "constant" is inversely proportional to the age of the universe. He has studied this evidence in greater detail elsewhere [P. Jordan, *Schwerkraft und Weltall*, zweite Auflage, Vieweg, Braunschweig, 1955; MR **17**, 1014; P. Jordan, J. Ehlers, W. Kundt, and R. K. Sachs, Akad. Wiss. Lit. Mainz Abh. Math.-Natur. Kl. **1961**, 791–837; MR **27** #6602]. *F. A. E. Pirani* (London)

Kilmister, C. W. 2070

The expression of field equations in terms of flux from sources. (French summary)

Les théories relativistes de la gravitation (*Royaumont*, 1959), *pp.* 45–55. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1962.

The author distinguishes "two-fields", in which the change of motion of test particles is independent both of the nature of the particles and of their motion, from "one-fields", in which this independence does not hold. He summarizes a development of the theory of Lorentz-invariant one-fields in terms of Eddington's E-numbers, or Dirac matrices (whose algebra is the direct square of quaternion algebra). Some Lorentz-invariant theories of gravitation and (almost) ordinary Maxwell theory appear as natural examples [cf. D. J. Newman, Proc. Roy. Irish Acad. Sect. A **59** (1958), 29–47; MR **21** #620; Mathematika **5** (1958), 146–160; MR **21** #3259].

The theory of two-fields, which may be developed in terms of the direct square of E-numbers, in turn leads naturally to field equations

$$R^{abcd}{}_{\|d}=J^{abc},\qquad R_{ab[cd\|e]}=0$$

for the tensor R^{abcd}, which is identified as the curvature tensor of Riemannian space-time. The source term J^{abc} remains to be identified. *F. A. E. Pirani* (London)

Ginzburg, V. L. 2071

Experimental verifications of the general theory of relativity.

Recent developments in general relativity, pp. 57–71. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw*, 1962.

The status of the three classical tests is surveyed from the viewpoint of September 1959; in addition, there are a few remarks on the effects of a rotating planet on its satellites and a short note added in proof which mentions the Pound Rebka experiment. (For more detail see Bertotti, Brill and Krotkov [*Gravitation*, pp. 1–48; Wiley, New York, 1962; MR **26** #1176].)

F. A. E. Pirani (London)

Weber, J. 2072

On the possibility of detection and generation of gravitational waves. (French summary)

Les théories relativistes de la gravitation (*Royaumont*, 1959), *pp.* 441–450. *Éditions du Centre National de la Recherche Scientifique, Paris*, 1962.

Very similar material, but with more details of calculations, is to be found in Chapter 8 of the author's book [*General relativity and gravitational waves*, Interscience, New York, 1961; MR **22** #10727], which in turn follows his ground-breaking paper [Phys. Rev. (2) **117** (1960), *306–313; MR* **22** *#1426*]. He proposes the use of linear mass quadrupoles and piezo-electric crystals for detection of gravitational waves, and of crystals and spinning rods for their generation, and discusses practical problems with care, but concludes that "a substantial gap still exists between what can be generated and what can be detected in a small laboratory", although he proposes also to search for gravitational waves arising from natural cosmical processes [cf. Forward, Zipoy, Weber, Smith and Benioff, Nature **189** (1961), 473; MR **22** #7828].

F. A. E. Pirani (London)

Costa de Beauregard, O. 2073

Une expérience macroscopique pour tester l'asymétrie du tenseur inertial des milieux doués de spin.

Recent developments in general relativity, pp. 171–174. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw,* 1962.

The test of the title requires a substance which exhibits a spin density on a macroscopic scale. The author proposes for this purpose to observe the oscillations of a small finite ring in an axial magnetic field. He does not give any estimates of practical orders of magnitude.

F. A. E. Pirani (London)

Bondi, H. 2074

Relativity and cosmology.

Recent developments in general relativity, pp. 47–55. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw,* 1962.

Short survey, mainly historical and without equations, of the relations between relativity and cosmology, seen from the author's characteristic viewpoint {which contrasts strongly with that of O. Klein [see #2075 following]}.

F. A. E. Pirani (London)

Klein, O. 2075

Mach's principle and cosmology in their relation to general relativity.

Recent developments in general relativity, pp. 293–302. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw,* 1962.

An illuminating qualitative discussion, leading to the conclusions that "there is hardly any argument for a direct meaning of the Mach principle in general relativity theory, nor any a priori argument in favour of any of the cosmological solutions". The arguments presented "point to the assumption that the world as we know it is to be regarded as a weak excitation of the vacuum state". {For other considerations about Mach's principle, see Bondi [#2074], Fabricius [#2077], Ozsváth and Schücking [#2015], and Costa de Beauregard [#2065].}

F. A. E. Pirani (London)

Gold, T. 2076

The arrow of time.

Recent developments in general relativity, pp. 225–234. *Pergamon, Oxford; PWN—Polish Scientific Publishers, Warsaw,* 1962.

This paper was published previously in the Report of the 1958 Solvay Conference [La Structure et l'Évolution de l'Univers (Inst. Internat. Phys. Solvay, Onzième Conseil Phys., Univ. Bruxelles, 1958), R. Stoops, Brussels, 1958].

It is pointed out that in an isolated system governed by time-symmetrical laws, a sense of time can never be defined with certainty; the uncertainty becomes nearly complete after the system has been isolated for a very long time. For a system interacting with its surroundings, on the other hand, a sense of time is defined by the capability of the universe at large to absorb any amount of radiation. This capability depends on the fact that the universe is expanding. The effects of the general expansion are observed on the small scale through the absorption of radiation. Thus, although the laws of nature may be time-symmetrical, the sense of time can be distinguished by the tendency of world-lines to spread out into the future.

The argument of the paper is qualitative; reference is made to a quantitative treatment by J. E. Hogarth [Proc. Roy. Soc. London Ser. A **267** (1962), 365–383; MR **27** #1293].

F. A. E. Pirani (London)

Fabricius, Charlotte 2077

Verallgemeinerter Thirring-Effekt zur Prüfung des Mach'schen Prinzips. (French summary)

Les théories relativistes de la gravitation (Royaumont, 1959), *pp.* 299–301. *Éditions du Centre National de la Recherche Scientifique, Paris,* 1962.

Following a suggestion of Hönl, the author considers an Einstein universe (closed static cosmological model) which is disturbed by assigning a (circular) velocity to a small piece of matter. The resulting perturbation of the motion of test particles, calculated in a slow motion linear approximation, is held to be more relevant to Mach's principle than previous calculations of the effects of a rotating shell in an otherwise empty space-time [H. Thirring, Phys. Z. **19** (1918), 33–39; ibid. **22** (1921), 29–30; H. Hönl and A.-W. Maue, Z. Physik **144** (1956), 152–167; MR **19**, 367; L. Bass and the reviewer, Philos. Mag. (7) **46** (1955), 850–856; MR **18**, 704].

F. A. E. Pirani (London)

McVittie, G. C. 2078

Cosmology and the interpretation of astronomical data. (French summary)

Les théories relativistes de la gravitation (Royaumont, 1959), *pp.* 253–274. *Éditions du Centre National de la Recherche Scientifique, Paris,* 1962.

From the author's introduction: "This paper deals with three problems in cosmology, namely, the applicability of uniform model universes to the observed universe, the present status of the expansion question, and the distribution in space of extragalactic Class II (faint) radio sources." In connection with the second problem, he concludes that ". . . a negative cosmical constant and an open hyperbolic space best fit the data" and that "A discrepancy between the observed and calculated values of (the present mean density) is inescapable on the steady-state theory". Data available up to (and including) the 1958 International Astronomical Union meeting, or thereabouts, are referred to.

In connection with the third problem, he concludes that the secular change in number density of these sources

cannot be accounted for simply by the expansion of the universe either in relativistic or in steady-state theories.
F. A. E. Pirani (London)

ASTRONOMY

See also 1865, 1927, 2068, 2069, 2071, 2072, 2074, 2075, 2076, 2077, 2078.

Veis, G. (Editor) 2079
★The use of artificial satellites for geodesy.
Proceedings of the First International Symposium on the Use of Artificial Satellites for Geodesy, Washington, D.C., April 26–28, 1962.
North-Holland Publishing Co., Amsterdam, 1963. xi+424 *pp.* (21 *plates*) $14.00.
Those papers of mathematical interest in the symposium described above will be reviewed individually.

Morrison, J. A. 2080
On the damping of a satellite motion.
Quart. Appl. Math. **22** (1964), 148–152.
Author's summary: "The motion in a circular orbit of a gravitationally oriented satellite, whose angular motions about the local vertical are damped by a roll-vee, gyro-stabilizer system, is considered, wherein the pitch axis of satellite remains perpendicular to the orbital plane. It is shown that no matter how large the initial local angular velocity of the satellite is, this velocity reaches any given smaller value in a finite time. It is also shown how it is possible to obtain a bound on this damping time."

Kovalevsky, Jean 2081
Sur le mouvement du satellite d'une planète à excentricité et à inclinaison quelconques.
C. R. Acad. Sci. Paris **258** (1964), 4435–4438.
The paper treats satellite theory when the ratio of the mean motion of the planet to that of the satellite is small and the eccentricity (e) and inclination (i) of the satellite orbit are arbitrary. A similar case was discussed by Delaunay in 1867 assuming small values for e and i.
Two classes of solutions are found, one in which the pericenter revolves around the planet and the other in which the pericenter performs librations around the maximum latitude. *V. Szebehely* (New Haven, Conn.)

Lagerstrom, P. A.; Kevorkian, J. 2082
Earth-to-moon trajectories with minimal energy.
J. Mécanique **2** (1963), 493–504.
This paper is an appendix to "Earth-to-moon trajectories in the restricted three-body problem" [same J. **2** (1963), 189–218; MR **28** #986] by the same authors. A detailed analytical discussion is offered of a singular perturbation problem within the framework of the restricted problem. Asymptotic formulas which are uniformly valid are derived by following a generalized boundary-value technique. The present paper treats the case when the energy content of a particle moving in the gravitational field of the earth and of the moon is minimum but sufficiently large so that trajectories exist connecting the neighborhood of the singularities.
V. Szebehely (New Haven, Conn.)

Genkin, I. L. 2083
The dynamics of the nonstationary galaxy.
Astronom. Ž. **40** (1963), 312–317 (*Russian*); *translated as Soviet Astronom. AJ* **7** (1963), 239–242.
Many of the observed properties of stellar motions have been explained by the dynamical theory of the steady-state galaxy, but the classical theory of stationary rotating stellar systems, which is based on only two integrals of motion, gives a value for the length of the z-axis of the velocity ellipsoid which is different from what is observed. Kuzmin formulated a theory for the third integral of stellar motion which satisfactorily explains the triaxiality of the velocity ellipsoid for the case of a stationary galaxy. The author generalizes Kuzmin's results for the dynamics of a nonstationary galaxy as well with success in the present paper. *A. C. Banerji* (Allahabad)

Hunter, C. 2084
The structure and stability of self-gravitating disks.
Monthly Notices Roy. Astronom. Soc. **126** (1963), 299–315.
Author's summary: "A method is presented for relating the law of rotation and density distribution of a plane disk of self-gravitating material. The method involves the use of oblate spheroidal coordinates, and the superposition of elementary solutions of Laplace's equation due to distributions of matter on the disk. General expressions for the law of rotation and the density-distribution are found as related series of Legendre functions. Simple particular disk-models involving only a finite number of terms in these series can be found, the most simple being the well-known one of a uniformly rotating disk with surface density $\sigma = \text{constant} \times (1 - r^2/R_0^2)^{1/2}$, where r is a radial coordinate and R_0 is the radius of the disk. A small-perturbation stability-analysis is carried out for this particular model by solving the zero-pressure hydrodynamic equations of compressible flow. The disk is found to be unstable to an infinite number of axisymmetric and non-axisymmetric modes of oscillation."

Hunter, C. 2085
The development of gravitational instability in a self-gravitating gas cloud.
Astrophys. J. **139** (1964), 570–586.
This is a continuation and correction of previous work by the author [same J. **136** (1962), 594–608; MR **26** #2313]. He considers an inviscid gas acted upon by gravity alone, and neglects pressure. Small perturbations are superposed on a spherically symmetric collapse, and it is shown that, according to linearised classical fluid dynamics, condensations can grow. Attention is then turned to nonlinear effects, and in particular to the question whether vorticity can prevent the growth of condensation. *W. B. Bonnor* (London)

Michie, Richard W. 2086
On the distribution of high energy stars in spherical stellar systems.
Monthly Notices Roy. Astronom. Soc. **125** (1963), 127–139.
Author's summary: "A distribution function $f(r, v, \mu; t)$ for isolated spherical stellar systems is obtained from the

Boltzmann equation with encounters described by the Fokker-Planck equation. The distribution function is believed to correspond rather closely to actual stellar systems since it is obtained from the Boltzmann equation, the potential is obtained from Poisson's equation, and the stellar orbits are not assumed to be isotropic everywhere but rather are more radial at greater distances from the centre. The paper emphasizes the importance of a careful analysis in the region of phase space at and near the energy of escape. In this region it is shown that the velocity space flux vector is constant, and it is this constancy which allows a solution for f. The distribution of high energy stars is depopulated for (i) those stars whose mass is small compared to the average stellar mass, (ii) regions close to the centre of the system, and (iii) large values of the model parameter C. It is proposed that the method of analysis presented in this study may be used for obtaining a distribution function for rotating stellar systems."

Lynden-Bell, D. 2087

Stellar dynamics. Only isolating integrals should be used in Jeans' theorem.

Monthly Notices Roy. Astronom. Soc. **124** (1962), 1–9.

Author's summary: "Jeans' theorem that the distribution function of an unrelaxed steady stellar system can be expressed as some (in general multivalued) function of elementary time-independent integrals of the motion has often been used restrictively. Only obvious integrals such as energy and angular momentum have been considered while there are always five time-independent integrals in all. This paper considers the role of the neglected integrals and shows that certain classes of integrals should indeed be omitted. A criterion is developed for finding some classes of potentials in which further integrals are important."

Wilson, P. R. 2088

Three-dimensional solutions of the transfer equation.

Monthly Notices Roy. Astronom. Soc. **126** (1963), 393–399.

Author's summary: "Methods of solving the equation of radiative transfer are obtained suitable for media whose diffusing and absorbing properties vary generally in three dimensions. The existing two-dimensional solutions are then deduced as special cases. As an illustration of the second and more important method, a model problem is solved."

Grant, I. P. 2089

Numerical approximations in radiative transfer.

Monthly Notices Roy. Astronom. Soc. **125** (1963), 417–436.

Author's summary: "The mathematical analysis of many astrophysical problems involving radiative transfer is so complicated that it is necessary to have recourse to numerical approximations to the radiative transfer equation. This paper describes an extension, for use in radiation problems, of Carlson's S_n difference approximation, which has been extensively used in neutron transport problems owing to its simplicity, its suitability for electronic computation, and its accuracy. Illustrations are provided to show the application of the new method in spherical problems with both scattering and non-scattering source terms, and the method has also been used to give an approximate solution to the Milne problem for a plane-parallel isotropic scattering grey atmosphere having constant net flux."

Dmitriev, N. A.; 2090
Zel'dovich, Ya. B. **[Zel'dovič, Ja. B.]**

The energy of accidental motions in the expanding universe.

Ž. Èksper. Teoret. Fiz. **45** (1963), 1150–1155 (*Russian. English summary*); *translated as Soviet Physics JETP* **18** (1964), 793–796.

Consider a Newtonian world model with scale function $R(t)$. Let the accidental velocity $\mathbf{v}$ of a particle whose position vector is $\mathbf{r}$ be defined by $\mathbf{v}=\dot{\mathbf{r}}-\mathbf{r}\dot{R}/R$, and introduce relative kinetic and potential energies by

$$T = \int \frac{1}{2}\rho v^2\, dV, \qquad F = -\frac{\kappa}{2}\iint \frac{(\rho_1-\bar{\rho})(\rho_2-\bar{\rho})}{|\mathbf{r}_1-\mathbf{r}_2|}\, dV_1 dV_2,$$

ρ_1 and ρ_2 being the densities at $\mathbf{r}_1$, $\mathbf{r}_2$, $\bar{\rho}$ being the average density, and κ the gravitational constant. The author proves the formula

$$\frac{1}{R}\frac{d}{dt}[R(T+F)] = -\frac{T}{R}\frac{dR}{dt},$$

and applies it to various aspects of cosmology.

W. B. Bonnor (London)

Schumacher, D. L. 2091

The direction of time and the equivalence of 'expanding' and 'contracting' world-models.

Proc. Cambridge Philos. Soc. **60** (1964), 575–579.

The necessity to discard, for empirical reasons, the contracting cosmological models implies that the laws of nature are too wide for nature itself [T. Gold, Amer. J. Phys. **30** (1962), 403–404]. The author argues that this difficulty of principle is avoided by the assumption that only the increase of entropy defines locally the sense of advance of time, because then there is an essential equivalence between contracting and expanding world models. *C. Gilbert* (Newcastle upon Tyne)

Thomas, T. Y. 2092

Outline of a theory of the universe.

Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 718–723.

It is assumed that the universe was initially devoid of matter and that material particles were accidentally created in regions where the strengths of gravitational discontinuities, propagated in the continuum, tended to become infinite. For further details, see the author's papers [J. Math. Anal. Appl. **3** (1961), 315–335; MR **25** #2906; Arch. Rational Mech. Anal. **10** (1962), 197–204; MR **26** #1192]. The explosion of an accumulation of material particles gave rise to a series of epochs of creation, which are related to the discretized galactic structure found by A. G. Wilson [Astronom. J. **68** (1963), 547].

C. Gilbert (Newcastle upon Tyne)

Narlikar, J. V. 2093
Neutrinos and the arrow of time in cosmology.
Proc. Roy. Soc. Ser. A **270** (1962), 553–561.

Although Maxwell's equations are time-symmetric, to explain the phenomenon of radiation it is necessary to discard the so-called advanced solutions and thus introduce a time asymmetry. Wheeler and Feynman developed a theory of electromagnetism from a single action principle based on direct inter-particle interaction. The field of a source 0 is given by the time-symmetric field

$$F_0 = \tfrac{1}{2}(F_{\mathrm{ret}} + F_{\mathrm{adv}}). \tag{1}$$

But experience shows that the field of a particle is given by F_{ret}. This is explained by assuming (1) to be the field of a particle in an empty universe and it is further shown that the contribution of the rest of the particles in a static Euclidean universe at 0 is a field given by $R = \frac{1}{2}(F_{\mathrm{ret}} - F_{\mathrm{adv}})$. This, when added to F_0, gives the familiar solution F_{ret}. By reversing the arrow of time throughout in the above argument one can obtain another possible solution F_{adv}. To avoid this solution Wheeler and Feynman postulated unsymmetrical initial conditions and argued on statistical grounds that this asymmetry would favor F_{ret} to F_{adv}. Hogarth proposed an alternative approach, based on the evidence that the universe is expanding. Hogarth showed that the correct observed field F_{ret} was given by some cosmological models such as the steady-state model, whereas F_{adv} was given by the Einstein-de Sitter model. In the present paper the analogous problem is considered in the case of neutrinos emitted by a source. It is shown that in some models, such as the steady-state model, the total contribution of the rest of the universe does give the correct neutrino field ψ_{ret}, but in some others, such as the Einstein-de Sitter model, this is not so. The physical consequence of this result is that in models in which one does not get ψ_{ret}, it should be possible to send neutrinos into the past.

J. N. Islam (College Park, Md.)

GEOPHYSICS

See 2079.

ECONOMICS, OPERATIONS RESEARCH, GAMES

See also 1578, 1728, 1730, 1732, 2115, 2121.

Hoffman, A. J.; Markowitz, H. M. 2094
A note on shortest path, assignment, and transportation problems.
Naval Res. Logist. Quart. **10** (1963), 375–379.

The shortest-route problem is formulated as an assignment problem, and the assignment problem is formulated as a succession of smaller shortest-route problems. Then, the transportation problem is formulated as a succession of shortest-route problems. The difference between this method and that of Ford and Fulkerson [*Flows in networks*, Princeton Univ. Press, Princeton, N.J., 1962; MR **28** #2917] lies in the fact that here the flows $X = (x_{ij})$ $(i = 1, \cdots, m;\ j = 1, \cdots, n)$ are determined one column at a time, each column requiring solution of a shortest-route problem.

E. L. Johnson (New Haven, Conn.)

Szwarc, Wlodzimierz [Szwarc, Włodzimierz] 2095
The stability of the transportation problem.
Mathematica (Cluj) **4 (27)** (1962), 397–400.

Examples are given to show that sufficiently small changes of the amounts produced and consumed at various centres in a transportation problem do not change (a) the set of positive elements of the optimal solution, and (b) the optimal solution. However, this problem was solved in 1961 in a joint paper of E. Balaş and the reviewer [Acad. R. P. Romîne Stud. Cerc. Mat. **12** (1961), 413–427; MR **27** #1306]. In the last part of the paper, the case of small changes in the unit transportation costs is examined on the same example.

P. L. Ivănescu (Bucharest)

Gassner, Betty Jane 2096
Cycling in the transportation problem.
Naval Res. Logist. Quart. **11** (1964), 43–58.

Author's summary: "The assignment problem, which is the most degenerate form of the transportation problem, and therefore the most likely to cycle, is investigated by means of linear graphs. It is proved that cycling is impossible in the 3×3 case, but possible in the 4×4 case. For the 5×5 case, it is shown both that there is a cycle independent of the one found for the 4×4 case and also that the cycle found in the 4×4 case can exist in the 5×5 case. Numerical examples are given."

Kerekesner, I. P.; Čehoviĭ, Ju. M. 2097
An algorithm for constructing a non-linear static model of a complex production process. (Ukrainian. Russian and English summaries)
Avtomatika **1964**, *no.* 2, 50–58.

Authors' summary: "A method is presented for constructing a nonlinear static model of a complex production process with parameters varying in time parameters. Recurrent formulae are presented, which permit—with continuous time weighting of the input information—obtaining a value for the static model coefficients at each step of renewal of the input information. Introduction of the recurrent formulae simplifies calculation and makes the process of constructing the variable model continuous. Criteria for selection of the weight function parameter p and the quality of items of the approximating polynomial k are discussed. This method can be applied to complex processes, approximated by an inertialess nonlinear component or a component with pure delay."

Sakamoto, Heihachi 2098
Contribution to the theory of systematic sampling and bedding methods in quality control. (French summary)
Proc. Fac. Engrg. Keio Univ. **13** (1960), 111–156.

Menges, G.; Behara, M. 2099
On decision criteria under various degrees of stability.
J. Indian Statist. Assoc. **1** (1963), 185–195.

The stability concept in decision theory is described with management science aspects. The orders of stability of states of nature are defined according to the existence of the moments of their probability distribution. Different decision criteria for choosing optimal strategies under various orders of stability are suggested.

S. Ichimura (Osaka)

Goldstein, A. A.; Kripke, B. R. 2100

Mathematical programming by minimizing differentiable functions.

Numer. Math. **6** (1964), 47–48.

Given a sequence $F_1, F_2, F_3, \cdots$ of continuous functions on real E^n, it is assumed that for some k, the set S_k, where $-1-(1/k) \leqq F_j(x_1, \cdots, x_n) \leqq 1+(1/k), j=1, \cdots, k$, is compact, and that the intersection S of the sets S_k is non-empty. Given a non-negative function G on E^n, the sequence of functions $H_k(x)=G(x)+(1/k^2)\sum_{j=1}^{k}(F_j(x))^{2k}$ is formed. If N is an integer such that S_k is compact for $k \geqq N$, it is assumed that there exists an integer $M \geqq N$ such that if $k \geqq M$, H_k achieves its minimum at a point z_k. The main result of the note is the following theorem. The sequence z_m has a cluster point, ζ. Every cluster point lies in S, and minimizes G thereon.

P. L. Ivănescu (Bucharest)

Altman, M. 2101

Stationary points in non-linear programming.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 29–35.

Let W be a closed convex set in a Banach space E formed by all points $x \in E$ satisfying the inequalities $f_i(x) \leqq a_i$ $(1 \leqq i \leqq m)$ and $g_k(x) \leqq b_k$ $(1 \leqq k \leqq n)$, where a_i, b_k are real numbers, f_i is a non-linear convex functional on E and continuously differentiable in the sense of Fréchet, and g_k is a continuous linear functional on E. Let F be a non-linear functional defined on W and continuously differentiable in the sense of Fréchet at every point of W. The Fréchet derivative of F at a point x is denoted by $F'(x)$. A point $x \in W$ is called a W-stationary point of F, if the maximum of $F'(x)s$, when s varies over $W-x$, is 0. Every local maximum point of F on W is necessarily a W-stationary point of F. If F is concave, then the W-stationary points of F are precisely the maximum points of F. Under certain regularity hypotheses, the author makes use of Farkas' theorem on linear inequalities to obtain a characterization of the W-stationary points.

Ky Fan (Evanston, Ill.)

Neuts, Marcel F. 2102

A multistage search game.

J. Soc. Indust. Appl. Math. **11** (1963), 502–507.

This is the development of an example in R. Bellman's book [*Dynamic programming*, Example No. 49, page 50, Princeton Univ. Press, Princeton, N.J., 1957; MR **19**, 820]: Player I hides an object in one of N boxes. Player II can search through the boxes at a certain cost each and receives a fixed reward if he finds the object. There is a positive probability of finding it if the right box is investigated. The author first derives stationary minimax strategies, i.e., assigns a constant probability of choosing each box at the next search, and hence does not use his information about past unsuccessful searches (as the author points out). The stationary case is used mere order to obtain the solution for the case where the disc factor in multi-stage search procedure equals one. U a Bellman functional equation, the author proves that sequence of functionals with a finite number of st converges uniformly in the strategy of the first player limit as the number of stages is increased indefinitely also proves that the sequence of functionals (the opt pay-off received by the first player) decreases m tonically if the discount factor is equal to one. Where discount factor lies between zero and one, a suffi condition is obtained. Finally, a generalisation is g where the object may be in none of the boxes and second player may claim that this is the case, wi corresponding reward [penalty] if he is right [wrong].

G. Morton (Lon

Saaty, Thomas L.; Bram, Joseph

★Nonlinear mathematics.

McGraw-Hill Book Co., New York, 1964. xv+381 $12.50.

The rather unrevealing title "Nonlinear Mathemat refers to the following topics: approximate solutio nonlinear equations, nonlinear programming, stabilit nonlinear ordinary differential equations, autom control, and prediction theory. The first chapter short introduction to functional analysis, including contraction principle and fixed-point theorems. To s extent this is window-dressing. The book is designe be accessible to a rather wide audience and for most the reader needs little more than standard adva calculus and linear algebra.

Under the heading nonlinear algebraic and trans dental equations comes a thorough discussion of Newt method, the method of steepest descent, and var generalizations of them. The chapter on nonli programming begins with some basic facts about extr of convex functions. The saddle-point condition theorems about duality are proved. Then eight diffe algorithms for nonlinear programming are described. authors have brought together a great deal of mate scattered through the literature, much of it less than years old.

The chapter on differential equations begins with usual existence and uniqueness theorem for first-o systems, followed by a review of linear, const coefficient systems. Then stability under small pertu tions, geometric theory (for two equations), Floc theory, and periodic solutions of nonlinear systems v periodic coefficients are discussed. An introduction Lyapunov stability is given, with applications to con systems. Pontryagin's maximum principle is stated, in general formulation of Berkovitz, as well as the functic analysis approach of Kulikowski in optimal con theory.

The last chapter introduces the linear theory of filte and prediction. Then the general estimation problen discussed, following Balakrishnan. Finally, a maxim likelihood technique related to the method of Kalr and Bucy is described.

The style is lucid and the choice of topics excellent. bibliography is extensive. The book should prove valus to student and researcher alike.

W. H. Fleming (Providence, R

Mangasarian, O. L. 2104
Equivalence in nonlinear programming.
Naval Res. Logist. Quart. **10** (1963), 299–306.
Author's summary: "The purpose of this work is to show the relation between the Kuhn-Tucker and the gradient-projection conditions for constrained maximization (minimization) problems. A similar relationship exists between Kuhn-Tucker-type conditions and gradient-projection-type conditions for certain constrained saddle-point problems. This latter relationship may lead to a computational algorithm for these constrained saddle-point problems." *R. W. Cottle* (Holmdel, N.J.)

Charnes, A.; Cooper, W. W.; Thompson, G. L. 2105
Critical path analyses via chance constrained and stochastic programming.
Operations Res. **12** (1964), 460–470.
Authors' summary: "Chance-constrained programming methods are applied to examine some statistical properties of PERT networks. Using duality, the PERT method is shown to be equivalent to use of the crudest linear decision rule and the confidence (or lack thereof) in meeting constraints is explicitly presented. The distribution of completion times (= Tintner's stochastic programming) follows easily and may often be multimodal, contrasting with (erroneous) central limit theorem usages in the literature. Possible extensions and developments of PERT using more adequate chance-constrained models and techniques are suggested and will be presented elsewhere."

Beniest, W. 2106
Jeux stochastiques totalement coopératifs arbitrés.
Cahiers Centre Études Rech. Opér. **5** (1963), 124–138.
Shapley's treatment of two-person stochastic games for the zero-sum case [Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 1095–1100; MR **15**, 887] is extended to nonzero-sum games with side payments, using two arbitration schemes, one due to Raiffa [*Contributions to the theory of games*, Vol. 2, pp. 361–387, Princeton Univ. Press, Princeton, N.J., 1953; MR **14**, 667].
J. H. Blau (Yellow Springs, Ohio)

BIOLOGY AND BEHAVIORAL SCIENCES

Rashevsky, N. 2107
Some remarks on Rosen's quantum-mechanical approach to genetics.
Bull. Math. Biophys. **26** (1964), 187–191.
Author's summary: "It has been suggested by R. Rosen [same Bull. **22** (1960), 227–255] that the multiple alleles or pseudoalleles correspond to multiple cites of degenerate states of some quantum mechanical observable which acts as a source of primary genetic information. It is pointed out here that if the quantum mechanical states are determined by the different sequences of the purine and pyrimidine bases in the DNA molecule, the expected number of pseudoalleles would be much too large. The expected number is considerably reduced if we assume that a quantum mechanical state determines the coupling between a molecule of transfer RNA and the corresponding amino acid." *A. Rapoport* (Ann Arbor, Mich.)

INFORMATION, COMMUNICATION, CONTROL

See also 1148, 1150, 1155, 1411, 1671, 1674, 1683, 1771, 1772, 2103.

Wolfowitz, J. 2108
On channels without a capacity.
Information and Control **6** (1963), 49–54.
Consider a noisy channel with $\{0, 1\}$ as input-output alphabet. Denote by $N(n, \lambda)$ the length of the longest code with words of length n and probability of error $\leq \lambda$. It is known [see the author, *Coding theorems of information theory*, Springer, Berlin, 1961; MR **26** #1227] that in many cases $\lim_{n\to\infty} n^{-1} \log N(n, \lambda) = C$ exists and is independent of λ. Some writers hold this property to be self-evident; to refute this opinion, the author gives an example of a simple channel satisfying

$$(*)\quad \limsup_n n^{-1} \log N(n, 5^{-k-2}) < \liminf_n n^{-1} \log N(n, 5^{-k}),$$

($k = 1, 2, \cdots$). Namely, first select a random integer $i \geq 1$ such that $\Pr(i = r) = 4/5^r$. Afterwards, transmit the code word over a memoryless binary symmetric channel with probability p_i of error for each letter. Here, $\{p_i\}$ is any fixed increasing sequence of numbers $< \frac{1}{2}$. As is indicated in the subsequent discussion, phenomenon (*) is related to the large variance of the number of incorrectly transmitted letters. *J. H. B. Kemperman* (Rochester, N.Y.)

Savčuk, A. P. 2109
On Shannon's estimates for the entropy of a language. (Russian. English summary)
Teor. Verojatnost. i Primenen. **9** (1964), 154–157.
Consider a language with alphabet $\{1, 2, \cdots, n\}$. In a random segment of length N, let H_N denote the average entropy of the last letter j, given the initial segment i of length $N-1$. Then one has the useful inequalities $H_N' \leq H_N \leq H_N''$, as pointed out by C. E. Shannon [Bell System Tech. J. **30** (1951), 50–64]. Here,

$$H_N' = \sum_k (k \log k)(q_k^N - q_{k+1}^N),\quad H_N'' = -\sum_k q_k^N \log q_k^N.$$

Further,

$$q_k^N = \sum_i p(i) p(j_{i,k}|i),$$

where $p(j_{i,k}|i)$ is the kth largest among the values $p(1|i), \cdots, p(n|i)$. In an obvious way, q_k^N equals the probability that a person, knowing i, will guess j right in exactly k trials.

In the present paper it is shown that $H_N = H_N'$ happens if and only if, for each i with $p(i) > 0$, the nonzero values $p(j|i)$ are all equal to each other (the actual value depending on i). Similarly, $H_N = H_N''$ if and only if, for each i with $p(i) > 0$, one has $p(j_{i,k}|i) = q_k^N$ for all k.
J. H. B. Kemperman (Rochester, N.Y.)

Csiszár, I. 2110
Über topologische und metrische Eigenschaften der relativen Information der Ordnung α.
Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (Liblice, 1962), pp. 63–73. Publ. House Czech. Acad. Sci., Prague, 1964.
Let (X, S) be a measurable space and E a given set of

probability measures on (X, S). The relative information of order α of a measure $P \in E$ with respect to the measure $Q \in E$ is defined by

$$I_\alpha(P\|Q) = \frac{1}{\alpha-1}\log\int_X\left(\frac{dP}{d\lambda}\right)^\alpha\left(\frac{dQ}{d\lambda}\right)^{1-\alpha}d\lambda \qquad (\alpha > 0),$$

where λ is any σ-finite measure on (X, S) with respect to which P and Q are absolutely continuous. Since $I_\alpha(P\|Q) \geqq 0$ for $\alpha > 0$ with equality only if $P = Q$, the relative information of order may serve as a measure of the difference between P and Q. The ε-neighborhood of a measure $Q \in E$ with respect to the information of order α is defined by

$$V_\alpha(Q, \varepsilon) = \{P : I_\alpha(P\|Q) < \varepsilon;\ P \in E\}.$$

The sets $V_\alpha(Q, \varepsilon)$ are called information neighborhoods of order α of Q. Let $f(u, v)$, $u \geqq 0$, $v \geqq 0$ be a non-negative function vanishing at the origin. A generalized information neighborhood is defined by

$$V_\alpha{}^f(Q, \varepsilon) = \{P : f(I_\alpha(P\|Q), I_\alpha(Q\|P)) < \varepsilon;\ P \in E\}$$

and called an f-neighborhood of order α. The first problem examined is whether the f-neighborhoods define a topology on E and whether they satisfy the Hausdorff axioms. There is also considered the problem whether there exist functions of the relative information of order α which determine a distance on E. It is shown that under minor restrictions on f the answer to both questions is affirmative for $0 < \alpha < 1$ and negative for $\alpha \geqq 1$.

S. Kullback (Washington, D.C.)

Rajski, C. 2111

On the normed information rate of discrete random variables.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 583–585. *Publ. House Czech. Acad. Sci.*, *Prague*, 1964.

Author's summary: "The properties of the quotient of information rate and joint entropy of two discrete random variables are discussed. It is shown that it has some advantages over correlation coefficients and thus may be used instead."

Papoulis, A. 2112

An estimate of the variation of a band-limited process.

IEEE Trans. Information Theory **IT-10** (1964), 72–74.

Author's summary: "Upper and lower bounds are established for the mean-square variation of a stationary process $X(t)$ whose power spectrum is bounded by ω_c, in terms of its average power P_0 and the average power P_1 of its derivative. It is shown that

$$\left(\frac{2}{\pi}\right)^2 P_1\tau^2 \leqq E\{|X(t+\tau)-X(t)|^2\} \leqq P_1\tau^2 \leqq \omega_c{}^2P_0\tau^2,$$

where the upper bounds are valid for any τ and the lower bound for $\tau < \pi/\omega_c$. These estimates are applied to the mean-square variation of the envelope of a quasi-monochromatic process."

Sakaguchi, Minoru 2113

Information pattern, learning structure, and optimal decision rule.

Information and Control **6** (1963), 218–229.

The author considers the usual decision-making situation of states of nature and actions, with a probability distribution over the former. Examples of different information patterns over the states of nature are given and the value of the pattern is defined as the difference between the expected utility given the pattern and the same quantity without it. For example, the states may consist of the pair (x, y) and information be given concerning x. The examples are extended to the case of two decision-makers controlling different actions with different patterns. The final section is supposed to apply the ideas to learning processes in adaptive control processes. The argument was not intelligible to the reviewer but again seemed to consist in evaluating difference between two expected utilities. *D. V. Lindley* (Aberystwyth)

Selin, Ivan 2114

The design of a sequential test for the detection of known signal in normal non-white noise.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 621–628. *Publ. House Czech. Acad. Sci.*, *Prague*, 1964.

Author's summary: "This paper considers the design of a sequential test for constant signals in correlated normal noise. Both discrete and continuous time parameters are considered. The noise is assumed to be stationary in the continuous case, but not in the discrete case. Through the transformation of random variables, it is shown that the test of given strength which minimizes expected test length is Wald's sequential probability (likelihood) ratio test."

Dobbie, James M. 2115

Search theory: A sequential approach.

Naval Res. Logist. Quart. **10** (1963), 323–334.

Author's summary: "Koopman and others have noted that the optimal distribution of effort, to maximize the detection probability with a given effort, has the property that it is the sum of the distributions obtained by optimizing conditionally when the effort is applied sequentially. The approach taken in this present investigation is to reverse the procedure by determining conditions under which the optimal distribution has this property, and then to use this property to find the optimal distribution. This method easily yields essentially all of the known results, both for the discrete and continuous cases, and for the two criteria of maximizing the detection probability with a given effort and minimizing the expected effort to attain a given attainable probability of detection. The method also yields some modest extensions. In the discrete case the method is equivalent to the simplest form of dynamic programming."

Picinbono, Bernard; de Suso Barba, José 2116

Sur une généralisation du filtrage optimal en détection-intégration.

C. R. Acad. Sci. Paris **259** (1964), 290–293.

Authors' summary: "La théorie du filtrage optimal en détection-intégration a été faite dans le cas d'un système à une seule entrée. On donne une généralisation dans le

cas d'un système à n entrées, les bruits étant caractérisés par leur intercorrélation. Diverses conséquences sont déduites des formules générales."

Montgomery, W. D. 2117
The extension to probability distributions for detection spatial filters.
IEEE Trans. Information Theory **IT-10** (1964), 2–5.
The author is interested in problems of "filtering out" spatial patterns from noise. In this context, he considers the problem of extending partial descriptions of the noise to a probability density function by choosing the minimal entropy (least biased) distribution compatible with the partial information. Several two-dimensional moment descriptions of the noise are thus extended and worked out in detail. *J. Hartmanis* (Schenectady, N.Y.)

Singleton, Richard C. 2118
Maximum distance q-nary codes.
IEEE Trans. Information Theory **IT-10** (1964), 116–118.
Author's summary: "A q-nary error-correcting code with $N=q^k$ code words of length $n=k+r$ can have no greater minimum distance d than $r+1$. The class of codes for which $d=r+1$ is studied first in general, then with the restriction that the codes be linear. Examples and construction methods are given to show that these codes exist for a number of values of q, k, and r."

Chien, R. T. 2119
On linear residue codes for burst-error correction.
IEEE Trans. Information Theory **IT-10** (1964), 127–133.
Author's summary: "Linear residue codes are useful for error control in both arithmetic operations and data transmission. When used in data transmission, they do not require special coding equipment, as encoding and decoding operations may be easily carried out in digital computers. In this paper, number-theoretic concepts are used to construct linear residue codes for the correction of burst errors. A general theory is developed for the class of codes derived from the multiplicative group $G(A)$ modulo A, an odd integer, which consists of all the integers smaller than and relatively prime to A. Construction schemes for various classes of burst-error-correcting binary linear residue codes are determined and discussed."

Strassen, V. 2120
Messfehler und Information.
Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2**, 273–305 (1964).
From the author's summary: "In Abschnitt 2 werden das Neyman-Pearsonsche Lemma und eine Verschärfung des bekannten Resultats von Shannon über das asymptotische Verhalten der Minimalmächtigkeit hochwahrscheinlicher Mengen für stationäre Quellen mit unabhängigen Zeichen auf totalmonotone Kapazitäten verallgemeinert (Satz 2.1, Satz 2.2). Die Methode ist repräsentativ in dem Sinne, daß im Falle des Neyman-Pearsonschen Lemmas zwei WVen, im Falle des Shannonschen Satzes eine Quelle konstruiert werden, die den jeweiligen totalmonotonen Kapazitäten im Hinblick auf das betrachtete Problem völlig äquivalent sind. Die Verallgemeinerung des Shannonschen Satzes auf nichtstationäre (z.B. fastperiodische) totalmonotone Kapazitäten mit unabhängigen Zeichen bietet keine Schwierigkeiten, die auf ergodische totalmonotone Kapazitäten wird in Abschnitt 5 behandelt. In Abschnitt 3 wird ein Coding-Theorem mit schwacher Umkehrung für stationäre Unterkanäle ohne Gedächtnis bewiesen (Satz 3.1): Die Definition eines Unterkanals erhält man aus der des Kanals, indem man alle auftretenden WVen durch totalmonotone Kapazitäten ersetzt. Den Beweis des Coding-Theorems kann man leider nicht mit einem im obigen Sinne repräsentativen Verfahren führen (wie einfache Beispiele zeigen, vgl. auch den Ausdruck (3.13), (3.15) für die Kapazität). Die Verallgemeinerung auf Unterkanäle mit endlichem Gedächtnis ist leicht."
J. Wolfowitz (Ithaca, N.Y.)

Aris, Rutherford 2121
★Discrete dynamic programming. An introduction to the optimization of staged processes.
Blaisdell Publishing Co. [*Ginn and Co.*], *New York-Toronto-London*, 1964. x + 148 *pp.* $5.00.
This book introduces the reader to methods of dynamic programming and deals mainly with discrete processes. An introductory chapter mentions various optimization methods, and Chapters 2 to 5 use an example from chemical engineering to discuss general concepts, the principle of optimality, and a graphical illustration. The next four chapters deal with other problems, e.g., in economics, information theory, and reliability theory. Chapter 9 treats continuous problems as limits of discrete ones, and the latter as arising through sampling from the former. Chapter 10 treats feedback as an example for cases, where the optimality principle cannot be applied without modification.

Some mathematical ideas, such as Lagrange multipliers (rather incompletely explained) and Jacobi matrices, are interpolated in these chapters, and Chapter 11 ("Some associated mathematical ideas") introduces more of them. On the whole, this is a useful elementary introduction to dynamic programming, demanding only a basic knowledge of differential calculus and equations, with good examples and with exercises (without solutions) after nine out of the eleven chapters. *S. Vajda* (Teddington)

Krutova, I. N.; Rutkovskiĭ, V. Ju. 2122
Dynamics of a self-adjusting first-order system with a model. (Russian. English summary)
Avtomat. i Telemeh. **25** (1964), 188–194.
Authors' summary: "The influence of error algorithms of the number of changed coefficients and the law of their changing on the performance of the self-adjusting system with a model is considered. It is shown that the introduction of the self-adjustment with coefficient change depending only on the model and system coordinate unbalance permits one to make stable the system with negative self-regulation coefficients."
J. Kuntzmann (Grenoble)

Savin, O. R. 2123
Improving the quality of extremal control of inertial processes. (Ukrainian. Russian and English summaries)
Avtomatika **1964**, *no.* 2, 33–41.

Author's summary: "In this article a method is discussed for changing the algorithm of the work of a step type extremal controller that makes it possible, at a constant step and control period, to increase the efficiency of the controller work."

Cypkin, Ja. Z. 2124

A criterion of absolute stability for sampled-data systems with monotone characteristics of the non-linear element. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 1029–1032.

This paper is a continuation of previous papers by the author [see, e.g., Automat. Remote Control **23** (1963), 1467–1472; MR **27** #5638]. The criterion of the absolute stability of a nonlinear, sampled-data control system given previously allows one to define a sufficient condition for absolute stability of such a system. The system is absolutely stable when its nonlinear characteristic $\phi(x)$ is restricted to lie within specified sectors in the first and third quadrants of the $[x, \phi(x)]$-plane $(0<\phi(x)/x<k)$. In many cases this criterion calls for quite conservative limits for k.

In this paper another possibility is considered by limiting the magnitude of the derivative of the characteristic of the nonlinear element. (A similar limitation was applied, for example, by E. I. Jury and B. W. Lee [IEEE Trans. Automatic Control **AC-9** (1964)].) This limitation happens to be less conservative and the formulation of the criterion turns out to be less complex from the practical point of view.

I. Cederbaum (Tel-Aviv)

Halkin, Hubert 2125

Liapounov's theorem on the range of a vector measure and Pontryagin's maximum principle.

Arch. Rational Mech. Anal. **10** (1962), 296–304.

The author outlines a new proof of the well-known generalization of the Euler-Weierstrass necessary condition by Pontrjagin et al. [Boltjanskiĭ, Gamkrelidze, and Pontrjagin, Izv. Akad. Nauk SSSR Ser. Mat. **24** (1960), 3–42; MR **22** #11191; Amer. Math. Soc. Transl. (2) **18** (1961), 341–382; MR **23** #A1908]. He gives an elegant characterization of the "cone of states attainable in the first approximation" ($\overset{+}{H}_v$), and proceeds to establish the theorem by geometric arguments, which depend critically on Ljapunov's lemma on the range of a vector measure [Izv. Akad. Nauk SSSR Ser. Mat. **4** (1940), 465–478; MR **2**, 315]. More detailed treatment may be found in the author's dissertation [Stanford Univ., Stanford, Calif., 1963].

{The passage from (4.15) to (4.16) in the proof of Theorem III lacks justification. There is a simple proof of this fact based on the Brouwer fixed-point theorem (due to S. Jones, unpublished).}

R. E. Kalman (Baltimore, Md.)

Novosel'tsev, V. N. [Novosel'cev, V. N.] 2126

Time-optimal control systems in the presence of random noise.

Avtomat. i Telemeh. **23** (1962), 1620–1630 (*Russian. English summary*); *translated as Automat. Remote Control* **23** (1963), 1519–1529.

Author's summary: "The question of time-optimality in the case of determinate input signals in the presence of random noise, is considered. It is shown that for a specific class of systems the control algorithm which ensures a minimum transient response time for the mathematical expectation also ensures the minimum mathematical expectation for the transient response time. An example of a time-optimal second-order system is considered."

A. Wouk (Madison, Wis.)

Gabasov, R.; Kirillova, F. M. 2127

Optimization of convex functionals on trajectories of linear systems. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 1007–1010.

Let a control process be described by the differential equations

$$(1)\qquad \dot{x} = A_1(t)x+B_1(t)y+C_1(t)u+f_1(t),\qquad \dot{y} = A_2(t)x+B_2(t)y+C_2(t)u+f_2(t),$$

where x and y are phase vectors in finite-dimensional spaces X and Y, the A_i, B_i, C_i, and f_i are matrix functions, and $u(t)$ is the "control" function. Given real numbers t_i $(i=0, 1, \cdots, m_i;\ t_i<t_{i+1})$, T $(t_m \leqq T)$, and L_i $(i=1, \cdots, m+4;\ L_i \geqq 0$ for $i>1)$, a functional $\phi(u, T)$ convex in u and defined for all control functions u, vectors x_0 and x_1 in X, and a curve $\gamma(t)$ $(t_0 \leqq t \leqq T)$ in Y, the authors consider the following problem: Find a necessary and sufficient condition for the existence of a function $u \in U$ (U being a given bounded, convex set in a function space) such that there is a corresponding solution of (1) satisfying the inequalities $\phi(u, T)<L_1$, $\|x(t_0)-x_0\| \leqq L_2$, $\|x(T)-x_1\| \leqq L_3 \|y(t_i)-\gamma(t_i)\| \leqq L_{i+4}$ for $i=0, 1, \cdots, m$. The condition given in this paper is that $\Lambda(L_1, \cdots, L_m, T) \geqq 0$, where Λ is a function obtained by minimizing another function of a finite number of variables (too complicated to describe here).

The authors also consider the optimization problem of finding a minimum value for L_k (with the remaining L_i fixed), or, for fixed L_i, finding the minimum value of T, for which the given problem has a solution. The preceding result indicates that this variational problem can be reduced to finding the minimum of a function of a finite number of variables. An iterative computational procedure for solving both the minimum T and minimum L_k problems is presented.

This paper in certain respects generalizes some work of the reviewer [Proc. Second IFAC Congr., Butterworths, London, 1964].

L. W. Neustadt (Ann Arbor, Mich.)

Fel'dbaum, A. A. [**Фельдбаум, А. А.**] 2128

★**Fundamentals of the theory of optimal automatic systems** [**Основы теории оптимальных автоматических систем**].

Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1963. 552 *pp.* 1.59 *r.*

One may say without reservation that this is a book intended for engineers; indeed, it is based on a graduate course the author teaches at the Institute of Automatics and Telemechanics. It is written in the simple, lucid and uncluttered style which seems to be the trademark of the scientists at the Institute and which the reader may have already encountered either in the author's other books or

else in books written by M. A. Aizerman or F. R. Gantmacher. The scope of the book is considerable, and this is mostly achieved by dispensing with rigor and time-consuming mathematical detail. On the other hand, the book brings the reader into constant contact with current research literature through copious examples drawn from it; it gives competent evaluations of the various aspects of the state of the art, and it is permeated with the author's own point of view. All but one chapter are heavily based on current research literature and, occasionally, entire sections consist of suitably presented and capably integrated versions of well-known papers. These papers are properly acknowledged and referred to in the text, while many others are listed in the rather extensive bibliography at the end of the book. A somewhat unusual feature for a Russian text is the thorough subject index, appended.

The book consists of six chapters. The first chapter is devoted to the formulation of various optimal control problems and covers such items as the choice of criteria of optimality, the classification of optimal systems into continuous, discrete time, and discrete state systems, as well as according to whether they do or do not involve stochastic processes, and finally, the manner of transmission of signals in a control system.

The second chapter is devoted to developing the necessary mathematical tools on a somewhat elementary level. There is a section on probability theory and stochastic processes, a section on the calculus of variations, one on dynamic programming which is rather well presented, and one on the maximum principle which is developed by means of dynamic programming.

The third chapter covers deterministic optimal control systems. It has an extensive section on the minimum time problem for linear differential systems, another one showing the application of classical variational techniques to problems with a quadratic cost function integrated from zero to infinity. It then goes further into dynamic programming applications for both differential and discrete state systems and, finally, it takes up the maximum principle once more.

The fourth chapter deals with the optimal control of systems required to follow a random input. It is essentially based on a paper by J. J. Florentin [J. Electronics Control (1) 10 (1961), 473–488; MR 27 #2372], on a paper by J. H. Eaton and L. A. Zadeh [Trans. ASME Ser. D. J. Basic Engrg. 84 (1962), 23–29; MR 27 #3476], and, to a lesser extent, on a few others.

Chapter 5 covers the basics of detection theory and gives an introduction to statistical decision theory. Most of this material is now classical and can be found in various other texts.

Chapter 6 is devoted to the adaptive control of systems in which the observations are corrupted by noise. The optimal computer is required both to control the system and to carry out computations for the identification of the noise characteristics. Because of this dual function of the controller, the author refers to these systems as "dual control systems". This chapter is heavily based on a number of papers written by the author himself.

In conclusion, one ought to stress again that this is a very well written book for the beginner in optimal control, and that it reveals a great deal about the nature and difficulty of a vast segment of the optimal control field.

E. Polak (Berkeley, Calif.)

Sarachik, P. E.; Kranc, G. M. 2129

Optimal control of discrete systems with constrained inputs.

J. Franklin Inst. **277** (1964), 237–255.

The authors consider control systems whose dynamic behavior is described by the equation

$$(1)\qquad \mathbf{y}(k) = \mathbf{y}_0(k) + \sum_{i=0}^{k} H(k, i)\mathbf{u}(i),$$

where i and k are non-negative integers, the "output" $\mathbf{y}$ is an m-vector, the "input" $\mathbf{u}$ is an r-vector, and the $H(k, i)$ and $\mathbf{y}_0(k)$ are given $m \times r$ matrices and m-vectors, respectively. Given a "desired" output sequence $\mathbf{y}^d(j)$, $j = 0, 1, \cdots$, the problem under consideration is that of finding a finite sequence of inputs $\mathbf{u}(0), \cdots, \mathbf{u}(N)$, satisfying certain constraints, such that the corresponding solution of (1) satisfies the relation $\mathbf{y}(N) = \mathbf{y}^d(N)$ and such that N is minimal. Either of the following two types of input constraints is considered:

$$(2)\qquad \sum_{i=0}^{N} \sum_{j=1}^{r} |u_j(i)|^p \leq L,$$

$$(3)\qquad \sum_{i=0}^{N} |u_j(i)|^{p_j} \leq L_j, \qquad j = 1, \cdots, r,$$

where L, p, or L_j and the p_j are given positive numbers and p (or the p_j) > 1. It is shown that the minimal N is the smallest integer k for which $H(k, 0), \cdots, H(k, k)$ satisfy a certain inequality, and a particular solution to the problem (in which L or the L_j are replaced by the smallest possible numbers) is constructed. If the constraints have the form (3), an unjustified limiting argument is used to extend the results to the case of one or more of the p_j equal to unity, and an erroneous form for the minimizing input sequence (equation (56)) is obtained. Finally, the case where some or all of the constraints (3) are replaced by inequalities of the form $\max_{0 \leq i \leq N} |u_j(i)| \leq L_j$ is considered. No proofs are presented (it is implied that the previously given ones carry over; they do not), but the stated conclusions are correct. If all the inequalities are of the latter form, the results agree with those of Krasovskiĭ [Prikl. Mat. Meh. 21 (1957), 670–677; MR 20 #4453] and the reviewer [Internat. Sympos. Nonlinear Differential Equations and Nonlinear Mechanics, pp. 267–283, Academic Press, New York, 1963; MR 26 #6104].

An ambiguous and often misleading notation and terminology make the paper difficult to read.

L. W. Neustadt (Ann Arbor, Mich.)

Edelbaum, Theodore N. 2130

Optimum low-thrust rendezvous and station keeping.

AIAA J. **2** (1964), 1196–1201.

Author's summary: "Analytic solutions are determined for the optimum correction of all six elements of elliptic satellite orbits with low-thrust, power-limited propulsion systems. The optimum direction and magnitude of thrust are determined as functions of time so as to minimize the fuel required to rendezvous in a given time or to minimize the fuel required for station keeping in the presence of known perturbations. The motion of the vehicle under the action of the optimum thrust program is also determined analytically. The solutions obtained are divided into two classes, depending on whether rendezvous requires few or many revolutions. In the

latter case, simple analytic solutions are obtained explicitly by neglecting short-period perturbations."

Y.-C. Ho (Cambridge, Mass.)

Warga, J. 2131

Minimizing variational curves restricted to a preassigned set.

Trans. Amer. Math. Soc. **112** (1964), 432–455.

The author's version of the theory of optimal controls follows the spirit of the reviewer's generalized curves, so that generalized (or relaxed) solutions exist and can be approximated by elementary admissible curves. The necessary conditions, which, in control theory, are presented in the elementary case in the book of Pontrjagin, Boltjanskii, Gamkrelidze, and Mishchenko [*The mathematical theory of optimal processes*, Interscience, New York, 1962; MR **29** #3316b], are then for the first time properly motivated. However, it is essential that these necessary conditions be first made applicable to generalized solutions (instead of merely to hypothetical elementary ones). In an earlier paper [J. Math. Anal. Appl. **4** (1962), 111–128; MR **25** #5415a; ibid. **4** (1962), 129–145; MR **25** #5415b], the author established this for solutions required to be in an open set, whereas in control problems the usual requirement is less strict. The present paper replaces the open set by one defined by an inequality of the form $a(x) \geqq 0$.

L. C. Young (Madison, Wis.)

Chang, Jên-Wei 2132

The problem of synthesizing an optimal controller in systems with time delay.

Avtomat. i Telemeh. **23** (1962), 133–137 (*Russian. English summary*); *translated as Automat. Remote Control* **23** (1962), 121–125.

The author presents an approximate method for synthesizing an optimal controller in systems with time delay given by

$$\frac{dx_i(t)}{dt} = \sum_{j=1}^{n} a_{ij}x_j(t) + \sum_{j=1}^{n} b_{ij}x_j(t-\tau) + b_i u \qquad (i = 1, 2, \cdots, n)$$

with the integral criterion

$$J = \int_0^\infty \left[\sum_{i=1}^{n} a_i x_i^2(t) + cu^2(t) \right] dt.$$

He shows that an optimal controller must satisfy the functional equation

$$\operatorname*{Min}_u \left\{ \sum_{i=1}^{n} a_i x_i^2(t) + cu^2(t) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i(t)} \left[\sum_{j=1}^{n} a_{ij}x_j(t) + \sum_{j=1}^{n} b_{ij}x_j(t-\tau) + b_i u \right] + \frac{1}{\tau} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i(t-\tau)} [x_i(t) - x_i(t-\tau)] \right\} = 0,$$

where f is a certain function of $x(t)$ and $x(t-\tau)$ derived from the original integral criterion. He assumes τ is very small and so approximates the initial function by a straight line. Then he applies the method of dynamic programming. He uses a first-order equation to prove the above relationship. An example of finding $f(x(t), x(t-\tau))$, and hence an optimal controller u, is given.

E. B. Lee (Minneapolis, Minn.)

★Switching theory in space technology. 2133

Edited by Howard Aiken and William F. Main.

Stanford University Press, Stanford, Calif., 1963. x+357 *pp.* $11.50.

Table of Contents: I: Introduction: Intuitive outline of the solution of a basic combinatorial problem (G. Pólya); II: Circuit Logics: On the mathematical theory of linear sequential networks (Günter Hotz); Totally sequential switching circuits (William H. Kautz); Threshold, majority, and bilateral switching devices (Eiichi Goto); Tryon's delay operator and the design of synchronous switching circuits (Franz E. Hohn); Minimum weights for threshold switches (David G. Willis); Switching and information (Heinz Zemanek); On the realizability of graphs with prescribed circuit matrices (V. Belevitch); The theory of algorithms (J. Paul Roth); Analysis and design confirmation of controlled-flow nets (G. W. Patterson); General E-algebras (Warren Semon); New techniques for the machining and shaping of ferrites (Frank F. Stucki); Magnetic comparators and code converters (Robert C. Minnick); III: New Switches and Storage Devices: Computer memories: Remarks on possible future developments (Jan A. Rajchman); Nonlinear resonance switching devices (José Garcia Santesmases); Superconductive switches and storage devices (A. E. Slade); Hydraulic switching devices (A. P. Speiser); Magnetic films: New possibilities, new problems (J. I. Raffel); Electron spin-echo storage (M. E. Browne, J. A. Cowen, and D. E. Kaplan); Switching and programming (A. van Wijngaarden); The application of system theory to space missions (W. R. Abbott); Asynchronous logics and application to information processing (David E. Muller); Information storage and retrieval as a switching system (L. B. Heilprin); Space-borne digital system for data bandwidth compression (Daniel Hochman); Recent advances in modular arithmetic (Nicholas Szabó).

Vavilov, È. M.; Osinskiĭ, L. M. 2134

Structural synthesis of automata operating with additional cycles. (Ukrainian. Russian and English summaries)

Avtomatika **1964**, *no.* 2, 42–49.

Authors' summary: "A method is presented for simplifying the structural designs of finite automata by introducing additional internal cycles."

Winograd, S. 2135

Input-error-limiting automata.

J. Assoc. Comput. Mach. **11** (1964), 338–351.

Author's summary: "Some properties of automata are investigated, which are capable of limiting the effect of input errors on their behavior. First, necessary and sufficient conditions are derived for an automaton to be capable of always being resynchronized within a bounded number of input letters after an error has occurred, and then the results are specialized to finite-state completely specified automata. Automata are investigated, which are capable of being resynchronized with probability one and

it is shown that a finite-state completely specified automaton possesses this property if and only if there exists a finite sequence which is a universal synchronizer for the automaton. Some connections with similar problems for variable-length codes are indicated."

J. Hartmanis (Schenectady, N.Y.)

Murskii, V. L. 2136

Transformations of certain types of circuits associated with finite automata. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 510–512.

Unter einer I-Schaltung (I-Sch.) im Alphabet $B=\{x_1, \cdots, x_k\}$ wird ein endlicher orientierter Graph S verstanden, in welchem gewisse Ecken als "Pole" ausgezeichnet sind und gewissen (nicht notwendig allen) Kanten eindeutig Buchstaben aus B zugeordnet sind. Für einen (orientierten) Weg π in S (der sich auch selbst durchdringen kann) bezeichne $Q(\pi)$ das Wort, das aus den zu den Kanten des Weges gehörigen Buchstaben — in Reihenfolge der Durchlaufung — gebildet wird. Für zwei Ecken α und β von S sei $M^S_{\alpha,\beta}$ die Menge aller Wörter $Q(\pi)$, welche zu Wegen π gehören, die in S die Ecken α und β verbinden. Zwei I-Schen. S und T heißen äquivalent, wenn es eine eineindeutige Abbildung f der Pole von S auf die Pole von T gibt, so daß für beliebige Pole α und β von S die Beziehung $M^S_{\alpha,\beta}=M^T_{f(\alpha),f(\beta)}$ gilt. Es sei S eine I-Sch. im Alphabet B und es seien $R_1, \cdots, R_k$ I-Schen. (in beliebigen Alphabeten) mit je zwei Polen, von denen jeweils der eine als Eingang und der andere als Ausgang markiert ist. Unter $S^{x_1\cdots x_k}_{R_1\cdots R_k}$ wird dann diejenige I-Sch. verstanden, die man aus S erhält, wenn man jede Kante von S, der der Buchstabe x_i entspricht, so durch die zugehörige I-Sch. R_i ersetzt, daß der Eingang von R_i mit dem Anfangspunkt der entsprechenden Kante und der Ausgang von R_i mit deren Endpunkt identifiziert wird ($i=1, \cdots, k$); als Pole von $S^{x_1\cdots x_k}_{R_1\cdots R_k}$ werden die ursprünglichen Pole von S beibehalten. Von den I-Schen. $R_1, \cdots, R_k$ wird dabei zusätzlich vorausgesetzt, daß sie in folgendem Sinne regulär sind: Es gibt in ihnen keinen (orientierten) Weg vom Ausgang zum Eingang und keinen nichtleeren Weg vom Eingang zum Eingang und vom Ausgang zum Ausgang. Ein Paar (S, T) von äquivalenten I-Schen. in B wird ein Regel-Schema zur äquivalenten Umformung von I-Schen. genannt. Sind $R_1, \cdots, R_k$ beliebige reguläre I-Schen., so heißt das Paar $(S^{x_1\cdots x_k}_{R_1\cdots R_k}, T^{x_1\cdots x_k}_{R_1\cdots R_k})$ eine Realisierung von (S, T). Jede solche Realisierung liefert offenbar eine Regel zur äquivalenten Transformation von I-Schen. Es wird ein Beweis für folgenden Satz skizziert: Es gibt kein endliches System von Regel-Schemata in B, so daß sich je zwei äquivalente I-Schen. in B mit Hilfe von Regeln, die Realisierungen von Regel-Schemata des betrachteten Systems sind, ineinander überführen lassen. Es werden Verallgemeinerungen dieses Resultats diskutiert und eine Anwendung auf endliche initiale Automaten gegeben, wobei die endlichen initialen Automaten mit dem Eingangsalphabet A und dem Ausgangsalphabet B als I-Schen. in $A\times B$ mit einem einzigen Pol (der dem Anfangszustand des Automaten entspricht) aufgefaßt werden. Es wird ein dem obigen Satz analoges Resultat für die Äquivalenz (im Sinne der Automatentheorie) von endlichen initialen Automaten hergeleitet.

G. Asser (Greifswald)

Kaszerman, Philip 2137

A geometric test-synthesis procedure for a threshold device.

Information and Control **6** (1963), 381–398.

A function $f\colon X_1{}^n\{-1, 1\}=\{(x_1, x_2, \cdots, x_n)\}\to\{0, 1\}$ is a threshold function if and only if there exist $A=(a_1, a_2, \cdots, a_n)$ and b (a_i, b real numbers) such that $a_1x_1+a_2x_2+\cdots+a_nx_n>b$ if and only if $f(x)=1$. The author describes an iterative procedure which determines (in a finite number of steps) A and b for f if it is a threshold function and oscillates otherwise. It is interesting to note that this result is a generalization of the perceptron convergence results [H. D. Block, Rev. Modern Phys. **34** (1962), 123–135; MR **25** #1057]. The generalization can be seen more easily after the non-mathematical jargon is stripped from the perceptron proof as it was done by A. Novikoff [Proc. Sympos. Math. Theory of Automata (New York, 1962), pp. 615–622, Polytechnic Press of Polytechnic Inst. of Brooklyn, Brooklyn, N.Y., 1963].

J. Hartmanis (Schenectady, N.Y.)

Jakubaĭtis, È. 2138

Formal synthesis and rules for simplifying multi-contact logical networks. (Russian)

Latvijas PSR Zinātņu Akad. Vēstis **1963**, *no.* 1 (186), 67–74.

This paper contains a sketchy discussion of some of the relations holding in the algebra generated by the logical operators and the delay operator, and illustrates the application of the relations to some simple examples.

R. M. Baer (Berkeley, Calif.)

Livšic, È. M. 2139

An asymptotic formula for the number of classes of isomorphic autonomous automata with n states. (Russian)

Ukrain. Mat. Ž. **16** (1964), 245–246.

Unter einem autonomen Automaten mit den Zuständen $\sigma_1, \cdots, \sigma_n$ wird eine eindeutige Abbildung der Menge $\{\sigma_1, \cdots, \sigma_n\}$ in sich verstanden. Es bezeichne A_n die Anzahl der untereinander nicht isomorphen autonomen Automaten mit n Zuständen. Durch Betrachtung der von Pólya [Acta Math. **68** (1937), 145–254] aufgestellten Funktion $G(x)=\sum_{n=1}^{\infty} A_nx^n$ wird mit Hilfe der von Murray [J. Assoc. Comput. Mach. **2** (1955), 61–82] angegebenen Formel für $G(x)$ folgende asymptotische Abschätzung für A_n gewonnen:

$$A_n \sim \frac{A}{\sqrt{(\pi n)}}\left(\frac{1}{\tau}\right)^n \qquad (n\to\infty).$$

Dabei bedeutet τ den Konvergenzradius der in Murrays Formel auftretenden Funktion $t(x)$ ($\tau\approx 0{,}34$), während A eine Konstante mit $A\approx 0{,}45\sqrt{\pi}$ ist.

G. Asser (Greifswald)

AUTHOR INDEX

Mathematical Reviews

Vol. 29, No. 3 | March 1965 | Reviews 2140–3328

GENERAL

See also 2157, 2265, 2686, 2823, 2881, 3028, 3267.

Pemberton, John E. 2140
★How to find out in mathematics.
Libraries and Technical Information Division, Vol. 2.
Pergamon Press, Oxford-Paris-Frankfurt; The Macmillan Co., New York, 1963. x + 158 *pp.* $2.45.
This is an up-to-date comprehensive guide to the use of libraries, journals, reference works, and so on for the purpose of obtaining information on mathematical topics at all levels. *S. S. Cairns* (Urbana, Ill.)

★American Mathematical Society Translations. 2141
Series 2, Vol. 35: 12 papers on analysis and applied mathematics.
American Mathematical Society, Providence, R.I., 1964. iv + 363 *pp.* $5.20.
This volume contains translations of two papers by I. P. Mysovskih, and one each by V. A. Zalgaller, M. I. Višik, S. N. Mergeljan and A. P. Tamadjan, L. A. Ljusternik, E. A. Volkov, V. K. Saul'ev, Ju. M. Berezanskiĭ, I. C. Gohberg and M. G. Kreĭn, K. K. Golovkin, and S. V. Iordanskii.

★American Mathematical Society Translations. 2142
Series 2, Vol. 36: 14 papers on groups and semigroups.
American Mathematical Society, Providence, R.I., 1964. iv + 395 *pp.* $5.40.
This volume contains translations of four papers by M. A. Naĭmark, three papers by V. V. Vagner, two by P. A. Gol'berg, one each by I. M. Gel'fand and D. A. Raĭkov, V. A. Kurbatov, A. I. Kostrikin, S. N. Černikov, and L. M. Gluskin.

★American Mathematical Society Translations. 2143
Series 2, Vol. 40: 9 papers on functional analysis and numerical analysis.
American Mathematical Society, Providence, R.I., 1964. iii + 290 *pp.* $4.40.
This volume contains translations of two papers by V. N. Kublanovskaja, and one each by V. N. Kublanovskaja and V. N. Fadeev, A. V. Štraus, Ju. I. Ljubič, O. V. Besov, M. M. Džrbašjan, Ju. L. Šmul'jan and V. B. Lidskiĭ.

★American Mathematical Society Translations. 2144
Series 2, Vol. 41: 4 papers on partial differential equations.
American Mathematical Society, Providence, R.I., 1964. iii + 317 *pp.* $4.40.
This volume contains five papers on differential equations: two by S. D. Èĭdel'man [Mat. Sb. (N.S.) **38 (80)** (1956), 51–92; MR **17**, 857; ibid. **53 (95)** (1961), 73–136; MR **24** #A925], and one each by L. N. Slobodeckiĭ [ibid. **46 (88)** (1958), 229–258; MR **22** #830], Yuh-lin Jou [ibid. **47 (89)** (1959), 431–484; MR **22** #1751], and A. A. Dezin [Trudy Mat. Inst. Steklov. **68** (1962), 88 pp.; MR **28** #2344].

Courant, Richard 2145
Mathematics in the modern world.
Sci. Amer. **211** (1964), 40–49.
An expository article on the nature of mathematics.

Dyson, Freeman J. 2146
Mathematics in the physical sciences.
Sci. Amer. **211** (1964), 128–146.
An expository article on the role of mathematics (in particular, group theory) in the physical sciences.

Davis, Philip J. 2147
Number.
Sci. Amer. **211** (1964), 50–59.
An expository article on the nature of numbers.

Bouligand, G. 2148
Sur les caractères heuristiques, latents ou tangibles, d'œuvres mathématiques récentes.
Math. Notae **19** (1964), 121–131.

Hawkins, David 2149
★The language of nature. An essay in the philosophy of science.
Drawings by Evan L. Gillespie.
W. H. Freeman and Co., San Francisco, Calif.-London, 1964. xii + 372 *pp.* $7.50.
This text is primarily intended for students preparing for a career in science whose general intellectual interests have been aroused by their professional studies and who are seeking an outlet for these interests in the general philosophical questions created by specific scientific topics. The book does, therefore, assume some familiarity with scientific concepts and modes of argument, a fact which enables it to proceed at a somewhat more advanced level than is normal in general treatments of this kind. Of particular interest is the attempt of the author to pay more attention than usual to thermodynamics and

information theory, subjects usually little discussed by philosophers. Apart from this, the broad pattern of discussion as shown by the chapter headings follows traditional lines. The first three chapters deal with the foundations of mathematics, number theory and geometry, including a discussion of the Erlangen Program, analysis, complex numbers and vectors, concluded by a discussion of the problem of universals, which follows a broadly "Aristotelian" viewpoint. This is succeeded by chapters on laws of motion (including the topic of determinism and causality), chance and probability, and a 20-page discussion of relativity and quantum mechanics. After the chapter on thermodynamics (Maxwell's Demon and the arrow of time), there is the usual enquiry into the justification of probable inference. The last hundred pages cover biological evolution, the philosophical problems raised by psychology, a chapter on ethics, and a concluding chapter on economic theory. Anyone pretending to depth in such a limited space would indeed be bold, but the author exhibits at least a careful indebtedness to such varied sources as Sartre, Dewey, Brunswick and Tolman. This seems to be a field where the "glorious amateur" is still permitted to have his say.

G. Buchdahl (Cambridge, England)

HISTORY AND BIOGRAPHY

Hamadanizadeh, Javad 2150

A medieval interpolation scheme for oblique ascensions.
Centaurus **9** (1963/64), 257–265.

The present paper is concerned with the section on oblique ascensions in the unpublished Zīj-i Ashrafī written in Persian in the beginning of the 14th century in Shiraz, Iran, by Muḥammad ibn abī 'Abdallāh Sanjar al-Kāmilī, known as Sayf-i Munajjim. Only one copy is known (in Paris).

This section first describes the classical manner of deriving the oblique-ascension function for a given latitude by means of spherical trigonometry, and then presents an interesting alternate "arithmetical method". Hamadanizadeh, who comes from the school of E. S. Kennedy in Beirut, gives an analysis of this method, as well as a translation of the relevant passage in the manuscript.

The aim of the method is to derive oblique ascensions, degree by degree, from rising times of whole signs, and it is an elegant adaptation of interpolation procedures, perhaps ultimately of Hindu origin, which employ constant second differences. *A. Aaboe* (New Haven, Conn.)

Busard, H. L. L. 2151

Ueber unendliche Reihen im Mittelalter.
Nieuw Tijdschr. Wisk. **51** (1963/64), 132–139.

This concise account concerns chiefly the work of Nicole Oresme (c. 1323–1382), with brief reference to earlier and later contributors. Oresme correctly summed several types of infinite series, using geometric and algebraic means, and he anticipated by some three centuries the proof of Pietro Mengoli (1626–1686) that the harmonic series is divergent. The author consequently points out that it is incorrect to impute to the Middle Ages the view that $\lim_{n\to\infty} u_n = 0$ is a sufficient criterion of convergence.

C. B. Boyer (Brooklyn, N.Y.)

Hofmann, Joseph Ehrenfried 2152

★Frans van Schooten der Jüngere. (English summary)
Boethius, Texte und Abhandlungen zur Geschichte der exakten Wissenschaften, Band II.
Franz Steiner Verlag GmbH, Wiesbaden, 1962. viii + 54 *pp.* (4 *plates*) *DM* 12.00.

The acceptance of the mathematical ideas of Descartes by his contemporaries is largely due to the great efforts of Frans van Schooten (1615–1660). He was an excellent connoisseur of the mathematics of his time and published (besides some books of his own) a collection of the best mathematical works of Viète in 1646 [*Francisci Vietae Opera mathematica*, Lugduni Batavorum, ex officinâ Bonaventurae & Abrahami Elzeviriorum, 1646], but is chiefly remembered as editor of the Latin editions (1649, ²1659/61) of Descartes' *Géométrie* [J. Maire, Leyden, 1637; Latin version published as *Geometria*, J. Maire, Leyden, 1649, second edition, two volumes apud Elzevirios, Amsterdam, 1659/61], originally published in French in 1637. The author gives first a survey of Schooten's life and work (pp. 1–8), then a small selection from Schooten's own publications and his commentaries on the *Géométrie*: on conic sections, cubic equations, the analytical method, and geometrical constructions avoiding the use of compasses (pp. 9–25). Twenty pages of copious notes supply the scholar with the mass of details and facts he is used to looking for in the author's studies. An index of names and letters and a list of the complete titles of the works that we have from Schooten's hand round off this useful though very compact sketch of a man to whose tireless efforts the progress of mathematics in the 17th century owes so much. *C. J. Scriba* (Hamburg)

Biermann, Kurt-R. 2153

Thomas Clausen: Mathematiker und Astronom.
J. Reine Angew. Math. **216** (1964), 159–197.

Diese ist eine auf das ganze einschlägige gedruckte Material und auf viele bisher unbekannte archivalische und daher ungedruckte Quellen gestützte Biographie und Würdigung des wertvollen wissenschaftlichen Werkes.

J. E. Hofmann (Ichenhausen)

Herrmann, Dieter 2154

★Joseph Louis Lagrange (1736–1813).
Archenhold-Sternwarte, Berlin-Treptow, 1963. 21 *pp.*

In dieser vorzugsweise auf die ältere einschlägige Literatur gestützten Kurzdarstellung werden vor allem die Leistungen Lagranges auf dem Gebiet der Himmelsmechanik behandelt. Die Beiträge zur Variationsrechnung und Funktionstheorie sind nur gestreift; die bedeutenden zahlentheoretischen Studien sind überhaupt nicht erwähnt.

J. E. Hofmann (Ichenhausen)

Frege, Gottlob 2155

★Begriffsschrift und andere Aufsätze.
Zweite Auflage. Mit E. Husserls und H. Scholz' Anmerkungen herausgegeben von Ignacio Angelelli.
Georg Olms Verlagsbuchhandlung, Hildesheim, 1964. xvi + 124 *pp.* *DM* 17.80.

Dieser unter Übernahme der von H. Scholz stammenden Verbesserungen durchgeführte Nachdruck der für die Entwicklung der mathematischen Logik so bedeutsamen

Begriffsschrift [L. Nebert, Halle, 1879] wird ergänzt durch die Kleinschriften: "Anwendung der Begriffsschrift" (1879), "Über den Briefwechsel Leibnizens und Huygens mit Papin" [Verlag Akad. Wiss., Berlin, 1881], "Über den Zweck der Begriffsschrift" (1882) und "Über die wissenschaftliche Berechtigung einer Begriffsschrift".

J. E. Hofmann (Ichenhausen)

Nadžakov, Georgi 2156

Academician Professor Nikola Obreškov. (Bulgarian)

Fiz.-Mat. Spis. Bŭlgar. Akad. Nauk **6 (39)** (1963), 232–235.

A brief account of Obreškov's professional career. No bibliography is given.

LOGIC AND FOUNDATIONS

See also 2149, 2155, 2190, 2192, 2290, 3011, 3013, 3014, 3319, 3325, 3328.

Quine, W. V. 2157

The foundations of mathematics.

Sci. Amer. **211** (1964), 112–127.

An expository article on the foundations of mathematics.

Mendelson, Elliott 2158

★Introduction to mathematical logic.

D. Van Nostrand Co., Inc., Princeton, N.J., 1964. x + 300 *pp.* $7.25.

The first half of this admirable text book presents a lucid account of classical two-valued sentence and predicate logic, and a formal system of arithmetic. Gödel's completeness theorem for predicate logic is proved by Henkin's saturation method, and there is a full account of Gödel's incompleteness theorem for formal arithmetic. This part of the book is well suited for a first course in mathematical logic; explanations are clear and motivation adequate. The second half proceeds at a faster rate and covers a great variety of topics, ranging over formalised set theory, Markov algorithms, Turing machines and partial recursive functions. An appendix outlines Schütte's proof of the consistency of formal arithmetic.

R. L. Goodstein (Leicester)

Novikov, P. S. 2159

★Elements of mathematical logic.

Translated by Leo F. Boron with a preface and notes by R. L. Goodstein.

Oliver & Boyd, Edinburgh-London; Addison-Wesley Publishing Co., Inc., Reading, Mass., 1964. xi + 296 *pp.* $7.95.

Das Lehrbuch gibt eine gut verständliche Einführung in die klassische Aussagen- und Prädikatenlogik mit Anwendungen zur beweistheoretischen Begründung der Arithmetik. Unter Verwendung von instruktiven Beispielen werden die Bedeutungen von Axiomensystemen und Logikkalkülen die Problematik mengentheoretischer Interpretationen und der finite Standpunkt von Hilbert sorgfältig erläutert. Die klassische Aussagenlogik wird zunächst auf der Grundlage von Wahrheitstafeln und dann in axiomatischer Weise dargestellt. Entsprechend wird die klassische Prädikatenlogik zuerst in ihrer nichtkonstruktiven mengentheoretischen Interpretation und dann in finiter Weise axiomatisch entwickelt. Die Begriffe der Widerspruchsfreiheit, Vollständigkeit und Unabhängigkeit sind eingehend behandelt, das Deduktionstheorem und einige Sätze zum Entscheidungsproblem abgeleitet. Ein Beweis für den Gödelschen Vollständigkeitssatz ist unter Benutzung der Skolemschen Normalformen durchgeführt. Der Kompaktheitssatz für beliebige Mengen aussagenlogischer Formeln ist als Satz von Maltsev bezeichnet. Der in dem Buch angegebene Beweis dieses Satzes bezieht sich aber nur auf abzählbare Formelmengen, für die Gödel den betreffenden Satz schon zugleich mit seinem Vollständigkeitssatz bewiesen hatte. Im Rahmen der Prädikatenlogik ist die Arithmetik mit den primitiv-rekursiven Funktionen axiomatisch entwickelt. Als Beispiel für beweistheoretische Untersuchungen ist ein neuartiger finiter Beweis für die Widerspruchsfreiheit der Arithmetik unter Ausschluß der vollständigen Induktion und ein Unabhängigkeitsbeweis für das Axiomenschema der vollständigen Induktion durchgeführt.

K. Schütte (Kiel)

Steiner, Hans Georg 2160

Frege und die Grundlagen der Geometrie. I, II.

Math.-Phys. Semesterber. **10** (1963/64), 175–186; *ibid.* **11** (1964), 35–47.

An excellent description of the historical conflict of Frege's ontological founding of mathematics with Hilbert's axiomatics and his use of the term "axiom".

H. Freudenthal (Utrecht)

Sayre, Kenneth M. 2161

Propositional logic in Plato's Protagoras.

Notre Dame J. Formal Logic 4 (1963), 306–312.

Boh, Ivan 2162

Walter Burleigh's hypothetical syllogistic.

Notre Dame J. Formal Logic 4 (1963), 241–269.

From the author's introduction: "It is the purpose of this paper (a) to give an exposition of Burleigh's treatment of those hypothetical syllogisms which involve explicitly hypothetical propositions—that is, conditional, disjunctive, and copulative (or conjunctive) propositions; (b) to examine the relation of special rules governing these syllogisms to the general rules of propositional logic; and (c) to point out the most obvious indebtedness of Burleigh to Boethius, whose writings so greatly influenced the medieval logical tradition."

Lakatos, I. 2163

Proofs and refutations. IV.

British J. Philos. Sci. **14** (1963/64), 296–342.

This is the concluding instalment of the author's discussion of proof and refutation [Part I, same J. **14** (1963), 1–25; MR **27** #4734; Part II, ibid. **14** (1963), 120–139; MR **27** #4735; Part III, ibid. **14** (1963), 221–245; MR **28** #1121], and is devoted to the notion of concept.

da Costa, Newton C. A. 2164

Calculs de prédicats avec égalité pour les systèmes formels inconsistants.

C. R. Acad. Sci. Paris **258** (1964), 1111–1113.

This is a sequel to two notes of the same author [same C. R. **257** (1963), 3790–3792; MR **28** #1123; ibid. **258** (1964), 27–29; MR **28** #2960], where calculi for application to inconsistent systems are considered. The new calculi, $C_n^{=}$ and $C_\omega^{=}$, are obtained from those in the second note by adjoining the usual equality axioms. Properties announced in the previous notes are stated here also for these calculi. For example, adding to $C_1^{=}$ the denial of contradiction yields $C_0^{=}$, which is the ordinary predicate calculus with identity; for each n there is some formula which, adjoined as an axiom in $C_n^{=}$, makes every formula provable, but this is not so for $C_\omega^{=}$. Also, a modified notion of the k-transform is introduced and applied.

H. Ribeiro (University Park, Pa.)

da Costa, Newton C. A. 2165

Calculs de descriptions pour les systèmes formels inconsistants.

C. R. Acad. Sci. Paris **258** (1964), 1366–1368.

See the preceding review [#2164]. In this note, after adding the axioms for descriptions, analogous properties are briefly stated for the new calculi. Consequences of adjoining comprehension axioms are also considered.

H. Ribeiro (University Park, Pa.)

Arruda, Ayda I.; da Costa, Newton C. A. 2166

Sur un théorème de Hilbert et Bernays.

C. R. Acad. Sci. Paris **258** (1964), 6311–6312.

In an earlier paper [#2164 above], the second author introduced the notion of a P_1-P_2-$\cdots$-P_n-k-transform of a formula in predicate calculus. The following theorem is stated without proof. If $\Gamma \vdash F$ in the predicate calculus with q predicate letters $P_1, \cdots, P_q$, then the P_1-P_2-$\cdots$-P_n-k-transform of F is deducible from the P_1-P_2-$\cdots$-P_n-k-transform of Γ in the predicate calculus with $q-n$ predicate letters and k individuals ($n \leqq q$). When $n=q$ this theorem reduces to a theorem of Hilbert and Bernays [*Grundlagen der Mathematik*, Bd. 1, pp. 119 ff., Springer, Berlin, 1934] concerning ordinary k-transforms.

H. J. Keisler (Madison, Wis.)

Leblanc, Hugues 2167

Proof routines for the propositional calculus.

Notre Dame J. Formal Logic **4** (1963), 81–104.

Es wird ein vollständiges System von 13 Schlußregeln des Aussagenkalküls angegeben. Neben die allgemeinen Strukturregeln (R) $A \vdash A$, (E) $A_1, \cdots, A_n \vdash B \Rightarrow A_1, \cdots, A_{n+1} \vdash B$, (P) (Prämissenvertauschung) treten je eine Einführungs- und Eliminationsregel für jede der Verknüpfungen $\sim$, $\supset$, &, $\vee$, $\equiv$; diese Regeln enthalten keine der anderen Verknüpfungen. Z.B. lautet die $\supset$-*Eliminations*-regel

$$A_1, \cdots, A_n \vdash B \supset C;$$

$$A_1, \cdots, A_n \vdash (B \supset D) \supset B \Rightarrow A_1, \cdots, A_n \vdash C.$$

Genauer gilt, daß jede gültige Konsequenz $A_1, \cdots, A_n \vdash C$ beweisbar ist mittels der allgemeinen Strukturregeln und allein derjenigen Regeln, welche die in dieser Konsequenz vorkommenden Verknüpfungen enthalten. Der Beweis für die Existenz solcher "reiner" Ableitungen wird in allen 32 Fällen durch Induktion und Reduktion auf frühere Fälle geführt, wobei die Einschachtelung der Verknüpfungen berücksichtigt wird.

Für Konsequenzen nur mit & und $\vee$ ergibt sich, da die für & und gegebenen Regeln auch intuitionistisch einsehbar sind, der Satz, daß die intuitionistisch und die klassisch gültigen Sätze, die nur & und $\vee$ enthalten, dieselben sind. Hieraus vermutet der Verfasser, daß man kein System der intuitionistischen Aussagenlogik allein durch Verstärkung der Regeln mit & und $\vee$ zu einem vollständigen System des klassischen AK ausbauen kann. Ein Beweis für diese Vermutung wird angekündigt.

Walter Oberschelp (Hannover)

Rieger, Ladislav 2168

Zu den Strukturen der klassischen Prädikatenlogik.

Z. Math. Logik Grundlagen Math. **10** (1964), 121–138.

In der Lindenbaum-Algebra der Klassen syntaktisch äquivalenter Formeln des PK wird die Klasse $[\exists x H]$ bzw. $[\forall x H]$ als Supremum bzw. Infimum der Klassen $[H(x/y)]$ (y beliebige Individuenvariable) gedeutet. Für diese von Mostowski [*Mathematical logic* (Polish), Monografie Mat., Vol. 18, Warsaw, 1948; MR **10**, 229], Sikorski und Rasiowa [Sikorski, Fund. Math. **35** (1948), 247–258; MR **10**, 437; Rasiowa und Sikorski, ibid. **37** (1950), 193–200; MR **12**, 661], Łoś [Studia Logica **2** (1955), 151–212; MR **18**, 785], Henkin [Fund. Math. **37** (1950), 63–74; MR **12**, 662] und anderen gegebene Algebraisierung der Logik beschreibt der Verfasser die nicht-elementare Struktur solcher Supremum- und Infimumbildungen genauer und beweist als Anwendung wiederum den Gödelschen Vollständigkeitssatz. Grundstruktur der Untersuchungen ist die "substitutiv indizierte Boolesche Algebra": Eine Boolesche Algebra **B** (mit Grundmenge B) zusammen mit einer Menge $\mathfrak{Z}_I$ von Indizierungen (d.h. Abbildungen von Indextupeln über I in B) von B heißt indizierte Algebra $(\mathbf{B}, \mathfrak{Z}_I)$, wenn neben gewissen Kompositionsregeln und Minimal- und Abgeschlossenheitseigenschaften der Indizierungen auch B abgeschlossen ist gegenüber einer charakteristischen Supremum- und Infimumbildung über gewisse Indizierungen. U.a. wird die Lindenbaum-Algebra des PK als indizierte Algebra nachgewiesen, und die polyadischen Algebren von Halmos [ibid. **43** (1956), 255–325; MR **19**, 112] und die zylindrischen Algebren von Tarski und Thompson [Bull. Amer. Math. Soc. **58** (1952), 65] werden hier subsumiert. Eine zusätzliche Invarianzeigenschaft der Indizierungsabbildungen gegenüber Ersetzungen von Indices kennzeichnet die indizierte Algebra als substitutiv. Ist ein duales Ideal J aus B abgeschlossen gegenüber einer Durchschnittsoperation bei einer festen Indizierung, so heißt J ein indiziertes Ideal von $(B, \mathfrak{Z}_I)$. Der Primidealsatz für substitutiv indizierte Algebren sagt die Existenz eines indizierter Ideals P aus, das als Ideal von B außerdem prim ist und oberhalb eines beliebig vorgegebenen, gegenüber gewissen Substitutionen abgeschlossenen indizierten Ideals J liegt—wenn nur die Indexmenge größer ist als die Menge der Indizierungen. Aus dem Primidealsatz folgt durch Konkretisierung auf eine geeignet konstruierte Lindenbaum-Algebra der Gödel-Mal'cev-Satz von der Erfüllbarkeit jeder widerspruchsfreien Menge von geschlossenen Formeln des PK über jedem hinreichend großen Individuenbereich. Man kann

die Geschlossenheit der Formeln entbehren und trotz Verringerung der Anforderungen an die Substitutivität von J im Primidealsatz diesen noch analog beweisen.

Ferner wird ein Darstellungssatz für substitutiv indizierte Algebren bewiesen. Die Arbeit schließt mit einigen Definitionen, die eine weitere Übersetzung von klassischen logischen Begriffsbildungen in diese algebraisierte Sprache gestatten. *Walter Oberschelp* (Hannover)

Žuravlev, Ju. I. 2169

Algorithms with finite memory over disjunctive normal forms. (Russian)

Diskret. Analiz. No. 1 (1963), 5–12.

Whereas originally the author described algorithms with fixed memories for the simplification of disjunctive normal forms [Dokl. Akad. Nauk SSSR **132** (1960), 260–263; MR **23** #A1515], he now studies and describes algorithms for the same purpose but with richer memories. {Closely related, but here not mentioned by the author, are his previous papers [ibid. **139** (1961), 1329–1331; MR **28** #25; Problemy Kibernet. **8** (1962), 5–44; MR **29** #4672; Dokl. Akad. Nauk SSSR **151** (1963), 1025–1028; MR **28** #26], and it appears that they must be studied in series.} *E. M. Fels* (Munich)

Žuravlev, Ju. I. 2170

On a class of functions of the algebra of logic which are not everywhere defined. (Russian)

Diskret. Analiz. No. 2 (1964), 23–27.

Building on, and in certain ways presupposing, his extensive previous paper, mainly about the simplification of disjunctive normal forms [Problemy Kibernet. **8** (1962), 5–44; MR **29** #4672], the author resumes the inquiry begun in his preceding note [#2169 above] and develops various combinatorial formulas pertaining to partial boolean functions. *E. M. Fels* (Munich)

Žuravlev, Ju. I. 2171

On a class of algorithms for choosing an element from a finite set. (Russian)

Diskret. Analiz. No. 2 (1964), 28–36.

This is an abstract treatment of what is commonly called "table lookup". Its implications seem to go rather beyond that. Algorithm-theoretic and information-theoretic considerations are blended for the purpose of optimizing search strategies (for certain problem classes).

Consider a matrix $T_{n\varphi}$ with n columns and φ rows [sic]. All entries consist of zeros and ones. No two rows are equal. A submatrix $T_{n\varphi'}$ of $T_{n\varphi}$ is obtained by deleting some row(s) of $T_{n\varphi}$. Row labels thereby remain unchanged. The problem is to find s_i, the ith row of $T_{n\varphi}$. By an operator A_i ($i=0, 1, \cdots, m$) a pair $(s, T_{n\varphi'})$ is carried into a new submatrix $T_{n\varphi''}$, where $s \subseteq T_{n\varphi''} \subseteq T_{n\varphi'}$ and s is one row of $T_{n\varphi'}$. A finite sequence $A_{j_1}, \cdots, A_{j_k}$, where $0 \leqq j_i \leqq m$ and $i=1, \cdots, k$, constitutes a (search) algorithm that realizes the row s of $T_{n\varphi}{}^0$ if $A_{j_1}(s, T_{n\varphi}{}^0) = T_{n\varphi}{}^1$, $A_{j_2}(s, T_{n\varphi}{}^1) = T_{n\varphi}{}^2, \cdots, A_{j_k}(s, T_{n\varphi}{}^{k-1}) = s$.

In a manner specified below, a positive number P is assigned to each application of an operator to a submatrix of $T_{n\varphi}$, say $P = P(A_i, T_{n\varphi'})$. Letting

$$\sum_{m=1}^{k} P(A_{j_m}, T_{n\varphi}{}^{m-1}) = P_s(K),$$

the weight $P_{T_{n\varphi}}(K)$ of K for $T_{n\varphi}$ is defined as $\max P_s(K)$, where the maximum is taken with respect to all $s \subseteq T_{n\varphi}$.

Consider $P(n) = \max(\min P_{T_{n\varphi}}(K))$, where the maximum is taken with respect to all matrices $T_{n\varphi}$ as characterized above and the minimum with respect to all somehow admissible (search) algorithms K. In what ensues, the behavior of $P(n)$ as $n \to \infty$ is studied relative to a special class of algorithms.

To make allowances for the information contained in a row, as the number of columns mirrors it, assignments are made as follows: $\beta(A_j, T_{n\varphi'}) = \log_2 n$, $j=1, \cdots, n$; $\beta(A_j) = \sum_{i=1}^{r} \beta(A_j, T_{n\varphi_i})$; and $\beta(K) = \sum_{j=0}^{m} \beta(A_j)$.

The operator A_j in the sequence of applications of $A_0, A_1, \cdots, A_n$ shall, for the time being, not be A_0. Then through A_j the jth entry α_j of row s is compared with all other entries of the jth column of the respective submatrix at hand. Those and only those rows whose jth entry equals α_j constitute the new submatrix. Further, put $P(A_j, T_{n\varphi'}) = P_1(\varphi')$, where P_1 is defined, nonnegative, nondecreasing, and continuous on all nonnegative real numbers. $P_1(x) = x$ is of particular interest.

Turning to A_0, it is demanded that $A_0(s, T_{n\varphi'}) = s$ for all submatrices $T_{n\varphi'}$. A distance $\rho(s_i, s_j)$ between rows $s_i = (\alpha_1 \cdots \alpha_n)$ and $s_j = (\beta_1 \cdots \beta_n)$ is defined as $\sum_{i=1}^{n} |\alpha_i - \beta_i|$. Suppose that $s_1, \cdots, s_{\varphi'}$ are the rows that constitute the submatrix $T_{n\varphi'}$. Put $\sum_{i=1}^{\varphi'} \rho(s_1, s_i) = \lambda_s$. Then $\theta = \min_s \lambda_s$ is said to be the type of the submatrix $T_{n\varphi'}$, the minimum being taken with respect to all its rows. Now a function P_2 is defined by the equation $P_2(A_0, T_{n\varphi'}) = P_2(\theta)$ and with the requirement that it possess the properties of P_1. Also, for A_0, one puts $\beta(A_0, T_{n\varphi'}) = \theta \log_2 n + n$.

It turns out that A_0 is an efficient search tool in those cases where $A_1, \cdots, A_n$, which "on the average" work as dichotomizers, are inefficient. Of special interest, incidentally, is the P_2 defined as $P_2(\theta) = \theta \log_2 n$.

An algorithm K for $T_{n\varphi}$ is regular if it consists of applications of $A_0, A_1, \cdots, A_n$ and $\beta(K)$ satisfies the condition that $\lim_{n\to\infty}(\beta(K)/n \cdot \varphi(n)) = 0$.

It is proved that for any matrix $T_{n\varphi}$ (as characterized at the beginning) there is a regular algorithm. Further, it turns out that if in each column of $T_{n\varphi}$ the number of zeros or the number of ones does not exceed χ ($0 < \chi < \varphi/2$), then the type θ of $T_{n\varphi}$ is not greater than $n\chi$, and $\inf_K P(K) \leqq P_2(\theta) \leqq P_2(n\chi)$, the infimum being taken with respect to regular algorithms only.

It is proved that, if $0 < \chi(n) < \psi(n) < \varphi(n)$ and

$$\lim_{n\to\infty}(\chi(n) \cdot \log_2 n/\psi(n)) = 0,$$

then K, fulfilling such requirements, is a regular algorithm. The author constructs such a regular K in extenso.

It is also proved that (i) if $\lim_{n\to\infty}(\chi(n) \cdot \log_2 n/\psi(n)) = 0$, $\lim_{n\to\infty} \psi(n)/\varphi(n) = 0$, and $\lim_{n\to\infty} \chi(n) = \infty$, then

$$P(n) < \frac{1}{\chi(n)} \int_{\chi}^{\varphi+\chi} P_1(x)\,dx + \max\left\{\int_1^{\psi+1} P_1(x)\,dx, P_2(n\chi)\right\}$$

and (ii)

$$P(n) < \sqrt{(2n \cdot \log_2 n) \cdot \varphi(n)}(1+\varepsilon), \text{ with } \varepsilon \to 0 \text{ as } n \to \infty.$$

{The author makes no reference at all to the pertinent literature, although there is bound to be some. At least one contribution that comes to mind in this context is the one by T. N. Hibbard [J. Assoc. Comput. Mach. **9** (1962), 13–28; MR **27** #2135].} *E. M. Fels* (Munich)

Žuravlev, Ju. I. 2172

On inessential variables of functions of a Boolean algebra which are not defined everywhere. (Russian)

Diskret. Analiz. No. 1 (1963), 28–31.

Consider the set of n-place partial boolean functions, viz., functions defined on some subset of all n-tuples of form $(x_1, \cdots, x_n)$, each of whose elements x_i is 0 or 1, and taking 0 or 1 as values, a set M_n of arguments $\{x_1, \cdots, x_n\}$, and an arbitrary nonempty subset $M = \{x_{i_1}, \cdots, x_{i_k}\}$ of M_n. Such a subset M is said to be admissible for F if there is a partial boolean function G such that $F(x_1, \cdots, x_n) = G(x_{i_1}, \cdots, x_{i_k})$. (The arguments that have been omitted from G as inessential are, so to speak, those on whose being 0 or 1 the value of F does not depend once zeros and ones have been assigned to the other argument places.) An admissible set M is also terminal for F if there is no proper subset of M that is admissible for F.

The number of distinct terminal sets for F is proved to be $C_n^{[n/2]}$. Incidentally, every boolean function has a unique terminal set.

Finally, let $K_1(F)$ be the cardinal number of the largest, and correspondingly $K_2(G)$ of the smallest, terminal set for F. Putting $K_1(F)/K_2(F) = m_n(F)$ and

$$\max_{F(x_1, \cdots, x_n)} m_n(F) = m(n),$$

it is proved that $m(n) = n - 1$. *E. M. Fels* (Munich)

Vasil'ev, Ju. L. 2173

On the number of terminal and minimal disjunctive normal forms. (Russian)

Diskret. Analiz. No. 2 (1964), 3–9.

This paper continues the line of inquiry begun by S. V. Jablonskiĭ [Problemy Kibernet. **2** (1959), 75–121; MR **23** #B2125] concerning algorithmic difficulties in network synthesis. The author aims at constructing examples of functions of the algebra of logic in which there are "many" terminal and minimal disjunctive normal forms, and he provides diverse combinatorial formulas for the assessment of their number. *E. M. Fels* (Munich)

Pieczkowski, A. 2174

On the systems Q and Q_f.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 137–139.

Die von Jáskowski [Studia Soc. Sci. Torunensis Sect. A **1** (1948), 1–15; MR **10**, 2; ibid. **1** (1948), 17–21; MR **10**, 2; ibid. **1** (1948), 23–35; MR **10**, 2] eingeführten Kalküle Q und Q_f erweisen sich als gegenseitig ineinander interpretierbar. Q ist dabei ein Kalkül, der sich durch Einsetzungen atomarer Formeln für Aussagevariablen in den üblichen Prädikatenkalkül PK verpflanzen läßt und dessen Satzmenge mittels des Satzbegriffes im PK definiert ist. In Q_f kann die Relevanz von Variablen ausgedrückt werden, und auch eine Art modaler Implikation (Faktorimplikation) ist dort definierbar. Jeder Formel γ aus Q_f wird in Bezug auf eine Variablenmenge $x_1, \cdots, x_n$ eine Übersetzung $T_n(\gamma)$ in den PK zugeordnet. Sätze von Q_f sind die Formeln γ, für die $T_n(\gamma)$ stets ein Satz des PK ist. Es genügt, für ein geeignetes s die Formel $T_s(\gamma)$ als Satz des PK nachzuweisen. Hierbei ist s als "Rang" von γ eine kombinatorisch ermittelte Anzahl, die mit den verschiedenen Quantifizierungsmöglichkeiten aller "wesentlichen" Systemvariablen zusammenhängt.

Walter Oberschelp (Hannover)

Pieczkowski, A. 2175

A set of axioms for the system Q_f of factorial implication.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 141–142.

Die in Q_f (vgl. vorstehendes Referat #2174] definierte Faktorimplikation $p > q$, die anschaulich besagt, daß für alle Belegungen der wesentlichen Variablen (Faktoren) von p die Implikation $p \to q$ gilt, wird in einem System Q_f^a als Grundbegriff mit eingeführt und axiomatisch so beschrieben, daß die Satzmenge von Q_f^a eine Untermenge derjenigen von Q_f ist und daß Q in Q_f^a interpretierbar ist, d.h., jeder Formel von Q eine Formel von Q_f^a entspricht. Ferner ist das System S5 von Lewis-Langford interpretierbar in Q_f^a: $p \prec q$ entspricht $\neg(p \to q) > (p \to q)$, $\Diamond p$ aber $\neg(p > \neg p)$. *Walter Oberschelp* (Hannover)

Pieczkowski, A. 2176

On the equivalence of the calculus of dependent sentential variables and the cylindrical algebra without diagonal elements.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 143–146.

Eine Variante des Systems Q (vgl. die beiden vorstehenden Referate #2174 und #2175) erweist sich als "absolut äquivalent" mit dem System der Gleichungen einer abstrakten zylindrischen ω-dimensionalen Algebra ohne Diagonalelemente [Henkin, *Mathematical interpretation of formal systems*, pp. 57–84, North-Holland, Amsterdam, 1955; MR **17**, 700; Henkin und Tarski, Proc. Sympos. Pure Math., Vol. II, pp. 83–113, Amer. Math. Soc., Providence, R.I., 1961; MR **23** #A1564]. Dies bedeutet, daß nicht nur die Übersetzungen von Sätzen der einen Theorie Sätze der anderen Theorie sind, sondern daß auch die Übersetzungsabbildungen wechselseitig umkehrbar sind. Dabei werden die Symbole 0, 1, +, ·, ′, C_i, = der zylindrischen Algebra bezüglich durch $q \wedge \neg q$, $q \vee \neg q$, Alternative $\vee$, Konjunktion $\wedge$, Negation $\neg$, Existenzquantifikation $\bigvee x_i$ und Deduktionsäquivalenz $(\vdash \alpha \leftrightarrow \beta)$ ersetzt. *Walter Oberschelp* (Hannover)

Shoenfield, J. R. 2177

The problem of predicativity.

Essays on the foundations of mathematics, pp. 132–139. *Magnes Press, Hebrew Univ., Jerusalem*, 1961.

The main result of this paper is that if $A(\alpha, \beta)$ is a Π_1^1-predicate and β_0 a constructible function, then either the set of α satisfying $A(\alpha, \beta_0)$ is empty or there is a constructible α with this property. Here $\alpha, \beta, \gamma, \cdots$ are functions from integers to integers. The phrase "α is constructible" means the same as $\alpha \in L$ in Gödel's monograph on the consistency of the axiom of choice and the continuum hypothesis. A Π_1^1-predicate is one which has the form $(\gamma)B(\alpha, \beta, \gamma)$, where B is arithmetical.

As corollaries the author obtains: (I) If A is an arithmetical predicate and γ_0 a constructible function, then $(\alpha)(\mathrm{E}\,\beta)A(\alpha, \beta, \gamma_0) \equiv (\alpha)_L(\mathrm{E}\,\beta)_L A(\alpha, \beta, \gamma_0)$ where $(\alpha)_L$ [$(\mathrm{E}\,\alpha)_L$] means: for every constructible α [there is a constructible α]. (II) Every Σ_2^1-set of integers (as well as every Π_2^1-set of

integers) is constructible. If there exist non-constructible sets of integers, then, as the author shows, this result cannot be improved to the case of three quantifiers.

An ordinal is called a Δ_2-ordinal if it is the order type of a relation R which belongs to the intersection $\Sigma_2^1 \cap \Pi_2^1$. The next corollary states that the $\Sigma_2^1 \cap \Pi_2^1$-sets of integers coincide with those constructible sets of integers whose orders in the Gödel hierarchy are the Δ_2-ordinals. This result is particularly interesting, as it shows how big the set of Δ_2-ordinals is.

Proofs of these theorems are preceded by a brief but illuminating discussion of the notion of predicativity.

A. Mostowski (Warsaw)

Maehara, Shôji; Takeuti, Gaisi 2178

A formal system of first-order predicate calculus with infinitely long expressions.

J. Math. Soc. Japan **13** (1961), 357–370.

The formal system discussed here is a generalization of Gentzen's **LK**. Namely, let Ω_0 be a fixed, strongly inaccessible number and let us call in the following any well-ordered sequence of an ordinal type less than Ω_0 admissible. Variables and predicate symbols of this system are ordinary ones, but formulas are those expressions formed by negations of formulas, generalized disjunctions of admissible sequences of formulas of the form $\bigvee_{\lambda<\alpha} \mathfrak{A}_\lambda$, and generalized existential quantifications on admissible sequences $\mathfrak{t}$ of mutually distinct variables of the form $\exists\, \mathfrak{t}\mathfrak{A}(\mathfrak{t})$. The antecedent and the succedent of any sequent are also admissible sequences of formulas. Inference rules are naturally modified, e.g., introduction of $\bigvee$ and $\exists$ in the antecedent are

$$\frac{\{\mathfrak{A}_{\lambda_\mu,\mu}\}_{\mu<\beta}, \Gamma \to \Theta \text{ for all } \{\lambda_\mu\}_{\mu<\beta} \text{ such that } \lambda_\mu < \alpha_\mu \ (\mu < \beta)}{\{\bigvee_{\lambda<\alpha_\mu} \mathfrak{A}_{\lambda,\mu}\}_{\mu<\beta}, \Gamma \to \Theta},$$

$$\frac{\{\mathfrak{A}_\lambda(\mathfrak{a}_\lambda)\}_{\lambda<\alpha}, \Gamma \to \Theta}{\{\exists\, \mathfrak{x}_\lambda \mathfrak{A}_\lambda(\mathfrak{x}_\lambda)\}_{\lambda<\alpha}, \Gamma \to \Theta},$$

where the $\mathfrak{a}$'s are sequences of distinct variables not occurring in the conclusion. It is assumed for any free variable occurring in a proof figure that an ordinal number called the height of the variable and denoting a certain order of dependency of the variable to other free variables in the proof figure is uniquely determined. The main theorem of the work is that every sequent which is valid in any non-empty domain is provable without cut. This theorem implies the completeness theorem in Gödel's sense, the cut-elimination theorem, the Löwenhein-Skolem theorem, and Craig's interpolation theorem for the system.

K. Ono (Nagoya)

Hájek, Petr 2179

Die durch die schwach inneren Relationen gegebenen Modelle der Mengenlehre.

Z. Math. Logik Grundlagen Math. **10** (1964), 151–157.

The author works on the basis of Gödel's axioms for set theory without the axiom of choice and with a weakened form of the axiom of regularity:

$$(\mathrm{E}a)(X)(\mathrm{E}u)\{(X = 0) \vee [(u \in X) \ \&\ (u \cap X \subset a)]\}.$$

Following P. Vopěnka [same Z. **8** (1962), 281–292; MR **26** #3610] he defines "half-models" of set theory as pairs (ψ, E), where ψ is a formula with one free class variable and E is a relation. Only classes satisfying $\psi(X)$ are taken as classes of the half-model. Sets of the half-model are classes of the form $\phi(z) = E''\{z\}$, where z is in the domain of E. A set $\phi(z)$ of the half-model bears the $\in$-relation of the half-model to a class X if and only if $z \in X$.

Vopěnka assumed in his paper that E is an "internal" relation, i.e., that $\phi(x) = \phi(y) \equiv x = y$ for arbitrary x, y in the domain of E. The present author weakens this assumption to $[\phi(x) = \phi(y)]$ & $(\langle x, z\rangle \in E) \to (\langle y, z\rangle \in E)$ for arbitrary x, y, z in the domain of E. Relations satisfying this condition are called weakly internal.

Theorems established in the paper give conditions (analogous to those given by Vopěnka) under which (some or all) axioms A-C are valid in a half-model with a weakly internal relation E. The following result is called the equivalence theorem: a half-model with a weakly internal relation is elementarily equivalent to a half-model with an internal relation. The author comments that because of this result, "die schwach inneren Relationen haben hauptsächlich methodische Bedeutung".

A. Mostowski (Warsaw)

Asenjo, F. G. 2180

Relations irreducible to classes.

Notre Dame J. Formal Logic **4** (1963), 193–200.

From the author's introduction: "The purpose of this paper is to formalize Whitehead's concept of internal relation, presenting it as a kind of variable independent of the two-place predicate definition of relation."

Chang, C. C. 2181

The axiom of comprehension in infinite valued logic.

Math. Scand. **13** (1963), 9–30.

The paradox of Russell shows that the axiom of comprehension is inconsistent in ordinary two-valued logic, and a similar argument shows that it is also inconsistent in every finite-valued logic. However, in the infinite-valued logic of Łukasiewicz, which the author denotes by Ł, the question of the consistency of the axiom of comprehension is still open. The full axiom of comprehension is the schema Σ_0 of all formulas of the form

$$(1) \qquad \forall x_1 \cdots x_n \exists y \forall t(t \in y \leftrightarrow U(t, x_1, \cdots, x_n)),$$

where U is an arbitrary formula of Ł. Skolem [Z. Math. Logik Grundlagen Math. **3** (1957), 1–17; MR **20** #3067] proved that the set Σ_1 of all formulas (1) such that U has no quantifiers is consistent in Ł (that is, there is a model for Ł in which every formula in Σ_1 has truth value 1). In this paper two other subsets of Σ_0, namely Σ_2 and Σ_3, are shown to be consistent in Ł. Σ_2 consists of all formulas (1) such that U contains no free variables other than t and y but may contain arbitrary quantifiers. Σ_3 is the set of all formulas (1) with the property that no bound variable u of U can occur in the first place of an atomic formula of the form $u \in v$ unless u is already the variable v. Thus Σ_3 is considerably larger than Σ_1. In the formulas of Σ_3, it is also permissible for the variable y to occur in U. The theorems hold whether or not Ł has an identity symbol. The proofs use Skolem's methods, the compactness theorem for Ł, and the Brouwer fixed-point theorem.

The author points out that most of the familiar axioms of Zermelo-Fraenkel set theory are not included in Σ_2 or Σ_3, so it is important to prove that larger subsets of Σ_0 are consistent. He also asks whether Σ_0 is consistent in

the weaker sense that there is a model for L in which every formula in Σ_0 has truth value greater than $\frac{1}{2}$.
H. J. Keisler (Madison, Wis.)

Halpern, J. D. 2182
The independence of the axiom of choice from the Boolean prime ideal theorem.
Fund. Math. **55** (1964), 57–66.

Let G be the axiomatic system of set theory based on axioms A, B, C of Gödel; thus axioms of choice and of regularity are excluded from G. Let AC be the abbreviation for the "axiom of choice" and EPI the abbreviation for the following axiom: There is a function F defined on the class of all Boolean algebras and such that $F(x)$ is a prime ideal of x for each Boolean algebra x. The main result of the paper states that if G is consistent, then the implication EPI→AC is not provable in G.

The proof is obtained by the permutation method. More specifically, the author proves that EPI is true in the model W^+ of the reviewer [cf. Fund. Math. **32** (1939), 201–252]. The proof that this is really so requires an ingenious application of a strengthened version of the Ramsay theorem due to Rado [J. London Math. Soc. **29** (1954), 71–83; MR **16**, 455]. *A. Mostowski* (Warsaw)

Caton, Charles E. 2183
A stipulation of a modal propositional calculus in terms of modalized truth-values.
Notre Dame J. Formal Logic **4** (1963), 224–226.

The author gives a definition of "logical truth" for a system containing the following "modalized truth-values": "logical truth", "contingent truth", "contingent falsity", and "logical falsity". He does not proceed axiomatically but relies upon the use of a non-standard stipulation in terms of 4-valued tables. The trick is to calculate standard 4-valued tables for well-formed expressions and then to rewrite the calculated tables according to specified rules. The latter are designed to yield a class of acceptable modal expressions which agree with the author's preconceived intuition regarding these expressions. Without raising questions concerning the authority of such intuition, there is still the question of the ad hoc character of the rewrite rules. It would seem desirable to develop, if at all possible, a theory of modal acceptability and rejectability which would be entirely free of all stigmas associated with the use of ad hoc rules.
A. Turquette (Champaign, Ill.)

Kirin, Vladimir G. 2184
On the polynomial representation of operators in the n-valued propositional calculus. (Serbo-Croatian summary)
Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II **18** (1963), 3–12.

Let T_n^k be the set of k-ary operators in an n-valued propositional calculus and $I_n[x_1, \cdots, x_k]$ the ring of polynomials in k indeterminates over the ring I_n of integers modulo n. A polynomial A of this ring represents an operator $f \in T_n^k$ if, for every $(x_1, \cdots, x_k) \in I_n^k$, we have $A(x_1, \cdots, x_k) = f(x_1, \cdots, x_k)$. It is shown that every $f \in T_n^k$ having a representative in $I_n[x_1, \cdots, x_k]$ is represented in a proper subset whose cardinality does not exceed that of T_n^k. Thus, in cases of smaller cardinality, there are necessarily operators in T_n^k not representable by polynomials. *E. J. Cogan* (Bronxville, N.Y.)

Keisler, H. Jerome 2185
Unions of relational systems.
Proc. Amer. Math. Soc. **15** (1964), 540–545.

Relational systems of a fixed similarity type are considered. A system $\mathfrak{A}$ is called an n-union of a set of systems M if each $\mathfrak{B} \in M$ is a subsystem of $\mathfrak{A}$ and each set of at most n elements of $\mathfrak{A}$ is included in some $\mathfrak{B} \in M$. The result of the paper is the following: For every elementary class K in the wider sense (i.e., $K \in EC_\Delta$) and every non-negative integer n, the following conditions are equivalent: (1) if $\mathfrak{A}$ is an n-union of some subset of K, then $\mathfrak{A} \in K$; (2) K is the class of all models of a set of sentences of the form $\forall v_1 \cdots \forall v_n \exists w_1 \cdots \exists w_k \Phi$, where Φ has no quantifiers or free variables other than $v_1, \cdots, v_n, w_1, \cdots, w_k$ $(k = 0, 1, \cdots)$. A generalization of this theorem consisting in "relativizing it to an arbitrary elementary theory" is also included.

As mentioned by the author, the easy case $n = 0$ of the theorem is well known (Tarski, Łoś, Henkin). Especially interesting is the case $n = 1$, since it is a characterization of elementary classes closed under unions. (A related characterization of elementary classes closed under unions of increasing chains of models was given by Łoś and Suszko [Fund. Math. **44** (1957), 52–60; MR **19**, 724] and Chang [Proc. Amer. Math. Soc. **10** (1959), 120–127; MR **21** #2576].) Methods of Morley and Vaught [Math. Scand. **11** (1962), 37–57; MR **27** #37] are employed in the proof. *J. Mycielski* (Wroclaw)

Železnikar, Anton 2186
Some arithmetic normal algorithms. (Serbo-Croatian summary)
Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II **17** (1962), 159–170 (1963).

Entirely in the confines of and derivative from the theory of algorithms of A. A. Markov [Trudy Mat. Inst. Steklov. **42** (1954); MR **17**, 1038], this paper specifies several translation algorithms for alphabets of numeral signs and affords proofs of a few concomitant theorems.
E. M. Fels (Munich)

Železnikar, Anton P. 2187
Some algorithm theory and its applicability. (Serbo-Croatian summary)
Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II **18** (1963), 141–158.

This paper, a paragon of expository clarity, presents another modification of the concept of normal algorithm (NA) in the sense of A. A. Markov [Trudy Mat. Inst. Steklov. **42** (1954); MR **17**, 1038], the modification apparently having been inspired by A. A. Ljapunov [Problemy Kibernet. **1** (1958), 5–22].

First, instead of one kind of concatenation for the formation of words from an alphabet, there is now a lateral and a frontal concatenation; e.g.,

$$P = \begin{matrix} aaa \\ bbb \\ aaa \end{matrix}$$

is a word.

Second, the replacement commands of NA's also take a kind of two-dimensional form, viz.,

$$\begin{matrix} a & & e \\ & bb \xrightarrow{\alpha} cd & \\ a & & e \end{matrix}$$

stands for the command to replace the word occurrence left to the arrow by the word occurrence right of the arrow (where α identifies the subscheme from which the replacement command is taken).

Third—and this is an arguable departure from the purposely local character of NA operations—such replacements are to be effected simultaneously in all places in which the word left of the arrow occurs in a given word.

Fourth, to avoid ambiguity, this calls for rules of dominance, e.g., that c dominates d. Taking this rule, the word P as specified above, and applying the mentioned replacement command to P in the two places where it is possible, one obtains the word

$$Q = \begin{matrix} aee \\ ccd \\ aee \end{matrix}$$

Fifth, instead of NA conventions that regulate the sequence of replacement-command executions, such a sequence is now specified explicitly, in the manner suggested by Ljapunov, by a characteristic.

Thus, a Železnikar algorithm is specified by an alphabet (with dominance convention), a scheme of replacement commands (coming in the form of subschemes), and a characteristic that governs the sequence of multiple-replacement operations.

After giving some examples of algorithms of this kind, the author proves an analogue of Markov's composition theorem [loc. cit., 101–119]. *E. M. Fels* (Munich)

Železnikar, Anton 2188

Behandlung logistischer Probleme mit Ziffernrechner. (Serbo-Croatian summary)

Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II **17** (1962), 171–179 (1963).

Neither claiming nor obtaining new results, this note expounds principles of digital-computer treatment of propositional logic, up to and including the problem of finding shortest disjunctive normal forms, with a method for which A. Svoboda [Stroje na Zpracování Informací **2** (1954), 157–208; MR **17**, 111] is given credit.

E. M. Fels (Munich)

SET THEORY

See also 2179, 2181, 2182.

Jones, R. E. D. 2189

Opaque sets of degree α.

Amer. Math. Monthly **71** (1964), 535–537.

Let Q denote the closed unit square in the plane. A set $A \subset Q$ is "opaque" [Bagemihl, Michigan Math. J. **6** (1959), 99–103; MR **21** #4396] provided every (straight) line which meets Q meets A. The author exhibits an opaque set of length only $\sqrt{2}+\frac{1}{2}\sqrt{6}$ $(=2.639)$ and gives the following theorem: Given a cardinal number α such that $2 \leq \alpha \leq c$, there exists an opaque set A such that every line which meets the interior of Q meets A in exactly α points. {The theorem is true, but the author's argument employs an illegitimate transfinite induction hypothesis and seems to need non-trivial alterations as well as a trivial alteration to make the set constructed be opaque.}

A. H. Stone (Rochester, N.Y.)

Bleicher, M. N. 2190

Some theorems on vector spaces and the axiom of choice.

Fund. Math. **54** (1964), 95–107.

Let (A, F) denote the statement: Every vector space V over the field F possesses a basis. Let (B, F) denote the statement: Every vector space V over the field F is such that if S is a subspace of V, then there is a subspace S' of V such that $S \cap S' = \{0\}$ and $S \cup S'$ generates V. The author shows that the assumption that the axiom of choice holds is equivalent to assuming that, for some field F, each of (A, F) and (B, F) holds. Several similar equivalents of the axiom of choice are also established. The author then asks similar questions about the wider class of generalized vector spaces (i.e., the generalized vector spaces defined by proper dependence relations of the author and the reviewer [Publ. Math. Debrecen **8** (1961), 55–63; MR **24** #A124]). It is shown, for example, that the axiom of choice is equivalent to the statement: Any generalized vector space possesses a basis.

G. B. Preston (Clayton)

Mullin, Albert A. 2191

On differences of certain structured sets.

Notre Dame J. Formal Logic 4 (1963), 158–160.

Robert, Pierre 2192

Sur un problème de disjonction (ensembliste, algébrique, etc.).

C. R. Acad. Sci. Paris **258** (1964), 6045–6048.

The author continues his investigation of φ-spaces (axiomatic dependence theory) [see same C. R. **258** (1964), 34–37]. Several existence theorems are given for φ-disjoint families in terms of natural cardinality conditions. Applications are indicated for field theory (algebraic independence) and geometry (direct unions of varieties).

D. Monk (Berkeley, Calif.)

COMBINATORIAL ANALYSIS

See also 2208, 3010, 3012, 3268.

Yaglom, A. M. [Jaglom, A. M.]; Yaglom, I. M. [Jaglom, I. M.] 2193

★Challenging mathematical problems with elementary solutions. Vol. I: Combinatorial analysis and probability theory.

Translated by James McCawley, Jr.; revised and edited by Basil Gordon.

Holden-Day, Inc., San Francisco, Calif.-London-Amsterdam, 1964. viii+231 *pp.* $5.95.

This is the first of a two-volume translation and adaptation of a Russian problem book [*Non-elementary problems*

in an elementary exposition (Russian), GITTL, Moscow, 1954; MR **17**, 18] by the twin brothers, Akiva and Isaak Yaglom (Volume II is subtitled *Problems from various branches of mathematics*). It contains 100 problems, many with several parts, their solutions, and their answers, in this order. Harder problems are marked by one, two, or three asterisks to indicate order of difficulty. It is intended that the reader should try to solve all problems (or at least the easier ones), using the answers, which include hints for solution, if necessary, before turning to the solutions. The latter, which occupy about eighty per cent of the book, may be read independently as a book of exercises in combinatorial analysis and probability. The character and scope of the problems are indicated by the several sections in which they are grouped and the number in each section, namely: introductory problems (11); the representation of integers as sums and products (22); combinatorial chessboard problems (9); combinatorial geometric problems (14); binomial coefficient identities (7); simple probabilities (22); probabilities for experiments with infinitely many possible outcomes (9); probabilities for experiments with a continuum of possible outcomes (6). Nineteen of the problems (in whole or in part) have appeared in the Moscow Mathematical Olympiads. A few remarks on individual problems may help to further characterize the book. Number 6 asks for the number of inequivalent ways of coloring a cube with six different colors when two colorings are equivalent if a rotation of the cube brings them to identity; hence it serves as an introduction to George Pólya's fundamental theorem in enumerative combinatorial analysis, which however seems to be beyond the scope the authors have intended. Number 13a asks for the number of integers less than 1000 and divisible by neither 3 nor 7, while 23a asks effectively for the denumerant $D_n(1, 2, 3)$ (the number of partitions of n into parts 1, 2 or 3). 34a asks for the number of ways of putting n non-attacking rooks on an n-by-n chessboard, and 34b for the smallest number of rooks on the same board so that every non-rook position is under rook attack, thus illustrating both enumerative and non-enumerative aspects. 54 is the problem of finding how many ways $2n$ points on the circumference of a circle may be joined in pairs by n chords which do not intersect within the circle; the answer is the Catalan number $c_n = (n+1)^{-1}\binom{2n}{n}$, which is also the number of two-candidate election returns with final vote (n, n) and such that a given candidate is never behind his opponent. 84 gives the mapping identifying these two problems. The binomial identity problems all have several parts, and many have distinct combinatorial flavor. The authors' solutions all use generating functions, but even this powerful tool may have its bad moments, as is evident in 57i. The identity here is a reflection of the fact that in one version of Banach's matchbox problem (which is problem 73) the probabilities must add to unity; the generating function solution is far from the two-line simplicity of the proof by recurrence given by Josef Kaucký [Mat.-Fyz. Časopis Sloven. Akad. Vied **12** (1962), 28–35; MR **28** #2056] (on the other hand, the authors' solution of 53, on which depends the solution of 54 mentioned above, is entirely by recurrence, possibly because of the stage of its appearance, and could have been appreciably shortened by the proper generating function). 77 is the problem of the four liars, and the solution gives also the generalization to n liars, unhappily without mention of William Feller [Amer. Math. Monthly **58** (1951), 606–608; MR **13**, 363]. 80 is the "problème des rencontres" in its probability guise; 82 is a special case of the "problème des ménages" in a similar odd context. 81 is a distribution problem in whose solution the Stirling numbers of the second kind are successfully hidden; indeed, what is worse, they are given a new notation conflicting with that for q-binomial coefficients. Despite these minor flaws, the book is exciting as well as coherent, the problems carefully chosen, or ingeniously created, the solutions clear and often elegant. Perhaps all collaborations should be by twin brothers.

J. Riordan (Murray Hill, N.J.)

McAndrew, M. H. 2194

A partition problem.

Math. Comp. **17** (1963), 291–295.

From the author's introduction: "The following theorem is proved: Given integers a, b, c, d, each $\geqq 2$, then either there exist integers m, n with $|m-n| \leqq 1$, a partition of a into m parts of which each part is coprime to b, and a partition of c into n parts, each part coprime to d; or the same conclusion holds with the roles of a and b reversed and the roles of c and d reversed. This question arises in the investigation of the minimum length of input strings required to distinguish two partial automata. Elgot and Rutledge [Finite Automata, Switching Circuit Theory and Logical Design Section (Proc. Second Sympos. Switching Circuit Theory and Logical Design, 1961), AIEE—Special Publ. S-134 (1961), 129–132] deduce an upper bound for the length of such strings and by using the theorem quoted above show that this upper bound can be attained. In Section 4 we demonstrate by an example that the restriction $a, b, c, d \geqq 2$ cannot be relaxed." *C. C. Elgot* (Yorktown Heights, N.Y.)

Abbott, H. L.; Moser, L. 2195

On a combinatorial problem of Erdős and Hajnal.

Canad. Math. Bull. **7** (1964), 177–181.

Let $m(n)$ denote the smallest integer for which there exists a family F_n of sets $A_1, A_2, \cdots, A_{m(n)}$ such that (i) each member of F_n has n elements, and (ii) if S is a set which meets each member of F_n, then S contains at least one member of F_n. We have $m(1)=1$, $m(2)=3$, $m(3)=7$, but $m(n)$ has not been evaluated for $n \geqq 4$. Erdős has asserted that it is not known whether or not $\lim_{n\to\infty} m(n)^{1/n}$ exists. The authors answer this question in the affirmative and, in addition, prove that for every $\varepsilon > 0$, $m(n) = O(\sqrt{7}+\varepsilon)^n$. Their main results utilize as lemmas the following inequalities for $m(n)$: $m(ab) \leqq m(a)m(b)^a$ and $m(n+1) \geqq m(n)$. *H. J. Ryser* (Syracuse, N.Y.)

Ryser, H. J. 2196

Matrices of zeros and ones in combinatorial mathematics.

Recent Advances in Matrix Theory (Proc. Advanced Seminar, Math. Res. Center, U.S. Army, Univ. Wisconsin, Madison, Wis., 1963), pp. 103–124. *Univ. Wisconsin Press, Madison, Wis.*, 1964.

A matrix A of 0's and 1's is called a tournament matrix if it is the incidence matrix of a round robin tournament, i.e., if $A + A^T$ has 0's on the main diagonal and 1's elsewhere. After reviewing some of the general theory of

(0, 1)-matrices [cf. the author's *Combinatorial mathematics*, Wiley, New York, 1963; MR 27 #51], various results about tournament matrices are obtained. In particular, the structure of those tournament matrices with given row and column sum vectors which have a minimal number of 1's above the main diagonal is investigated. Also, it is shown that a tournament matrix of order n has term rank $n-1$ or n according as whether or not any of its irreducible components are zero matrices of order 1.

J. W. Moon (Edmonton, Alta.)

Erdős, P.; Ginzburg, A. 2197

On a combinatorial problem in Latin squares. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. 8 (1963), 407–411 (1964).

Let S_n be an arbitrary Latin square based on n elements $a_1, a_2, \cdots, a_n$. A row and column of this square intersecting on the main diagonal beginning at the lower left corner are called corresponding. A principal minor T_c is obtained from S_n by striking out $n-c$ arbitrary rows and corresponding columns. Let $k_{i_1, i_2, \cdots, i_q}$ $(i_1, i_2, \cdots, i_q = 1, 2, \cdots, n$, all different) denote the number of columns in T_c containing the elements $a_{i_1}, a_{i_2}, \cdots, a_{i_q}$ simultaneously, and let $k^{(q)}$ denote the minimum of $k_{i_1, i_2, \cdots, i_q}$. Now let n and $k^{(q)}$ be two given positive integers. The authors study the minimal c (denoted by b) such that for an arbitrary S_n at least one T_c can be obtained with the prescribed $k^{(q)}$.

From the definition of b it follows directly that

$$\binom{b}{q} b \geqq k^{(q)}\binom{n}{q},$$

and assuming that the main diagonal is occupied by one element a_1, this may be improved to

$$\binom{b-1}{q} b \geqq k^{(q)}\binom{n-1}{q}.$$

The main theorem gives an upper bound for b when $k^{(q)}=1$. "In any given n-by-n Latin square there can be found a principal minor of order not more than

$$Cn^{q/q+1}(\log n)^{1/q+1}$$

(C a sufficiently large absolute constant) containing every q-tuple $(a_{i_1}, a_{i_2}, \cdots, a_{i_q})$ $(i_1, i_2, \cdots, i_q = 1, 2, \cdots, n$, all different) in some column." The authors mention a number of unsolved problems arising out of these investigations.

H. J. Ryser (Syracuse, N.Y.)

Kislicyn, S. S. 2198

On the selection of the kth element of an ordered set by pairwise comparisons. (Russian)

Sibirsk. Mat. Ž. 5 (1964), 557–564.

Let $V_k(n)$ denote the minimum number of pairwise comparisons necessary in order to locate the kth largest element of a set of n different elements. The author shows that

$$(*)\quad V_k(n) \leqq n-1+\sum_{i=1}^{k-1}[\log_2(n-i)], \qquad k \leqq \left[\frac{n+1}{2}\right].$$

For $k=1$ and 2 the equality holds. This result was also obtained by J. Schreier [Mathesis Polska 7 (1932), 154–160]. The author adds, in proof, that for $k=3$ the equality does not hold in general, because, e.g., $V_3(5)=6$, while the right-hand side of (*) is equal to 7. It is possible, however, and seems to be an open question whether the equality holds provided that one requires also a complete ordering of the k elements. The author remarks that for $k=n$ the right-hand side of (*) is also an upper bound for the number of comparisons needed for the complete ordering of the set. This result was also obtained by H. Steinhaus [*Mathematical snapshots*, pp. 38–40, Oxford Univ. Press, New York, 1950; MR 12, 44]. Steinhaus conjectured that the equality holds in this case. This problem seems to be open yet.

E. Seiden (E. Lansing, Mich.)

Ostrowski, A. M. 2199

On some determinants with combinatorial numbers.

J. Reine Angew. Math. 216 (1964), 25–30.

Using $\Gamma(x+1)/\Gamma(\chi+1)\Gamma(x-\chi+1)$ in place of the binomial coefficient $\binom{x}{\chi}$, the author shows that the determinant $\left|\binom{x}{k_\mu-\nu}\right|$ equals

$$\frac{\Gamma(x+1)^{m+1}\prod_{\mu=1}^{m}(x+\mu)^{m+1-\mu}\Delta(k_0, \cdots, k_m)}{\prod_{\mu=0}^{m}[\Gamma(k_\mu+1)\Gamma(x+1+m-k_\mu)]},$$

where Δ is the Vandermonde determinant and $\mu, \nu = 0, 1, \cdots, m$. In particular, when the k_i's are non-negative different increasing integers, then the determinant equals $\Delta\prod_{\mu=0}^{m}(k_\mu-\mu)!/k_\mu!\prod_{\mu=0}^{m}\binom{x+m-\mu}{k_\mu-\mu}$, which is positive when $x>k_m-m-1$. Thus he shows the positiveness of all $(m+1)$-rowed minors selected from the first $m+1$ columns of $\left\|\binom{x}{\mu-\nu}\right\|$ $(\mu, \nu=0, 1, \cdots, n)$ provided $1 \leqq m \leqq n$ and $x>n-2$, as well as the positiveness of those in the first $m+1$ columns of $\left\|\binom{n-\nu}{\mu-\nu}\right\|$ and the last $m+1$ columns of $\left\|\binom{\nu}{\mu}\right\|$.

B. Vinograde (Ames, Iowa)

ORDER, LATTICES

See also 2318, 2764.

Varlet, Jules 2200

Contribution à l'étude des treillis pseudo-complémentés et des treillis de Stone.

Mém. Soc. Roy. Sci. Liège Coll. in-8° (5) 8 (1963), *no.* 4, 71 *pp.*

This is a thoroughgoing study of pseudo-complemented, relatively pseudo-complemented and Stone lattices. The author's main concern seems to be to establish many elementary properties of these types of lattices in general and under certain conditions as well. In Chapter 3 the author introduces the interesting notion of semi-convex sublattices ($x \cup y$ and $x \cap y$ are in it if and only if x and

y are in it), e.g., in a modular lattice the elements which have complements form a semi-convex sublattice.
G. Grätzer (University Park, Pa.)

Varlet, J. [Varlet, Jules] 2201
Idéaux dans les lattis pseudo-complémentés.
Bull. Soc. Roy. Sci. Liège **33** (1964), 143–150.
Let L be a pseudo-complemented lattice, I an ideal of L, $I^* = \{x | x \cap i = 0$ for all $i \in I\}$, $\tilde{I} = \{x | x \leqq i^{**}$ for some $i \in I\}$. Then both I^{**} and $\tilde{I}$ are closure-operations of the lattice of all ideals of L. A typical result: Let L be a complete pseudo-complemented modular lattice. Then $\tilde{I} = I^{**}$ holds for all ideals if and only if every dense ideal contains a dense element. *G. Grätzer* (University Park, Pa.)

Kuntzmann, Jean; Lamoitier, J. 2202
Forme arborescente abrégée pour une fonction booléenne.
C. R. Acad. Sci. Paris **257** (1963), 1829–1830.
The authors sketch the description of a canonical form for Boolean polynomials and state some of its properties. The paper is not self-contained.
P. R. Halmos (Ann Arbor, Mich.)

Magari, Roberto; Mangani, Piero 2203
Alcune osservazioni sugli assiomi delle "Algebre monadiche" di Halmos.
Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **98** (1963/64), 265–278.
The authors study equivalences and non-equivalences among various conditions that an operator on a Boolean algebra may satisfy. Typical positive result: If C is an operator on a Boolean algebra such that (i) $C0 = 0$, (ii) $Cx \geqq x$ for all x, and (iii) $Cx \leqq Cy$ whenever $x \leqq y$, then a necessary and sufficient condition that C be an existential quantifier (that is, $C(x \cdot Cy) = Cx \cdot Cy$ for all x and y) is that $C(-Cx) = -Cx$ for all x. Typical negative result: If the hypothesis (iii) is omitted, the conclusion no longer holds. *P. R. Halmos* (Ann Arbor, Mich.)

Magari, Roberto; Mangani, Piero 2204
Sulle topologie "compatibili" con una data algebra monadica.
Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **98** (1963/64), 299–314.
If C is an existential quantifier [see #2203 above] on a Boolean algebra A of subsets of a set S, then there exists a topology for S such that, for each set X in A, the closure of X is equal to CX. Among all such topologies the one for which the range of C is a base is minimal. Can the topology be chosen so that its closure operator is an existential quantifier? The authors study this question and offer some answers; the answers are expressed in terms of the equivalence relation induced by C, according to which two elements of X are equivalent just in case they belong to the same sets in the range of C.
P. R. Halmos (Ann Arbor, Mich.)

GENERAL MATHEMATICAL SYSTEMS

See also 2288.

Gavrilov, Mihail; Čobanov, Ivan 2205
The index of non-associativity of multiplicative structures. (Bulgarian. German summary)
Annuaire Univ. Sofia Fac. Sci. Phys. Math. Livre 1 Math. **56** (1961/62), 23–26 (1963).
A multiplicative structure is any pair (S, R) where S is a non-void set and R is a binary operation on S (i.e., $R \subset S^3$, where S^3 is the Cartesian cube of S, and there exists for every $(a, b) \in S^2$ a single $c = ab$ such that $(a, b, c) \in R$). Let N be the set of all $(a, b, c) \in S^3$ such that $(ab)c \neq a(bc)$. The cardinal of N is called the index of non-associativity of (S, R). The following problem is considered: when S is given and R may vary, what indices of non-associativity can be obtained? G. Szász [Acta Sci. Math. (Szeged) **15** (1953), 20–28; MR **15**, 95] has proved that for every S with more than three elements the index of non-associativity may be one. Al. C. Climescu [Bull. École Polytech. Jassy **2** (1947), 347–371; MR **10**, 100] has proved that if S is finite and contains more than three elements (say, n), then the index of non-associativity takes all values from 0 to n^3. The authors prove that the last result is valid for infinite S also. The proof is based on the following lemma: If S has more than one element, then there exists R such that the index of non-associativity of (S, R) is equal to n^3, where n is the cardinal of S.
B. M. Schein (Šain) (Saratov)

Gluskin, L. M. 2206
Positional operatives. (Russian)
Dokl. Akad. Nauk SSSR **157** (1964), 767–770.
An n-operative is a set S with an n-ary operation $(x_1, \cdots, x_n)$ defined on it. By assuming an associative law, the author studies elements which are invertible in a certain sense, ideals, and representation theorems. The results are too technical to be reproduced here.
G. Grätzer (University Park, Pa.)

Ivanova, O. A. 2207
On direct powers of unary algebras. (Russian. English summary)
Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, *no.* 3, 31–38.
A unary algebra is a set on which is defined a single unary operation $x \to x'$. For a unary algebra A, let A^n denote the direct product of n copies of A. Marcia and Bryant [Pacific J. Math. **10** (1960), 1347–1359; MR **22** #9463] showed that if A and B are finite unary algebras such that $A^2 \cong B^2$, then $A \cong B$. The paper under review generalizes this result in two ways. First, it is shown that if A and B are finite unary algebras such that $A^n \cong B^n$ for some $n \geqq 1$, then $A \cong B$. Second, the same result is obtained with various relaxations of the condition that A and B be finite. In order to give an example of these generalizations, some definitions are needed. Define inductively $x^0 = x$, $x^n = (x^{n-1})'$. An element x is called cyclic if $x^n = x$ for some $n > 0$. For $k \geqq 0$, let A/k denote the set of all elements x in A such that x^k is cyclic. Theorem: If A and B are unary algebras such that (i) A/k is finite for every $k \geqq 0$ and (ii) $A = \bigcup_{k \geqq 0} A/k$, then $A^n \cong B^n$ for $n \geqq 1$ implies that $A \cong B$. *R. S. Pierce* (Seattle, Wash.)

THEORY OF NUMBERS

See also 2247, 2258, 2324, 2390, 2697, 2841.

Avanesov, E. 2208

Elementary solution of an arithmetic problem. (Bulgarian)

Fiz.-Mat. Spis. Bŭlgar. Akad. Nauk. **6 (39)** (1963), 206–208.

Let n be a positive integer, and let $f(n)$ $[F(n)]$ be the number of solutions in non-negative integers of the equation $x+2y+5z=n$ $[x+2y+5z+10(u+2v+5t)=n]$. Then $f(n)=1+[20^{-1}n(n+8)]$, and $F(n)=\sum f(i)f(n-10i)$, i running from 0 to $[n/10]$. An obvious application is the determination of the number of ways in which a given integral dollar sum can be made up by using one-, two-, five-, ten-, twenty-, and fifty-dollar bills.

F. Haimo (Belmont, Mass.)

Rotkiewicz, A. 2209

Sur les chiffres initiaux et finals des nombres a^n et $a^n \pm b^n$.

Rend. Circ. Mat. Palermo (2) **12** (1963), 150–154.

Let a be an integer greater than 1, relatively prime to 10, such that $2^\alpha \| a^4-1$, $5^\beta \| a^4-1$, and $\gamma=\max(\alpha,\beta)$. Let $a_1,\cdots,a_m$; $b_1,\cdots,b_k$ be ordered arrays of digits. The author proves that there is an integer s such that the first m digits of a^s are $a_1,\cdots,a_m$ (in that order) and the $\gamma+1,\gamma+2,\cdots,\gamma+k$ digits (measured either from the left or the right) are $b_1,\cdots,b_k$ (in that order). The proof relies on simple congruence arguments. This extends similar results of Sierpiński [Acta Arith. **5** (1959), 265–266; MR **22** #6961].

D. J. Lewis (Ann Arbor, Mich.)

Gandhi, J. M. 2210

Congruences for Genocchi numbers.

Duke Math. J. **31** (1964), 519–527.

The Genocchi numbers G_r are defined by

$$2t/(e^t+1) = \sum_1^\infty G_r t^r/r!.$$

In analogy to results due to Kummer, Voronoi, Frobenius and to Carlitz [Amer. J. Math. **75** (1953), 163–172; MR **14**, 539] the author derives a number of congruences for the G_r. Most of his theorems are of the type

$$\xi^m(\xi^b-1)^n \equiv 0 \pmod{p^r},$$

where m, b, n, r satisfy certain inequalities and divisibility properties. After expansion of $\xi^m(\xi^b-1)^n$ into powers of ξ, we have to replace ξ^k by G_k in some theorems, and by $G_{k(p-1)}/k$ or by G_{kb}/k in some others.

N. G. de Bruijn (Eindhoven)

Sekanina, Milan 2211

On the factorization of the set of non-negative integers. (Russian. English summary)

Czechoslovak Math. J. **14 (89)** (1964), 161–170.

See the author's earlier paper [same J. **9 (84)** (1959), 485–495; MR **22** #6776] for related material. For an index class I, consider functions f on I to $\mathfrak{A}$, the set of non-negative integers, functions which are almost always 0. For a system of subsets A_i of $\mathfrak{A}$, indexed by I, consider the set F of those f above for which $f(i) \in A_i$ for each $i \in I$; and let $A=\sum_{i\in I} A_i$ mean that $A=\{\sum_{i\in I} f(i) : f \in F\}$, calling each A_i a factor of A. If each $x \in A$ can be written in the form $\sum f(i)$ for precisely one f (depending on x), write $A=\sum^* A_i$. A typical result is that if non-void $A=\sum^* A_i$, then $A_k \cap A_j$ $(k \neq j)$ has no more than one member. If I is binary, write $A=A_1+^*A_2$ (in the unique representation case). Then $A+^*B=A+^*C$ and A non-void imply $B=C$. A subset A of $\mathfrak{A}$ is said to be almost periodic if there exists $k \in \mathfrak{A}$ such that $a+k \in A$ for every $a \in A$. Suppose that A is a factor of $\mathfrak{A}$ itself. Then A is almost periodic if and only if $A=\sum^* C_i$, where each C_i is a subset of $\mathfrak{A}$ consisting of some multiples of a fixed integer (with further specifications involving a chain of integral divisors, too long to include here).

F. Haimo (Belmont, Mass.)

Cohen, Eckford 2212

Arithmetical notes. XI. Some divisor identities.

Enseignement Math. (2) **10** (1964), 248–254.

Part X appeared in Proc. Amer. Math. Soc. **15** (1964), 534–539 [MR **29** #3424]. New proofs are given of various identities involving unitary divisors of a positive integer r, that is, divisors d of r such that $(d, r/d)=1$. These identities were originally proved in a series of four papers by the author [Math. Z. **74** (1960), 66–80; MR **22** #3707; Duke Math. J. **28** (1961), 183–192; MR **25** #3916; ibid. **28** (1961), 475–485; MR **24** #A3124; Publ. Math. Debrecen **9** (1962), 99–104; MR **25** #3894].

W. H. Mills (Princeton, N.J.)

Pellegrino, Franco 2213

Teorema di Wilson e numeri primi gemelli.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **35** (1963), 258–262.

Using Wilson's theorem and the fact that if p is composite and $\neq 4$ then $(p-1)! \equiv 0 \pmod p$, the author proves that p and $p-2$ are both primes if and only if $4[(p-3)!/(p-2)] \equiv -5 \pmod p$.

H. W. Brinkmann (Swarthmore, Pa.)

Brillhart, John; Lehmer, D. H.; Lehmer, Emma 2214

Bounds for pairs of consecutive seventh and higher power residues.

Math. Comp. **18** (1964), 397–407.

Using a computing machine, the authors show that every prime of the form $7n+1$, except 29, 71, 113, and 491, has a pair n, $n+1$ of positive seventh power residues with $n \leqq 1649375$. Moreover, this bound is best possible since there exist an infinite number of primes with 1649375, 1649376 as the first pair of consecutive seventh power residues. It is not known how to prove a theorem of this type without the use of high-speed computers. The corresponding question for sixth and smaller power residues has been previously settled by the Lehmers and others [D. H. Lehmer, E. Lehmer and W. H. Mills, Canad. J. Math. **15** (1963), 172–177; MR **26** #3660; R. G. Bierstedt and W. H. Mills, Proc. Amer. Math. Soc. **14** (1963), 628–632; MR **27** #4787]. For kth power residues with $k \geqq 8$, it is not known whether or not such a bound exists. For k odd and greater than 3, or for k twice a prime of the form $4N+3$, the authors show that if such

a bound exists, it must be at least 5^k. There is a brief discussion of the cases $k=8$, 9, and 10.
W. H. Mills (Princeton, N.J.)

Sierpiński, W. 2215
Sur les suites d'entiers deux à deux premiers entre eux.
Enseignement Math. (2) **10** (1964), 229–235.

The author proves several interesting theorems, and we cite a few. Every integer >6 is a sum of two relatively prime integers >1; there exist arbitrarily long arithmetic progressions of natural numbers >1 which are also mutually co-prime; if $f(x)$ is a polynomial with integer coefficients with positive leading coefficient, then it cannot happen that the infinite sequence $f(1)$, $f(2)$, $f(3)$, $\cdots$ consists of mutually co-prime numbers; if a, b, c are positive integers such that the equation $ax+by=c$ has a solution in integers x, y, then this equation has an infinite number of solutions in relatively prime integers x, y.

A. Schinzel remarked that the last theorem can be deduced without difficulty from a theorem of Th. Skolem [Norsk Mat. Tidskr. **15** (1933), 25–27].
S. Chowla (University Park, Pa.)

Dimovski, Ivan 2216
On the rational solutions of the equation $1+x+x^2=y^3$. (Bulgarian)
Fiz.-Mat. Spis. Bŭlgar. Akad. Nauk. **6 (39)** (1963), 209–211.

Starting from results due to Nagell [Norsk Mat. Forenings Skr. (1) No. 2 (1921)], Ljunggren proved in 1942 [Acta Math. **75** (1942), 1–21; MR **8**, 135] that the only integral solutions of the equation

$$(1)\qquad 1+x+x^2 = y^3$$

are (0, 1), (−1, 1), (18, 7), and (−19, 7). In this paper the author gives an elementary geometric proof of the theorem that equation (1) has infinitely many solutions in rational numbers. *R. Finkelstein* (Tucson, Ariz.)

Moessner, Alfred; Sinha, T. N. 2217
Some Diophantine problems.
Math. Student **31** (1963), 35–39 (1964).

Using some identities and relations the authors find some solutions to certain systems of Diophantine equations, e.g., $2(x^m+y^m+z^m)=u_1^m+\cdots+u_6^m$ $(m=1, 2, 3, 4, 5)$.
D. J. Lewis (Ann Arbor, Mich.)

Baker, A. 2218
On an analogue of Littlewood's Diophantine approximation problem.
Michigan Math. J. **11** (1964), 247–250.

Littlewood has asked whether for each pair of real numbers θ and ϕ and each $\varepsilon>0$ there exist integers $n\neq 0$, m, q, such that $|n|\cdot|n\theta-m|\cdot|n\phi-q|<\varepsilon$. This question remains unanswered. However, Davenport and the reviewer [same J. **10** (1963), 157–160; MR **27** #4794] obtained a negative answer for the analogous question concerning formal power series fields over infinite fields. In this paper the author gives specific examples of θ and ϕ for which a negative answer results. Specifically, he shows that for all polynomials $u(t)\neq 0$, $v(t)$, $w(t)$ with real coefficients, one has

$$|u(t)|_K|u(t)\theta-v(t)|_K|u(t)\phi-w(t)|_K \geq e^{-5},$$

where $|a_mt^m+a_{m+1}t^{m+1}+\cdots|_K=e^m$ if $a_m\neq 0$.
D. J. Lewis (Ann Arbor, Mich.)

Ivanov, V. K. 2219
Higher-dimensional generalizations of the Euler summation formula. (Russian)
Izv. Vysš. Učebn. Zaved. Matematika **1963**, *no.* 6 (37), 72–80.

C. Müller [Abh. Math. Sem. Univ. Hamburg **19** (1954), no. 1–2, 41–62; MR **15**, 939] has generalized the well-known Euler summation formula to the functions of r variables. The role of Bernoulli polynomials is played by the so-called basic functions (Grundfunktionen).

The author shows that the summation formulas of this kind are connected with the differential operators. There exist corresponding summation formulas for many linear differential operators with constant coefficients. The analogues of the Bernoulli polynomials are the distributions, which cannot be expressed by the usual functions.

Let D be an r-dimensional space region, which is limited by a smooth surface S, not containing integer points. Let $f(x)=f(x_1,\cdots,x_r)$ be a function in $D+S$, having continuous derivatives of order p. Denote by $L=L(\partial/\partial x)$ the linear differential operator with constant real coefficients, which is a polynomial in $\partial/\partial x_1,\cdots,\partial/\partial x_r$. Then

$$\sum_{n\subset D} f(n) = \int_D f(x)H(x)\,dx+\int_S P(f,G)\,ds+\int_D GM(f)\,dx,$$

where the summation is extended over integer points $n=(n_1,\cdots,n_r)$. Here M is the operator conjugate to L. $P(u,v)$ is a polynomial in u, v and in their partial derivatives. The polynomial $P(u,v)$ is defined by Green's formula

$$\int_D (uLv-vMv)\,dx = \int_S P(u,v)\,ds.$$

This formula is also true in the case when v or some of its derivatives are distributions if the products under the integral sign make sense, and on the surface S there are no singular points of the distribution v and its derivatives. The function $H(x)$ is equal to

$$\sum_{L(2\pi i n)=0} e^{2\pi i(n_1x_1+\cdots+n_rx_r)},$$

where n runs over all integer points such that $L(2\pi in)=0$ and $L(2\pi in)$ is the polynomial which is obtained from $L(\partial/\partial x)$ by replacement of $\partial/\partial x_j$ by $2\pi in_j$. $G(x)$ is a basic function, which is a distribution, having the period 1 and satisfying the equation $LG=\Delta(x)-H(x)$. Here $\Delta(x)$ denotes the distribution $\Delta(x)=\sum_n\delta(x-n)$, where the summation extends over all integer points, and $\delta(x-n)$ is the delta-function having singularity at the point n. The basic function $G(x)$ is equal to

$$\sum_{L(2\pi in)\neq 0}\frac{e^{2\pi i(n_1x_1+\cdots+n_rx_r)}}{L(2\pi in)}.$$

The necessary and sufficient condition for the existence

the basic function $G(x)$ is the existence of k such that e Diophantine inequality

$$|L(2\pi in)| < (n_1^2+\cdots+n_r^2)^k$$

is only a finite number of integer solutions $(n_1, \cdots, n_r)$. The simplest and most complete results are obtained in e case when L is an elliptic operator. If the elliptic erator L is of the type $L=(L_0+(-\lambda)^m)^p$, where $L_0(\omega)$ a homogeneous positive definite polynomial of degree n and $\lambda \geqq 0$, then the basic function of the operator L is usual function.
Some special cases are examined in detail.

J. Kubilius (Vilnius)

ang, Serge 2220

Diophantine approximations on toruses.

Amer. J. Math. **86** (1964), 521–533.

uthor's introduction: "The main purpose of this paper to make precise certain analogies between diophantine pproximations on the multiplicative group and elliptic urves and to show how these are related to approximaons on the additive group by means of the exponential apping. We shall state two conjectures."
The first conjecture is as follows. Let Γ be a finitelyenerated subgroup of the group G_K of points defined over e number field K on the multiplicative group G and let be a nonconstant function on G defined over K. Then e logarithmic heights $h(P)$ of points P in Γ which are ounded away from 0 and ∞ and satisfy the inequality $\phi(P)|<|h(P)|^{-rm-\epsilon}$ are bounded, where m is the rank of and r is the maximum multiplicity of a zero of ϕ. (The garithmic height is the logarithm of the more usual eight.) On taking logarithms, this conjecture is seen to e substantially equivalent to the conjecture that a set f logarithms of m elements of K behaves for linear iophantine approximation like "almost any" set of m eal numbers provided that they are linearly independent ver the rationals. The second conjecture relates to unctions on an elliptic curve defined over an algebraic umber field and is similarly motivated.
Of interest are (1) a new type of "Lipschitz principle" cf. Davenport, J. London Math. Soc. **26** (1951), 179–183; MR **13**, 323], which is said to be more useful in applications nd which is used to obtain an asymptotic estimate of he number of units of K with height less than a given umber; and (2) a proof, due to Tate, that the (logarithmic) eight of a rational point on an abelian variety is essenially given by a quadratic form (conjecture of Néron cf. Lang, *Diophantine geometry*, Interscience, New York, 962; MR **26** #119]).

J. W. S. Cassels (Cambridge, England)

züsz, Péter 2221

Über einen Satz der Kettenbruchlehre. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 341–346 (1964).

Let p and q, $0<p<q$, be relatively prime integers. There is an interval $I(p,q)$ of maximal length such that, for each $\alpha \in I(p,q)$, the fraction p/q is an approximant of the simple continued fraction expansion of α. The author proves $\frac{4}{5}<q^2|I(p,q)|<\frac{3}{2}$, where $|I(p,q)|$ is the length of the interval, and

$$\lim_{q\to\infty}[\varphi(q)]^{-1}\sum_{q^2|I(p,q)|\leqq t}1=(9-12/t)^{1/2}, \qquad \tfrac{4}{5}\leqq t\leqq \tfrac{3}{2},$$

$$\lim_{q\to\infty}[\varphi(q)]^{-1}\sum q^2|I(p,q)| = 2\ln 2,$$

where $\varphi(q)$ is the Euler function and the summations are for all integers p, $0<p<q$, $(p,q)=1$. The last of these results answers a question proposed by Erdős.

E. P. Merkes (Cincinnati, Ohio)

Froda, Alexandre 2222

Critères paramétriques d'irrationalité.

Math. Scand. **12** (1963), 199–208.

Let $\alpha=\lim (y_r/x_r)$, where x_r and y_r are positive integers, be the limit of an increasing sequence of rational numbers; let $q_1, q_2, q_3, \cdots$ be positive numbers, and let $p_{r+1}=q_{r+1}/q_r<x_{r+1}/x_r$ for all r, $\lim (x_r/q_r)=\infty$. Put

$$\xi_r = 1-\frac{x_r}{q_r}+\left[\frac{x_r}{q_r}\right], \quad \eta_r=\frac{y_r}{q_r}-\left[\frac{x_r}{q_r}\right] \quad (r=1,2,3,\cdots).$$

Assume that for sufficiently large r

$$\frac{y_{r+1}-p_{r+1}y_r}{x_{r+1}-p_{r+1}x_r} > \frac{y_{r+2}-p_{r+2}y_{r+1}}{x_{r+2}-p_{r+2}x_{r+1}}$$

and that there exists an increasing sequence of suffixes $r_1, r_2, r_3, \cdots$ such that $\xi_{r_1}\geqq\xi_{r_2}\geqq\xi_{r_3}\geqq\cdots$, $\eta_{r_1}\geqq\eta_{r_2}\geqq\eta_{r_3}\geqq\cdots$. Then α is irrational. A similar criterion holds for the limits of decreasing sequences. When $q_r=1$ for all r, a result due to V. Brun is obtained [Arch. Math. Naturvid. **31** (1910), no. 3].

K. Mahler (Canberra)

Fraenkel, Aviezri S. 2223

Transcendental numbers and a conjecture of Erdős and Mahler.

J. London Math. Soc. **39** (1964), 405–416.

Das Hauptanliegen dieser Arbeit ist es, aus Nebenbedingungen für die Näherungsbrüche (im Sinne der Kettenbruchlehre) reeller Zahlen auf die arithmetische Natur und die Verteilung dieser Zahlen zu schliessen [vgl. auch Erdős und Mahler, dasselbe J. **14** (1939), 12–18]. Neben anderem wird gezeigt, dass es eine überall dichte Menge von Liouville-Zahlen gibt derart, dass für jede dieser Zahlen sowohl der Zähler als auch der Nenner von unendlich vielen Näherungsbrüchen gewisse Bedingungen der Teilbarkeit und der Grössenordnung erfüllen. Bei den Beweisen werden der Approximationssatz von Kronecker und der Satz von Linnik über die kleinste Primzahl in einer primen Restklasse herangezogen.

G. J. Rieger (Munich)

Rankin, R. A. 2224

On the minimal points of perfect quadratic forms.

Math. Z. **84** (1964), 228–232.

A positive definite quadratic form $f(\mathbf{x})=\sum_{i,j=1}^n a_{ij}x_ix_j$ is said to be perfect if the set of integer points $\mathbf{x}$ for which $f(\mathbf{x})$ is minimal, say $f(\mathbf{x}_\nu)=M$ $(\nu=1,\cdots,p)$ with $\mathbf{x}_\nu\neq\mathbf{x}_\mu$ for $\nu\neq\mu$, is such that the equations $f(\mathbf{x}_\nu)=M$ completely determine the a_{ij}. For a perfect form we have $\frac{1}{2}n(n+1)\leqq p\leqq 2^n-1$. Form all possible n-by-n determinants Δ_μ having the Δ_ν's as row vectors. Then $0\leqq\Delta_\mu^2\leqq\gamma_n^n$, where

γ_n is Hermite's constant. Davenport and Watson [Mathematika 1 (1954), 14–17; MR 16, 18] showed that this upper bound could not be improved appreciably. Let Δ^2 be the least positive value of the Δ_μ^2. The author [ibid. 3 (1956), 15–24; MR 18, 114] showed $\Delta^2 \leqq (1/n)\{\gamma_n(1+1/n)\}^n < e\gamma_n{}^n/n$. He now shows that $\Delta = 1$ if $n \leqq 6$ and

$$\Delta^2 \leqq \frac{n(n!)}{(n+1)(2n)!}\{(n+1)\gamma_n\}^n \quad \text{for } n > 6.$$

For large n, this last bound represents a considerable improvement. The proof is essentially combinatorial.

D. J. Lewis (Ann Arbor, Mich.)

Gerl, Peter 2225

Zur Gleichverteilung auf Flächen.

J. Reine Angew. Math. **216** (1964), 113–122.

The author studies the asymptotic behavior of sequences on r-dimensional surfaces Φ. A sequence of points $u(n) \in \Phi$ is said to be uniformly distributed with density ρ on a Jordan domain $Q \subseteq \Phi$ with area $\|Q\| < \infty$ if for every complex-valued Riemann integrable function f on Q the relation

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n \leq N} f(u(n)) = \frac{1}{\|Q\|} \int_Q f(u)\rho(u)\,du$$

holds. The author characterizes sequences $u(n)$ with density ρ on Q in terms of uniformly distributed sequences in the r-dimensional unit cube E_r and obtains generalizations of a theorem of H. Weyl. An application is made to the description of the structure of uniformly distributed sequences in the group G of rotations in 3-dimensional space: They are of the form $u(n) = (\varphi_1(n), \vartheta(n), \varphi_2(n))$ if $\varphi_1, \vartheta, \varphi_2$ denote the Eulerian angles of an element $g \in G$, where $\varphi_1(n) = 2\pi y_1(n)$, $\vartheta(n) = \arccos(1 - 2y_2(n))$, $\varphi_2(n) = 2\pi y_3(n)$, and $(y_1(n), y_2(n), y_3(n))$ is uniformly distributed in E_3.

J. E. Cigler (Vienna)

Srinivasan, B. R. 2226

On the fundamental identity of the theory of primes.

Math. Student **30** (1962), 163–169 (1963).

The author investigates the well-known theorem, called by Ingham [*The distribution of prime numbers*, p. 16, Cambridge Univ. Press, London, 1932] "the fundamental identity of the theory of primes", viz., the following: Let $f(n)$ be a "multiplicative" function of n (positive integer), with

$$(1)\quad f(m)\cdot f(n) = f(mn) \quad \text{if } (m, n) = 1 \quad (\to f(1) = 1),$$

then $\sum_1^\infty f(m) = \prod_p \{\sum_{k=0}^\infty f(p^k)\}$, provided that the series on the left is absolutely convergent, in which case the product is also absolutely convergent. If $f(n)$ is "completely multiplicative", i.e., if (1) holds for all m and n, co-prime or not, we can write $\sum_1^\infty f(m) = \prod_p \{1 - f(p)\}^{-1}$, where p runs through all primes. By application of the Möbius function $\mu(n)$ we have, under general conditions about convergence, the inversion formula

$$(2)\qquad g(r) = \sum_{m=1}^\infty f(mr) \to f(r) = \sum_{n=1}^\infty \mu(n) g(nr)$$

[cf. Hardy and Wright, *An introduction to the theory of numbers*, Theorem 270, Clarendon, Oxford, 1938], e.g.,

$$(3)\ f(n) = x^n,\ \ |x| < 1;\ \ g(m) = \frac{x^m}{1-x^m},\ \ x = \sum_{n=1}^\infty \frac{\mu(n)x^n}{1-x^n}.$$

The author aims at a generalization of these results to the case in which the indices only run through a set of positive integers prime to the finite set of primes $p_1, p_2, \cdots, p_n$. So he finds (Theorem 6) as a generalization of (2)

$$\sum_{r=1}^\infty \sum_{m \mid p_1 p_2 \cdots p_n} \mu(m) f(mr) = \sum_{(r_n; p_1 p_2 \cdots p_n)=1} f(r_n) = f(1) + f(p_{n+1}) + \cdots,$$

with Corollary 2 (generalization of (3)) for $f(n) = x^n$

$$S_n(x) = \sum_{m \mid p_1 p_2 \cdots p_n} \frac{x^m}{1-x^m} = \sum_{(r_n; p_1 p_2 \cdots p_n)=1} x^{r_n} = x + x^{p_{n+1}} + \text{higher powers}.$$

The deduction of these results (which should be very simple by means of the classical properties of the μ-function) is here needlessly sophisticated and hidden behind a heavy use of generalised set-theory.

S. C. van Veen (Delft)

Srinivasan, B. R. 2227

Lattice points in a circle.

Proc. Nat. Inst. Sci. India Part A **29** (1963), 332–346.

Let $\mathscr{E}(x)$ denote the excess of the number of lattice points inside and on the circle $u^2 + v^2 = x$ over its area. Hua [Quart. J. Math. Oxford Ser. **13** (1942), 18–29; MR **4**, 190] proved that

$$\mathscr{E}(x) = O(x^{13/40}(\log x)^{9/8}).$$

The author proves that $\mathscr{E}(x) = O(x^{13/40})$. His Lemma 1 is a two-dimensional analogue of a famous lemma of van der Corput. It runs as follows. Suppose D is a finite region in the Euclidean plane subject to some complicated restrictions that we do not repeat. Let $f(u, v)$ possess continuous partial derivatives up to second order in D. Let $|f_{uu}| \geqq r_1 > 0$, $|f_{vv}| \geqq r_2 > 0$, $|f_{uu}f_{vv} - f_{uv}{}^2| \geqq r^2 > 0$ throughout D, where $r \leqq A\sqrt{(r_1 r_2)}$. Then

$$\iint_D e^{2\pi i f(u,v)}\,du\,dv \ll \frac{1}{r}.$$

The author also states that Yin [Sci. Sinica **11** (1962), 10–15; MR **28** #3008] has claimed a further improvement in the circle-problem, namely,

$$\mathscr{E}(x) = O(x^{12/37+\epsilon}).$$

S. Chowla (University Park, Pa.)

Foster, D. M. E. 2228

On generating points of a lattice in the region

$$|x_1{}^2 + x_2{}^2 - x_3{}^2| \leqq 1.$$

Proc. Glasgow Math. Assoc. **6**, 141–155 (1964).

If Λ is a lattice of determinant Δ in 3-space, it is of interest to know for what values of k there is a basis for Λ contained in the set

$$|x_1{}^2 + x_2{}^2 - x_3{}^2| \leqq k.$$

In this paper the value $k = \Delta^{2/3}$ is found for the case when Λ has points other than 0 on the cone

$$x_1{}^2 + x_2{}^2 - x_3{}^2 = 0$$

and the value $k=(27\Delta^2/25)^{1/3}$ for the case when Λ has no points on the cone other than 0. The results are best possible in the sense that in each case a lattice can be found which does not satisfy the conclusion of the theorem if k is replaced by any smaller value k'. The proof makes use of continued fraction theory, classical theory of quadratic forms and geometric methods. Subsidiary results about bases for two-dimensional lattices are obtained. *A. M. Macbeath* (Birmingham)

Chowla, S. 2229

Some applications of a method of A. Selberg.

Norske Vid. Selsk. Forh. (Trondheim) **36** (1963), 194–200.

Let α be the root of the equation $\int_0^{3\pi/2}\cos x\cdot x^{-\alpha}\,dx=0$ $(0<\alpha<1)$. If (b_n) is a sequence of non-negative numbers and $\sum_{n=0}^{t} b_n \cos nx \geqq 0$ for all $t\geqq 0$ and all x, then the series $\sum_{n=1}^{\infty} b_n n^{\beta-1}$ is convergent for $\beta<\alpha$. This is due to A. Selberg. This theorem is here proved by his method, and many theorems and conjectures of this type are stated and discussed. *S. Izumi* (Tokyo)

Lehner, Joseph 2230

On automorphic forms of negative dimension.

Illinois J. Math. **8** (1964), 395–407.

Let Γ be a horocyclic group and assume that the function $F(\tau)$ is regular in the upper half-plane and is an automorphic form (with respect to Γ) of dimension $-r$ and multiplier system v. At the finite parabolic cusps p_k, $F(\tau)$ admits a (generalized) Fourier expansion of the form

$$(\tau-p_k)^r e(-\kappa_k A_k\tau/\lambda_k)F(\tau) = \sum_{m=-\mu_k}^{\infty} a_m^{(k)} e(mA_k\tau/\lambda_k) \quad (\mu_k \geqq 0)$$

(the exact definitions of the constants κ_k, A_k, λ_k are lengthy and may be found in the author's book [*Discontinuous groups and automorphic functions*, Amer. Math. Soc., Providence, R.I., 1964; MR **29** #1332]). Generalizing the classical circle method, the author [see Michigan Math. J. **4** (1957), 265–279; MR **21** #5013; ibid. **6** (1959), 173–193; MR **21** #5014; ibid. **7** (1960), 65–74; MR **23** #A3846] has obtained explicit expressions for the Fourier coefficients $a_m^{(k)}$ as functions of the principal parts of $F(\tau)$ at the parabolic cusps, provided that the dimension is positive or zero, in the latter case with a bounded error term this is still, in a sense, a "best possible" result, the forms of dimension zero being determined by their principal parts only up to an additive constant. No such results could be expected in the case of forms of negative dimensions, because these are not completely determined by their principal parts at the cusps; indeed, there exist non-constant automorphic forms of negative dimension that are regular everywhere. In the present paper the author shows that the circle method can still be used to determine the Fourier coefficients of forms of negative dimension, with an error term of the order of magnitude of the Fourier coefficients of an everywhere regular form. The Cauchy integral representation of the $a_m^{(k)}$ is obtained easily, and the major part of the paper is devoted to the way in which the path of integration has to be partitioned and to the estimation of the corresponding integrals. *E. Grosswald* (Paris)

Knopp, Marvin Isadore; Smart, John Roderick 2231

On Kloosterman sums connected with modular forms of half-integral dimension.

Illinois J. Math. **8** (1964), 480–487.

Es sei Γ die elliptische Modulgruppe, r eine reelle Zahl und v ein Multiplikatorsystem für Γ und die Dimension r. Insbesondere ist dann $v(M)$ für jedes $M\in\Gamma$ eine komplexe Zahl mit $|v(M)|=1$. Speziell für $U=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ sei $v(U)=e^{2\pi i\kappa}$, $0\leqq\kappa<1$. Mit vorgegebenen ganzen Zahlen c, n, μ, $c>0$ wird die Exponentialsumme

$$W(c,n,\mu,v) = \sum_{\substack{0<d\leqq c\\(c,d)=1}} \bar v(M_d)\exp\left[\frac{2\pi i}{c}\{(n+\kappa)a+(\mu+\kappa)d\}\right]$$

gebildet. Dabei ist M_d für jedes d mit $0<d\leqq c$, $(c,d)=1$, irgendeine Matrix $M_d=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma$ mit $(c\quad d)$ als zweiter Zeile. Die Summe $W(c,n,\mu,v)$ ist unabhängig davon, in welcher Weise $M_d\in\Gamma$ so gewählt wird. Die Verfasser beweisen für halbganze Dimensionen r und für $c\to+\infty$ die asymptotische Abschätzung

$$(*)\qquad W(c,n,\mu,v) = O(c^{1/2+\varepsilon}),\quad \varepsilon>0.$$

Die dabei eingehende Konstante hängt von μ und v, aber nicht von n ab. Der Beweis kombiniert eine Methode von H. Petersson [Abh. Deutsch. Akad. Wiss. Berlin Kl. Math. Allg. Natur. **1954**, no. 2; MR **17**, 129] mit einem Resultat von A. V. Malyšev [Vestnik Leningrad. Univ. **15** (1960), no. 13, 59–75; MR **23** #A2391]. Die Verfasser formulieren ferner die folgende Vermutung: Zu jeder reellen nicht halbganzen Dimension r gibt es ganze Zahlen n, μ und ein Multiplikatorsystem v derart, daß (*) nicht gilt. Schließlich wird noch eine Anwendung von (*) auf die Abschätzung der Fourierkoeffizienten von Spitzenformen erwähnt. *E. Gottschling* (Berlin)

Mitrović, Dragiša 2232

The signs of some constants associated with the Riemann zeta-function.

Michigan Math. J. **9** (1962), 395–397.

Briggs and Chowla have proved the following theorem. If $\zeta(s)=(s-1)^{-1}+\sum_{n=0}^{\infty}a_n(s-1)^n$, then $a_n=(-1)^n\gamma_n/n!$, where

$$\gamma_n = \lim_{N\to\infty}\left[\sum_{k=1}^{N}\frac{\log^n k}{k}-\frac{\log^{n+1}N}{n+1}\right]$$

[Amer. Math. Monthly **62** (1955), 323–325; MR **16**, 999]. {Remark of the reviewer: This result was discovered by Stieltjes (1885) and it has been discussed thoroughly by Stieltjes and Hermite [cf. *Correspondance d'Hermite et de Stieltjes*, Tome I, pp. 153, 155, 158, lettre 75, Gauthier-Villars, Paris, 1905]. In *Correspondance*, II, p. 451, it is stated by Gram that this result has been rediscovered by Jensen.} Briggs has established that infinitely many γ_n are positive and infinitely many are negative [Michigan Math. J. **3** (1955/56), 117–121; MR **17**, 955].

This result has been completed by the author in the present paper as follows: Each of the inequalities $a_{2n}>0$, $a_{2n}<0$, $a_{2n-1}<0$, $a_{2n-1}>0$ holds for infinitely many n; each of the inequalities $\gamma_{2n}<0$, $\gamma_{2n}>0$, $\gamma_{2n-1}<0$, $\gamma_{2n-1}>0$ holds for infinitely many n.

The method of the proofs rests on the supposition that

the cardinal numbers of the sets of the first category, as well as those of the last category $< \aleph_0$, will lead to a contradiction. *S. C. van Veen* (Delft)

Apostol, Tom M. 2233

A short proof of Shô Iseki's functional equation.

Proc. Amer. Math. Soc. **15** (1964), 618–622.

Notations: $\lambda_p(x) = \sum_{m=1}^{\infty} m^{-p} e^{-2\pi m x}$,

$$\Lambda_p(\alpha, \beta, z) = \sum_{r=0}^{\infty} \{\lambda_p((r+\alpha)z - i\beta) + \lambda_p((r+1-\alpha)z + i\beta)\},$$

$$R_p(\alpha, \beta, z) = -(2\pi z)^p\{(p+1)!\}^{-1} \sum_{n=0}^{p+1} \binom{p+1}{n} (iz)^{-n} B_{p+1-n}(\alpha) B_n(\beta).$$

$B_n(t)$ stands for the Bernoulli polynomial of degree n, and the parameters involved are subject to certain restrictions. Iseki proved [Duke Math. J. **24** (1957), 653–662; MR **19**, 943] the functional equation

$$\Lambda_p(\alpha, \beta, z) = (iz)^{p-1} \Lambda_p(1-\beta, \alpha, z^{-1}) + R_p(\alpha, \beta, z).$$

Using a method going back to Rademacher [J. Reine Angew. Math. **167** (1932), 312–336] and based on the use of the Mellin transform, the author gives a new proof of Iseki's functional equation. *E. Grosswald* (Paris)

Uchiyama, Miyoko; Uchiyama, Saburô 2234

On the representation of large even integers as sums of a prime and an almost prime.

Proc. Japan Acad. **40** (1964), 150–154.

It was proved by C. T. Pan [Acta Math. Sinica **12** (1962), 95–106; MR **29** #4727; Sci. Sinica **11** (1962), 873–888; MR **27** #1427] that every sufficiently large even integer is the sum of a prime and a product of at most five prime factors. The authors replace "five" by "four", but seem to be unaware of the fact that this was already achieved meanwhile by Pan [ibid. **12** (1963), 455–473; MR **28** #73]. *N. G. de Bruijn* (Eindhoven)

Schnabl, Roman 2235

Zur Theorie der homogenen Gleichverteilung modulo 1.

Österreich. Akad. Wiss. Math.-Natur. Kl. S.-B. II **172** (1963), 43–77.

P. Erdős and G. G. Lorentz [Acta Arith. **5** (1958), 35–44; MR **21** #37] called a sequence (x_n) homogeneously equidistributed mod 1 if for every positive integer d the sequence (x_{nd}/d) is uniformly distributed mod 1. The author generalizes this concept to the case of uniform distribution with respect to weighted means (M, λ_n) and constructs examples of such sequences. He also generalizes the concept of normal number to the case of weighted means and proves that there are numbers which are $(M, 1/n)$-normal but not normal in the usual sense. Some metrical theorems and a generalization of a theorem of Erdős and Lorentz conclude the paper. *J. E. Cigler* (Vienna)

Koch, Helmut 2236

Über den 2-Klassenkörperturm eines quadratischen Zahlkörpers. I.

J. Reine Angew. Math. **214/215** (1964), 201–206.

Let P and k be the rational number field and the absolute quadratic number field with discriminant Δ. Let S be the set of the prime rational integers dividing Δ. We define $k^{(n,2)}$ and $\mathsf{P}_S^{(n)}$ for $n \geqq 0$ inductively such that $k^{(0,2)} = k$, $\mathsf{P}_S^{(0)} = \mathsf{P}$, and $k^{(n+1,2)}$ and $\mathsf{P}_S^{(n)}$ are the composite field of all unramified quadratic extensions of $k^{(n,2)}$ central over k and the composite field of all cyclic extensions of $\mathsf{P}_S^{(n)}$ with 2-power degree unramified at all prime divisors not belonging to S and central over P. Let $k^{(\infty)} = \bigcup_{n=0}^{\infty} k^{(n,2)}$.

Proving $k^{(\infty)} \cap \mathsf{P}_S^{(2)} = k^{(2,2)}$ and using group-theoretic lemmas, the author determines the structure of the Galois group $G(k^{(2,2)}/k)$ explicitly. Then he shows that $G(k^{(2,2)}/k)$ is determined only by the values of the Legendre symbols (a/p) for $a|\Delta$, $p|\Delta$. It is shown that $k^{(\infty)} \neq k^{(2,2)}$ if the number of elements of S is greater than 4. *K. Masuda* (Okayama)

Gel'fand, I. M.; Graev, M. I.; Pjateckiĭ-Šapiro, I. I. 2237

Representations of adèle groups. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 487–490.

Es sei G die Gruppe der unimodularen Matrizen 2-ten Grades mit Koeffizienten aus dem rationalen Zahlkörper Q. Man denke sich G eingebettet in die zugehörige "Adelgruppe" G_A im Sinne von A. Weil [Séminaire Bourbaki, 1958/59, Exp. 186, Secrétariat mathématique, Paris, 1959; MR **28** #1091]. G_A wird folgendermassen konstruiert: Für jede Primstelle p von Q (einschliesslich der unendlichen Stelle) sei G_p die Gruppe der unimodularen Matrizen 2-ten Grades über der p-adischen Komplettierung Q_p von Q; wenn p endlich ist, bedeute U_p diejenige Untergruppe von G_p, die aus allen Matrizen mit p-adisch ganzen Koeffizienten besteht. Dann besteht G_A aus allen formalen Produkten $g = \prod_p g_p$, wobei $g_p \in G_p$ und $g_p \in U_p$ für alle bis auf endlich viele p. Für eine endliche Menge S von Primstellen p, welche die unendliche Primstelle enthält, wird $G_A{}^S = \prod_{p \in S} G_p \cdot \prod_{p \notin S} U_p$ gesetzt. G_A ist die Vereinigung aller dieser Untergruppen $G_A{}^S$. Die Topologie von G_A wird so definiert, dass die $G_A{}^S$ (versehen mit der Produkt-Topologie) als offene, topologische Untergruppen von G_A erscheinen. G_A ist lokal-kompakt, und G ist diskret in G_A [Weil, loc. cit.]. In der vorliegenden Arbeit wird die Darstellung untersucht, die von G_A im Raum $L_2(G_A/G)$ der quadratisch-integrierbaren Funktionen auf dem homogenen Raum G_A/G vermittelt wird. Und zwar handelt es sich um die Zerlegung dieser Darstellung in irreduzible Komponenten. Es sei Z die Untergruppe der Matrizen der Form $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$ und D die Untergruppe der Diagonalmatrizen aus G; sei $X = G_A/Z_A D$. Die Darstellung von G_A im Raume $L_2(X)$ zerlegt sich in irreduzible Darstellungen T_π, wobei π die Charaktere von Λ_A/Λ durchläuft und Λ die multiplikative Gruppe von Q bezeichnet. Der Darstellungsraum von T_π wird dabei gebildet durch diejenigen Funktionen $f_\pi(x)$ auf X, welche der Relation genügen: $f_\pi(\lambda x) = \pi(\lambda)|\lambda|^{-1} \cdot f_\pi(x)$, $\lambda \in \Lambda$, $|\lambda| = \prod_p |\lambda_p|_p$, und für welche $\|f_\pi\|^2 = \int |f_\pi(x)|^2 \, d\omega < +\infty$, wobei das Integral erstreckt wird über die Fläche $\Omega: |x_p|_p = 1$ und $d\omega$ das natürliche Mass auf Ω bezeichnet. Die Darstellungen T_{π_1} und T_{π_2} sind äquivalent wenn und nur wenn $\pi_1 = \pi_2$ oder $\pi_1 = \pi_2{}^{-1}$. Eines der Hauptergebnisse der Verfasser besagt, dass diese Darstellungen T_π auch in einer gewissen Teildarstellung der Darstellung von

G_A in $L_2(G_A/G)$ vorkommen, und zwar mit der Multiplizität 1. π kann als Grössencharakter im Sinne von Hecke angesehen werden; zu ihm gehört ein Charakter χ der multiplikativen Gruppe χ von Q mit endlichem Führer. Die Funktionalgleichung der Dirichletschen L-Funktion $L(s, \chi)$ ist damit gleichbedeutend, dass T_π in der in Rede stehenden Teildarstellung nur einfach vorkommt. Ein Teil der Resultate der Verfasser bezieht sich auf allgemeine algebraische Gruppen über einem beliebigen algebraischen Zahlkörper. Beweise werden nicht gegeben.

{Literatur: [1] A. Weil, Adèles and algebraic groups, [Lecture notes, Princeton, 1961], [2] A. Borel [Inst. Hautes Études Sci. Publ. Math. No. 16 (1963), 5–30], [3] I. M. Gel'fand and I. I. Pjateckiĭ-Šapiro [Trudy Moskov. Mat. Obšč. 12 (1963), 389–412; MR 28 #3115].}

P. Roquette (Tübingen)

Moroz, B. Z. 2238

On the continuability of the scalar product of Hecke series of two quadratic fields. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 1265–1267.

The author extends the results of his earlier paper [same Dokl. **150** (1963), 752–754; MR **27** #1423] to any two imaginary quadratic fields.

Let K_1 and K_2 be imaginary quadratic fields with discriminants d_1, d_2 $(d_1 \neq d_2)$, let $\eta^{w_1 n}$ and $\zeta^{w_2 m}$ be Hecke characters of the fields K_1 and K_2 accordingly, where w_j is the number of unities of K_j. Denote by N_j the norm of the divisors in K_j. The scalar product of Hecke series of the fields K_1, K_2 is defined by the series

$$Z(s, \eta^{w_1 n}, \zeta^{w_2 m}) = \sum_{N_1\mathfrak{A} = N_2\mathfrak{B}} \eta^{w_1 n}(\mathfrak{A})\zeta^{w_2 m}(\mathfrak{B})(N_1\mathfrak{A})^{-2s},$$

where the summation is extended over all integer divisors in K_1 and K_2 with equal norms $N_1\mathfrak{A} = N_2\mathfrak{B}$, and s is a complex variable.

The author proves the analytic continuability of the function $Z(s, \eta^{w_1 n}, \zeta^{w_2 m})$ in the half-plane $\operatorname{Re} S > \frac{1}{2} - \gamma$, where γ is an absolute positive constant.

Let $\hat{\mathfrak{A}}$ be a class of divisors in K_1, and $\hat{\mathfrak{B}}$ a class of divisors in K_2. Denote by $f_{\hat{\mathfrak{A}},\hat{\mathfrak{B}}}(X)$ the number of pairs of prime divisors $(\mathfrak{p}, \mathfrak{q})$ satisfying the conditions $N_1\mathfrak{p} = N_2\mathfrak{q} < X$, $\mathfrak{p} \in \hat{\mathfrak{A}}$, $\mathfrak{q} \in \hat{\mathfrak{B}}$. Then the number of pairs of prime divisors $(\mathfrak{p}, \mathfrak{q})$ for which $\mathfrak{p} \in \hat{\mathfrak{A}}$, $\mathfrak{q} \in \hat{\mathfrak{B}}$, $N_1\mathfrak{p} = N_2\mathfrak{q} < X$, $\varphi_1 \leq \operatorname{arc} \alpha \leq \varphi_2$, $\tilde{\varphi}_1 \leq \operatorname{arc} \beta \leq \tilde{\varphi}_2$ is

$$\frac{w_1 w_2}{4\pi^2}(\varphi_2 - \varphi_1)(\tilde{\varphi}_2 - \tilde{\varphi}_1) f_{\hat{\mathfrak{A}},\hat{\mathfrak{B}}}(X) + O(X \exp(-\gamma_1 \sqrt{\log X})),$$

where γ_1 is an absolute positive constant, α and β are the ideal numbers of Hecke corresponding to the divisors $\mathfrak{p}, \mathfrak{q}$.

The proof is only sketched. The continuability of $Z(s, \eta^{w_1 n}, \zeta^{w_2 m})$ is founded on the properties of integer point distribution on hyperboloids.

J. Kubilius (Vilnius)

Yokoi, Hideo 2239

A note on the Galois cohomology group of the ring of integers in an algebraic number field.

Proc. Japan Acad. **40** (1964), 245–246.

Paraphrased slightly from author's introduction: "Let K be a finite Galois extension of a finite algebraic number field F and let $G = G(K/F)$ be the Galois group of K/F. Denote by σ_K and σ_F the rings of integers of K and F, respectively. In an earlier paper [Nagoya Math. J. **16** (1960), 83–90; MR **23** #A888] the author proved Theorem 1: If $H^0(G, \sigma_K)$ is trivial then $H^r(H, \sigma_K)$ is trivial for every r and every subgroup H of G. Later the author obtained [Proc. Japan Acad. **38** (1962), 499–501; MR **28** #83; Acta Arith. **8** (1962/63), 243–250; MR **28** #82] the following Theorem 2. Let K/F be a cyclic extension of prime order. Then the groups $H^r(G, \sigma_K)$ are isomorphic for all r. It is generally conjectured that the groups $H^r(G, \sigma_K)$ have the same order for all r. In this note the author proves that this is the case if K/F is a cyclic extension of any finite degree." *G. B. Seligman* (New Haven, Conn.)

Kubota, Tomio 2240

Some arithmetical applications of an elliptic function.

J. Reine Angew. Math. **214/215** (1964), 141–145.

The author considers $f(z) = \operatorname{sn}(2-2i)\omega z$, where $\omega = \int_0^1 dx/(1-x^4)^{1/2}$, and $z = \int_0^x dx/(1-x^4)^{1/2}$ defines $x = \operatorname{sn} z$. The period module of f coincides with the Gaussian integers $\mathfrak{o}$ (in $Q(i)$) and the divisor structure of f (written additively) is $(0) + (\frac{1}{2} + \frac{1}{2}i) - (\frac{1}{2}) - (\frac{1}{2}i)$. The author proves Gauss's biquadratic reciprocity formula by the Eisenstein method [G. Eisenstein, same J. **29** (1845), 177–184] based on

$$\left(\frac{\nu}{\mu}\right)_4 = \prod_{\alpha \in M} f(\nu\alpha/\mu)/f(\alpha/\mu),$$

where M is a quarter-residue system modulo μ (a prime in $\mathfrak{o}$). {Reviewer's note: Attention should be drawn to L. Koschmieder [Math. Ann. **83** (1921), 280–285], in addition to the author's reference to G. Herglotz [Ber. Verh. Sächs. Akad. Wiss. Leipzig Math.-Phys. Kl. **73** (1921), 303–310].} The author further considers the biquadratic Gaussian sum

$$\tau_4(\pi) = \sum_{a \bmod \pi} (a/\pi)_4 \zeta^a,$$

where $\pi \equiv 1 \bmod l^3$ $(l^2 = (2))$, and $N(\pi) = p$ (prime), $\zeta^p = 1$ $(\zeta \neq 1)$. The author notes $\tau_4(\pi) \cong \mathfrak{P}^{p-1}$ for a prime ideal $\mathfrak{P}$ of $Q(\zeta)$. Actually, $\mathfrak{P}^{(p-1)/4}$ is shown to be principal. Finally, the author shows an analogue of Gauss's formula for $\tau(p) = \sum_{a \bmod p} (a/p)\zeta^p$ (p prime). In one form,

$$\tau(p) = (\zeta - \zeta^{-1})(\zeta^3 - \zeta^{-3}) \cdots (\zeta^{p-2} - \zeta^{2-p}).$$

Actually, $\tau_4(\pi) = \varepsilon_\pi \sigma(\pi)$, where ε_π is a fourth root of unity and $\sigma(\pi) = \prod f(mc/\pi)^3 f(m'c'/\bar{\pi})^3$, where π is defined as above, while m goes over a reduced residue system of rational integers mod π while $mm' \equiv 1 \pmod p$. The value of ε_π still requires discussion. *H. Cohn* (Tucson, Ariz.)

Ramachandra, K. 2241

Some applications of Kronecker's limit formulas.

Ann. of Math. (2) **80** (1964), 104–148.

This paper contains some remarkable new results on the construction of the ray class fields of an imaginary quadratic number field. The proofs depend largely on calculations involving special modular functions in connection with Kronecker's limit formulae. Let

$$\vartheta(z; u, v) = \sum_{n=-\infty}^{+\infty} e^{\pi i z(n-v)^2 + 2\pi i n u - \pi i u v}$$

with rational u, v of common denominator f and

$$\Phi(z;u,v) = \{\vartheta(z;u,v)\eta(z)^{-1}\}^{12},$$

where $\eta(z)$ is the classical eta function. The functional equation due to Kronecker,

$$\Phi^f(M(z);M(u,v)) = \Phi^f(z;u,v),$$

for each substitution M of the elliptic modular group, is the basis of the following considerations.

Notations: P is the field of rational numbers; Σ an imaginary quadratic extension of P with discriminant d and different $\mathfrak{d}$; $\mathfrak{f}$, $\mathfrak{g}$ integral ideals of Σ; $\mathfrak{g}$ a divisor of $\mathfrak{f}$; $\bar{\mathfrak{f}}$ the complex conjugate ideal; $\Sigma_{\mathfrak{f}}$, $\Sigma_{\bar{\mathfrak{f}}}$ the class fields over Σ for the ray class groups mod $\mathfrak{f}$, $\bar{\mathfrak{f}}$; f the smallest positive rational integer divisible by $\mathfrak{f}$.

For a ray class R mod f put

$$\Phi_{\mathfrak{f},\mathfrak{g}}(R) = \Phi^f\left(\frac{\beta_2}{\beta_1}; s(\beta_1), s(\beta_2)\right),$$

where β_1, β_2 is a basis of $\mathfrak{b}\mathfrak{g}/\mathfrak{d}\mathfrak{f}$, $\mathfrak{b}$ being an ideal of the ray class R, and where s denotes the trace with respect to P. This is an invariant of R. The author also defines $\Phi_{\mathfrak{f},\mathfrak{f}}(R)$; however, the reviewer does not understand why that expression is also an invariant of R. Perhaps a correction is necessary. Kronecker's limit formula now yields (1) $\sum_R \bar{\chi}(R)\log|\Phi_{\mathfrak{f},\mathfrak{g}}(R)| = 0$ if $\mathfrak{g}$ does not divide $\mathfrak{f}/\mathfrak{f}_\chi$, $r(\mathfrak{f},\mathfrak{g},\chi)L(1,\chi_0)\neq 0$ otherwise, the sum to be extended over all ray classes R mod $\mathfrak{f}$, where $\chi\neq 1$ is a character of the ray class group, $\mathfrak{f}_\chi$ the conductor of χ, χ_0 the proper character mod $\mathfrak{f}_\chi$ defined by χ, and $r(\mathfrak{f},\mathfrak{g},\chi)$ an elementary factor. Similarly for the principal character

$$(2) \qquad \sum_R \log|\Phi_{\mathfrak{f},\mathfrak{g}}(R)| = 0$$

if $\mathfrak{f}/\mathfrak{g}$ is not a power of a prime ideal, and is $r(\mathfrak{f},\mathfrak{g})L_d(1)$ otherwise, where $L_d(s) = \sum (d/n)n^{-s}$.

The $\Phi_{\mathfrak{f},\mathfrak{g}}(R)$ are conjugate algebraic numbers contained in $\Sigma_{\bar{\mathfrak{f}}}$. As a consequence of (2) they are units if $\mathfrak{g}=(1)$ and $\mathfrak{f}$ is a power of a prime ideal. Otherwise, suitable power products $\varepsilon(R)$ of them are units. Under the Galois group G of $\Sigma_{\bar{\mathfrak{f}}}/\Sigma$ they behave as follows:

$$\Phi_{\mathfrak{f},(1)}(R)^\sigma = \Phi_{\mathfrak{f},(1)}(R\bar{S}_\sigma), \quad \varepsilon(R)^\sigma = \varepsilon(R\bar{S}_\sigma), \quad \sigma\in G,$$

where S_σ is the ray class mod $\bar{\mathfrak{f}}$ associated with σ in the sense of class field theory. The proof uses Hasse's foundation of complex multiplication [J. Reine Angew. Math. **157** (1926), 115–139; ibid. **165** (1931), 64–88]. As a consequence, the relative norms $\eta_{\Sigma_{\mathfrak{f}}/K}(\varepsilon(R))$ with respect to an intermediate field K between Σ and $\Sigma_{\bar{\mathfrak{f}}}$ generate that field.

The units $\varepsilon(R)^{\sigma-1}$ ($\sigma\in G$) generate a subgroup of finite index in the total unit group of $\Sigma_{\mathfrak{f}}$. Combining (1) and (2) with the class number formula, this index turns out to be the class number $H_{\mathfrak{f}}$ of $\Sigma_{\mathfrak{f}}$ times a factor consisting of the numbers h of absolute ideal classes, $h_{\mathfrak{f}}$ of ray classes mod $\mathfrak{f}$ in Σ, and of elementary expressions. This formula generalizes one given by Fueter [Rend. Circ. Mat. Palermo **29** (1910), 380–395] and Siegel [*Lectures on advanced analytic number theory*, Tata Inst. Fund. Res., Bombay, 1961].

In the last section the author proves that the values taken by a modular form $f(z)$ and its derivatives $f'(z)$, $f''(z)$, $\cdots$ for $z\in\Sigma$ are rationally expressible by algebraic numbers, by π, and by $\Delta^{1/u}(\sqrt{d})$, i.e., a suitable root of the delta function. Furthermore, $\Delta(\sqrt{d})$ is expressed in terms of $\Gamma(m/d)$, $m=1,\cdots,|d|-1$. The paper closes with some applications on the algebraic nature of certain elliptic integrals, one of which was first stated by Chowla and Selberg [Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 371–374; MR **11**, 84]. *M. Eichler* (Basel)

FIELDS AND POLYNOMIALS

See also 2236, 2239, 2241, 2262, 2517, 2702.

Rodriquez, Gaetano 2242

Forme canoniche quadratiche e cubiche nei domini d'integrità p-adici.

Ist. Lombardo Accad. Sci. Lett. Rend. A **97** (1963), 881–898.

Using simple p-adic approximations, the author discusses necessary and sufficient conditions for the solvability of equations like $x^2-\alpha y^2=\beta$ and $x^3+\alpha y^3=\beta$ for given p-adic integers α, β by p-adic integers. (The author is dealing with power series in one variable.)

O. F. G. Schilling (Lafayette, Ind.)

Rowlinson, Elizabeth 2243

New proofs for two theorems of Capelli.

Canad. Math. Bull. **7** (1964), 431–433.

The theorems in question may be stated as follows: Let $g(x)$, $g_1(x)$, $h(x)$, $h_1(x)$ be polynomials over a field R of characteristic 0, such that degree $g(x)=m=$ degree $h_1(x)$, degree $g_1(x)=n=$ degree $h(x)$, $(m,n)=1$, and $f(x)=g(h(x))=g_1(h_1(x))$. Then $f(x)$ is irreducible over R if and only if $g(x)$ and $g_1(x)$ are irreducible over R if and only if $g(x)$ is irreducible over R and $h(x)-\beta$ is irreducible over $R(\beta)$ where β is a root of $g(x)$.

The proofs use only elementary properties of the degree of an extension of a field.

R. J. Crittenden (Providence, R.I.)

McConnel, R. 2244

Pseudo-ordered polynomials over a finite field.

Acta Arith. **8** (1962/63), 127–151.

The aim of this paper is to provide the algebraic background to the generalization of Pierce's method of constructing "Moulton planes" [Pierce, Canad. J. Math. **13** (1961), 427–436; MR **23** #A1281], namely, to construct on a finite field one-to-one functions with order-preserving properties, generalizing those constructed by Carlitz [Proc. Amer. Math. Soc. **11** (1960), 456–459; MR **22** #8005; Acta Arith. **7** (1961/62), 167–172; MR **25** #66].

The main theorem is the following. F is a field of order $q=p^n$, and $d|q-1$, say $q-1=dm$; $\Psi_d(x)=_{\text{def}}x^m$, $x\in F$; $f(x)$ is a permutation polynomial on F constructed to satisfy $f(0)=0$, $f(1)=1$, $\Psi_d\{f(x)-f(y)\}=\Psi_d(x-y)$ for all x, y. It is proved that $f(x)=x^{p^j}$, where $d|p^j-1$, $0\leq j<n$.

In the latter part of the paper the restrictions $f(0)=0$, $f(1)=1$ are removed, and $f(x)$ is generalized to a function of m variables. *T. G. Room* (Sydney)

Okuzumi, Motoichi 2245

Generating elements in a field.

Kōdai Math. Sem. Rep. **16** (1964), 127–128.

Suppose that F/k is a finite separable extension. An intermediate field M is said to have property (P) if every primitive element of F/M is also a primitive element of F/k. It is proved that the maximal subfield Δ of F/k with property (P) is the intersection of all maximal subfields of F/k. Thus, in case F/k is Galois, the field Δ belongs to the subgroup of the Galois group which is generated by all elements of prime order.

O. F. G. Schilling (Lafayette, Ind.)

Jensen, C. U. 2246

A remark on relative integral bases for infinite extensions of finite number fields.

Mathematika **11** (1964), 64–66.

Let K be an algebraic number field (i.e., a finite extension of the rationals), K' an infinite algebraic extension of K, R and R' the rings of algebraic integers in K and K'. Then R' is a free R-module (compare with the fact that "big" projective modules are free). Given a sub-R'-module A' of K', conditions that A' be free over R are given.

P. Samuel (Waltham, Mass.)

Robinson, Raphael M. 2247

Conjugate algebraic integers in real point sets.

Math. Z. **84** (1964), 415–427.

The diameter d_n of order n of a closed bounded point set S in the complex plane is the maximum possible value of the geometric mean of the distances between each pair of points chosen from n points in S. M. Fekete [same Z. 17 (1923), 228–249] showed that $d_2 \geqq d_3 \geqq d_4 \geqq \cdots$. Hence there is a number $d_\infty \geqq 0$ such that $d_n \to d_\infty$ as $n \to \infty$. d_∞ is called the transfinite diameter of S. The present paper generalizes a result of the author [*Studies in mathematical analysis and related topics*, pp. 305–315, Stanford Univ. Press, Stanford, Calif., 1962; MR **26** #2433]. The author proves that any real point set E of transfinite diameter greater than one which consists of a finite number of closed intervals must contain infinitely many sets of conjugate algebraic integers. The problem is unresolved for point sets of transfinite diameter one. If the origin is an inner point of E, it is shown that E must contain infinitely many sets of conjugate algebraic units.

F. Supnick (New York)

Kunz, Ernst 2248

Arithmetische Anwendungen der Differentialalgebren.

J. Reine Angew. Math. **214/215** (1964), 276–320.

Pour simplifier, nous n'exposerons les résultats de l'auteur que dans le cas des corps de fonctions algébriques. Soit k un corps, et soit K une extension de type fini de k; on notera $A^r(K/k)$ l'espace vectoriel sur K formé des formes différentielles extérieures de degré r de l'extension K/k. Pour toute extension L/K algébrique de degré fini, l'auteur définit des opérateurs de trace $\sigma^r_{L/K}: A^r(L/k) \to A^r(K/k)$ caractérisés par les propriétés suivantes: (i) $\sigma^r_{M/K} = \sigma^r_{L/K} \circ \sigma^r_{M/L}$ pour $K \subset L \subset M$; (ii) pour x dans L de trace t et de norme n, on a $\sigma^0_{L/K}(x) = t$ et $\sigma^1_{L/K}(x^{-1}\,dx) = n^{-1}dn$; (iii) $\sigma^r(da) = d\sigma^{r-1}(a)$ pour a dans $A^{r-1}(K/k)$; (iv) si ι est l'homomorphisme de $A(K/k)$ dans $A(L/k)$, défini par l'inclusion de K dans L, on a $\sigma^{r+s}(\iota(a)\cdot b) = a\cdot\sigma^r(b)$ pour a dans $A^s(K/k)$ et b dans $A^r(L/k)$.

L'auteur définit ensuite le résidu de Poincaré comme suit. Soit R un anneau de valuation discrète dans K, contenant k, de corps des restes $\mathbf{K}$ avec $d = \dim_k K = \dim_k \mathbf{K} + 1$, et soit t une uniformisante de R. Lorsque K est de caractéristique 0, toute forme différentielle $\omega \in A^r(K/k)$ avec $d\omega = 0$ admettant un pôle d'ordre $\leqq 1$ en R est de la forme $\omega = a\cdot dt/t + b$, avec a et b sans pôle en R, et le résidu de ω en R est l'image de a dans $A^{r-1}(\mathbf{K}/k)$. Supposons maintenant K de caractérisque $p \neq 0$; choisissons un sous-corps k_0 de k contenant k^p, avec $[k:k_0]$ fini, linéairement disjoint de K^p et $\mathbf{K}^p$ sur k^p avec $k_0[R^p] = k_0(K^p) \cap R$; si $\omega \in A^r(K/k_0)$ admet un pôle d'ordre $\leqq 1$ en R et si $d\omega = 0$, on peut définir son résidu $\mathrm{res}_R\,\omega \in A^{r-1}(\mathbf{K}/k_0)$. Le théorème central du mémoire est la formule $\mathrm{res}_R\,(\sigma^r_{L/K}\omega) = \sum_i \sigma^r_{\mathbf{L}_i/\mathbf{K}}(\mathrm{res}_{S_i}\,\omega)$ où L est une extension algébrique finie de K, où les S_i sont les anneaux de valuation discrète de L dominant R, et $\mathbf{L}_i$ le corps des restes de S_i; cette formule est due à Hasse lorsque k est parfait et $d = 1$.

A titre d'application, l'auteur généralise le théorème de la différente de Dedekind aux extensions non-séparables, et complète les résultats obtenus par lui-même [Math. Z. **76** (1961), 56–74; MR **25** #72] et par Zariski et Falb [Amer. J. Math. **83** (1961), 542–556; MR **26** #1316] sur la classe canonique des corps de fonctions algébriques d'une variable dans le cas non-séparable.

P. Cartier (Strasbourg)

Nakano, Takeo 2249

A generalized valuation and its value groups.

Comment. Math. Univ. St. Paul. **12** (1964), 1–22.

A multiplicative group Σ is called a v-group if there is given on it a second operation "+" which is associative, commutative and distributive on the left and right. For example, all positive elements $\Sigma(F)$ of an ordered skew field form a v-group. A subgroup Γ of the v-group Σ is called (left) convex if $\gamma \in \Gamma$ and $\lambda + \mu = 1$ imply $\lambda + \mu\gamma \in \Gamma$. It follows immediately that a left convex normal subgroup is also right convex. Hence, Σ/Γ is seen to be a v-group, and any homomorphic image of Σ onto a v-group is obtained in this manner. Furthermore, a subgroup Γ is called a v-subgroup if $\alpha, \beta \in \Gamma$ implies $\alpha + \beta \in \Gamma$. It is seen that convex subgroups of l-groups (in the sense of Birkhoff) are v-subgroups, and that a factor group Σ/Γ is an l-group (Γ is convex normal) if and only if Γ is a v-subgroup. Next, suppose that R is a ring such that to each $x \in R$ there is associated a non-void subset $\|x\|$ of Σ such that (1) $\|x\| = \Sigma$ if and only if $x = 0$, (2) $\|x \pm y\| \supset \|x\| + \|y\|$ and $\|xy\| \supset \|x\|\,\|y\|$ for $x, y \in R$. The function $x \to \|x\|$ is called a valuation of R with value group Σ. Note that (1) and (2) imply that the "values" $\|x\|$ are upper classes of Σ. Furthermore, all $x \in R$ such that $\|x\| \cap \Gamma \neq \varnothing$, Γ a v-subgroup, form a subring $R(\Gamma)$ of R. Then a generalization of Krull's splitting theorem for valuations yields: "(A) Let Γ be a convex normal subgroup of Σ. If $\|x\|\cdot\Gamma \neq \Sigma$ for every non-zero $x \in R$, then $|x| = \|x\|\cdot\Gamma$ is a valuation of R with value group Σ/Γ. (B) If Γ is a v-group of Σ, then $R(\Gamma)$ is a subring of R and $J(\Gamma) = \{x;\ \|x\| \supset \Gamma\}$ is an ideal of $R(\Gamma)$. Let X be any residue class of $R(\Gamma)/J(\Gamma)$. Then $\|X\| = (\|x\| \cap \Gamma) + \Gamma$ $(x \in X)$ is a valuation of $R(\Gamma)/J(\Gamma)$ with value group Γ". The sets $U_\alpha = \{x;\ \|x\| + \Sigma \ni \alpha\}$, considered as members of a neighborhood base of $0 \in R$, determine a topology on R. Then two valuations are termed equivalent if they determine the same topology

on R, and two v-groups Σ_1 and Σ_2 are termed equivalent if there exists, for any ring R and any valuation of R with value group Σ_1, an equivalent valuation of R with value group Σ_2, and vice versa. The author shows that for a bounded convex normal subgroup Γ (there exists $\beta \in \Gamma$ such that $\gamma \leqq \beta$ for all $\gamma \in \Gamma$) the factor group Σ/Γ is equivalent to Σ. Furthermore, the existence of a "strong unit" $\nu \in \Sigma$, $\nu^n > \alpha$, with positive $n = n(\alpha)$ for all $\alpha \in \Sigma$, implies that any valuation with value group Σ is equivalent to a pseudo-valuation in the sense of Mahler. A v-group is called additive if the cancellation law holds for addition. Then, as a final theorem the author proves that any simple (absence of non-trivial convex normal subgroups) v-group is isomorphic to the v-group $\Sigma(F)$ for some subfield of the real number field. Also, the existence of completions by means of Cauchy sequences is established.

O. F. G. Schilling (Lafayette, Ind.)

Ribenboim, P. 2250

On the existence of fields with few discrete valuations.

J. Reine Angew. Math. **216** (1964), 43–49.

This note deals with an extension of Hasse's existence theorem for algebraic number fields $\bar{K}/K$ in which a finite set of prime ideals of K have prescribed decompositions [H. Hasse, Math. Ann. **95** (1925), 229–238; for generalizations see W. Krull, Abh. Math. Sem. Univ. Hamburg **23** (1959), 23–35; MR **21** #3406; P. Ribenboim, Trans. Amer. Math. Soc. **105** (1962), 278–294; MR **25** #5059]. Let T be a field with n pairwise non-equivalent, discrete, rank 1 valuations w_i with corresponding residue class fields T_i of characteristic χ. If K_i/T_i are algebraic extensions, then there exists a field K which has exactly n pairwise non-equivalent discrete rank 1 valuations v_i whose residue class fields are the given fields K_i. An example shows that equality of the characteristic χ cannot be dropped in general. This theorem furnishes the basis for numerous examples of fields with curious properties, to wit: "For every prime number p there exist $2^{\aleph_0}$ fields K of algebraic numbers satisfying the following conditions: (a) K has exactly one discrete valuation w; (b) the residue class field of K with respect to w is the prime field of characteristic p; (c) K is henselian, but not complete with respect to w; (d) K is neither algebraically closed nor maximally ordered, however, if F is a subfield such that $[K:F] < \infty$, then necessarily $F = K$".

O. F. G. Schilling (Lafayette, Ind.)

Ribenboim, P. 2251

On the completion of a valuation ring.

Math. Ann. **155** (1964), 392–396.

Suppose that v is a valuation of the field K with corresponding valuation ring A. The author investigates various relations and the lack of such between the valuation ring A^* belonging to the natural extension v^* of v to an immediate maximal extension K^* of K. Among other results the following are established. (1) If A^* is a finitely generated A-module, then $K^* = K$. The proof depends on the author's work on multiply complete fields [Math. Z. **69** (1958), 466–479; MR **20** #3861] and on the validity of the relation $\sum e_i f_i = n$ [see O. Zariski and P. Samuel, *Commutative algebra*, Vol. II, p. 80, Van Nostrand, Princeton, N.J., 1960; MR **22** #11006]. (2) If v is a valuation of rank 1 and its rational rank is at most countable, then K^*/K is not a purely transcendental extension.

O. F. G. Schilling (Lafayette, Ind.)

Tai, Zhi-zhong [Tai, Chih-chung] 2252

On discrete valuations.

Acta Math. Sinica **13** (1963), 17–22 (*Chinese*); *translated as Chinese Math.* **4** (1963), 18–23.

In this note certain results of P. Ribenboim [Math. Z. **68** (1957), 1–18; MR **19**, 1035; ibid. **69** (1958), 466–479; MR **20** #3861] and F. K. Schmidt [Math. Ann. **108** (1933), 1–25] are extended to fields K admitting a valuation w of the following type. The value group of w is to be such that the factor groups Δ'/Δ of the isolated subgroups of the value group of w which belong to immediately adjacent prime ideals of the valuation ring are isomorphic to the additive group of integers. It is shown that, for such valuations, completeness of K in the sense of Krull implies maximal completeness in the sense of Ostrowski and Kaplansky. Two valuations w and w' of K are called comparable if $A \supseteq A'$ or $A' \supseteq A$ for the corresponding valuation rings; they are called independent if $A \cdot A' = K$. Assuming that A has the zero ideal as the only limit prime ideal, among other generalizations of theorems of the two authors [loc. cit.] the following results are established. (1) If K is complete with respect to w, then K possesses no non-trivial discrete valuation v of finite rank which is comparable with w or is independent of w. (2) Suppose that K is complete with respect to w and v and w are independent. If $\bar{K}$ is the complete closure of K with respect to w', then $\bar{K}$ is a separably algebraically closed field. (3) If K is complete with respect to a discrete valuation w, then (i) K cannot possess any discrete valuation of finite rank; (ii) if w' is another discrete valuation of K which possesses the same order type as w and with respect to which K is complete, then w' and w must be comparable.

O. F. G. Schilling (Lafayette, Ind.)

Nishimura, Hajime 2253

On differentially integral elements.

Proc. Japan Acad. **40** (1964), 145–149.

Let P be a differential subring of an ordinary differential ring R. J. Brzezinski [Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **10** (1962), 325–328; MR **26** #1313] called an element $x \in R$ differentially integral over P if some finitely generated submodule of the P-module R contains all the derivatives of x, and showed that the set $C_R(P)$ of all such elements is a differential subring of R. When P is a differential integral domain and R is its differential field of fractions, he called P differentially integrally closed if $C_R(P) = P$. He showed that if R is a unique factorization domain (i.e., a factorial ring) of which the field of rational numbers is a subring, and if each irreducible element of R fails to divide its derivative, then R is differentially integrally closed. The present author extends these definitions and results to differential rings, ordinary or partial, with "higher differentiations", as recently used by K. Okugawa [J. Math. Kyoto Univ. **2** (1962/63), 295–322; MR **27** #5754]. In this more general setting he shows that if R is a factorial ring of which some field is a subring, then R is differentially integrally closed if and only if every irreducible element of R fails to divide at least one of its higher derivatives.

E. R. Kolchin (New York)

Sopov, Petr 2254

On the solution of the binomial equation in a differential field. (Bulgarian. French summary)

Annuaire Univ. Sofia Fac. Sci. Phys. Math. Livre 1 *Math.* **56** (1961/62), 1–6 (1963).

Let K be a field of characteristic 0 and with an operator d such that $d(u+v)=du+dv$ and $d(uv)=vdu+udv$. Further, there is to be at least one $x \in K$ such that $dx \neq 0$ but $d^2x=0$. Let a be a non-zero constant ($da=0$), and let n be an integer $\geqq 2$. The author shows that there is at least one constant y such that $y^n=a$. Since K includes the rational field, the equation $y^2=-1$ has a solution in K. For definitions and methods, see the author's earlier paper [same Annuaire **55** (1960/61), 175–197; MR **26** #6163].

F. Haimo (Belmont, Mass.)

Kazimirskiĭ, P. S. 2255

A theorem on elementary divisors for a ring of differential operators. (Russian. English summary)

Ukrain. Mat. Ž. **16** (1964), 309–318.

The author considers a differential field τ, ordinary or partial, the ring $\Delta(\tau)$ of linear differential operators over τ, and matrices over $\Delta(\tau)$. Given a matrix A, he seeks invertible square matrices P, Q such that PAQ has a special form. His Theorem 2 asserts that if A is $n\times n$ and of rank n, then there exist P, Q such that PAQ is diagonal with $n-1$ of the coordinates on the diagonal being unity, and his Theorem 3 concerns an $m\times n$ matrix of rank $\geqq 2$. However, there seems to be an error in the proof of his vital Lemma 2. Namely, in line 6 of page 314 he asserts that if $\varphi \in \tau$, $a\in\Delta(\tau)$, then $a\cdot\varphi=\varphi a+(a\varphi)$, where $(a\varphi)\in\tau$; as there are no grounds for supposing that the operator a has order 1 or that φ is a constant, this assertion appears to be unjustified.

E. R. Kolchin (New York)

ABSTRACT ALGEBRAIC GEOMETRY

See also 2220, 2237, 2248, 2307, 2697, 2822.

Demaria, Davide Carlo 2256

Sulla dimensione virtuale di un divisore sopra una varietà algebrica normale.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **98** (1963/64), 761–772.

Let V be a normal projective variety, E a hyperplane section, and D a divisor on V. It is known that dim $|D+nE|$ is a polynomial $P(n)$ for n large. One sets $\delta(D)=P(0)$. If V is a curve of genus g, we have $\delta(D)=d(D)-g$ (d denoting the degree).

For h large, we have dim $|D+hE|=\delta(D+hE)$. The virtual arithmetic genus $p_a(D)$ is related to $\delta(D)$ by the formula

$$p_a(D) = (-1)^d(1+\delta(-D))+p_a(V)$$

(where $d=\dim(V)$). The use of $\delta(D)$ gives a short and smooth treatment of the theory of the virtual arithmetic genus.

P. Samuel (Waltham, Mass.)

Nishimura, Toshio 2257

On the arithmetic normality of the extended Grassmann variety. III.

Bull. Kyoto Gakugei Univ. Ser. B No. 22 (1963), 1–4.

Part II appeared in same Bull. No. 17 (1961), 51–52. By an extended Grassmann variety the author denotes the totality of pairs L^r, L^s in L^n with $L^r\subset L^s$, where L^i is the i-dimensional projective space over the universal domain K. A point-model V of this totality is easily obtained as a subvariety of the product $G_{n,r}\times G_{n,s}$ of the Grassmannians $G_{n,r}$ and $G_{n,s}$ [see the reviewer's paper, Rend. Circ. Mat. Palermo (2) **3** (1954), 244–269; MR **16**, 852]. The author now proves the following theorem on the arithmetic normality of V: The ring of polynomials $K[Q]$, where Q is the set of products of all Grassmann coordinates of L^r and L^s, is integrally closed. The proof of this theorem is similar to that of the arithmetic normality of $G_{n,s}$ by Igusa [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 309–313; MR **15**, 824]. Denoting by $S^{n-r-1,r}$ in $G_{n,r}$ and $S^{n-s-1,s}$ in $G_{n,s}$ the Schubert varieties of all L^r intersecting a fixed L^{n-r-1} (or L^{n-s-1}), the author proves that all primal intersections of V are arithmetically equivalent to an integer multiple of the sum

$$S^{n-r-1,r}\times G^{n,s}+S^{n-s-1,s}\times G^{n,r}$$

lying in V.

W. Burau (Hamburg)

Igusa, Jun-ichi 2258

On the graded ring of theta-constants.

Amer. J. Math. **86** (1964), 219–246.

Es sei g eine natürliche Zahl. Die Menge $\mathfrak{S}_g$ der komplexen Matrizen g-ten Grades mit positiv-definitem Imaginärteil bildet eine konvexe offene Teilmenge des $\frac{1}{2}g(g+1)$-dimensionalen komplexen Vektorraumes. $\mathfrak{S}_g$ heisst die "Siegelsche obere Halbebene vom Grade g". Die reelle symplektische Gruppe Sp$(g,\mathbf{R})$ operiert folgendermassen auf $\mathfrak{S}_g$: Man schreibe $M\in\mathrm{Sp}(g,R)$ in der Form $M=\begin{pmatrix}a&b\\c&d\end{pmatrix}$, wobei a,b,c,d Matrizen g-ten Grades sind; dann ist $M\cdot\tau=(a\tau+b)(c\tau+d)^{-1}$ für $\tau\in\mathfrak{S}_g$. Der Verfasser betrachtet diskrete Untergruppen Γ von Sp$(g,\mathbf{R})$, die mit der "Siegelschen Modulgruppe" Sp$(g,\mathbf{Z})$ kommensurabel sind in dem Sinne, dass $\Gamma\cap\mathrm{Sp}(g,\mathbf{Z})$ endliche Indizes in Γ und in Sp$(g,\mathbf{Z})$ besitzt. Eine analytische Funktion ψ auf $\mathfrak{S}_g$ heisst "Modulform vom Gewicht k bezüglich Γ", wenn $\psi(M\cdot\tau)=\det(c\tau+d)^k\psi(\tau)$ für $M\in\Gamma$, $\tau\in\mathfrak{S}_g$; im Falle $g=1$ kommt hierzu noch in geläufiger Weise eine Analytizitätsbedingung im Unendlichen. Es bedeute $A(\Gamma)_k$ den komplexen Vektorraum der Modulformen vom Gewicht k. Durchläuft k die positiven ganzrationalen Zahlen, so ist die direkte Summe $A(\Gamma)=\sum_k A(\Gamma)_k$ ein graduierter ganzabgeschlossener Integritätsbereich. Er ist endlich erzeugt über dem Körper $\mathbf{C}=A(\Gamma)_0$ der komplexen Zahlen und bestimmt daher eine projektive Mannigfaltigkeit proj $A(\Gamma)$ über $\mathbf{C}$. Man nennt proj $A(\Gamma)$ die "Modulmannigfaltigkeit" bezüglich Γ; sie enthält die Quotientenmannigfaltigkeit $\Gamma\backslash\mathfrak{S}_g$ als offene Teilmenge im Sinne der Zariskischen Topologie. Der Verfasser stellt sich die Aufgabe, die Struktur der Modulmannigfaltigkeit proj $A(\Gamma)$ zu untersuchen.

Das Hauptergebnis der vorliegenden Arbeit bezieht sich auf die Kongruenzuntergruppe $\Gamma_g(4,8)$ von Sp$(g,\mathbf{Z})$; sie besteht aus denjenigen Matrizen $M=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{Sp}(g,\mathbf{Z})$, für welche $M\equiv 1 \bmod 4$, $(a^tb)_0\equiv(c^td)_0\equiv 0 \bmod 8$ (hier bedeutet s_0 den Vektor der Diagonalkoeffizienten der quadratischen Matrix s). Der Verfasser zeigt: Der graduierte Ring

$A(\Gamma_g(4, 8))$ ist die ganzabgeschlossene Hülle des graduierten Ringes $\mathbf{C}[\theta_m\theta_n]$, der über $\mathbf{C}$ durch die Produkte von "Thetakonstanten" θ_m erzeugt wird. Letztere werden wie folgt erklärt: m sei ein Vektor in $\mathbf{Z}^{2g}$; m', m'' $\in \mathbf{Z}^g$ sind die Vektoren der ersten bzw. letzten g Komponenten von m. Durch

$$\theta_m(\tau, z) = \sum_{p\in Z^g} \exp 2\pi i[\tfrac{1}{2}\cdot{}^t(p+\tfrac{1}{2}m')\tau(p+\tfrac{1}{2}m') + {}^t(p+\tfrac{1}{2}m')(z+\tfrac{1}{2}m'')]$$

wird eine analytische Funktion der beiden Variablen $\tau \in \mathfrak{S}_g$, $z \in \mathbf{C}^g$ definiert: die Thetafunktion der Charakteristik m. Durch $\theta_m(\tau) = \theta_m(\tau, 0)$ wird eine analytische Funktion θ_m der Variablen $\tau \in \mathfrak{S}_g$ definiert; man nennt θ_m die Funktion der Thetakonstanten der Charakteristik m. Zum Beweis des Hauptergebnisses wird erstens gezeigt, dass $A(\Gamma_g(4, 8))$ ganzalgebraisch ist über $\mathbf{C}[\theta_m\theta_n]$; hierzu wird das folgende Lemma verwendet: Seien $\psi_1, \cdots, \psi_n$ Modulformen bezüglich Γ ohne gemeinsame Nullstelle in der Quotientenmannigfaltigkeit $\Gamma\backslash\mathfrak{S}_g$ und ihrem Rande. Dann ist $A(\Gamma)$ ganzalgebraisch über $\mathbf{C}[\psi_1, \cdots, \psi_n]$. Zweitens wird gezeigt, dass der (homogene) Quotientenkörper $F(\Gamma_g(4, 8))$ übereinstimmt mit dem durch die Thetaquotienten erzeugten Körper $\mathbf{C}(\theta_m/\theta_n)$. Vermöge der Galoisschen Theorie von $F(\Gamma_g(4, 8))$ über $F(\mathrm{Sp}(g, \mathbf{Z}))$ wird diese Behauptung zurückgeführt auf den Satz, dass $F(\mathrm{Sp}(g, \mathbf{Z}))$ in $\mathbf{C}(\theta_m/\theta_n)$ enthalten ist. Der Beweis dieses Satzes bildet den Hauptteil dieser Arbeit; die Beweisschritte beruhen auf Ideen von Wirtinger [*Untersuchungen über Thetafunctionen*, Teubner, Leipzig, 1895]. Für vorgegebenes g kann die ganzabgeschlossene Hülle von $\mathbf{C}[\theta_m\theta_n]$ und damit auch $A(\Gamma_g(4, 8))$ explizit bestimmt werden. Hieraus kann dann auch $A(\mathrm{Sp}(g, \mathbf{Z}))$ als Teilring davon bestimmt werden. In den Fällen $g = 1$, 2 ist $\mathbf{C}[\theta_m\theta_n]$ selbst ganzabgeschlossen: die hieraus folgenden Konsequenzen sollen für $g = 2$ in einer weiteren Arbeit des Verfassers diskutiert werden. Die Fälle $g = 1, 2, 3$ wurden mit einer anderen Methode vom Verfasser bereits früher behandelt [der Verfasser, Amer. J. Math. **84** (1962), 175–200; MR **25** #5040; Proc. Internat. Congr. Mathematicians (Stockholm), 1962, pp. 522–525, Inst. Mittag-Leffler, Djursholm, 1963]. *P. Roquette* (Tübingen)

Tits, J. 2259

Algebraic and abstract simple groups.

Ann. of Math. (2) **80** (1964), 313–329.

This paper goes a long way towards the realization of the hope expressed by the reviewer in 1951 that some general method be found which would give the structure of all "isotropic" classical groups without having to examine separately each type of group. It is well known that the first breakthrough in that direction was made in the famous paper of Chevalley in 1955 [Tôhoku Math. J. (2) **7** (1955), 14–66; MR **17**, 457], which bridged in a spectacular way the gap between Lie algebras and finite groups. The originality of the author has been to realize that the gist of Chevalley's arguments could be expressed in a purely group-theoretical way, namely, the existence in a group G of two subgroups B, N generating G, such that $H = B \cap N$ is normal in H, and that $W = N/H$ (the "Weyl group") is generated by a set R of involutory elements satisfying two simple conditions (and corresponding to the "roots" in Chevalley's case). This he calls a (BN)-pair, and in the first part of the paper he shows how, under some additional conditions, one can then exhibit in G a normal subgroup G_1 such that, if B_0 is the intersection of all the conjugates of B, every normal subgroup of G either contains G_1 or is contained in B_0, and $G_1/(G_1 \cap B_0)$ is simple. In the second part, the author applies the results of the first part to the case where the group under consideration is the group G_k of rational points of an algebraic group G defined over k; k is a field having at least 4 elements, and G has no connected proper normal algebraic subgroup defined over k; in order to show that the axioms of the first part are satisfied in this set-up, the author has to rely heavily on a forthcoming paper by A. Borel and himself. He thus is able to cover all the groups considered in the reviewer's monograph [*La géométrie des groupes classiques*, seconde édition, revue et corrigée, Springer, Berlin, 1963; MR **28** #1239] which are defined over commutative fields, or sfields having finite rank over their center; his main theorem also applies to the groups previously considered by Chevalley, Steinberg, and himself (i.e., algebraic groups having a Borel subgroup defined over k). Finally, the author stresses the fact that his abstract axioms also cover groups which cannot be obtained from algebraic groups, such as the Suzuki and Ree groups, and he uses them to settle one of the remaining open questions about those groups, namely, the structure of the Ree group of type F_4 over the field with two elements. *J. Dieudonné* (Paris)

Mayer, A. L. 2260

Special divisors and the Jacobian variety.

Math. Ann. **153** (1964), 163–167.

Let C be a complete non-singular curve of genus $g \geqq 2$, J the Jacobian variety of C, P a fixed point of C, and φ the canonical map of C into J such that $\varphi(P)$ is the neutral element of J. Denote by Θ the corresponding Θ-divisor on J and Θ^m the set of points $\varphi(\mathfrak{A})$, $\mathfrak{A} \succ 0$, $\deg(\mathfrak{A}) = g-1$, such that $l(\mathfrak{A}) \geqq m$. Assuming that the universal domain is of characteristic zero, it is shown that Θ^m coincides with the set of points of order m on the variety Θ. *T. Matsusaka* (Waltham, Mass.)

Morikawa, Hisasi 2261

On a definition of Abelian variety.

Ann. Scuola Norm. Sup. Pisa (3) **18** (1964), 161–163.

Suppose a projective variety possesses an everywhere defined binary composition law, together with a neutral element for this law and an automorphism serving as a right inverse, and suppose that each right multiplication is an automorphism. Then the variety is an abelian variety. *M. Rosenlicht* (Berkeley, Calif.)

Eichler, M. 2262

Eine Vorbereitung auf den Riemann-Rochschen Satz für algebraische Funktionenkörper.

J. Reine Angew. Math. **214/215** (1964), 268–275.

This paper gives a sketch of some definitions, arguments, and results whose detailed statements and proofs will be given later. The Riemann-Roch theorem discussed here is a postulation formula for vector bundles over projective n-space, and there are commentaries on the coefficients. Everything is purely algebraic. *M. Rosenlicht* (Berkeley, Calif.)

Seshadri, C. S. 2263

Algebraic vector bundles over the product of an affine curve and the affine line.

Proc. Amer. Math. Soc. **10** (1959), 670–673.

Let k be an algebraically closed field, k^1 the affine line, and C a nonsingular irreducible affine curve defined over k. The author uses results of Serre on projective modules to prove that every algebraic vector bundle over $C \times k^1$ can be obtained as the inverse image of an algebraic vector bundle over C by the projection map $C \times k^1 \to C$.

This generalises the well-known result of the author [Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 456–458; MR **21** #1318] concerning the case $C = k^1$. It has itself been further generalised by Bass [Trans. Amer. Math. Soc. **102** (1962), 319–327; MR **25** #3960]. Other subsequent and related results are due to Horrocks [Proc. London Math. Soc. (3) **14** (1964), 714–718] and to Newstead and the reviewer [the reviewer, Proc. Cambridge Philos. Soc. **58** (1962), 209–216; MR **25** #2611; corrected in Newstead and the reviewer, ibid. **60** (1964), 421–424; MR **29** #4066].

R. L. E. Schwarzenberger (Liverpool)

Seshadri, C. S. 2264

Quotient space by an abelian variety.

Math. Ann. **152** (1963), 185–194.

Let X be an abstract variety and A an Abelian variety. Assume that X is normal and is a transformation space with respect to the group variety A, such that for $(x, y) \in X \times X$, $y = g(x)$, $g \in A$, the correspondence $(x, y) \to g$ is a morphism. Then it is shown that $X/G = Y$ is an algebraic variety and $p: X \to Y$ is a locally isotrivial fibre space, where p is the canonical morphism of X onto Y.

T. Matsusaka (Waltham, Mass.)

LINEAR ALGEBRA

See also 2190, 2196, 2199, 2255, 2281, 2317, 2631, 2737, 2824, 2950, 2951, 2955, 2957, 2962, 3018.

Schneider, Hans (Editor) 2265

★Recent advances in matrix theory.

Proceedings of an Advanced Seminar Conducted by the Mathematics Research Center, United States Army, at the University of Wisconsin, Madison, October 14–16, 1963.

The University of Wisconsin Press, Madison, Wis., 1964. xi + 142 *pp.* $4.00.

This volume contains the text of the proceedings mentioned in the title and consists of six papers, each of which will be reviewed individually.

López de Cicileo, Perla; Iséki, Kiyoshi 2266

On the logarithmic property of the indices of endomorphism on a linear space.

Proc. Japan Acad. **40** (1964), 29–30.

Let E be a linear space, and let u be an endomorphism on E. Denote the dimension of $u^{-1}(0)$ by $\dim(\mathrm{Ker}(u))$, and the codimension of $u(E)$ by $\mathrm{cod}(\mathrm{Im}(u))$. If one of these values is finite, $d(u) = \dim(\mathrm{Ker}(u)) - \mathrm{cod}(\mathrm{Im}(u))$ is called the index of u. Using a lemma of A. Deprit [Acad. Roy. Belg. Cl. Sci. Mém. Coll. in 8° **31** (1959), no. 2; MR **21** #2910], the following theorem is proved. Let u, v be endomorphisms on E. If one of the indices of u, v is finite, then $d(v \circ u) = d(u) + d(v)$. This theorem was previously proved, in a different way, by M. Audin [Publ. Sci. Univ. Alger. Sér. A **4** (1957), 5–76; MR **20** #4758]. *S. Marcus* (Bucharest)

López de Cicileo, Perla; Iséki, Kiyoshi 2267

On adjoint maps between dual systems.

Proc. Japan Acad. **40** (1964), 72–73.

Let E, F be a dual system on α, and let G, H be a dual system on β. In this note the authors study the algebraic adjoint map u^* of a linear map u from E to G, defined by $\alpha(x, u^*(y')) = \beta(u(x), y')$ for all $x \in E$ and $y' \in H$.

R. Cristescu (Bucharest)

Marcus, Marvin 2268

On two classical results of I. Schur.

Bull. Amer. Math. Soc. **70** (1964), 685–688.

Es sei G eine Permutationsgruppe des Grades n und M eine Darstellung von G durch unitäre lineare Operatoren auf dem unitären Vektorraum U der endlichen Dimension m. Für eine beliebige (n, n)-Matrix $A = (a_{ik})$ werde gesetzt

$$M_A = \sum_{\sigma \in G} M(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

Es wird folgender Satz bewiesen: Für zwei beliebige Vektoren x, y aus U und zwei beliebige (n, n)-Matrizen A, B gilt

$$|(M_{AB}x, y)|^2 \leqq (M_{AA^*}x, x)(M_{B^*B}y, y).$$

Der Beweis erfolgt durch übersichtliche Schlüsse unter Heranziehung entsprechender Tensorprodukte. Aus diesem Satz erhält man leicht die im Titel genannten Ergebnisse von I. Schur [Math. Z. **1** (1918), 184–207], nämlich: Ist H eine positiv definite hermitesche Matrix, so ist auch M_H positiv definit. Es bezeichne χ den Charakter von M und $d_\chi(H)$ die Spur von M_H; dann gilt $d_\chi(H) \geqq m \cdot \det H$. Aus der vom Verfasser bewiesenen Ungleichung ergibt sich für beliebige (n, n)-Matrizen A, B insbesondere $|d_\chi(AB)|^2 \leqq d_\chi(AA^*) \cdot d_\chi(B^*B)$.

R. Kochendörffer (Rostock)

Taussky, Olga 2269

On the variation of the characteristic roots of a finite matrix under various changes of its elements.

Recent Advances in Matrix Theory (Proc. Advanced Seminar, Math. Res. Center, U.S. Army, Univ. Wisconsin, Madison, Wis., 1963), *pp.* 125–138. *Univ. Wisconsin Press, Madison, Wis.,* 1964.

In this summary, the following headings appear: Perturbations; Majorization; Symmetrization; AB, PAQ; $p(A, B)$, A & B, Monotonicity, Convexity; Other Transformations; Replacing of Some Elements, Bordering, Subdeterminants; Results Concerning the Elementary Symmetric Functions of the Characteristic Roots; Rearrangements of Elements. Many results are stated and sources named. Unfortunately, there is no bibliography.

A. S. Householder (Oak Ridge, Tenn.)

Poljak, S. S. 2270

Triangular similarity of triangular matrices. (Russian)

Ukrain. Mat. Ž. **16** (1964), 539–544.

The problem considered is the following. If A and B are unit upper triangular matrices, for what conditions does there exist a unit upper triangular matrix X such that $XA = BX$. An obvious necessary condition is that $a_{i,i+1} = b_{i,i+1}$. Beyond that they become complicated.
A. S. Householder (Oak Ridge, Tenn.)

Naĭman, P. B. 2271
On the spectral analysis of non-Hermitian periodic Jacobi matrices. (Ukrainian. Russian and English summaries)
Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1307–1311.
Author's summary: "The spectrum on the n-periodic Jacobi matrix T with complex elements is a complete λ-prototype of some μ-ellipse σ with a polynomial representation $P(\lambda) = \mu$ degenerating in special cases to a segment. If the equation of the nth degree $P(\lambda) = \mu$ has no multiple roots with any $\mu \in \sigma$, the matrix T is reduced to diagonal form by means of the linear transformation V, limited together with V^{-1} and, in the general case, to a triangular form by means of the unitary transformation U."

Chipman, John S.; Rao, M. M. 2272
Projections, generalized inverses, and quadratic forms.
J. Math. Anal. Appl. **9** (1964), 1–11.
The authors derive a number of results on linear combinations of idempotent matrices (for projections in Hilbert space, see also an earlier paper by Mizel and Rao [Pacific J. Math. **12** (1962), 343–357; MR **25** #4365]). The results are used to give, in a unified treatment, new proofs and generalizations for the fundamental theorems concerning the generalized inverse of a matrix [see Penrose, Proc. Cambridge Philos. Soc. **51** (1955), 406–413; MR **16**, 1082]. Finally, applications to the distribution theory of quadratic forms in Gaussian random variables are given. (Proposition 1.5 could also be seen via the Jordan normal form; an alternative proof for Proposition 1.7 follows by use of compound matrices.) {The authors asked to add to the end of Proposition 2.3: $U_jU_j^\dagger$ is the (symmetric) idempotent of AA' corresponding to α_j and $U_j^\dagger = U_j'$.}
O. Taussky-Todd (Pasadena, Calif.)

Vivier, Marcel 2273
Sur l'emploi de certains opérateurs linéaires.
C. R. Acad. Sci. Paris **259** (1964), 280–283.
The author denotes by $\mathscr{M}_a$ the set of matrices over a ring a. For $\mu \in \mathscr{M}_a$, $\mathscr{F}^n(\mu)$ denotes the direct sum of n matrices μ, and $\mathscr{T}^n(\mu)$ is the matrix obtained by replacing each element u_{ij} of μ by $\mathscr{F}^n(u_{ij})$. If M_i are the columns of a $p \times q$ matrix in $\mathscr{M}_a$, $\mathscr{S}M$ denotes the $pq \times 1$ matrix composed of the M_i in order. It is very easy to show:

$$\mathscr{S}(AB) = \mathscr{F}^r(A)\mathscr{S}B = \mathscr{T}^p(B^*)\mathscr{S}A,$$

where the asterisk denotes the transpose. A is $p \times q$ and B is $q \times r$.

These ideas can be used to represent matrix equations of the form $ax - xb = y$. In particular, the Cayley-Hamilton theorem follows immediately from the identity $x - x = 0$.
Burton W. Jones (San Jose)

Wan, Zhe-xian [Wan, Cheh-hsian]; Li, Gen-dao [Li, Ken-tao] 2274
The two theorems of Schur on commutative matrices.
Acta Math. Sinica **14** (1964), 143–150 (*Chinese*); *translated as Chinese Math.* **5** (1964), 156–164.
Let A be the algebra of $n \times n$ matrices over a field F. In 1905 I. Schur investigated the maximal commutative subalgebras of A, that is, commutative subalgebras of maximal dimension $N(n)$ as a vector space over F, assuming that F was the field of complex numbers. His results were as follows: When $n = 2\nu \geqq 4$, every maximal commutative algebra is similar to the algebra consisting of all matrices of the form $\alpha I + \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$, where α is a scalar, I is the $n \times n$ unit matrix and A runs through all $\nu \times \nu$ matrices. When $n = 2\nu - 1 \geqq 5$, there are two possible canonical forms: either A runs through all $\nu \times (\nu - 1)$ matrices or else through all $(\nu - 1) \times \nu$ matrices. In each case $N(n) = [n^2/4]$. Schur's classical results were generalised by N. Jacobson for arbitrary fields except non-complete fields of characteristic 2 [Bull. Amer. Math. Soc. **50** (1944), 431–436; MR **6**, 33]. The present authors have given an elementary proof which is valid for any ground field. *W. Ledermann* (Brighton)

de Albuquerque, Luís Mendonça [de Albuquerque, Luís G. Mendonça] 2275
Matrices with non-negative elements. Stochastic matrices. (Portuguese)
Rev. Fac. Ci. Univ. Coimbra **31** (1962), 5–160.
This rather comprehensive and self-contained expository treatise on the theory of non-negative matrices as of 1961 begins with a historical sketch of the development of the subject beginning with Perron's initial paper [Math. Ann. **64** (1907), 248–263]. Chapter and section headings of the main body of the paper are: Chapter I, Fundamental Concepts, § 1. Introduction, § 2. Characteristic polynomial proper values, § 3. Other spectral properties; Chapter II. Positive and Non-negative Matrices, § 1. Introduction § 2. Definitions. Primary properties, § 3. Perron's theorem § 4. Frobenius's theorem, § 5. Primary consequences of Frobenius's theorem, § 6. Bounds for the dominant root. § 7. Determination of the maximum root of a positive matrix, § 8. Localization of the characteristic roots of non-negative matrices, § 9. Cyclic form of non-negative, irreducible matrices; Chapter III, Primitive and Imprimitive Matrices. Reducible Matrices, § 1. Primitive matrices. § 2. Non-negative reducible matrices, § 3. Observations. Chapter IV, Stochastic Matrices, § 1. Definitions. Stochastic matrices and Markov chains, § 2. Some properties of stochastic matrices, § 3. Normal form of stochastic matrices, § 4. Positive matrices and stochastic matrices. § 5. Sequences of powers of a stochastic matrix, § 6. Considerations in the construction of stochastic matrices.

An extensive bibliography (70 citations) is given, spanning a period from 1902 up to 1961.
R. F. Rinehart (Cleveland, Ohio)

Bhatnagar, R. K. 2276
On the invariants of the powers of a matrix.
Math. Student **31** (1963), 65–72 (1964).
The author gives explicit expressions for the coefficients of the characteristic equation of A^m as functions of the

coefficients of the characteristic equation of A, where A is a matrix of order three or four.
D. E. Rutherford (St. Andrews)

Kremneva, Ju. P. 2277
Determination of the number of solutions of a homogeneous bilinear system (2, m). (Russian)
Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 3 (40), 70–74.
The author discusses the system of bilinear homogeneous equations

$$(1) \qquad \sum_{i=1}^{2} \sum_{j=1}^{m} a_s^{ij} x_i y_j = 0, \qquad s = 1, \cdots, m+1.$$

Let $A=[a_s^{1j}]$, $B=[a_s^{2j}]$, $s=1,\cdots,m+1$, $j=1,\cdots,m$, and $C=[AB]$. Let $M_m = Ax_1 + Bx_2$, and let $\varphi_i(x_1, x_2)$ be the determinant of the matrix obtained from M_m by deleting the ith row.

Non-trivial solutions of (1) are taken to have the form $(c(q, 1); \mathfrak{U})$ (or $(c(1, 0); \mathfrak{U})$: the author does not mention this possibility), where $c(q, 1)$ is a ray of solutions of the system $\varphi_i(x_1, x_2)=0$, $i=1,\cdots,m+1$, and $\mathfrak{U}\neq 0$ is the vector space of solutions $(y_1,\cdots,y_m)$ of the linear system obtained from (1) by setting $x_1=q$, $x_2=1$.

The author assumes that the $\varphi_i(x_1, x_2)$ are not all identically zero; otherwise each $(q, 1)$ would give rise to a solution of (1). He also assumes that C has rank $m+1$ (if C had rank m, he shows that there are (allowing multiplicity) m solutions of (1)).

Define the $m(m+1)$-by-$m(m+1)$ matrix

$$\Delta_m = \begin{bmatrix} A & B & & & \\ & A & B & & \\ & & \ddots & \ddots & \\ & & & A & B \\ & & & & A \quad B \end{bmatrix}$$

The author proves the following theorem: If Δ_m has rank $m(m+1)-k$, then the greatest common divisor of the $\varphi_i(x_1, x_2)$ has degree k, and there are k solutions (allowing multiplicity) of (1).
D. Carlson (Corvallis, Ore.)

Tarasov, I. S. 2278
The inductive theory of non-commutative determinants. (Russian)
Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 4 (41), 152–161.
The author develops a theory of determinants over a sfield F of characteristic zero. He defines four different determinants $D_i(A)\in F$ $(i=1, 2, 3, 4)$ of a square matrix A (over F) and gives many properties. In the case of a commutative field F, the four values $D_i(A)$ coincide and the whole theory becomes identical with the classical theory.
Z. Janko (Canberra)

Bandemer, Hans 2279
Erweiterung des Satzes von Binet-Cauchy.
Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe **11** (1962), 581–584.
This paper extends to a product of three matrices a result of Pascal for a product of two. Let A, B, C be matrices where n is the number of rows of A and the number of columns of C, while the other dimensions may be infinite provided convergence implies the existence of the product matrix ABC. Then the determinant of the product matrix is:

$$\sum\sum \det A \begin{pmatrix} 1, 2, \cdots, n \\ k_1, k_2, \cdots, k_n \end{pmatrix} \times \det B \begin{pmatrix} k_1, k_2, \cdots, k_n \\ r_1, r_2, \cdots, r_n \end{pmatrix} \det C \begin{pmatrix} r_1, r_2, \cdots, r_n \\ 1, 2, \cdots, n \end{pmatrix},$$

where the parenthesis after the B denotes the choice of the rows numbered $k_1, k_2, \cdots, k_n$ and the columns numbered $r_1, r_2, \cdots, r_n$ (similarly for A and C), and the double sum is over all combinations of n rows and/or n columns.
Burton W. Jones (San Jose)

Petersen, G. M.; Baker, Anne C. 2280
Solvable infinite systems of linear equations.
J. London Math. Soc. **39** (1964), 501–510.
In this paper the authors show that a certain set of conditions on a real matrix with elements $a(i,j)$ is sufficient to guarantee that the infinite system $\sum_j a(i,j)u_j = b_i$ $(i=1, 2, \cdots)$, with $\{b_i\}$ real and arbitrary, will have a solution such that $\sum_j a(i,j)u_j$ is absolutely convergent for each i. The reader should consult the references cited in the paper for related results [e.g., G. M. Petersen and A. C. Thompson, same J. **38** (1963), 335–340; MR **27** #3652].
B. E. Rhoades (Berkeley, Calif.)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 2246, 2253, 2315, 2328.

Keller, Roy F. 2281
Motions of matrix rings.
Canad. J. Math. **16** (1964), 159–165.
The ring R of all linear transformations of a vector space over a division ring D has an isomorphic representation by means of row-finite matrices with elements in D. This ring becomes a lattice-valued ring with a valuation in the lattice of all right ideals of R on assigning to every element A of R the principal right ideal $[A]_r$ generated by A. In terms of this valuation a lattice-valued distance function $d(X, Y)$ having the expected properties may be defined as $[X-Y]_r$, making R a lattice metric space.

It is then natural to define a motion of R to be a one-to-one mapping of R onto R which preserves distances. The author's principal theorem states that the most general motion of R is given by the formula: $Y=XA+B$, where A is any non-singular element (unit) of R. The row-finite matrix representation is used in determining the elements A and B.
O. Frink (University Park, Pa.)

Goldie, A. W. 2282
Torsion-free modules and rings.
J. Algebra **1** (1964), 268–287.
Let R be a ring and let M be a right R-module. For each submodule N of M, define the closure of N by $\mathrm{cl}\, N = \{x\in M \mid R$ is an essential extension of the r-ideal $x^{-1}N\}$. It is shown that $\mathrm{cl}\,\mathrm{cl}\,\mathrm{cl}\, N = \mathrm{cl}\,\mathrm{cl}\, N$. Thus, it makes sense to define the torsion submodule of M by cl cl 0. If R is a right Noetherian ring with zero singular ideal and W is the

Wedderburn radical of R, then it is shown that cl W is the tertiary radical of R. The rank of a Noetherian ring is studied in some detail. Much of the paper is concerned with an Artinian ring R with zero singular ideal. The following theorems are proved in case R is also generalized uniserial. (1) R is a direct sum of irreducible generalized uniserial rings. (2) If R is irreducible, then R is isomorphic to a complete blocked triangular matrix ring over a division ring. *R. E. Johnson* (Rochester, N.Y.)

Kupisch, Herbert 2283

Bemerkung über unzerlegbare Darstellungen von Algebren.

Arch. Math. **15** (1964), 189–190.

Let A be an algebra over an algebraically closed field with radical N. The author gives a short simple proof of the following theorem about such algebras: If A has a primitive idempotent e such that at least one of the modules Ae/Ne, $Ne/N^2e, \cdots, N^ne/N^{n+1}e, \cdots$ contains two isomorphic simple modules, then A has an infinite number of non-isomorphic indecomposable modules.

It can be shown that algebras satisfying the hypotheses of the theorem have infinite two-sided ideal lattices. In this case a result of the reviewer [Ann. of Math. (2) **66** (1957), 418–429, Theorem 2.1; MR **19**, 526] could be used to show that algebra also has indecomposable modules of arbitrarily large (but finite) dimension over the field as well.

J. P. Jans (Seattle, Wash.)

Behrens, Ernst-August 2284

Unendliche quasi-einreihige Halbgruppen.

Math. Ann. **155** (1964), 343–368.

A quasi-uniserial semigroup is a semigroup S with zero, 0, such that for all $a, b \in S$, $a \in (a)$, $(ab) = (a)(b)$, $(a) = (b)$ implies $a = b$, $a \neq 0$ implies that $a \in (b_i)$ for only finitely many $b_i \in S$, every nonzero idempotent of S is primitive, and if the nonzero idempotents e_j ($j \in J$) provide a decomposition of S into subsemigroups e_jSe_k ($j, k \in S$) disjoint except for 0 and if $e_je_k \neq 0$ for $j, k \in J$, then there is exactly one $w \in S$ such that $w \in (e_je_k)$, $w \neq e_je_k$ and $w \in (a) \subseteq (e_je_k)$ implies $w = a$ or $a = e_je_k$. Generalizing the structure theorem of D. Rees [Proc. Cambridge Philos. Soc. **36** (1940), 387–400; MR **2**, 127], the author completely determines the structure of quasi-uniserial semigroups. Next, (1) the large semigroup algebra $K[S]$ of S over a field K [N. Bourbaki, *Les structures fondamentales de l'analyse*, Livre II, *Algèbre*, Chapitre II, Actualités Sci. Indust., No 1032, Hermann, Paris, 1947; MR **9**, 406] is shown to be linearly compact in the topology given by ideals of $K[S]$ as neighborhoods of 0. (2) The lattice of closed ideals of $K[S]$ is distributive. (3) The closures of the powers of the Jacobson radical of $K[S]$ converge to 0. Using results of D. Zelinsky on the raising of idempotents [Duke Math. J. **21** (1954), 315–322; MR **15**, 928], it is proved that (4) $K[S]$ has an identity element. In the final section a partial converse is obtained. If an algebra $\mathfrak{o}$ over a field K with radical $\mathfrak{n}$ satisfies (1)–(4), then $\mathfrak{o}/\mathfrak{n}$ is the complete direct sum of simple Artin rings $\mathfrak{o}_i/\mathfrak{n}_i$ for $i \in I$. If also K is the center of $\mathfrak{o}_i/\mathfrak{n}_i$ and $\mathfrak{o}_i/\mathfrak{n}_i$ is finite-dimensional over K for all $i \in I$, then there is a quasi-uniserial semigroup S such that the lattices of closed ideals of $\mathfrak{o}$ and of $K[S]$ are isomorphic.

M. F. Smiley (Riverside, Calif.)

Bovdi, A. A. 2285

Crossed products of a semigroup and a prime ring. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 465–467.

The paper contains proofs that certain theorems in an earlier work of the author [same Ž. **4** (1963), 481–499; MR **27** #2527] for a crossed product of a semigroup and a field remain valid for a crossed product of a semigroup and a prime ring. *J. Wiegold* (Cardiff)

NON-ASSOCIATIVE ALGEBRA

See also 2701.

McCrimmon, K. 2286

Jordan algebras of degree 1.

Bull. Amer. Math. Soc. **70** (1964), 702.

For finite-dimensional special Jordan algebras, A. A. Albert [Trans. Amer. Math. Soc. **69** (1950), 503–527; MR **12**, 475] proved that if A is a special Jordan algebra over a field Φ of characteristic $\neq 2$ with an identity 1 and such that every a in A has the form $a = \lambda 1 + z$ for λ in Φ, z nilpotent, then $A = \Phi 1 + Z$, where Z is a nil subalgebra of A. N. Jacobson [Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 140–147; MR **17**, 822] obtained this result for general finite-dimensional Jordan algebras. In this short note Jacobson's theory of inverses as developed in the above reference and in the same Proc. **48** (1962), 1154–1160 [MR **25** #3971] is used to prove the above result without any dimensionality restriction. *L. A. Kokoris* (Chicago, Ill.)

Ginzburg, A. 2287

A note on Cayley loops.

Canad. J. Math. **16** (1964), 77–81.

Let the Cayley numbers be written in the form $a + bc$, a and b real quaternions, e an indeterminate. Then, as is well known, the 8 units and their opposites form a loop C_1 of order 16 with respect to multiplication. C_1 is diassociative (i.e., every two of its elements generate a subgroup), and has three generators x_1, x_2, x_3 satisfying: (4) $x_\alpha^4 = 1$, $x_\alpha^2 = x_\beta^2 \neq 1$, $x_\alpha x_\beta = x_\beta^3 x_\alpha$, $\alpha \neq \beta$; $x_\alpha x_\beta \cdot x_\gamma = x_\alpha^3 \cdot x_\beta x_\gamma$, α, β, γ all different.

The author defines a C-loop as a loop of order 16 with the above properties and shows (Proposition 1) that there exist exactly two non-isotopic C-loops, C_1 and C_2, C_2 being obtained by the following modification of the multiplication of the Cayley numbers: $(a+be)(c+de) = ac - b\bar{d} + (ad + \bar{c}b)e$ (C_1 is a Moufang loop; C_2 does not satisfy Moufang's identity; therefore they are non-isotopic). Every C-loop is isomorphic either to C_1 or to C_2: the first [second] case occurs if and only if: $(x_\alpha x_\beta)(x_\beta x_\gamma) = x_\alpha x_\gamma$ [respectively $= x_\gamma x_\alpha$; Proposition 2]. In the second section, the author shows (Proposition 3) that C_2 is a quasi-associative loop (QA-loop), but C_1 is not a QA-loop (for the definition of a QA-loop, see D. Tamari [Notices Amer. Math. Soc. **7** (1960), 77]). In Section 3, the author studies the algebra A obtained from the set $\{a + be\}$ of the Cayley numbers preserving addition, and modifying multiplication as shown above. A is simple, flexible (i.e., $(ab)a = a(ba)$), and power-associative, but A is not alternative. A class of algebras with the same properties can be obtained by putting $(a+be)(c+de) = ac + \mu b\bar{d} + (ad + \bar{c}b)e$, μ real.

L. Lombardo-Radice (Rome)

HOMOLOGICAL ALGEBRA

See also 2239, 2626a–d, 2803.

Livšic, A. H. 2288

Category-theoretic foundations of the duality of radicality and semi-simplicity. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 319–336.

The author studies the notions of radical and semisimplicity in the context of category theory. Thus a class R of objects in a category $\mathfrak{K}$ is called radical if the following hold. (1) The image of an R-object by a normal epic is an R-object; (2) every object g has a unique maximal normal sub-object $R(g)$ in R; (3) if $g \to \bar{g}$ is a normal epic with kernel $R(g)$, then $\bar{g}$ contains no non-zero normal R-subobject. An object g is called semisimple if $R(g)=0$. If the category $\mathfrak{K}$ satisfies certain mild assumptions on the factorization of normal monics and epics, then the class S of semisimple objects determined by R satisfies conditions dual to (1)–(3), so that R and S determine each other. Some conditions for a class R to be radical are given, e.g., R is a radical class if and only if any object having an ascending invariant series with factors in R itself belongs to R. This leads to a construction of the minimal radical class containing a given class M of objects, which is closed under quotients by normal epics. Define a degree for certain objects by assigning degree zero to the M-objects, and for any ordinal α, assign degree α to g if g has an ascending invariant series with factors of degree less than α. Now the class $R(M)$ of all objects with some (finite or transfinite) degree is the minimal radical class containing M. Dual considerations apply to semisimple objects.

P. M. Cohn (Chicago, Ill.)

GROUP THEORY AND GENERALIZATIONS

See also 2142, 2259, 2268, 2284, 2287, 2703, 3328.

Turner-Smith, R. F. 2289

Marginal subgroup properties for outer commutator words.

Proc. London Math. Soc. (3) **14** (1964), 321–341.

If φ is a word in n variables and G a group, let $\varphi(G)$ denote the subgroup of G generated by the values of φ as the variables range over G. P. Hall [a research lecture "Nilpotent groups", Summer Seminar, Proc. Fourth Canadian Math. Congr. (Banff, 1957), Univ. Toronto Press, Toronto, Ont., 1959, p. 3; MR **28** #1074] has defined an "associated marginal subgroup" $\varphi^*(G)$ and has made a number of conjectures generalizing the results of Baer [Math. Ann. **124** (1952), 161–177; MR **13**, 622] on finiteness criteria for commutator subgroups. One of these conjectures states that if G has the maximal condition on subgroups and $\varphi(G)$ is finite, then so is $[G:\varphi^*(G)]$. Among other things, the present paper proves a much stronger version of this conjecture in the case where φ is a so-called "outer commutator word", essentially a repeated commutator of commutators of independent variables.

M. Rosenlicht (Berkeley, Calif.)

Frenkel', V. I. 2290

On the effective partial ordering of finitely defined groups. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 651–670.

The author considers groups $\mathfrak{A}_P$ which are defined as follows: The group is generated by $a_1, a_2, \cdots, a_n$; $p_1, p_2, \cdots, p_t$. The (non-trivial) defining relations are

$$\text{(I)}\qquad p_j = u_j{}^i p_j v_j{}^i, \qquad i = 1, \cdots, \nu_j,\ j = 1, \cdots, t;$$

$$\text{(II)}\qquad A_k = B_k, \qquad k = 1, \cdots, \mu.$$

$u_j{}^i$, $v_j{}^i$, A_k, B_k contain only the generators a_l. Let $\mathfrak{A}_P$ be the subgroup generated by $a_1, \cdots, a_n$. It is required that for each j the mapping

$$L_{p_j}: u_j{}^i \to (v_j{}^i)^{-1}, \qquad i = 1, \cdots, \nu_j,$$

define an isomorphism in $\mathfrak{A}_P$. The system $P=(p_1, \cdots, p_t)$ is called a system of stable letters. There are certain cancellation rules for stable letters which possess an analogy with cancellation rules in free groups.

Groups with stable letters were introduced and investigated by P. S. Novikov [Izv. Akad. Nauk SSSR Ser. Mat. **18** (1954), 485–524; MR **17**, 706], and independently by W. W. Boone in a series of articles [cf. for example, Nederl. Akad. Wetensch. Proc. Ser. A **57** (1954), 231–237; ibid. **57** (1954), 492–497; ibid. **58** (1955), 252–256; MR **16**, 564]. Groups $\mathfrak{A}_P$ as defined above were also introduced and studied by J. L. Britton [Proc. London Math. Soc. (3) **8** (1958), 493–506; MR **23** #A2326; Ann. of Math. (2) **77** (1963), 16–32]. Indeed, the existence of the above isomorphic mapping L_{p_j} is the isomorphism condition in the sense of Britton.

Using results of Novikov, the author develops a theory of cancellation of stable letters. Here there is a considerable overlapping with Britton's work. The pivotal Lemma 4 of the present paper is to be found as Lemma 4 in Britton's last-mentioned paper.

In a previous paper [Dokl. Akad. Nauk SSSR **152** (1963), 67–70; MR **27** #3715] the author introduced the notion of an effective partial ordering of a group with respect to a regular subsemigroup. Let $\tilde{\mathfrak{A}}_P{}^+$ be a subsemigroup which is invariant (under conjugacy) and pure (does not contain 1). Then $\mathfrak{A}_P{}^+ = \tilde{\mathfrak{A}}_P{}^+ \cup 1$ is a subsemigroup which is called regular. The partial ordering is defined as follows: let $a, b \in \mathfrak{A}_P$, then $a \geq b$ if and only if $ab^{-1} \in \mathfrak{A}_P{}^+$ or $b^{-1}a \in \mathfrak{A}_P{}^+$. The ordering is called effective if there is an algorithm for deciding whether, for arbitrary words $a, b \in \mathfrak{A}_P$, $a \geq b$ does or does not hold.

The author's main concern in the present paper is the connection between the notion of effective partial ordering and other algorithmic notions. Consider the equation

$$(*)\qquad Z_0 Y_1 Z_1 Y_2 Z_2 \cdots Z_{\lambda-1} Y_\lambda Z_\lambda = 1,$$

where the Z_i are arbitrary elements of $\mathfrak{A}_P$ and the Y_i are unknowns. Only such words Y_i are admitted as a solution which do not contain any stable letters.

The main results of the paper are the following theorems. (1) If in a group $\mathfrak{A}_P$ the solvability of all equations (*) is (algorithmically) decidable, then there is a certain invariant pure subsemigroup $\tilde{\mathfrak{A}}_P{}^+$ which is explicitly constructed and which is such that it is decidable whether a given word α is contained in $\tilde{\mathfrak{A}}_P{}^+$ or not; (2) If in a group $\mathfrak{A}_P$ (a) the solvability of all equations (*) is decidable, and (b) the word problem is decidable, then $\mathfrak{A}_P$ may be effectively partially ordered with respect to the regular subsemigroup $\mathfrak{A}_P{}^+$, which stems from $\tilde{\mathfrak{A}}_P{}^+$ in Theorem (1).

The pure subsemigroup $\tilde{\mathfrak{A}}_P{}^+$ in Theorem (1) consists of

all words of $\mathfrak{A}_P$ for which the exponential sums for all stable letters are not negative, and their sum is strictly positive.

Then the author considers free products $\mathfrak{A}_P' = \mathfrak{A}_P * \{\bar{p}\}$, where $\bar{p}$ generates a free cyclic group. The group $\mathfrak{A}_P'$ is again of type $\mathfrak{A}_P$, a system of stable letters for $\mathfrak{A}_P'$ is obtained by adding $\bar{p}$ to the system P of $\mathfrak{A}_P$. The following theorem is proved: $\mathfrak{A}_P'$ may be effectively partially ordered with respect to a regular subsemigroup which is defined as above if and only if the following holds for the free factor $\mathfrak{A}_P$: (a) the solvability of all equations (*) in $\mathfrak{A}_P$ is decidable; (b) the word problem in $\mathfrak{A}_P$ is decidable.

Using the results and methods sketched so far, the author studies special problems of algorithmic decidability. He constructs a number of counter-examples, all of which are variants of examples given by Novikov.

There is a finitely related group which may be effectively partially ordered with respect to a regular subsemigroup and which is such that the conjugacy problem is not decidable. There is a finitely related group $\mathfrak{A}_0$ for which the word problem is decidable and which contains a regular subsemigroup $\mathfrak{A}_0{}^+$ such that $\mathfrak{A}_0$ cannot be effectively partially ordered with respect to $\mathfrak{A}_0{}^+$.

The exposition is clear and the paper is self-contained as far as possible. *J. Mennicke* (Princeton, N.J.)

Sergienko, V. I. 2291

On a factorization criterion for finite groups. (Russian)

Dokl. Akad. Nauk BSSR 8 (1964), 357–360.

The author gives a criterion for a finite group G to be the direct product of certain Hall subgroups. In order to state his results, we introduce the following definitions. A group G is biprimary if the order of G is of the form p^aq^b, where p, q are distinct primes. Let π be a set of primes. π is invariant in G if every biprimary subgroup H of G of order p^aq^b, where $p \in \pi$, $q \notin \pi$, has a normal Sylow p-subgroup. G satisfies condition $D_\pi{}^d$ if G has a Hall S_π-subgroup G_π such that G_π contains a conjugate in G of every π-subgroup of G, and G_π is dispersed, i.e., G_π has a Sylow tower. Theorem: Suppose G has a normal series whose factors satisfy $D_\pi{}^d$, and suppose the set π' of primes complementary to π is invariant in G. Then G satisfies D_π and $D_{\pi'}$, and G has a normal $S_{\pi'}$-subgroup. Among several consequences is the following generalization of a theorem of Čunihin. If G is a π-separable group, and if $\pi \cup \pi' = \pi_0 \cup \pi_1 \cup \pi_2 \cup \cdots \cup \pi_k$ is a disjoint union of invariant sets of primes with $\pi_0 \supseteq \pi'$, then G is a direct product of its S_{π_i}-subgroups. *P. Fong* (Berkeley, Calif.)

Di Fiore, Lora 2292

Una osservazione sui t-sottogruppi normali di un gruppo finito.

Matematiche (Catania) 18 (1963), 12–14.

A t-group is one for which all chains of normal subgroups are principal. The author proposes the problem of studying the t-groups which are normal subgroups of a given finite group G. To do this she considers the group U generated by all normal t-subgroups of G and proves the following theorem: Under the assumption that the intersection of all maximal subgroups of G is the identity, it follows that G is soluble if and only if the center of U is identical with the centralizer of U.

H. W. Brinkmann (Swarthmore, Pa.)

Watts, Charles E. 2293

A Jordan-Hölder theorem.

Pacific J. Math. 14 (1964), 731–734.

A class of finite groups $\mathfrak{D}$ is a genetic class provided that if it contains a group G it contains every isomorphic group and also every subgroup and quotient group of G. F is the free additive Abelian group generated by the set of all isomorphism classes of all finite groups, $[G]$ denotes the isomorphism class of a group G. $N(\mathfrak{D})$ is the subgroup of F generated by all elements of the form $[G]-[H]-[G/H]$ with H a normal subgroup of G, G/H belonging to $\mathfrak{D}$. $K(\mathfrak{D}) = F/N(\mathfrak{D})$.

A group G is $\mathfrak{D}$-simple if it contains no proper quotient group belonging to $\mathfrak{D}$. The author proves the following theorem. Given any genetic class $\mathfrak{D}$, $K(\mathfrak{D})$ is the free Abelian group freely generated by the elements $k[S]$, where S is $\mathfrak{D}$-simple.

Given a group G there is a sequence $G = G_0, G_1, \cdots, G_n$ for which G_{i+1} is normal in G_i, G_i/G_{i+1} is simple and in $\mathfrak{D}$, and G_n is either 1 or is $\mathfrak{D}$-simple and not in $\mathfrak{D}$.

If $\mathfrak{D} = \mathfrak{G}$, the class of all finite groups, the theorem becomes the Jordan-Hölder theorem. In the general case G_i/G_{i+1} are among the composition factors of G, but the group G_n, if $\neq 1$, is something new. It is a subnormal subgroup of G which depends, up to isomorphism, only on G and on $\mathfrak{D}$. *D. E. Littlewood* (Bangor)

Iséki, Kiyoshi 2294

On existence of linear functionals on Abelian groups.

Proc. Japan Acad. 40 (1964), 70–71.

By reiterating the proof of V. Pták [Studia Math. 15 (1956), 365–366; MR 18, 320] with a theorem of G. Aumann [Arch. Math. 8 (1957), 422–427; MR 20 #6468] in place of the Hahn-Banach theorem, the author proves the theorem of S. Mazur and W. Orlicz [Studia Math. 13 (1953), 137–179; MR 16, 932] for Abelian groups instead of linear spaces. *A. Hulanicki* (Wrocław)

Myškin, V. I. 2295

Homogeneous separable Abelian torsion-free groups. (Russian)

Mat. Sb. (N.S.) 64 (106) (1964), 3–9.

A torsion-free Abelian group G is said to be separable if every finite subset of G can be embedded in a completely decomposable direct summand of G. The author considers the following problem: Is every separable torsion-free group isomorphic to an interdirect sum of groups of rank 1, i.e., to a subgroup of the complete direct sum of groups of rank 1 which contains the direct sum. The paper gives a partial answer to this problem. The author says that a torsion-free group G satisfies the axiom of countability if it contains a countable family of subgroups $\{C_i\}$ such that (1) for every i, $G = F_i + C_i$ and F_i has finite rank; (2) $\bigcap_i C_i = 0$. Theorem 1: A homogeneous torsion-free Abelian group which satisfies the axiom of countability and is separable is isomorphic to an interdirect sum of countably many groups of rank 1. Theorem 2: Suppose G is an interdirect sum of a family $\{A_\mu\}_{\mu \in \mathfrak{M}}$ of groups of rank 1. If G is homogeneous and it is a pure subgroup in the complete direct sum of the groups A_μ, $\mu \in \mathfrak{M}$, then G is separable.

A. Hulanicki (Wrocław)

Šunkov, V. P. 2296
On groups which are decomposable into a uniform product of their p-subgroups. (Russian)
Dokl. Akad. Nauk SSSR **154** (1964), 542–544.

The groups of the title are generated by their p-subgroups, and cyclic subgroups of distinct p-subgroups commute. The structure of such groups is determined in the periodic case, and it is shown that the property in question is not inherited by subgroups. The subclass of groups defined by replacing "p-subgroups" by "Sylow p-subgroups" is closed under the taking of subgroups. The structures of the periodic and the finite groups in this subclass are also determined. It is noted that these finite groups are MacLane groups [cf. G. Zappa, Atti Sesto Congr. Un. Mat. Ital. (Napoli, 1959), pp. 262–263; Edizioni Cremonese, Rome, 1960; MR **23** #A753]. A structure theorem for finite MacLane groups is given and is used to construct a counter-example to an assertion of G. Zappa [loc. cit.]. Quasi-uniform products are defined by imposing the commutation requirement only on subgroups generated by a member of a chosen set of generators of a p-subgroup. Theorem: A finite group is supersolvable if and only if it is decomposable into a quasi-uniform product of its Sylow p-subgroups. *M. Greendlinger* (Ivanovo)

Migliorini, Franco 2297
Sistemi di rappresentanti di laterali e complementi normali.
Matematiche (Catania) **18** (1963), 54–58.

Mit den Begriffen eines ausgezeichneten Systems von Elementen einer Gruppe [R. Kochendörffer, Math. Nachr. **17** (1958), 189–194; MR **22** #4768] und der r-Abgeschlossenheit von Baer [Math. Z. **71** (1959), 325–334; MR **21** #7250] formulierte G. Zappa [Matematiche (Catania) **13** (1958), 61–64; MR **23** #A1703] die Vermutung, die in vorliegender Arbeit bewiesen wird: Seien π das System der Primteiler der Ordnung der Gruppe G, π' das komplementäre System. Sei H eine auflösbare Untergruppe von G. Es gibt dann und nur dann ein ausgezeichnetes System von Repräsentanten der Nebengruppen von H in G, wenn G π'-abgeschlossen ist. *J. J. Burckhardt* (Zürich)

Krause, Eugene F. 2298
Groups of exponent 8 satisfy the 14th Engel congruence.
Proc. Amer. Math. Soc. **15** (1964), 491–496.

Let G be a group of exponent 8, n a positive integer, $(y,x)=(y,x;1)=y^{-1}x^{-1}yx$ and $(y,x;n+1)=((y,x;n),x)$. Let G_m be the mth term of the lower central series of G, i.e., $G_1=G$, $G_{m+1}=(G_m,G)$. The author shows that in G, $(y,x;14)\subset G_{16}$. This improves a result of Sanov, namely, $(y,x;23)\in G_{25}$ [Izv. Akad. Nauk SSSR Ser. Mat. **15** (1951), 477–502; MR **14**, 722] and a conjecture of R. H. Bruck, namely, $(y,x;15)\in G_{17}$ [Arch. Math. **13** (1962), 179–186; MR **27** #2552].

The author also proves that if $y_t\in G_t$, $t\geqq 3$, then $(y_t,x;4)^4$, $(y_t,x;8)^2$, $(y_t,x;12)\in G_m$, where

$$m=\min(t+13,\,2t+10,\,3t+7,\,4t+4,\,5t+1).$$

The author's method consists in obtaining relations among basic commutators by using the Hall collection formula:

$$(1)\qquad (xy)^8=x^8y^8(y,x)^{28}\cdots.$$

Further relations are obtained by replacing x by x^2 and x^4 in (1). Results from the following paper [#2299] are used. *R. R. Struik* (Pueblo, Colo.)

Krause, Eugene F. 2299
On the collection process.
Proc. Amer. Math. Soc. **15** (1964), 497–504.

P. Hall showed [Proc. London Math. Soc. (2) **36** (1933), 29–95] that for any a, b in a nilpotent group G,

$$(ab)^n=\prod u^{e_u(n)},$$

where n is a positive integer, the u are basic commutators in a and b, and $e_u(n)$ is a polynomial of the form $a_1n+a_2\binom{n}{2}+\cdots+a_k\binom{n}{k}$, a_i non-negative integers, where k is less than or equal to the weight of u in a and b. It is well known [see, e.g., M. Hall, Jr., *The theory of groups*, p. 326, Macmillan, New York, 1959; MR **21** #1996] that if $u=(b,a;k)$, then $e_u(n)=\binom{n}{k+1}$. Here $(b,a)=(b,a;1)=b^{-1}a^{-1}ba$ and $(b,a;k+1)=((b,a;k),a)$. This paper describes an algorithm which in principle makes it possible to compute $e_u(n)$ for any u.

Let $c_k=(b,a;k)$, $d_{K,L}=(c_K,b;L)$. The author gives explicit formulas for $e_u(n)$ for u equal to $d_{K,L}$, $(d_{K,L},(b,a))$ and $(((b,a),b),((b,a),a))$. The author's algorithm is based on the observation that

$$(ab)^{n+1}=(ab)(ab)^n=ab\prod u^{e_u(n)}=\prod u^{e_u(n+1)}.$$

Hence, if $d_u(n)=e_u(n+1)-e_u(n)$, then $e_u(n)=\sum_{m=0}^{n-1}d_u(m)$. The problem of computing $e_u(n)$ is then reduced to the problem of computing $d_u(m)$.

Results from this article are used to prove that a group of exponent 8 satisfies the 14th Engel condition [see #2298 above]. *R. R. Struik* (Pueblo, Colo.)

Janko, Zvonimir 2300
Finite simple groups with short chains of subgroups.
Math. Z. **84** (1964), 428–437.

The following theorem is proved. Let G be a finite non-abelian simple group all of whose chains of subgroups have length at most five. Then G is isomorphic to $\mathrm{LF}(2,q)$ for some prime power q. The proof of this theorem is surprisingly complicated and makes use of a great many recently proved results concerning the structure of simple groups whose Sylow 2-groups have various special forms. *W. Feit* (New Haven, Conn.)

Wong, W. J. 2301
On linear p-groups.
J. Austral. Math. Soc. **4** (1964), 174–178.

In a previous paper [same J. **3** (1963), 180–184; MR **27** #2562] the author defined a quasi-permutation group of degree n to be a finite group with a faithful representation of degree n whose characters had only non-negative integral values. If G is such a group, he proved that it also possessed the following simple properties of permutation groups of degree n: (i) the order of G is a divisor of the order of the symmetric group of degree n; (ii) if G is a p-group and $n<p^2$, then G has exponent at most p and derived length at most 1.

In this paper (ii) is generalised so that if $n < p^{a+1}$, then G has exponent $\leqq p^a$ and derived length at most a. As a corollary he proves (i) with the word "order" replaced by the word "exponent".

These results are deduced from four more general theorems on the derived length and exponent of a finite group with a faithful representation of given degree.

D. E. Littlewood (Bangor)

Kemhadze, Š. S. 2302

On the definition of the Baer nil-group. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **33** (1964), 279–284.

A group is said to be strongly locally nilpotent (s.l.n.) if each subgroup with a finite number of generators lies in a subnormal (i.e., accessible) nilpotent subgroup. Each s.l.n. group is a Baer nilgroup [Baer, Math. Z. **62** (1955), 402–437; MR **17**, 124]. Theorems 1 and 2: The subgroup $N(G)$ generated by all the normal s.l.n. subgroups of a group G is likewise normal and s.l.n., is the sole maximal characteristic s.l.n. subgroup of G, and is generated by all the accessible s.l.n. subgroups of G. Theorems 3 and 4: The following are equivalent: (a) G is a Baer nilgroup; (b) each subgroup of G with a finite number of generators is nilpotent and accessible; (c) each subgroup of G with a finite number of generators satisfies an Engel condition of the form $[G, A, A, \cdots, A] = 1$. *F. Haimo* (Belmont, Mass.)

Stonehewer, S. E. 2303

Abnormal subgroups of a class of periodic locally soluble groups.

Proc. London Math. Soc. (3) **14** (1964), 520–536.

Let G be a periodic locally soluble group with radical of finite index. A subgroup L of G is abnormal if, for every g in G, g belongs to the group generated by L and $g^{-1}Lg$. The author proves that G possesses self-normalizing maximal locally nilpotent subgroups, and that all such subgroups are conjugate, abnormal in G, and the only abnormal locally nilpotent subgroups of G. This generalizes R. W. Carter's theorem [Math. Z. **75** (1960/61), 136–139; MR **23** #A928] for finite soluble groups. Examples are given by the author to show that "periodic", "with radical of finite index", and "maximal" are necessary for his results.

The author also establishes "relationships between the self-normalizing maximal locally nilpotent subgroups and the basis normalizers" of G, thereby extending results of P. Hall [Proc. London Math. Soc. (2) **43** (1937), 316–323; ibid. (2) **43** (1937), 507–528] on finite soluble groups. Properties of G, under additional restrictions on its radical, are also investigated. *S. Lipschutz* (Levittown, N.Y.)

Djumin, N. V. 2304

On a class of linear groups. (Russian. English summary)

Ukrain. Mat. Ž. **16** (1964), 3–12.

The main result is as follows. The subgroup A of a full linear group (the vector space is to have dimension at least 3 and the field characteristic not 2) is a partition group if and only if (i) *A is an intersection of complete triangular groups*, and (ii) there exists a diagonal subgroup normalising A. *J. Wiegold* (Cardiff)

Tamaschke, Olaf 230

S-Ringe und verallgemeinerte Charaktere auf endliche Gruppen.

Math. Z. **84** (1964), 101–119.

Let H be a finite group and A the group algebra of H ov the complex number field. Let $H = T_1 \cup \cdots \cup T_t$ be partition of H, and denote the sum of the elements of by τ_i. The subspace T of A generated by $\tau_1, \cdots, \tau_t$ is calle an S-ring if (i) T is a subalgebra of A, and (ii) the mappin $x \to x^{-1}$ ($x \in H$) permutes $\tau_1, \cdots, \tau_t$. This notion originate in the theory of permutation groups. In the paper unde review, the author considers the following dual situatio Let $D_k = (d_{ij}^{(k)})$, $k = 1, \cdots, n$, be a complete set of inequiva lent irreducible matrix representations of H, the degree c D_k being denoted by δ_k. Let the functions $\delta_k d_{ij}^{(k)}$ b divided into disjoint sets $R_1, \cdots, R_r$, and denote the su of the elements of R_i by ρ_i. The vector space generated b $\rho_1, \cdots, \rho_r$ is called a generalized character algebra if (i) is a (commutative) algebra with respect to the usual mult plication of functions, and (ii) the mapping $f \to f^*$, whe $f^*(x) = f(x^{-1})$ for x in H, maps R into itself. If T is an ring as above, which is commutative and has the same un element as A, then the set $T^\#$ of all complex-valued fun tions on H which are constant on each T_i is a generalize character algebra. When T is the centre of A (and the are the conjucacy classes of H), $T^\#$ is the algebra generate by the irreducible characters of H. It is shown, furthe more, that every generalized character algebra R co taining the constant function 1 has the form $T^\#$ for som S-ring T. The paper also includes certain "orthogonalit relations" and some discussion of the applications to th theory of permutation groups. *H. K. Farahat* (Sheffield

Coleman, Donald B. 230

On the modular group ring of a p-group.

Proc. Amer. Math. Soc. **15** (1964), 511–514.

Let $F(G)$ be the group ring of a finite p-group G over field F of characteristic p. Denote by $Z(G)$ the center of G $Z\{F(G)\}$ the center of $F(G)$, G' the commutator subgrou of G, and by U the multiplicative group of units of $F(G)$ Then G is a subgroup of U, and it is shown that (i) th normalizer of G in U is precisely $G \cdot Z(U)$, and (ii) $G' = G \cap U'$.

Next, let $C_1, \cdots, C_t$ denote the non-central conjugat classes of G, and denote by K_i the sum of all elements o C_i, $1 \leqq i \leqq t$. The author proves that the vector spac spanned over F by $\{K_1, \cdots, K_t\}$ is an ideal in $Z\{F(G)\}$ Furthermore, each $K_i{}^p = 0$, provided that G satisfies th condition: for $a, b \in G$,

$$(*) \qquad ab^p = b^p a \quad \text{if and only if} \quad a^p b = ba^p.$$

(This is valid for regular p-groups, and hence certainly for abelian p-groups.)

This result is used in proving the main theorem: If G, H are finite p-groups satisfying (*), and if $Z\{F(G)\} \cong Z\{F(H)\}$, then

$$\{x^p : x \in Z(G)\} \cong \{y^p : y \in Z(H)\}.$$

If $Z(G)$ and $Z(H)$ have the same order, this implies that $Z(G) \cong Z(H)$.

The special case in which G and H are abelian p-groups yields a theorem of Deskins [Duke Math. J. **23** (1956), 35–40; MR **17**, 1052]: If $F(G) \cong F(H)$, then $G \cong H$, where G and H are abelian p-groups.

The author remarks that $F(G) \not\cong F(H)$ when G is the quaternion group of order 8, H the dihedral group of order 8, and F the prime field of characteristic 2.

I. Reiner (Urbana, Ill.)

Iwahori, Nagayoshi 2307

On the structure of a Hecke ring of a Chevalley group over a finite field.

J. Fac. Sci. Univ. Tokyo Sect. I **10**, 215–236 (1964).

If G is a group and B a subgroup of G such that $[B : B \cap x^{-1}Bx] < \infty$ for any $x \in G$, then the free Z-module generated by the double cosets $B \times B$ has an obvious natural multiplication and forms a ring $H(G, B)$ called the Hecke ring of the pair G, B.

In the paper under review, the structure of $H(G, B)$ is determined completely when G is a Chevalley group over a finite field and B a Borel subgroup of G. More explicitly, if l is the rank of G, then $H(G, B)$ is characterized by the generators $S_1, \cdots, S_l$ and the defining relations:

$$S_i^2 = q\cdot 1 + (q-1)S_i \qquad (i = 1, \cdots, l),$$
$$S_iS_j = S_jS_i \quad \text{if} \quad \theta_{ij} = \pi/2,$$
$$S_iS_jS_i = S_jS_iS_j \quad \text{if} \quad \theta_{ij} = 2\pi/3,$$
$$(S_iS_j)^2 = (S_jS_i)^2 \quad \text{if} \quad \theta_{ij} = 3\pi/4,$$
$$(S_iS_j)^3 = (S_jS_i)^3 \quad \text{if} \quad \theta_{ij} = 5\pi/6,$$

where θ_{ij} denotes the angle between the fundamental roots α_i and α_j, and where q denotes the order of the underlying finite field. These relations resemble closely those of the Weyl group W of G, from which they are in fact derived.

Rimhak Ree (Vancouver, B.C.)

Berman, S. D. 2308

On the theory of integral representations of finite groups. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 506–508.

Let G be a cyclic group of order mp^α, where p is a prime not dividing m. The following results are known about the indecomposable representations of G over the ring Z of rational integers or the ring R_p of p-local rational integers. If $\alpha \geq 3$, then G has such representations of arbitrarily high degree. If $\alpha = 1$, then all such representations are already realized in the group ring ZG or R_pG. This article contains statements of results in the remaining case $\alpha = 2$, when the ring of coefficients is R_p. A classification is given which is based on the irreducible constituents of the indecomposable representation. Included at the end of the article is an example, without proof, of a non-abelian p-group, $p > 3$, having indecomposable representations of arbitrarily high degree, whose irreducible rational constituents are all equivalent. The ring of coefficients in this case can be either Z, R_p, or J_p, the ring of p-adic integers.

P. Fong (Berkeley, Calif.)

Matuljauskas, A. 2309

On the number of indecomposable representations of the group Z_8. (Russian. Lithuanian and German summaries)

Litovsk. Mat. Sb. **3** (1963), *no.* 1, 181–188.

Consider representations of a cyclic group G of order 8 by matrices with rational integral entries. The author exhibits a family of indecomposable representations of degrees $8n$, $n = 1, 2, 3, \cdots$. This shows that the number of indecomposable representations of G is infinite.

The author does not seem to be aware that more general results of this type have been published recently. In fact, for any cyclic group of order p^e, $e > 2$, there are infinitely many inequivalent indecomposable representations by matrices with rational integral entries. This theorem has been proved independently by Heller and the reviewer [Ann. of Math. (2) **77** (1963), 318–328; MR **26** #2520], Dade [ibid. (2) **77** (1963), 406–412; MR **26** #2521], and Berman and Gudivok [Dokl. Akad. Nauk SSSR **145** (1962), 1199–1201; MR **25** #3095].

I. Reiner (Urbana, Ill.)

Baer, Reinhold 2310

Irreducible groups of automorphisms of abelian groups.

Pacific J. Math. **14** (1964), 385–406.

Let A be an abelian group and Γ a group of automorphisms of it; then Γ is said to be irreducible if A has no proper non-trivial Γ-invariant subgroups. The main results of the paper are as follows. (1) If Γ is locally finite, then A is an elementary abelian p-group for some prime p, and the centraliser Δ of Γ in the ring of endomorphisms of A is a commutative absolutely algebraic field of characteristic p. (2) If Γ has a periodic abelian subgroup of finite index, then the rank of A (as a vector space over Δ) is finite and Γ is a group of finite rank. (3) If, in addition, Γ is of finite exponent, then A and Γ are both finite.

J. Wiegold (Cardiff)

Šneperman, L. B. 2311

Semigroups of endomorphisms of quasi-ordered sets. (Russian)

Leningrad. Gos. Ped. Inst. Učen. Zap. **238** (1962), 21–37.

A quasi-order is any reflexive and transitive binary relation. It is called non-trivial if it is different from the identical and from the universal quasi-orders. A quasi-ordered semigroup is a semigroup endowed with a quasi-order that is compatible with the semigroup operation.

Let ρ be a quasi-order on a set Ω. The set E of all endomorphisms (i.e., isotonic transformations) of Ω_ρ is a semigroup under the natural multiplication of endomorphisms. Let us define a quasi-order $\bar\rho$ on E: $\bar\rho = \{(s_1, s_2) : (s_1\alpha, s_2\alpha) \in \rho$ for all $\alpha \in \Omega\}$. Then E_ρ is a quasi-ordered semigroup. The main problem solved is the following: What is an abstract characterization of the semigroup E and of the quasi-ordered semigroup $E_{\bar\rho}$?

A quasi-ordered semigroup B_ρ is isomorphic to the quasi-ordered semigroup of all endomorphisms of some non-trivially quasi-ordered set if and only if (1) B contains a minimal two-sided ideal T; (2) T is a left zero semigroup (i.e., each of its elements is a left zero of T); (3) a quasi-order ρ induces on T a non-trivial quasi-order; (4) T is a right reductor of B, i.e., if $a_1, a_2 \in B$, $a_1t = a_2t$ for all $t \in T$, then $a_1 = a_2$. B_ρ is maximal relative to the property: T is a left ideal and right reductor of B.

The author shows that E can be quasi-ordered in an inner way (using the multiplication of E only) and that this quasi-order coincides with $\bar\rho$. He also gives an abstract characterization of E (for non-trivial ρ) using only the operation of E. There are also found abstract characterizations of E when ρ satisfies some special conditions (e.g., is an order, linear order, or lattice order, etc.).

As a corollary, Gluskin's result [L. M. Gluskin, Uspehi Mat. Nauk **16** (1961), no. 5 (101), 157–162; MR **24** #A1336] is obtained: The semigroups of all endomorphisms of two non-trivially quasi-ordered sets are isomorphic if and only if these sets are isomorphic or anti-isomorphic. *B. M. Schein (Šain)* (Saratov)

Lesohin, M. M. 2312

Galois connections in systems with exterior multiplication. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 1 (38), 84–86.

The author continues his study of systems (of semigroups) with exterior multiplication (hereafter called "systems"), using his earlier terminology [same Izv. **1964**, no. 1 (38), 84–86; MR **28** #148]. An ideal I of a semigroup is called isolated if $x^n \in I$, $n>0$, implies $x \in I$. The main theorem states that if C is a group with zero adjoined, and if (A, B, C) is a system, complete with respect to B, then the lattice of isolated ideals of A (with a minimal lattice element, if one is not already present, adjoined) is anti-isomorphic to the lattice of all ideals of B. This generalizes results of Št. Schwarz [Czechoslovak Math. J. **4** (**79**) (1954), 219–247; MR **16**, 1085] concerning complex characters of commutative semigroups. There are several corollaries concerning systems, and also a necessary and sufficient condition for every ideal of a commutative semigroup to be prime. *E. J. Tully, Jr.* (Davis, Calif.)

Seidel, H. 2313

Nilpotenzkriterien für Halbgruppen.

Monatsb. Deutsch. Akad. Wiss. Berlin **5** (1963), 599–603.

Throughout this paper, S is a semigroup with zero. An element a of S is nilpotent if $a^n=0$ for some n; a semigroup S is a nilsemigroup if any element of S is nilpotent; S is nilpotent if $S^n=\{0\}$ for some n. Let a be an element of a semigroup S; $(a)_R$, $(a)_L$ and $(a)_F$ be the principal right, left and two-sided ideals of S generated by a, respectively. The author defines the following conditions. Let A be a subsemigroup of S. (1) The right power chain condition for A holds in S if, for any $a \in A$, the sequence $(a)_R \supseteq (a^2)_R \supseteq \cdots \supseteq (a^n)_R \supseteq \cdots$ ends in a finite number of terms. (2) The right product chain condition for A holds in S if, for some elements, $a_1, \cdots, a_n, \cdots$ of A, the sequence $(a_1)_R \supseteq (a_1a_2)_R \supseteq \cdots \supseteq (a_1\cdots a_n)_R \supseteq \cdots$ ends in a finite number of terms. The left, two-sided power or product chain conditions are defined by using $(\ )_L$ or $(\ )_F$ instead of $(\ )_R$. The main theorems are: (I) a one- or two-sided ideal A of S is a nil-ideal if and only if (1) $a, b \in A$ and $ab=a$ imply $a=0$, and (2) the one-sided power chain condition for A holds in S. (II) A one- or two-sided ideal A of S is nilpotent if and only if (1) $a, b \in A$, $ab=a$ imply $a=0$; (2) the one-sided product chain condition for A holds in S; and (3) A has a finite index (i.e., there is an n such that $A^n=A^{n+1}$). In (II), other necessary and sufficient conditions are given, that is, (1) of (II) is replaced by (1′): $a, b \in A$ and $a \in (ab)_F$ imply $a=0$. (2) of (II) is replaced by the two-sided product chain condition. The remaining part is devoted to application of the above theorems to Jacobson radicals [Hoehnke, Math. Nachr. **26** (1963), 1–13; MR **28** #4051] and nilradicals [Bruck, *A survey of binary systems*, p. 45, Springer, Berlin, 1958; MR **20** #76]. Namely, the author gives necessary and sufficient conditions for subsemigroups of Jacobson radicals or nilradicals to be nil or nilpotent. *T. Tamura* (Davis, Calif.)

Sasaki, Morio; Inoue, Reiko 2314

Semigroups whose any subsemigroup contains a definite element.

Proc. Japan Acad. **40** (1964), 57–58.

The authors show that a semigroup S contains an element e belonging to every subsemigroup if and only if S is an (ideal) extension (in the sense of A. H. Clifford [Trans. Amer. Math. Soc. **68** (1950), 165–173; MR **11**, 499]) of a periodic group (with identity element e) by a nilpotent semigroup. *E. J. Tully, Jr.* (Davis, Calif.)

Šutov, È. G. 2315

Potential divisibility in commutative semigroups and rings. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 4 (41), 162–168.

Let a and b be elements of a commutative semigroup S. Then a is called a potential divisor of b [cf. the author, Leningrad. Gos. Ped. Inst. Učen. Zap. **166** (1958), 75–103; MR **21** #2018] if S can be embedded in a semigroup S' with an element c such that $ac=b$. Potential divisors in rings are defined similarly. The author shows that in a commutative semigroup S, a is a potential divisor of b if and only if (1) $ax=ay$ implies $bx=by$ for all $x, y \in S \cup \{1\}$ (where 1 acts as identity on S). Secondly, if R is a commutative ring, then a is a potential divisor of b if and only if (2) $ax=ay$ implies $bx=by$ for all $x, y \in R \cup Z$ (where Z is the ring of integers, acting on R in the usual way). Examples are given of rings satisfying (1) but not (2) for some pair (a, b) and of rings with two pairs of potential divisors which cannot be realized as actual divisors in any semigroup containing the ring. *P. M. Cohn* (Chicago, Ill.)

Petrich, Mario 2316

Produit cartésien de demi-groupes complètement simples.

C. R. Acad. Sci. Paris **259** (1964), 277–279.

Let S_k, $k \in K$, be semigroups and Z the set of all k such that S_k has a zero 0_k. The Cartesian product of S_k in the ordinary sense is denoted by $\prod S_k$, each element of which is written as (a_k). In the case where Z is not empty, the Rees factor semigroup of $\prod S_k$ modulo the ideal $\{(a_k) \in \prod S_k \mid a_k = 0_k \text{ for some } k \in K\}$ is denoted by $\prod^0 S_k$, which the author calls the Cartesian 0-product of S_k. The main results are: (1) If Z is not empty, $\prod^0 S_k$ is completely 0-simple if and only if S_k $(k \in Z)$ is completely 0-simple and S_k $(k \notin Z)$ is completely simple; if Z is empty, $\prod S_k$ is completely simple if and only if S_k $(k \in K)$ is completely simple. As an immediate consequence: (2) If $\prod^0 S_k$ is completely 0-simple, a non-zero minimal right ideal of $\prod^0 S_k$ is the Cartesian 0-product of non-zero minimal right ideals of S_k, and vice versa. For $\prod S_k$, similar statements are true. Let $S_k = M^0(G_k; I_k, \Lambda_k; P_k)$ for $k \in Z$, $S_k = M(G_k; I_k, \Lambda_k; P_k)$ for $k \notin Z$; $\prod^0 S_k = M^0(G; I, \Lambda; P)$ and $\prod S_k = M(G; I, \Lambda; P)$ [see Clifford and Preston, *The algebraic theory of semigroups*, Vol. I, p. 88, Amer. Math. Soc., Providence, R.I., 1961; MR **24** #A2627]. The author shows that G, I, Λ, and P are expressed in terms of G_k, I_k, Λ_k, and P_k, respectively. *T. Tamura* (Davis, Calif.)

Tully, Edward J., Jr. 2317
Representation of a semigroup by row-monomial matrices over a group.
Proc. Japan Acad. **40** (1964), 157–160.

A new method is given (without proofs) for constructing homomorphisms of a semigroup S into a semigroup of matrices over a group with zero, where each matrix has at most one non-zero element in each row. The construction is more general than that given by Schützenberger [C. R. Acad. Sci. Paris **244** (1957), 1994–1996; MR **19**, 249; ibid. **246** (1958), 865–867; MR **20** #2384] and Preston [Quart. J. Math. Oxford Ser. (2) **9** (1958), 169–176; MR **20** #5814].
P. F. Conrad (Canberra)

Čupona, Ǵorǵi [Čupona, G.] 2318
On associative congruences. (Macedonian. English summary)
Bull. Soc. Math. Phys. Macédoine **13** (1962), 5–12.

Theorem 1: The associative congruences on a groupoid G form a lattice with a universal infimum. Here, a congruence α is called associative if $x(yz)\alpha(xy)z$ for all x, y, and z in G. The infimum τ turns out to be the congruence such that $x\tau y$ if there exist x_i such that $x\omega x_1\omega x_2\omega\cdots\omega x_n\omega y$, where $u\omega v$ means that u and v are both products $\prod a_i$ but, possibly, with different placing of parentheses. Theorem 2 (in part): Let N be a set with an $(n+1)$-ary associative operation called $*$. (N is an n-semigroup under $*$.) Let G be the set of all j-tuples of elements of N for all possible j, $1\leq j\leq n$. Define the product of a j-tuple by a k-tuple (j may or may not equal k) by juxtaposition, using $*$ for left-contraction if the juxtaposition has length greater than n. Let $\tau=\tau(G)$ be as in Theorem 1. Then G/τ has the property that N is an n-subsemigroup of G/τ generating G/τ (G/τ is a covering of N). Further, if S is any covering of N, there exists an N-homomorphism of G/τ upon S.
F. Haimo (Belmont, Mass.)

Koskas, Maurice 2319
Sous-hypergroupes complets d'un hypergroupe. Théorie de Jordan-Hölder.
C. R. Acad. Sci. Paris **258** (1964), 3801–3804.

The author applies the results of a previous paper [same C. R. **258** (1964), 1965–1967; MR **29** #1281]. Let H be a hypergroup with composition law $*$, complete closure θ_H, and let ϕ be the canonical homomorphism of H onto its finest group homomorph g. The complete subhypergroups of H are then the subhypergroups of its associativity hypergroup $\bar{H}$, defined as the set H with composition law $\times$ given by $x\times y=\theta_H(x*y)$. The homomorphism ϕ defines an isomorphism between the lattice of complete subhypergroups of H and the lattice of subgroups of g, and is used to characterise the strongly right regular equivalences of H. A complete subhypergroup of H is invariant if and only if its image under ϕ is a normal subgroup of g, and a similar result holds for quasi-invariance. The Jordan-Hölder theorem for groups leads to a corresponding theorem for hypergroups.
J. A. H. Shepperd (Manchester)

Geddes, A. 2320
Power-free groups.
Proc. Cambridge Philos. Soc. **60** (1964), 393–408.

A power-free group is a set P of elements $a, b, c, \cdots$ and a particular element e, satisfying the following conditions. (I) For any two distinct elements a and b, there is a unique element ab (the product of a and b) in P. Also, there is a unique element ee in P. (II) $(ab)c=a(bc)$ for all elements a, b, c in P, provided all indicated products are defined. (III) $ea=a=ae$ for all a in P. (IV) To each element a in P there corresponds an element a' (distinct from a) in P such that $a'a=e=aa'$.

The author considers the question of embedding a power-free group in a group.
F. A. Sherk (Toronto, Ont.)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 2142, 2237, 2259, 2307, 2664, 2738, 2869.

Hofmann, Karl Heinrich 2321
Zerfällung topologischer Gruppen.
Math. Z. **84** (1964), 16–37.

Let G be a topological group and N a closed normal subgroup. If G contains a closed subgroup C such that the natural projection $\pi: G\to G/N$ is an isomorphism (onto) when restricted to C, then N is said to split in G over C. This paper is a continuation of the joint work of the author and the reviewer [Mem. Amer. Math. Soc. No. 43 (1963); MR **27** #1529] in that the techniques used there are continued and sharpened, and new areas of application suggested by that work are explored. The questions take two essential directions: (I) Given a group G and a closed normal subgroup N of G, when does N split in G; (II) Given that N splits in G and that Γ is a group of automorphisms of G such that $\Gamma N\subset N$, when does there exist a complement C with $\Gamma C\subset C$ such that N splits in G over C. Classical examples of (I) are Schur's theorem that if G is finite and the order of N is relatively prime to its index in G, then N splits, and Iwasawa's theorem that if N is a finite-dimensional vector group and G/N is compact, then N splits. The techniques of proof of these two results are similar, using as they do cross-sections and invariant means on the set $C(G/N, G)$ of all (continuous) functions with compact support from G/N to G, and in fact these two properties are in general sufficient to give a splitting [ibid., Theorem IV]. (The author's definition of invariant mean is not precisely the same as that used in the paper mentioned above.) The author seeks a unified and generalized treatment to arrive at sufficient conditions for obtaining a mean, and further he wishes to soften the cross-section requirement. To this end, he introduces the following concepts. (1) Let P be a set of prime numbers. A compact group G is called a P-group if it is totally disconnected and, if p is a prime divisor for some factor group of G, then $p\in P$. (2) An abelian group G is P-divisible if it is p-divisible for every $p\in P$, and it is uniquely P-divisible if it is uniquely p-divisible for every $p\in P$. (3) G is strongly P-divisible if it is uniquely P-divisible and if there is a basis $\mathfrak{B}$ of open sets of G such that if $p\in P$, then $V+\cdots+V\subset\{p\cdot v: v\in V\}$ for p summands V, $V\in\mathfrak{B}$. Examples are given to illustrate the range of this definition, and the following theorem is proved. Let G be a locally compact group, N a closed normal subgroup which is P-divisible if $P\neq\varnothing$ and a vector group otherwise, and G/N

compact. If $(G/N)/(G/N)_0$ is a P-group and there is a cross-section for N/N_0 in G/N_0, then N splits in G and all complements are conjugate under inner automorphisms. (In the statement of this theorem (Theorem III), the local compactness of G is inadvertently omitted from the hypotheses. However, it is clear from the proof that it is intended. Also, in 1.9, p must be replaced by P, since this stronger version is used in the proof of Theorem III. No essential modification of the proof of 1.9 is necessary to obtain this.) In the direction of question (II), the author proves the following. Let G be a splitting extension of a normal subgroup N and Γ a compact group of automorphisms of G such that $\Gamma N = N$. If there is an invariant mean for $C(\Gamma, N)$, then there is a complement C of N with $\Gamma C = C$. In several further results, he weakens the conditions on N and strengthens those on G/N to obtain the same conclusion. These come mainly from clever applications of Theorem I, which gives sufficient conditions for the existence of a mean. The techniques used, even if not the specific results obtained, should be of interest to anyone seeking information about splitting, whether from a purely algebraic or from a topological algebraic point of view.

P. S. Mostert (New Orleans, La.)

Ušakov, V. I. 2322

Topological $\overline{FC}$-groups. (Russian)

Sibirsk. Mat. Ž. **4** (1963), 1162–1174.

The paper is a continuation of previous ones [Ukrain. Mat. Ž. **14** (1962), 366–371; MR **27** #1531; Sibirsk. Mat. Ž. **4** (1963), 689–694; MR **27** #5859]. An $\overline{FC}$-element in a topological group is an element of which the closure of its conjugacy class is compact. An element is called compact if the subgroup it generates has compact closure. Main theorems: A connected locally compact group in which the set of $\overline{FC}$-elements is dense is an extension of a compact group by a vector space (in particular, if the group is a connected semisimple Lie group, it is compact); in a locally compact $\overline{FC}$-group the set of its compact elements is a closed characteristic subgroup of which the factor group is abelian with no compact elements except the unit element; a locally compact group is a compactly generated $\overline{FC}$-group if and only if it is an extension of a compact group by the direct product of a vector space and a free abelian group of finite rank. *H. de Vries* (Amsterdam)

Helgason, S. 2323

Fundamental solutions of invariant differential operators on symmetric spaces.

Amer. J. Math. **86** (1964), 565–601.

Let G/K be a globally Riemannian symmetric space of non-compact type, and let $\mathbf{D}(G/K)$ be the ring of all real linear G-invariant differential operators on G/K. A fundamental solution for $D \in \mathbf{D}(G/K)$ is, by definition, a distribution T satisfying $DT = \delta$, where δ is a delta function on G/K. In this paper the existence of a fundamental solution for such a $D \neq 0$ is proved (Theorem 4.2). Let F be a C^∞-function on G/K with compact support, then a global solution of the differential equation $DU = F$ is given easily by a fundamental solution T (Proposition 4.1). When G is a complex Lie group, more detailed information is obtained in § 5. In § 6, an explicit formula for the Green's function on a compact semisimple Lie group is given. Returning to G/K, let $\mathbf{Z}(G)$ be the ring of all real bi-invariant linear differential operators on G. The natural mapping from $\mathbf{Z}(G)$ into $\mathbf{D}(G/K)$ is surjective if G is a classical group (real or complex) but not surjective if G is one of the exceptional simple groups $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$ (Theorem 7.5).

M. Sugiura (Osaka)

Weil, André 2324

Sur certains groupes d'opérateurs unitaires.

Acta Math. **111** (1964), 143–211.

This is the first of two memoirs in which the author proposes to throw light on the role played by the symplectic group in the celebrated work of C. L. Siegel on quadratic forms. To a large extent the results of this installment may be formulated and proved as theorems about a certain group of unitary operators invariantly associated with a locally compact commutative group G. These general results are proved in Chapter I, which occupies slightly less than half of the paper. In Chapters II and III G is specialized to the local and adélic groups of modern number theory. The main result of Chapter II is a deduction of the quadratic reciprocity law from the theorems of Chapter I and the main result of the paper is proved in Chapter III. Chapters IV and V contain supplementary material of various kinds; in particular, a formulation of the main result of the projected second installment.

Let G be a commutative locally compact group and let G^* be its dual. For each $w = u, u^* \in G \times G^*$ let $U(w)$ be the unitary operator in $L^2(G)$ which maps ϕ into ϕ', where $\phi'(x) = \phi(x+u)\langle x, u^*\rangle$. Then $U(w_1)U(w_2) = F(w_1, w_2)\times U(w_1 + w_2)$, where $F(u_1, u_1^*; u_2, u_2^*) = \langle u_1, u_2^*\rangle$, so that U is a projective representation of $G \times G^*$ with $1/F$ as multiplier. Let $\mathbf{A}(G)$ denote the group of all unitary operators in $L^2(G)$ of the form $tU(w)$, where t is a complex number of modulus one, and let $\mathbf{B}_0(G)$ denote the group of all unitary operators V in $L^2(G)$ such that $V\mathbf{A}(G)V^{-1} = \mathbf{A}(G)$. $\mathbf{A}(G)$ is clearly isomorphic to the group $A(G)$ of all pairs w, t, where $w \in G \times G^*$, t is as above and $(w_1, t_1)(w_2, t_2) = (w_1 + w_2, F(w_1, w_2)t_1t_2)$. Thus each member of $\mathbf{B}_0(G)$ induces an automorphism of $A(G)$ which is readily seen to have the form $w, t \to (w)\sigma, tg(w)$, where σ is an automorphism of $G \times G^*$, g is a continuous function from $G \times G^*$ to the complex numbers of modulus one and g and σ are related by the following identity: $F((w_1)\sigma, (w_2)\sigma) = F(w_1, w_2) \times g(w_1 + w_2)/g(w_1)g(w_2)$. Conversely, any pair σ, g with σ and g so related defines an automorphism of $A(G)$, and the group of all such is denoted by $B_0(G)$. Let π_0 denote the homomorphism of $\mathbf{B}_0(G)$ into $B_0(G)$ defined by the foregoing considerations. Theorem 1 asserts that π_0 is surjective and has the group of constant multiples of the identity as its kernel. {Reviewer's remark: The identity connecting g and σ says that $1/F$ and its transform by σ are equivalent multipliers in the sense of the theory of projective representations and that g defines the trivial multiplier which is their ratio. It is known that $G \times G^*$ has a unique irreducible $1/F$ representation and that U is irreducible. Thus for each pair σ, g in $B_0(G)$ the mapping $w \to g(w)U((w)\sigma)$ is an irreducible $1/F$ representation equivalent to U. Let $V_{\sigma,g}$ be any unitary operator setting up the equivalence. Then $V_{\sigma,g}$ is unique up to a multiplicative constant and $\sigma, g \to V_{\sigma,g}$ defines a projective representation of $B_0(G)$ whose range (choosing the constant in all possible ways) coincides with $\mathbf{B}_0(G)$. π_0 is then the inverse of this projective representation, that is, $\pi_0(V_{\sigma,g}) = \sigma, g$.}

The author defines a continuous complex-valued function f on G to be a character of the second degree if its values lie on $|z|=1$ and $f(u_1+u_2)/f(u_1)f(u_2)$ is an ordinary character in u_1 for each fixed u_2. If f is any such, then there clearly exists a unique homomorphism ρ of G into G^* such that $\langle u_1, (u_2)\rho\rangle \equiv f(u_1+u_2)/f(u_1)f(u_2)$. When ρ is an isomorphism of G with G^*, then f is said to be non-degenerate. For each non-degenerate f let W_f be the unitary operator which carries ψ into ψ', where $\psi'(u)=cf(u)\psi^*((u)\rho)$. Here c is a positive real number chosen so as to make W_f unitary and ψ^* is the Fourier transform of ψ. Straightforward calculations show that $W_f U(u, u^*) W_f^{-1} = aU((u^*)\rho^{-1}, (-u)\rho - u^*)$, where $a=(-u, u^*)/f((u^*)\rho^{-1})$, and that $W_f^3 U(u, u^*) W_f^{-3} = U(u, u^*)$. Thus $W_f \in \mathbf{B}_0(G)$, and W_f^3 is a constant multiple of the identity. This constant, which the author denotes by $\gamma(f)$, plays a central role in the proof of the quadratic reciprocity law given in Chapter II. A simple calculation shows that if we use ρ to identify G and G^*, then the Fourier transform of convolution by f is multiplication by b/f, where b is a constant and $b/|b|=\gamma(f)$. In other words, the (generalized) Fourier transform of f is a constant b times $1/f$ and $\gamma(f)$ is $b/|b|$. That $\gamma(f)$ may be so defined is the content of Theorem 2. ($|b|$ is 1 whenever ρ preserves Haar measure.) Theorem 3 presents still another interpretation of the important invariant $\gamma(f)$.

Now let Γ be a closed subgroup of G and let Γ^* be its annihilator in G^*. Let $B_0(G, \Gamma)$ be the subgroup of $B_0(G)$ consisting of all σ, g such that g is identically one on $\Gamma \times \Gamma^*$ and $(\Gamma \times \Gamma^*)\sigma = \Gamma \times \Gamma^*$. Let $H(G, \Gamma)$ be the Hilbert space of all locally integrable functions θ on $G \times G^*$ which satisfy the identity $\theta(u+\xi, u^*+\xi^*) = \theta(u, u^*)\langle \xi, -x^*\rangle$ for $\xi, \xi^* \in \Gamma \times \Gamma^*$ and are such that $\int_{G\times G^*/\Gamma\times\Gamma^*} |\theta(u, u^*)|^2 < \infty$. The proofs of Theorems 4 and 5 are based on the observation that the $1/F$ representation U of $G \times G^*$ may be defined in a "natural" way in $H(G, \Gamma)$ and that this realization of U leads to a natural unitary representation of $B_0(G, \Gamma)$ which inverts π_0. Indeed, the mapping $\phi \to \theta$, where $\theta(u, u^*) = \int_\Gamma \phi(u)\langle \xi, u^*\rangle\, d\xi$ is defined for all continuous ϕ with compact support on G, has a unique continuous extension Z which is a unitary map of $L^2(G)$ on $H(G, \Gamma)$. Moreover, if $U^\Gamma(u, u^*)$ is the operator mapping θ into θ', where $\theta'(x, x^*) = \theta(x+u, x^*+u^*)F(x, x^*; u, u^*)$, then $Z^{-1}U^\Gamma(u, u^*)Z = U(u, u^*)$. For each σ, g in $B_0(G, \Gamma)$ let $r_\Gamma(\sigma, g)$ map θ into θ'', where $\theta''(x, x^*) = \theta((x, x^*)\sigma)g(x, x^*)$. One verifies at once that r_Γ is a homomorphism of $B_0(G, \Gamma)$ into the group of unitary operators in $H(G, \Gamma)$ and that $\pi_0(Z^{-1}r_\Gamma(\sigma, g)Z) = \sigma, g$. Now when ϕ is continuous with compact support, then $Z(\phi)(e, e) = \int_\Gamma \varphi(\xi)\, d\xi$. Since $\theta'(e, e) = \theta(e, e)$ whenever θ' is the transform by $r_\Gamma(\sigma, g)$ of θ and both θ and θ' are continuous at e, e, it follows that $\int_\Gamma \varphi(\xi)\, d\xi = \int_\Gamma s\varphi(\xi)\, d\xi$ whenever φ is suitably restricted $s = \sigma, g \in B_0(G, \Gamma)$ and $s\varphi$ means the transform of φ by $Z^{-1}r_\Gamma(\sigma, g)Z$. Let $S(G)$ denote the Bruhat generalization (from R^n to G) of the space of all rapidly decreasing infinitely differentiable functions. The author shows that $S(G)$ is invariant under the operators $Z^{-1}r_\Gamma(s)Z$ and that the above cited identity $\int_\Gamma \varphi(\xi)\, d\xi = \int_\Gamma s\varphi(\xi)\, d\xi$ holds for φ in $S(G)$ and s in $B_0(J, \Gamma)$. This is Theorem 4. It reduces to the classical Poisson summation formula when s and Γ are suitably chosen. In addition, the author calls attention to the following corollary. For each $\phi \in S(G)$ the function $s \to \int_\Gamma s\varphi(\xi)\, d\xi$ defined for all $s \in \mathbf{B}_0(G)$ is constant on the right $\mathbf{B}_0(G, \Gamma)$ cosets. Here $\mathbf{B}_0(G, \Gamma)$ is $r_\Gamma(B_0(G, \Gamma))$. {Reviewer's remark: As shown by the reviewer [same Acta **99** (1958), 265–311; MR **20** #4789] the "inducing" construction allowing one to pass from an arbitrary unitary representation of a closed subgroup of a locally compact group to a well-defined representation of the whole group may be extended so as to apply to projective representations with a fixed multiplier. The multiplier $1/F$ is identically one on $\Gamma \times \Gamma^*$, and U^Γ is just the $1/F$ representation induced by the one-dimensional identity representation of $\Gamma \times \Gamma^*$. When $\Gamma = G$, U^Γ reduces to U. That these induced projective representations are all irreducible and mutually equivalent is a consequence of the general theory developed in the work cited above—at least when G is separable.}

Now let f be a non-degenerate character of the second degree on G which reduces to one on Γ and is such that $(\Gamma)\rho = \Gamma^*$. The formula for $W_f U(u, u^*) W_f^{-1}$ shows at once that $\pi_0(W_f) \in B_0(G, \Gamma)$ and that $r_\Gamma(\pi_0(W_f)) = ZW_fZ^{-1}$. It follows that W_f^3 is the identity, that is, that $\gamma(f) = 1$. This is Theorem 5.

The derivation of the quadratic reciprocity law in Chapter II is based upon the following considerations. (a) If χ is a character of the additive group of a finite-dimensional vector space X over a locally compact field k and f is a quadratic form on X, then $\chi \circ f$ is a character of the second degree on the additive group of X. (b) Let k be as in (a) but not discrete, not of characteristic two and not the complex field. Let a and b be non-zero members of k and let f be the quadratic form taking x, y, z, t into $x^2 - ay^2 - bz^2 + abt^2$. Then for any non-trivial χ we have $\gamma(\chi \circ f) = (a/b)$, where (a/b) is one if a is the norm of an element in $k(b^{1/2})$ and is minus one otherwise. (c) Applying (b) to the locally compact completions k_v of a suitable discrete field k, one can reduce the problem of showing that $\prod_v (a/b)_v = 1$ to showing that $\gamma(f) = 1$ for a certain character of the second degree on the appropriate adèle group. That $\gamma(f) = 1$ follows on applying Theorem 5 with G the adèle group, and Γ the subgroup of principal adèles.

The first half of Chapter III is concerned with the special case of the general theory of Chapter I in which the group G is a vector space X over a local field. In this special case it is possible to replace $B_0(X)$ and $\mathbf{B}_0(X)$ by closely related groups which may be topologized so as to be locally compact. For any field k let X be a finite-dimensional vector space over k and let $B(x_1, x_1^*, x_2, x_2^*) = [x_1, x_2^*]$, where x_1^* and x_2^* are in the dual of X. Using B instead of F, the author defines a group $\mathrm{Ps}(X)$ by analogy with the definition of $B_0(G)$. Now let k be local and let χ be a non-trivial character on its additive group. The vector space dual of X may be identified with the dual of its additive group by setting $\langle x, x^*\rangle = \chi([x, x^*])$. The mapping μ which takes σ, f into σ, $\chi \circ f$ is clearly a homomorphism of $\mathrm{Ps}(X)$ into $B_0(X)$. Consider the subgroup of $\mathrm{Ps}(X) \times B_0(X)$ consisting of all triples σ, f, V with $\mu(\sigma, f) = \pi_0(V)$. The author calls it the metaplectic group and denotes it by $\mathrm{Mp}(X)$. It is to the groups $\mathrm{Ps}(X)$ and $\mathrm{Mp}(X)$ that he gives locally compact topologies. The mapping μ is injective whenever k is not of characteristic 2, and then $\mathrm{Ps}(X)$ and $\mathrm{Mp}(X)$ may be identified with subgroups of $B_0(X)$ and $\mathbf{B}_0(X)$, respectively. In any case the mapping π which takes σ, f, V into σ, f is a surjective homomorphism of $\mathrm{Mp}(X)$ into $\mathrm{Ps}(X)$ whose kernel is isomorphic to the one-dimensional torus.

In the second half of the chapter analogues of $\mathrm{Ps}(X)$ and $\mathrm{Mp}(X)$ are defined with X the adèle group X_A of a finite-dimensional vector space X_k over a field k which is

either an algebraic number field or an algebraic function field in one variable over a finite field. These groups are also given locally compact topologies. The chapter concludes with a discussion of the result of applying the corollary of Theorem 4 of Chapter I to the special case in which $G = X_A$ and $\Gamma = X_k$ (which, of course, has a natural imbedding in X_A). Replacing $\mathbf{B}_0(X_A)$ by $\mathrm{Mp}(X_A)$ and using a natural map of $\mathrm{Ps}(X_k)$ in $B_0(X_A, X_k)$, the author gives additional arguments to show that the functions on $\mathrm{Mp}(X_A)$, invariant under the action of $\mathrm{Ps}(X_k)$, provided by the corollary in question, are actually continuous with respect to the topology of $\mathrm{Mp}(X_A)$. This is Theorem 6. It is the main result of the paper and, according to the author, contains as a special case the classical construction of automorphic functions by means of theta series.

The homomorphism π described above shows that $\mathrm{Ps}(X)$ is an extension of $\mathrm{Mp}(X)$ by the group T of complex numbers of modulus one. In Chapter IV it is shown that this extension need not be trivial but can be reduced to an extension of $\mathrm{Mp}(X)$ by the two-element group. The situation is discussed in detail in the local case and sketched in the adélic case.

Let X be a finite-dimensional vector space over a field k which is also a left A module for a finite-dimensional associative algebra A over k. It is assumed that A has an identity e and comes equipped with the following further elements of structure: (a) an involutary anti-automorphism $t \to t^i$ which is the identity on ke, (b) a distinguished linear functional τ invariant under i such that $t, u \to \tau(tu)$ is a non-degenerate symmetric bilinear form. Under these circumstances X^* and $X \oplus X^*$ become left A modules in a natural way, and one obtains a subgroup $\mathrm{Ps}(X/A)$ of $\mathrm{Ps}(X)$ by considering the set of all σ, f such that (i) σ commutes with all left multiplications by members of A and (ii) f may be written in the form $f(x) = \tau(F(x, x))$, where F is a function from $X \times X$ to A which is linear in the first variable and i-linear in the second. The statement of the main theorem of the projected sequel to this paper concerns a group of the form $\mathrm{Ps}(X/A)$ and its subgroup $\mathrm{P}(X/A) = \mathrm{Ps}(X, A) \cap \mathrm{P}(X)$. Here $\mathrm{P}(X)$ is the set of all σ, f in $\mathrm{Ps}(X)$ with $(X^*)\sigma = X^*$. Chapter V is mainly concerned with the definition and formal properties of $\mathrm{Ps}(X/A)$ and $\mathrm{P}(X)$ and concludes with the following statement of the main theorem in question. Let k be an algebraic number field and let A be simple. Let G be the group of all elements a in A for which $aa^i = e$. Let G_A be the corresponding adèle group, and normalize Haar measure so that G_A/G has measure one. Let X_A be the adèle group corresponding to the left A module X and let ϕ be an element in $S(X_A)$. Then

$$\int_{G_A/G} \sum_{\xi \in X} \phi(a\xi)\, da = \sum_s r_k(s)\phi(0),$$

where r_k is the r_Γ of Theorem 4 of Chapter I adapted to the circumstances of Theorem 6 of Chapter III, Γ is the subgroup X of X_A and the summation is over a complete set of representatives of the left cosets of $\mathrm{P}(X/A)$ in $\mathrm{Ps}(X/A)$. The formula is valid whenever both members are absolutely convergent and the second member is absolutely convergent if and only if $\dim(X) > 4 \dim \mathrm{Q}(X/A)$. Here $\mathrm{Q}(X/A)$ is the vector space of all functions on X of the form $x \to \tau(F(x, x))$ as described under (ii) above, and it is assumed that no member of the center of A is invariant except members of ke.

G. W. Mackey (Cambridge, Mass.)

Mackey, George W. 2325

Ergodic theory, group theory, and differential geometry.
Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 1184–1191.

In this research announcement, the author proposes in outline a theory of virtual groups, a preliminary version of which appears in Section 7 of the notes to his 1961 Colloquium Lectures [Bull. Amer. Math. Soc. **69** (1963), 628–686; MR **27** #3745]. A virtual group is a rather complicated mathematical object. The definition is contained in the following two paragraphs:

Let $\mathfrak{F}$ be a groupoid (i.e., a small abstract category in which every morphism is invertible), which is endowed with an analytic Borel structure such that the domain of composition is a Borel subset of $\mathfrak{F} \times \mathfrak{F}$ and such that composition and inversion are Borel maps. Let $S_{\mathfrak{F}}$ denote the set of units of $\mathfrak{F}$ and π the map $x \to xx^{-1}$ of $\mathfrak{F}$ onto $S_{\mathfrak{F}}$. Let C be a measure class in $\mathfrak{F}$, $\tilde{C}$ the measure class in $S_{\mathfrak{F}}$ induced by π, and C_s ($s \in S_{\mathfrak{F}}$) the components of a direct integral decomposition of C over $(S_{\mathfrak{F}}, \tilde{C})$ with respect to the fibration of $\mathfrak{F}$ by π. C will be called ergodic if (1) it is invariant under inversion; (2) for C-almost all $z \in \mathfrak{F}$, the map $x \to zx$ of $\pi^{-1}(z^{-1}z)$ onto $\pi^{-1}(zz^{-1})$ carries $C_{z^{-1}z}$ onto $C_{zz^{-1}}$; (3) every real-valued Borel function f on $S_{\mathfrak{F}}$ satisfying $f(zz^{-1}) = f(z^{-1}z)$ for C-almost all $z \in \mathfrak{F}$ is constant $\tilde{C}$-almost everywhere. If C is ergodic, $(\mathfrak{F}, C)$ will be called an ergodic groupoid. If S is a co-null Borel subset of $S_{\mathfrak{F}}$, then $\mathfrak{F} \restriction S = \{z \in \mathfrak{F} : \pi(z), \pi(z^{-1}) \in S\}$ is called an inessential contraction of $\mathfrak{F}$, and $(\mathfrak{F} \restriction S, C \mid \mathfrak{F} \restriction S)$ is ergodic.

If $(\mathfrak{F}', C')$ is another ergodic groupoid, a functor $\varphi : \mathfrak{F} \to \mathfrak{F}'$ is called a strict homomorphism if it is a Borel map and if $(\varphi \mid S_{\mathfrak{F}})^{-1}(E)$ is $\tilde{C}$-null whenever E is a $\tilde{C}'$-null set in $S_{\mathfrak{F}'}$ contained in the union of all those sets $\pi((\pi^{-1}(z))^{-1})$ which are of $\tilde{C}'$ measure zero. {This modification of the definition on p. 1186 was communicated to the reviewer by the author.} A homomorphism is a Borel map $\varphi : \mathfrak{F} \to \mathfrak{F}'$ whose restriction to some inessential contraction of $\mathfrak{F}$ is a strict homomorphism. Two strict homomorphisms from $\mathfrak{F}$ to $\mathfrak{F}'$ are strictly similar if they are naturally equivalent as functors via a Borel map from $S_{\mathfrak{F}}$ to $\mathfrak{F}'$. Two homomorphisms are similar if they have strictly similar restrictions to a common inessential contraction. Two ergodic groupoids $\mathfrak{F}$ and $\mathfrak{F}'$ are similar if there exist homomorphisms $\varphi : \mathfrak{F} \to \mathfrak{F}'$ and $\psi : \mathfrak{F}' \to \mathfrak{F}$ such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are similar to identity homomorphisms. A similarity class of ergodic groupoids is called a virtual group. A morphism in the category $\mathfrak{B}$ of virtual groups is a similarity class of homomorphisms of ergodic groupoids.

Every separable locally compact group is an ergodic groupoid and hence determines a virtual group. Every homomorphism of separable locally compact groups determines a morphism in $\mathfrak{B}$. Two separable locally compact groups determine the same virtual group if and only if they are isomorphic. Two homomorphisms of separable locally compact groups determine the same morphism in $\mathfrak{B}$ if and only if they are conjugate.

Let G be a separable locally compact group acting on an analytic Borel space S. Let μ be a σ-finite measure on S which is quasi-invariant and ergodic under G. Let $C = C_1 \times C_2$, where C_1 is the measure class of μ and C_2 is the Haar measure class in G. Then $(S \times G, C)$ becomes an ergodic groupoid if composition is defined by $(s, x)(t, y) = (s, xy)$ [respectively, is undefined] if $sx = t$ [$sx \neq t$]. The map $\varphi : S \times G \to G$ given by $\varphi(s, x) = x$ is a homomorphism. Let P [Q] be the virtual group determined by G [$S \times G$], and let h be the morphism in $\mathfrak{B}$ determined by φ. Then S and the

action of G on S are determined up to null sets by Q and h. Finally, Q is determined by a locally compact group H if and only if G is essentially transitive on S, in which case H is isomorphic to the stability group of a point in the non-trivial orbit and h is determined by the inclusion map.

There is a section showing how the present theory ties in with the author's notion of a C^∞-ergodic lacing [Seminars on Analytic Functions, Vol. 2, pp. 259–272, U.S. Air Force, Washington, D.C., 1958]. Finally, the author discusses connections with fibre-space theory and with work of Ehresmann and of Y. H. Clifton and J. W. Smith.

The announcement under review contains no proofs; proofs and further developments are promised for subsequent publications. *R. J. Blattner* (Princeton, N.J.)

Mackey, George W. 2326

Group representations and analysis.

Rice Univ. Studies **49** (1963), *no.* 4, 13–27.

This is a series of three expository lectures given by the author at Rice University. The novice to the field of group representations will find them an excellent introduction and outline. *F. Hahn* (New Haven, Conn.)

Figà-Talamanca, Alessandro 2327

Multipliers of p-integrable functions.

Bull. Amer. Math. Soc. **70** (1964), 666–669.

Let G be a locally compact abelian group. The author establishes by a straightforward proof that the set of translation-invariant bounded operators in L_p is isometrically isomorphic to a conjugate space. This result is then used to establish other properties of the set of translation-invariant operators. The paper concludes with a theorem establishing the equivalence of a number of definitions of a type of lacunary set.

J. T. Schwartz (New York)

Roman, P. 2328

A remarkable connection between Kemmer algebras and unitary groups.

J. Mathematical Phys. **5** (1964), 777–783.

Author's summary: "Starting from a Kemmer algebra generated by n elements, a new algebra generated by $\frac{1}{2}(n^2+n)$ elements is derived. This algebra has only one irreducible representation which is of dimension $(n+1)$. This representation is equivalent to the first elementary representation of the infinitesimal operators of the unitary group in a space of $(n+1)$ dimensions. Omitting the identity element, one of course obtains the first elementary representation of the unitary unimodular group in a space of $(n+1)$ dimensions. The procedure is first illustrated for the case of $n=3$ and explicit representations of the infinitesimal operators of U_4 (SU_4) are obtained. Next, the proof for arbitrary n is outlined. Finally, the degenerate case of U_3 (SU_3) is discussed."

Many references to research and expository papers on the algebra of field equations, as well as a short outline of the fundamental facts about Kemmer's algebras and the connection of their representations and the representations of the unitary groups, are given.

A. Hulanicki (Wrocław)

Solomon, Louis 2329

Invariants of Euclidean reflection groups.

Trans. Amer. Math. Soc. **113** (1964), 274–286.

The paper contains detailed results on the structure of Euclidean reflection groups. Theorem 1: Let G be a Euclidean reflection group (containing reflections) and let M be an irreducible G-module. Let $\gamma \in G$ be a reflection, let M_γ^- be the subspace of all $x \in M$ such that $\gamma x = -x$ and suppose that $\dim M_\gamma^-$ is independent of γ. Let S be the C-algebra of complex polynomials on C^n, and F the ideal generated by the G-invariant homogeneous polynomials of positive degree. If M of dimension 1 is an irreducible constituent of the submodule $(S/F)_q$ of degree q of S/F for the values $q_1(M), \cdots, q_l(M)$, then the average of the q_i is

$$\frac{\dim M_\gamma^-}{\dim M} r.$$

Theorem 2: Let L be a complex simple Lie algebra and let V be a Cartan subalgebra of L. Let G be the Weyl group of L acting in V. Let E_p be the G-module of alternating p-forms over V. Then E_p is irreducible and occurs in $(S/F)_q$ for precisely $q=m_{i_1}+\cdots+m_{i_p}$, $i_1<\cdots<i_p$, where the m_i+1 are the degrees of the n algebraically independent functions f_i which generate the algebra of G-invariant polynomials. A basis for the isotypic component of S/F of type E_p is given by the minors of order p of the Jacobian matrix of the f_i, reduced mod F. In a last theorem, a basis of the module of alternating 1-forms over the algebra of invariant polynomials is obtained if the minors of the Jacobian obtained by deleting the partial derivatives of the f_i of highest degree are algebraically independent. {Bibliographical note: There is no reference given for Molien's formula counting the number of irreducible representations of a subgroup of the symmetric group. The reference in the paper of Shephard and Todd [Canad. J. Math. **6** (1954), 274–304; MR **15**, 600] is incorrect. It should read: Molien, Math. Natur. Mitt. S.-B. Preuss. Akad. Wiss. Berlin **1897**, 689–693.}

H. W. Guggenheimer (Minneapolis, Minn.)

Želobenko, D. P. 2330

On the theory of representations of complex and real Lie groups. (Russian)

Trudy Moskov. Mat. Obšč. **12** (1963), 53–98.

The paper is devoted to an explicit elaboration of a method of Godement [Trans. Amer. Math. Soc. **73** (1952), 496–556; MR **14**, 620] for obtaining irreducible representations.

Let G be a topological group. A subgroup Z of G is said to be solvably subordinate (s.s.) if Z is contained in the commutator of a connected solvable subgroup of G. The triple of subgroups Y, D, Z is said to be a triangular decomposition of G if (i) Y, Z are s.s. subgroups, (ii) $D \subset$ (normalizer Y) $\cap$ (normalizer Z), (iii) $Y \cap DZ = D \cap Z = (1)$, (iv) $G =$ closure of YDZ. For a complex semisimple group, for instance, Y [Z] is generated by the e_α belonging to the simple negative [positive] roots α, and D is the Cartan subgroup.

Let Y, D, Z be a triangular decomposition of G and Λ a representation of D in a finite-dimensional vector space $\mathscr{F}$. Define, for any $z \in Z$, $g \in G$ with $zg = y'd'z' \in YDZ$, $z\cdot g$ to be z' and $\Lambda(z, g)$ to be $\Lambda(d')$. Let, furthermore, $\mathscr{F}^Z$ denote the space of continuous $\mathscr{F}$-valued functions f on Z for which the partial map $G \times Z \to \mathscr{F}$ defined by $(g, z) \to$

$\Lambda(z, g)f(z \cdot g)$ extends continuously on all of $G \times Z$. Define $T_g f$ to be the function $z \to \Lambda(z, g)f(z \cdot g)$ (extended continuously to all of Z). Then T is a representation of G in $\mathcal{F}^Z$. Any representation induced by T in a G-stable subspace of $\mathcal{F}^Z$ is called an induced representation of G with character Λ. Theorem (Godement and the author): Any irreducible (finite-dimensional) representation of G is induced by some irreducible representation Λ of D in some space $\mathcal{F}$.

For complex semisimple groups the space $\mathcal{F}$ is always one-dimensional (i.e., $\mathcal{F} = C$) and the theorem just says that any irreducible representation of G is induced by its highest weight Λ. The subspaces of C^Z in which the irreducible representations of G are induced turn out to be spaces of polynomials which arise as solutions of differential equations $D_\alpha^{m_\alpha + 1} f = 0$, where D_α are the differential operators associated with the e_α, and m_α is the coefficient of the root α in the highest weight Λ. Various other questions are discussed. *W. T. van Est* (Leiden)

Steinberg, Robert 2331

A closure property of sets of vectors.

Trans. Amer. Math. Soc. **105** (1962), 118–125.

Let S be the system of roots of a simple Lie algebra over the complex number field. Imbed S in a real Euclidean space in the usual way. Then S is said to have the property P if $x, y \in S$, $x \neq -y$ and $(x, y) < 0$, then $x + y \in S$; $0 \notin S$. The author's purpose is to study this property. There are many interesting results. One of them is Theorem 1.1, which states: "Let S be finite and have property P, and let A be a subset with property P. For each $x \in S$, assume that at most one [respectively, exactly one, at least one] of x and $-x$ is in A. Then there is an ordering of the finite-dimensional Euclidean space such that A is contained in [A is, A contains] the set of positive elements of S". Harish-Chandra [Amer. J. Math. **77** (1955), 743–777, Lemma 4; MR **17**, 282] proved the second case, and Borel and Hirzebruch [ibid. **80** (1958), 458–538, pp. 471–473; MR **21** #1586] proved all three cases. But they make the assumption that if $x, y \in A$ and $x + y \in S$, then $x + y \in A$. The assumption is somewhat stronger than the author's property P.

Theorem 1.4 is a generalization of some classical results. Theorem 1.5 states that if S satisfies certain conditions, then S is infinite. When S is infinite, the author proves some drastic consequences; the main one (Theorem 1.6) is that if S is infinite, symmetric and indecomposable, and has property P and $\dim \geqq 2$, then $S \cup \{0\}$ is a group (under the addition of property P). *C. Y. Chao* (Pittsburgh, Pa.)

FUNCTIONS OF REAL VARIABLES

See also 2354, 2427, 2556, 2711.

Haaser, Norman B.; LaSalle, Joseph P.; Sullivan, Joseph A. 2332

★A course in mathematical analysis. Vol. II: Intermediate analysis.

Blaisdell Publishing Co. [Ginn and Co.], New York-Toronto-London, 1964. xii+677 *pp.* $11.75.

This is a useful textbook: the student is led gently from the comfortable surroundings of elementary calculus up to the point of departure into modern mathematics, either pure or applied as his inclination may be. The material is quite classical, the approach subtly modern. The book evolved out of classroom use by two of the authors (Haaser and Sullivan) and they make excellent use of their experience.

Chapter 1 deals with real vector spaces of finite dimension. The rudiments are given in algebraic form; the theory is developed by codifying the student's geometrical intuition, then abstracting to algebraic language. Chapter 2 treats solid analytic geometry in a modern setting. Vector notation is used, leading in a natural way to the concepts of orthogonality, linear dependence and basis. Chapter 3 presents the standard introduction to differential geometry, followed by an introduction to vector methods in mechanics.

Chapter 4 is entitled "Real-valued functions of a vector", and the viewpoint implied by this title is consistently followed. The material is the classical theory of multiple limits and partial derivatives, but the approach taken prepares the student for a relatively painless transition to abstract functional analysis. Chapter 5 is the natural next step: "Vector-valued functions of a vector". Chapter 6 covers multiple integrals, but there is one unfortunate omission, viz., surface integrals.

Chapter 7 is the most pronounced departure of the book from customary approaches. Change of variable in multiple integration is approached by means of set functions. The theorems then appear as analogues of their one-dimensional counterparts. This should prepare the student for his future excursions into probability theory and modern approaches to integration.

Chapters 8, 9, 10 are entitled "Sequences", "Series", and "Improper integrals". The customary material is adequately covered. Chapters 11 and 12, which treat differential equations and functions defined by them, are J. P. LaSalle's explicit contribution to the volume; this could be inferred were it not stated in the preface. There is no need to review these chapters beyond noting that those familiar with LaSalle's work will not be disappointed. *W. Langlois* (San Jose, Calif.)

Ostrowski, A. 2333

★Aufgabensammlung zur Infinitesimalrechnung. Erster Band: Funktionen einer Variablen.

Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften, Mathematische Reihe, Band 28. *Birkhäuser Verlag, Basel-Stuttgart,* 1964. 336 *pp.* *sFr.* 38.50.

The author has collected and augmented the problems contained in his *Vorlesungen über Differential- und Integralrechnung* [Birkhäuser, Basel, 1945; MR **13**, 540] and has issued them as a separate volume with complete solutions. The material covered ranges from the fundamental properties of real numbers through Taylor's formula. The flavor of the book is somewhat reminiscent of Pólya and Szegö's *Aufgaben und Lehrsätze aus der Analysis* [Dover, New York, 1945; MR **7**, 418], though at a more elementary level.

What makes the present volume unique is a section, 55 pages in length, containing hints to a good many of the problems. The user can therefore obtain his problem in the first section, verify his answer in the third, passing if

necessary through the second section for a hint. This book should prove very useful both to teachers and to students.
P. J. Davis (Providence, R.I.)

Timan, O. P. [Timan, A. F.] 2334
On the mutual deviation of classes of real functions defined on an arbitrary compactum and with given majorants of the modulus of continuity. (Ukrainian. Russian and English summaries)
Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1416–1418.

Author's summary: "Let Q be an arbitrary compact metric space with metric $\rho(x, y)$; $\omega_1(t)$ and $\omega_2(t)$ are any two continuity moduli determined on the section $0 \leqq t \leqq d(Q)$, where $d(Q)$ is the diameter of Q, and $H_{\omega_1}(Q)$ [respectively, $H_{\omega_2}(Q)$] is the class of all definite real functions $f(x)$ at points $x \in Q$ satisfying

$$|f(x)-f(y)| \leqq \omega_1\{\rho(x, y)\}$$

[respectively, $|f(x)-f(y)| \leqq \omega_2\{\rho(x, y)\}$]. In the general case the solution is given of the problem of the deviation of classes $H_{\omega_1}(Q)$ and $H_{\omega_2}(Q)$, determined explicitly. For the case when $Q = \bar{G}^n$ is some closed limited region of a finite-dimensional space of points $x(x_1, \cdots, x_n)$ with an arbitrary metric $\rho(x, y)$, a homeomorphic sphere $\rho(x, 0) \leqq 1$ with $n \geqq 2$ for any two continuity moduli, the principle of the boundary holds, expressed explicitly, also if $\bar{G}^n$ is replaced in its left part by any part containing the boundary Γ."

Kingman, J. F. C. 2335
A note on limits of continuous functions.
Quart. J. Math. Oxford Ser. (2) **15** (1964), 279–282.

The author in an earlier paper [Proc. London Math. Soc. (3) **13** (1963), 593–604; MR **27** #4283] generalized a theorem of Croft [Eureka No. 20 (1957), 11–13] by proving that if $f(t)$ $(t > 0)$ is a continuous real-valued function such that, as n runs through the positive integers, and if the limit $\lim_{n\to\infty} f(nh)$ exists for all positive h, then the limit $\lim_{t\to\infty} f(t)$ exists. Equivalently, by making the transformation $t = e^x$, one obtains that for f a continuous function from the reals to the reals, then if for each x the limit $\lim_{n\to\infty} f(\log n + x)$ exists, then so does the limit $\lim_{x\to\infty} f(x)$. In the present paper the author shows, roughly speaking, that the result continues to hold if $\{\log n\}$ is replaced by $\{c_n\}$ provided that $c_{n+1} - c_n \to 0$ $(n \to \infty)$, where $\{c_n\}$ is a strictly increasing sequence converging to infinity.
R. V. Chacon (Columbus, Ohio)

Bögel, Karl 2336
Bedingungen für die Existenz einer Punktmenge, auf der die Gleichung $f_{xy}'' = f_{yx}''$ erfüllt ist.
Math. Nachr. **22** (1960), 143–144.

In J. Reine Angew. Math. **173** (1935), 5–30, bewies der Verfasser u. a. folgenden Satz: Hat $f(x, y)$ im Definitionsbereich $\mathfrak{R}$ Ableitungen f_x', f_y', f_{xy}'', ist f_{xy}'' beschränkt in $\mathfrak{R}$ und $f(x, y)$ daselbst (zwei-dimensional) stetig, so existiert f_{yx}'' fast überall in $\mathfrak{R}$ und ist fast überall gleich f_{xy}''. Hier wird gezeigt, daß die Stetigkeit von $f(x, y)$ sich durch zweimalige Anwendung des Mittelwertsatzes aus den übrigen Bedingungen ableiten läßt.
J. Ridder (Zbl **98**, 265)

Neugebauer, C. J. 2337
Symmetric and smooth functions of several variables.
Math. Ann. **153** (1964), 285–292.

The concepts of symmetric functions and of smooth functions of one variable are well known and are widely used in the theory of trigonometric series. As for many other concepts, various extensions of these concepts to several variables suggest themselves "a priori". One such extension, namely that of "angular symmetry" and "smoothness", is studied in the paper, and it is shown that in this case certain important theorems for one variable carry over.

Let Q be the unit square and f a real-valued function on Q; let $p = (x, y)$, $h = (h_1, h_2)$ and

$$\Delta^2 f(p, h) = f(p+h) + f(p-h) - 2f(p).$$

By "angular symmetry" or "smoothness" for f is meant that $\Delta^2 f(p, h) = o(1)$ or $o(h)$, respectively, where $p+h$ is restricted to lie in an angular region $S(p)$ with vertex p and positive vertex angle; further, $S(p)$ is allowed to vary "measurably" from point to point. The following theorems are proved, which are extensions of known theorems in the case of one variable. (1) A measurable angular symmetric function is continuous a.e. in Q. (2) If $E_{\rm ap}$ [respectively, E] denotes the set of points at which f has an approximate [ordinary] total differential, then for a measurable angular smooth function, $|E_{\rm ap}| = |E|$. (3) Under a stronger "smoothness" condition, a measurable smooth function has an ordinary total differential on a set E which is of the power of the continuum in every non-empty open subset of Q. *T. P. Srinivasan* (Chandigarh)

Nakanishi, Shizu 2338
Sur une fonction continue qui est partout non dérivable.
Proc. Japan Acad. **40** (1964), 14–18.

Cette note concerne l'intégrale E.R. introduite par Kinjirô Kunugi [Fund. Appl. Aspects Math. **1** = Res. Inst. Appl. Electricity Monograph **7** (1959), 1–30; MR **27** #261]. L'auteur donne un exemple de fonction réelle f définie et intégrable E.R. sur [0, 1], dont l'intégrale indéfinie

$$F(x) = (\text{E.R.}) \int_0^x f(t)\, dt$$

est une fonction continue n'ayant de dérivée en aucun point de [0, 1]. L'exemple repose sur une construction laborieuse et d'une finesse remarquable. *S. Marcus* (Bucharest)

Nasibov, M. H. 2339
Reconstruction of a function of two variables from its Schwarzian derivative. (Russian. Azerbaijani summary)
Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Tehn. Mat. Nauk **1964**, no. 1, 25–34.

Let F be a function of two real variables. Put

$$\delta_{uv} F(x, y) = F(x+u/2, y+v/2) - F(x+u/2, y-v/2) - F(x-u/2, y+v/2) + F(x-u/2, y-v/2),$$

$$D^2 F(x, y) = \lim \delta_{uv}{}^2 F(x, y)/u^2 v^2 \quad (u \to 0, v \to 0),$$

$$D_{x^2}{}^2 F(x, y) = \lim\, (F(x+u, y) - 2F(x, y) + F(x-u, y))/u^2 \quad (u \to 0).$$

The main result is the following theorem.

Let F, f, g be finite functions on the interval $R=[a_1, b_1; a_2, b_2]$. Suppose that F is continuous and that $D^2F(x, y)=f(x, y)$, $D_x{}^{2}F(x, y)=g(x, y)$. Let f, g be summable on R, let $f(x, y)$ be summable in y on $[a_2, b_2]$ for each $x\in[a_1, b_1]$, and let $g(x, y)$ be summable in x on $[a_1, b_1]$ for each $y\in[a_2, b_2]$. Then

$$F(x, y) = \int_{a_1}^{x} d\xi \int_{a_1}^{\xi} \left[\int_{a_2}^{y} d\eta \int_{a_2}^{\eta} f(t, \tau)\, d\tau\right] dt + x\cdot\varphi_1(y)+y\cdot\varphi_2(x)+\varphi_3(x)+\varphi_4(y).$$

J. Mařík (Prague)

Valabrega, Elda Gibellato 2340

Il teorema di esistenza degli zeri delle funzioni continue nell'analisi moderna.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **98** (1963/64), 437–444.

Historical remarks concerning the relationship between the intermediate value property and continuity for a real-valued function on a real interval.

T. A. Botts (Charlottesville, Va.)

Obreškov, Nikola 2341

On regularly monotone functions. (Bulgarian. French summary)

Annuaire Univ. Sofia Fac. Sci. Phys. Math. Livre 1 *Math.* **56** (1961/62), 27–34 (1963).

Author's summary: "L'auteur démontre la généralisation suivante d'un théorème de S. N. Bernstein: Soit $f(x)$ une fonction complexe de la variable réelle x, définie dans l'intervalle (a, b) et indéfiniment dérivable dans cet intervalle. Supposons que les valeurs de $f^{(n)}(x)$ à partir d'un nombre n_0 se trouvent dans un angle A_n d'ouverture $\omega_n<\pi$ et que la fonction $|f^{(n)}(x)|$ est monotone dans l'intervalle (a, b). On suppose aussi pour les nombres ω_n que

$$\liminf_{n\to\infty}\left(\cos\frac{\omega_n}{2}\right)^{1/n} = \mu > 0.$$

Alors la fonction $f(x)$ est prolongeable analytiquement autour du segment ab dans une fonction régulière."

Garg, K. M. 2342

Applications of Denjoy analogue. I. (Sufficient conditions for a function to be monotone).

Ann. Polon. Math. **15** (1964), 159–165.

Several previously given conditions for a continuous function to be monotone, involving derivatives or Dini derivates, are generalized by slightly weakening one or more of the conditions. *P. Franklin* (Cambridge, Mass.)

Laha, R. G. 2343

On the decomposition of a class of functions of bounded variation.

Canad. J. Math. **16** (1964), 479–484.

Let B be the class of real-valued functions $G(x)$ of bounded variation in $(-\infty, \infty)$ having at least two points of increase and such that $G(-\infty)=0$. B_0 is the subclass consisting of those $G(x)$ for which $\int_{-\infty}^{\infty} \exp(vx)|dG|<\infty$ and whose Fourier-Stieltjes transforms are of the form $\exp P(t)$, where $P(t)$ is a polynomial of degree ≥ 2. It is then shown that if $G(x)\in B_0$ and $G(x)=G_1(x) * G_2(x)$, where each $G_j(x)$ satisfies the conditions $G_j(x)\in B$ and $V[G_j(x)]_{-y}{}^{\infty}$, $V[G_j(x)]_{-\infty}{}^{-y}$ are both $O(\exp y^{1+\delta})$ as $y\to\infty$ for some $\delta>0$, then $G_j(x)\in B_0$. This is a generalization of decomposition theorems in the theory of probability, where $G(x)$ is assumed to be a distribution function.

J. Blackman (Syracuse, N.Y.)

Fréchet, Maurice 2344

Sur diverses définitions de la différentiabilité.

Enseignement Math. (2) **10** (1964), 177–228.

Le but et le contenu de ce travail très utile sont clairement présentés par l'auteur même: "La définition usitée dans notre jeunesse, de la différentielle d'une fonction $f(x, y)$ supposait seulement l'existence des dérivées partielles de f au point considéré. Mais le recours à une rigueur croissante avait montré que pour étendre au cas de plusieurs variables les propriétés si utiles de la différentielle d'une fonction d'une variable, il était nécessaire d'introduire des hypothèses (variées avec les différents cas). C'est alors que plusieurs essais, couronnés de succès, ont montré qu'on pouvait rétablir ce parallélisme en donnant à la différentielle totale une définition plus stricte. Il est alors curieux de constater que plusieurs de ces essais, partant de considérations totalement différentes, donnant des définitions de la différentiabilité de formes essentiellement différentes (comme on va pouvoir s'en assurer plus loin), fournissent cependant des définitions équivalentes comme nous avons pu le prouver aux pages 198 à 206. Nous énoncerons les quatre définitions qui sont parvenues à notre connaissance. Comme plusieurs de ces définitions semblent être assez connues et comme la convergence de quatre d'entre elles est un cas assez rare en mathématique, il nous a semblé que leur rappel pourrait intéresser les lecteurs d'une revue du type de *L'Enseignement Mathématique* . . . quelques résultats, quelques raisonnements présentés ici (en particulier la démonstration des équivalences et celle de certaines propriétés des différentielles secondes) sont inédits". *S. Marcus* (Bucharest)

Denny, J. L. 2345

A continuous real-valued function on E^n almost everywhere 1-1.

Fund. Math. **55** (1964), 95–99.

The purpose of this paper is to define a function $f: E^n\to(0, 1)$ such that (i) it is uniformly continuous; (ii) there is a Borel set $D\subset E^n$ with $|E^n-D|=0$ and the restriction of f to D is (1, 1); and (iii) each function of a real variable obtained by fixing $(n-1)$ of the coordinate variables in f is nondecreasing. *S. J. Taylor* (London)

Waliszewski, Włodzimierz 2346

Condition nécessaire et suffisante pour qu'un certain système de deux équations à deux inconnues ait des solutions.

Bull. Soc. Sci. Lettres Łódź **14** (1963), *no.* 7, 11 *pp.*

If Z is a finite subset of the unit circle, let $R(Z)$ be the collection of sets of mutually disjoint line segments whose end-points lie in Z. Two disjoint sets Z, Z' are ρ-related if each set $\tau\in R(Z)$ contains a segment which divides Z' into

two sets each having an odd number of points. (The relation ρ is symmetric.)

If $F(x, y)$ is a homogeneous polynomial of degree s, call a point (x_0, y_0) of the circle a k-tuple direction of F if $F(x, y) = (x_0 y - y_0 x)^k P(x, y)$, where $P(x_0, y_0) \neq 0$ (the exponent k is missing in the paper), and simply a direction if $F(x_0, y_0) = 0$. Let (F) be the class of C^1-functions f in the plane such that

$$F(x, y) + f(x, y) = F_1(x, y) + f_1(x, y) = F_2(x, y) + f_2(x, y) = 0$$

has no solutions, and which satisfy

$$\lim_{(x,y)\to\infty} f(x, y)(x^2+y^2)^{-s/2} = 0,$$

$$\lim_{(x,y)\to\infty} f_i(x, y)(x^2+y^2)^{-(s-1)/2} = 0, \quad i = 1, 2$$

(numerical subscript indicates partial differentiation). Finally, denote by $Z(F)$ the set of directions of the polynomial F which have odd multiplicity. Then, using five lemmas, the author proves the following theorem: If F and G are homogeneous polynomials such that $Z(F) \cap Z(G)$ is empty, a necessary and sufficient condition that the system

$$F(x, y) + f(x, y) = 0,$$

$$G(x, y) + g(x, y) = 0$$

have a solution for each pair $f \in (F)$ and $g \in (G)$ is that $Z(F)$ and $Z(G)$ be ρ-related.

J. V. Ryff (Cambridge, Mass.)

Donoghue, William F., Jr. 2347

Functions which are polynomials on a dense set.

J. London Math. Soc. **39** (1964), 533–536.

The author constructs a function $f(x)$ with the following properties: (a) $f(x)$ has infinitely many derivatives for all real x; (b) every interval contains a subinterval in which $f(x)$ is identical with some polynomial; (c) $f(x)$ is itself not a polynomial. He defines $f(x)$ to be "locally a polynomial at the point x_0" if $f(x)$ coincides with a polynomial in some neighborhood of x_0. His construction thus produces a function $f(x)$ with properties (a) and (c) which is locally a polynomial on a set everywhere dense on the real line. In fact, the exceptional set (where $f(x)$ fails to be locally a polynomial) can be made to be of Hausdorff measure zero, corresponding to any given function $h(t)$ which is increasing in $[0, 1]$ and continuous and equal to zero at $t = 0$.

F. Herzog (E. Lansing, Mich.)

Rama Mohana Rao, M. 2348

A note on an integral inequality.

J. Indian Math. Soc. (N.S.) **27** (1963), 67–69.

Let $|\Delta[r(t) - \psi(t)]| \leqq [\geqq] \int_t^{t+\Delta t} f(s, r(s))\, ds$ for $\Delta t \geqq 0$, where $f(t, r) \geqq 0$ is defined for $r \geqq 0$ and $t \geqq t_0$, is measurable in t for fixed r and continuous in r for fixed t; r and ψ are continuous on $t \geqq t_0$. Then $r(t) \leqq [\geqq] \psi(t) + m(t)$, where $m(t)$ is the maximal [minimal] solution of $z' = f(t, z + \psi(t))$ through $(t_0, 0)$. This result is related to those of Viswanatham [Proc. Amer. Math. Soc. **14** (1963), 15–18; MR **26** #379] but is obtained without the requirement made there that $f(t, r)$ be monotone non-decreasing in r for fixed t.

C. E. Langenhop (Carbondale, Ill.)

Džanašija, G. A. 2349

On the superposition of two functions of the Gevrey class. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **33** (1964), 257–262.

Let G_α $(\alpha > 1)$ be the class of infinitely differentiable functions $f(x)$ on $(-\infty, +\infty)$ which satisfy

$$|f^{(n)}(x)| \leq CA^n n^{n\alpha}, \qquad n = 0, 1, 2, \cdots,$$

where the constants C and A depend on the function f and $n^{n\alpha}$ equals 1 for $n = 0$. The author proves that if $f \in G_\alpha$ and $g \in G_\beta$, then their superposition $F = fg$ belongs to G_γ with $\gamma = \max(\alpha, \beta)$. The proof consists in direct estimations of the magnitude of $F^{(n)}(x)$. A formula for the nth derivative of the composite function is also obtained. {In this connection, cf. M. A. McKiernan [Amer. Math. Monthly **63** (1956), 331–333; MR **17**, 953].}

M. Kuczma (Katowice)

MEASURE AND INTEGRATION

See also 2325, 2338, 2663, 2853, 2869.

Areškin, G. Ja. 2350

On the compactness of the family of completely additive set functions. (Russian)

Leningrad. Gos. Ped. Inst. Učen. Zap. **238** (1962), 102–118.

Let **K** be the σ-algebra generated by an algebra **M**. All signed measures to be considered are finite. A family Φ of signed measures on **K** is said to be uniformly additive on **M** if and only if, for a given sequence $\{M_k\}$ of disjoint sets $M_k \in$ **M** and $\varepsilon > 0$, there exists an index N such that $n > N$ implies $|\sum_n^\infty \varphi(M_k)| < \varepsilon$ for each $\varphi \in \Phi$. A sequence φ_n of signed measures on **K** converges to a signed measure φ if $\varphi_n(K) \to \varphi(K)$ holds for each $K \in$ **K**. A family Φ of signed measures on **K** is compact if and only if each subsequence of Φ contains a subsequence which converges to some signed measure. The author proves the following theorems. A sequence $\{\mu_n\}$ of measures on **K** converges to a measure μ on **K** if and only if $\mu_n(M) \to \mu(M)$ for $M \in$ **M** and $\{\mu_n\}$ is uniformly additive on **M**. If **M** is countable, then a family Φ of measures on **K** is compact if and only if Φ is uniformly bounded and uniformly additive on **M**, and the compactness of the family of total variations of the signed measures belonging to a family implies the compactness of the family itself. If μ is a measure on **K**, then a sequence $\{\varphi_n\}$ of signed measures converges to a signed measure φ if and only if the absolutely continuous and singular parts of φ_n (with respect to μ) converge to the corresponding part of φ, and a family Φ of signed measures is compact if and only if the families of the absolutely continuous and singular parts of the functions $\varphi \in \Phi$ are both compact. As applications, convergence and compactness of probability distribution functions are studied.

Á. Császár (Budapest)

Fishel, B.; Papert, D. 2351

A note on hyperdiffuse measures.

J. London Math. Soc. **39** (1964), 245–254.

A measure κ is hyperdiffuse if $\kappa(R) = 0$ for every R which is of first category and belongs to a σ-ring S of subsets of a topological space, and a measure λ is ordinary if for every $E \in S$ there exists $R \subset E$ such that $R \in S$, R is of

first category and $\lambda(R)=\lambda(E)$. The authors show that any measure is decomposable into the sum of a hyperdiffuse measure and an ordinary measure, and that in a certain category of spaces, which includes all locally connected spaces and all metric spaces without isolated points, any inner regular measure must be ordinary. Also, results of J. Dixmier [Summa Brasil. Math. **2** (1951), 151–182; MR **14**, 69] for normal measures defined on Stonian spaces are proved for completely regular spaces, and a construction for hyperdiffuse measures is given.

U. D'Ambrosio (Providence, R.I.)

Maĭkov, E. V. 2352

τ-polynomials and τ-analytic functionals. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 1025–1028.

Cette note est la suite d'une note antérieure [mêmes Dokl. **155** (1964), 266–269; MR **28** #3317]; mêmes notations dans l'analyse présente. Un τ-polynôme est une fonctionnelle de la forme

$$(1)\quad f(\xi)=\int_0^1\cdots\int_0^1 \varphi(t_1, \xi(t_1), \cdots, t_n, \xi(t_n))\,dt_1\cdots dt_n,$$

φ étant une fonction continue. Définition: Un τ-polynôme est dit ξ^0-simple ($\xi^0\in S$) si dans (1) la fonction φ peut être choisie telle que, pour $j=1, \cdots, n$,

$$\varphi(t_1, x_1, \cdots, t_{j-1}, x_{j-1}, t_j, \xi^0(t_j), \cdots, t_n, x_n)\equiv 0.$$

Un τ-polynôme arbitraire est alors uniquement une somme de τ-polynômes ξ^0-simples. (Développement de Taylor autour de la trajectoire ξ^0.) Une fonctionnelle $f(\xi)$, $\xi\in S$, est dite τ-analytique si elle est représentée par son développement de Taylor (infini). Par exemple, $f(\xi)=\exp(\int_0^1 v(\xi(t))\,dt)$ est τ-analytique si la fonction v est continue. L'auteur introduit une topologie dans l'ensemble des fonctionnelles τ-analytiques, et il constate que le dual de cet espace vectoriel topologique consiste de "τ-mesures" qui servent à prolonger certaines répartitions de masse sur des sous-espaces de dimension finie lorsque ces répartitions ne sont pas prolongeables en mesures ordinaires [Maĭkov, Uspehi Mat. Nauk **18** (1963), no. 3 (111), 243–244]. *E. J. Akutowicz* (Bologna)

Novák, Josef 2353

On a topological relation between a σ-algebra A of sets and the system of all A-measurable functions. (Russian summary)

Czechoslovak Math. J. **14 (89)** (1964), 267–270.

Author's summary: "In the present paper the following statement is proved: Every σ-algebra A of subsets of a given non-void point set X and the system of all A-measurable real functions on X are non-homeomorphic sequentially regular convergence spaces."

Kozlovcev, S. G. 2354

On the structure of measurable functions. (Russian)

Mat. Sb. (N.S.) **64 (106)** (1964), 275–285.

If f is a finite real function on (0, 1), $x_0\in(0,1)$, and if a, b are arbitrary real numbers, we denote by $E_{a,b}(f, x_0)$ the set of all $x\in(x_0, 1)$ such that $a<(f(x)-f(x_0))/(x-x_0)<b$. The author constructs a finite measurable function f on (0, 1) such that, for almost all $x_0\in(0,1)$, the following assertions are true. (i) If $0<a<b<\infty$, then x_0 is a point of dispersion for $E_{a,b}(f, x_0)$ and the right upper density of $E_{-b,-a}(f, x_0)$ at x_0 is $\geq(b-a)/b$. (ii) There exists a set $S(x_0)$ such that x_0 is a point of the right and left upper density of $S(x_0)$, that $f(x)<f(x_0)$ for every $x\in S(x_0)$, and that $(f(x)-f(x_0))/(x-x_0)\to 0$ for $x\to x_0$, $x\in S(x_0)$.

J. Mařík (Prague)

Kubota, Yôto 2355

The mean continuous Perron integral.

Proc. Japan Acad. **40** (1964), 171–175.

The author defines a Perron type of integral of a function $f(x)$ on an interval $[a, b]$ by assuming that upper [lower] functions $U(x)$ $[L(x)]$ satisfy conditions: (1) $U(a)=L(a)=0$; (2) $U(x)$ $[L(x)]$ are M-continuous in the sense that they are general Denjoy integrable and

$$\lim_{h\to 0}(1/h)\int_c^{c+h}U(x)\,dx = U(c)$$

for all c of $[a, b]$; (3) $U(x)$ $[L(x)]$ is $(\underline{\mathrm{ACG}})$ $[(\overline{\mathrm{ACG}})]$ in the sense that $[a, b]$ is the sum of a countable number of closed sets relative to each of which $U(x)$ $[L(x)]$ is $\underline{\mathrm{AC}}$ $[\overline{\mathrm{AC}}]$; and (4) AD $U(x)\geq f(x)$ a.e. [AD $L(x)\leq f(x)$ a.e.]. $f(x)$ is then MP-integrable if $f(x)$ has upper and lower functions and $\inf U(b)=\sup L(b)$. It is then shown that the MP-integral is equivalent to the MG-integral of H. W. Ellis [Canad. J. Math. **1** (1949), 113–124; MR **10**, 520], where $f(x)$ is MG-integrable if there exists an $F(x)$ such that (1) $F(x)$ is M-continuous on $[a, b]$; (2) $F(x)$ is ACG on $[a, b]$; and (3) AD$F(x)=f(x)$ a.e.

T. H. Hildebrandt (Ann Arbor, Mich.)

Appling, William D. L. 2356

Interval functions and absolute continuity.

Illinois J. Math. **8** (1964), 439–445.

Given two non-decreasing real-valued functions h and m on $[a, b]$, we say that h is absolutely continuous with respect to m if the Stieltjes measure generated by h is absolutely continuous (a.c.) with respect to that generated by m. The object of the present paper is to extend a result of the author [Canad. J. Math. **15** (1963), 752–754; MR **27** #4903] to obtain conditions in terms of the Hellinger integral, each of which is necessary and sufficient for h to be a.c. with respect to m. If $\int_I (dh)^p(dm)^{1-p}$ is the limit of

$$\sum_{k=1}^{n}\{h(x_k)-h(x_{k-1})\}^p\{m(x_k)-m(x_{k-1})\}^{1-p}$$

$$\text{as } \max|x_k-x_{k-1}|\to 0,$$

where $x_0<x_1<\cdots<x_n$ is a dissection of I, these conditions are: (i) For real bounded interval functions H defined on subintervals of $[a, b]$, the existence of $\int_{[a,b]}H(I)\,dm$ implies the existence of $\int_{[a,b]}H(I)\,dh$;

$$\text{(ii)}\quad \int_{[a,b]}(dh)^p(dm)^{1-p}\to h(b)-h(a)\quad \text{as } p\to 1-;$$

$$\text{(iii)}\quad \int_{[a,b]}\left|dh-\int_I (dh)^p(dm)^{1-p}\right|\to 0\quad \text{as } p\to 1-.$$

S. J. Taylor (London)

Varberg, Dale E. 2357
On Gaussian measures equivalent to Wiener measure.
Trans. Amer. Math. Soc. **113** (1964), 262–273.

The author investigates the question of when two Gaussian measures are equivalent in the case where one of the measures is Wiener measure. Necessary and sufficient conditions are given for equivalence, and an explicit formula is found for the Radon-Nikodým derivative. These results are based on a paper of Woodward [same Trans. **100** (1961), 459–480; MR **24** #A1017]. The sufficiency results are obtained by examining the question of when it is possible to represent a Gaussian process by means of a linear transformation of the Wiener process, and this in turn has led the author to the consideration of Gaussian processes having factorable covariance functions.

R. V. Chacon (Columbus, Ohio)

Hajian, Arshag 2358
Strongly recurrent transformations.
Pacific J. Math. **14** (1964), 517–523.

A recurrent transformation is a measurable transformation ϕ defined on a σ-finite measure space $(X, \mathscr{B}, m)$ with the following property: If A is a measurable set of positive measure, then for almost all $x \in A$, $\phi^n x$ belongs to A for infinitely many integers n. The author introduces the notion of a strongly recurrent set and defines a strongly recurrent transformation, and shows that a transformation is strongly recurrent if and only if there exists a finite invariant measure μ equivalent to m. This is accomplished by showing the connection between strongly recurrent sets and weakly wandering sets.

The definitions are the following: A measurable set S is said to be strongly recurrent (with respect to ϕ) if the set of all integers n such that $m(\phi^n S \cap S) > 0$ is relatively dense, i.e., if there exists a positive integer k such that $\max_{0 \leq i \leq k-1} m(\phi^{n-i} S \cap S) > 0$ for $n = 0, \pm 1, \pm 2, \cdots$. The transformation ϕ is said to be strongly recurrent if every set of positive measure is strongly recurrent. The author notes that the property of a transformation ϕ being strongly recurrent is preserved under equivalent measures.

R. V. Chacon (Columbus, Ohio)

Parry, W. 2359
On Rohlin's formula for entropy
Acta Math. Acad. Sci. Hungar. **15** (1964), 107–113.

The author presents a formula for computing the entropy of an ergodic stationary non-atomic stochastic process with a finite number of states. The formula is then applied to compute the entropy of certain specific processes yielding a correction of an arithmetical error of Rohlin in computing the entropy of a β-transformation. The correction is said to have been acknowledged by Rohlin in private correspondence. *R. V. Chacon* (Columbus, Ohio)

Parry, W. 2360
Note on the ergodic theorem of Hurewicz.
J. London Math. Soc. **39** (1964), 202–210.

Let T be a measurable non-singular transformation of a probability space $(X, \mathscr{A}, p)$ onto itself. Let $\mathscr{A}^n = T^{-n}(\mathscr{A}) = \{T^{-n}(E) : E \in \mathscr{A}\}$ and $\mathscr{A}^\infty = \bigcap_{n=0}^\infty \mathscr{A}^n$, so that $\mathscr{A} = \mathscr{A}^0 \supset \mathscr{A}^1 \supset \cdots \supset \mathscr{A}^\infty$. By the Radon-Nikodým theorem and the non-singularity of T, there is a positive $\mathscr{A}^1$-measurable function ω such that $pT(E) = \int_E \omega\, dp$ for all $E \in \mathscr{A}^1$. Define $\omega_n = \omega(T^{n-1}) \cdots \omega(T)\omega$ and

$$f^n = \sum_{i=0}^{n} f(T^i)\omega_i,$$

for measurable finite-valued f.

The author gives an affirmative answer to the question whether or not the incompressibility of T implies $\lim_{n\to\infty} E(g^n/\mathscr{A}^n) = \infty$ a.e. for $g > 0$ a.e., where $E(f/\mathscr{A})$ denotes the conditional expectation of f with respect to $\mathscr{A}$. This result is a corollary of a decomposition theorem which appeals to a general ergodic theorem for its proof.

R. V. Chacon (Columbus, Ohio)

Baxter, Glen 2361
An ergodic theorem with weighted averages.
J. Math. Mech. **13** (1964), 481–488.

Let $\{f_k, k = 1, 2, \cdots\}$ be a sequence of non-negative numbers with $\sum_k f_k = 1$, and define the sequence $\{u_n, n = 0, 1, \cdots\}$ recursively by

$$u_0 = 1, \qquad u_n = \sum_{i=1}^{n} f_i u_{n-i}.$$

Let $(S, \mathscr{F}, m)$ be a finite measure space and T a one-one measure-preserving point transformation of S into S. Let f be an integrable function and define

$$R_n(s) = \sum_{m=0}^{n} u_m f(T^m s) \Big/ \sum_{m=0}^{n} u_m.$$

The author proves that if $R_n(s) - R_n(Ts)$ converges to zero a.e., then $\lim_n R_n(s)$ exists a.e. The special case $f_1 = 1$, $f_k = 0$, $k > 1$, is the individual ergodic theorem. Other examples are given.

J. R. Blum (Albuquerque, N.M.)

Chacon, R. V. 2362
A class of linear transformations.
Proc. Amer. Math. Soc. **15** (1964), 560–564.

Given positive constants a_1 and a_2, it is shown that there exists an invertible positive isometry T of $L_1(0, 1)$ and a positive function $f \in L_1 \cap L_\infty$ such that

$$\|f\|_1 \leqq a_1, \qquad \|f\|_\infty \leqq a_2,$$

$$\liminf\, (1/n) \sum_{0}^{n-1} T^k f = 0,$$

$$\limsup\, (1/n) \sum_{0}^{n-1} T^k f = \infty$$

a.e. The construction also provides a new example of a measurable point transformation having no σ-finite equivalent invariant measure. For any $\delta > 0$ the transformation T of $L_1(0, 1)$ can be made to satisfy the condition $\|T\|_\infty \leqq 1 + \delta$. *J. C. Oxtoby* (Bryn Mawr, Pa.)

Hadwiger, H.; Rätz, J. 2363
Zur Deckungsmonotonie von Inhaltsoperatoren.
Math. Nachr. **27** (1963/64), 145–161.

Let Γ be a commutative group of one-to-one mappings of a set R and E a non-empty subset of R. The set E is referred to as the set of units and the triple $\langle R, \Gamma, E \rangle$ is called a basis. A subset A of R is called E-bounded if it is contained in a finite union of Γ-images of E. A volume

field ("Inhaltsfeld") is a class of E-bounded sets closed under the operations of union and intersection, mapped among themselves by Γ, and including E and the empty set. A volume operator J defined on a volume field is a real-valued non-negative Γ-invariant additive set function with $J(E)=1$. Analogous concepts of integral field (a set of functions on R with E-bounded support) and integral are defined, and it is shown that a volume operator can be extended to an integral if and only if it has the property of being covering monotonic, that is, if $A_1, \cdots, A_n$ and $B_1, \cdots, B_m$ are subsets belonging to the volume field and if $[C]$ denotes the characteristic function of the set C, then $\sum_{\nu=1}^{n}[A_\nu] \leqq \sum_{\mu=1}^{m}[B_\mu]$ implies $\sum_{\nu=1}^{n} J(A_\nu) \leqq \sum_{\mu=1}^{m} J(B_\mu)$. Sufficient conditions are given for a volume operator to be covering monotonic, and an example is given of a volume operator which is monotonic without being covering monotonic. Construction of outer and inner volumes associated with a volume operator is also considered and is used to define the closure or completion of a volume operator. *A. M. Macbeath* (Birmingham)

Miranda, Mario 2364

Distribuzioni aventi derivate misure insiemi di perimetro localmente finito.

Ann. Scuola Norm. Sup. Pisa (3) **18** (1964), 27–56.

The perimeter of a set E of R^n, as defined by E. De Giorgi in 1954 [Ann. Mat. Pura Appl. (4) **36** (1954), 191–213; MR **15**, 945], turns out to be $P(E)=\sup\{\int_E \operatorname{div} g(x)\,dx : g \in \mathscr{D}^{(n)}, |g| \leq 1\}$, where $\mathscr{D}^{(n)}$ is the space of functions from R^n in R^n with components in C_0^{∞}. Given an open set A in R^n, let $\mathscr{K}(A)$ denote the family of all open bounded sets, excluding the empty set, which are, together with their closure, contained in A; $\varphi(x, E)$ denotes the characteristic function of E.

The author says a set E has locally finite perimeter on A if $P(E \cap C)$ is finite for any sphere C in $\mathscr{K}(A)$. Thus a set E has locally finite perimeter on A if and only if $\varphi(x, E)$ has derivatives that are measures in A. Then, several results concerning sets of locally finite perimeter are given, some generalizing those of E. De Giorgi [op. cit.; Ricerche Mat. **4** (1955), 95–113; MR **17**, 596] and some related to results obtained by W. H. Fleming [Illinois J. Math. **4** (1960), 452–478; MR **24** #A202].

Among the main results we distinguish the following. (i) If γ is a real function defined in $\mathscr{K}(A)$, and if $\mathscr{E}$ is the family of sets of locally finite perimeter on A such that $\int_K |D\varphi| \leq \gamma(K)$ for every K in $\mathscr{K}(A)$ (where $D\varphi$ is the vector measure which is the derivative of φ and $|D\varphi|$ is its total variation measure), then $\mathscr{E}$ is compact in $L^1_{\mathrm{loc}}(A)$. (ii) The study of the trace of sets of locally finite perimeter over spherical surfaces. (iii) The following isoperimetric inequality: If E has locally finite perimeter on A and C is a sphere and belongs to $\mathscr{K}(A)$,

$$\inf_{\lambda \in R} \int_C |\varphi(x, E)-\lambda|\,dx \leq \gamma_n \cdot 2^{1/(n-1)}(2^{1/n}-1)^{-n/(n-1)} \cdot \left\{\int_C |D\varphi(x, E)|\right\}^{n/(n-1)}$$

(γ_n is the isoperimetric constant of the n-sphere); another inequality involving the boundary of C is given. (iv) The following regularity theorem: If E has locally finite perimeter on A, and $D\varphi(x, E)/|D\varphi(x, E)|$ is continuous in $\mathscr{F}E \cap A$, then $\mathscr{F}E \cap A$ is a hypersurface of class C^1.

U. D'Ambrosio (Providence, R.I.)

FUNCTIONS OF A COMPLEX VARIABLE

See also 2230, 2258, 2341, 2417, 2418, 2423, 2517, 2669, 2758, 2790, 2942, 2943, 2944, 3023.

Ridder, J.; Westermann, L. R. J. 2365a

Hinreichende Bedingungen für analytische, harmonische und subharmonische Funktionen. II, III.

Nederl. Akad. Wetensch. Proc. Ser. A **64** = *Indag. Math.* **23** (1961), 261–271, 272–279.

Westermann, L. R. J. 2365b

Hinreichende Bedingungen für analytische, harmonische und subharmonische Funktionen. IV, V.

Nederl. Akad. Wetensch. Proc. Ser. A **65** = *Indag. Math.* **24** (1962), 266–282, 283–290.

Westermann, L. R. J. 2365c

Hinreichende Bedingungen für analytische, harmonische und subharmonische Funktionen. VI, VII, VIII, IX.

Nederl. Akad. Wetensch. Proc. Ser. A **66** = *Indag. Math.* **25** (1963), 97–109, 110–120, 328–336, 337–348.

This is a continuation of the development, begun in Part I of this sequence [by Ridder and Westermann, same Proc. **64** (1961), 188–196; MR **24** #A2655], of refinements of the Looman-Menchoff theorem and its counterparts for harmonic and subharmonic functions. The results (which are too extensive to summarize here) are obtained by a combination of analytical and geometrical methods.

M. G. Arsove (Seattle, Wash.)

Skof, Fulvia 2366

Sull'andamento delle somme parziali delle serie di potenze con integrale assolutamente convergente.

Boll. Un. Mat. Ital. (3) **18** (1963), 405–419.

Let $f(z)=\sum_0^{\infty} a_k z^k$ converge in $|z|<1$ and set $f_n(z)=\sum_0^{n} a_k z^k$. The author is concerned with functions $f(z)$ for which $\sum_0^{\infty} |a_k|/(k+1)$ is convergent and $|f(z)|$ remains bounded along a set Θ of radii $(0, e^{i\theta})$ of $|z|=1$. Following the ideas of L. Fejér [S.-B. Bayer. Akad. Wiss. Math.-Phys. Kl. **1910**, 3. Abh.], the author first proves that if $|f(re^{i\theta})| \leqq K(\theta)$, where $\theta \in \Theta$, for $0 \leqq r < 1$, then

$$(*) \qquad |f_n(e^{i\theta})| \leqq 2K(\theta)+\left[\sum_0^n k|a_k| \cdot \sum_{n+1}^{\infty} |a_k|/k\right]^{1/2}$$

for $n \geqq n_0(f)$, independent of θ. Cases in which the coefficients are subject to the growth condition $|a_n| \leqq \psi(n)/n$, under hypotheses concerning $\psi(t)>0$ for $t \geqq 0$, are then discussed. These latter regulate the behaviour of $\psi(t)$, the two simplest cases being $\psi(t)$ monotonic and $\psi(t)$ slowly oscillating in such a way that, for every $c>0$, $\psi(ct) \sim \psi(t)$. The hypotheses are of a fairly general character and result in estimates of the second term in (*) of the form $c\Psi(n)$, where $c=c(\psi)$ and $\Psi(n)=\{\int_1^{n+1} \psi(t)\,dt \cdot \int_{n+1}^{\infty} (\psi(t)/t^2)\,dt\}^{1/2}$.

R. Wilson (London)

Suprun, N. N. 2367

On the reconstruction of an analytic function from the values of its generalized Gel'fond derivatives at a point. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 806–809.

Let $f(z)=\sum_{k=0}^{\infty} a_k z^k$, $a_0=1$, $a_k \neq 0$, $\limsup_{k\to\infty} \sqrt[k]{|a_k|}=1/r$,

$0<r<\infty$, and $\liminf_{k\to\infty} \sqrt[k]{|a_k|}>0$, be some fixed analytic function. Let $F(z)$ be analytic in a circle of finite radius with center at $z=0$. Represent $F(z)$ in the form $F(z)=\sum_{k=0}^{\infty} a_k b_k z^k$, where $\limsup_{k\to\infty} \sqrt[k]{|b_k|}=1/\beta$, $\beta>0$. The Gel'fond derivative of order n is defined by the relation $D_f^n F(z)=\sum_{k=0}^{\infty} b_{n+k} a_k z^k$, $n=0, 1, 2, \cdots$. This function is regular in the circle $|z|<\beta r$.

Let $\{A_n\}_{n=0}^{\infty}$ be an arbitrary sequence of complex numbers such that $1/\limsup_{n\to\infty} \sqrt[n]{|A_n|}=\beta>0$, and let z_0 be a point in the circle $|z|<\beta r_1$, $r_1=\min\{\rho, r\}$, ρ being the distance from the origin to the nearest zero of $f(z)$. The author shows that there is a unique function $F(z)$ analytic in $|z|<\beta r$ for which $D_f^n F(z_0)=A_n$, $n=0, 1, \cdots$, and in this circle that $F(z)$ can be expanded in a generalized power series $F(z)=\sum_{k=0}^{\infty} A_k (z, z_0)_f^k$, where $(z, z_0)_f^n$ is defined by the expansion

$$\phi(\zeta) = f(\zeta z)/f(\zeta z_0) = \sum_{n=0}^{\infty} (z, z_0)_f^n \zeta^n.$$

By defining $D_f^{-n}F(z)$ in a natural way the author generalizes the above result to Laurent-type expansions.

W. C. Royster (Lexington, Ky.)

Szekeres, G. 2368

Fractional iteration of entire and rational functions.

J. Austral. Math. Soc. **4** (1964), 129–142.

The author considers the fractional iterates of analytic functions $f(z)$ which have a fixpoint ζ with multiplier 1 ($f(\zeta)=\zeta, f'(\zeta)=1$). If n is a non-negative integer, then $f_n(z)$, the nth iterate of $f(z)$, is defined as follows: $f_0(z)=z$, $f_{n+1}(z)=f\circ f_n(z)$ ($f\circ g(z)\equiv f(g(z))$) and $f\circ f_{-1}(z)=f_{-1}\circ f(z)=z$; and, in general, for every s and t, $f_s\circ f_t(z)=f_t\circ f_s(z)=f_{s+t}(z)$ [I. N. Baker, same J. **2** (1961/62), 265–294, esp. p. 268; MR **25** #4082]. The following theorem is proved. Suppose that $f(z)$ is an entire or rational function which has a fixpoint ζ (finite or infinite). Suppose, furthermore, that $f(z)$ has a family of fractional iterates $f_s(z)$ with the property that for every real σ, $f_\sigma(z)$ is holomorphic at $z=\zeta$. Then $f(z)$ is a bilinear function of the form $f(z)=\zeta+(z-\zeta)/(1+a(z-\zeta))$ with $f_s(z)=\zeta+(z-\zeta)/(1+as(z-\zeta))$, or a linear function $f(z)=z+a$ with $f_s(z)=z+as$. This theorem extends the following result of Baker [loc. cit., p. 292]: If $f(z)$ is a nonlinear entire function of the form

$$f(z) = z+\sum_{k=m+1}^{\infty} a_k(z-\zeta)^k, \qquad a_{m+1}\neq 0 \quad (m \geqq 1),$$

then there can only be countably many s for which the fractional iterates

$$f_s(z) = z+\sum_{k=m+1}^{\infty} a_k^{(s)}(z-\zeta)^k, \qquad a_{m+1}^{(s)} = sa_{m+1},$$

are entire functions. *S. M. Shah* (Lawrence, Kans.)

Baker, I. N. 2369

Fractional iteration near a fixpoint of multiplier 1.

J. Austral. Math. Soc. **4** (1964), 143–148.

G. Szekeres [#2368 above] raised the question whether his theorem remains true for meromorphic functions, or more generally, for single-valued analytic functions. It is proved in this paper that if $f(z)$ is a meromorphic function, and if $f(z)$ has the expansion $f(z)=z+\sum_{m+1}^{\infty} a_k(z-\zeta)^k$, $a_{m+1}\neq 0$, $m\geqq 1$, and S is the set of those s-values for which the series

$$f_s(z) = z+\sum_{k=m+1}^{\infty} a_k^{(s)}(z-\zeta)^k, \qquad a_{m+1}^{(s)} = sa_{m+1},$$

has a positive radius of convergence, then S is the whole plane only when the function f has the form $f(z)=\zeta+(z-\zeta)/(1+a(z-\zeta))$. *S. M. Shah* (Lawrence, Kans.)

Kamthan, Pawan Kumar 2370

On entire functions of order (R) zero represented by Dirichlet series.

Math. Japon. **8** (1963), 17–22.

Let $f(s)$ be a nonconstant entire function represented by a Dirichlet series with exponents λ_n such that $\limsup (n/\lambda_n)=D$, $\liminf(\lambda_{n+1}-\lambda_n)=h\leqq 1/D$, and let $f(s)$ have Ritt order zero. As usual, $M(\sigma)$, $\mu(\sigma)$, $\lambda_{\nu(\sigma)}$ denote the maximum modulus on vertical lines, the maximum term, and the rank of this term. The author proves three results; the second and third each imply a known connection between μ and λ. (1) $\log M(\sigma)=O(\sigma^{\alpha-\beta})$ if and only if

$$e^\sigma \int_\sigma^\infty e^{-x}\lambda_{\nu(x)}\,dx = O(\sigma^{\alpha-\beta-1}),$$

$$2\leq \alpha<\infty,\ \alpha-2\leqq\beta\leqq\alpha-1.$$

(2) $$\limsup \{e^{-\varepsilon\sigma}/\lambda_{\nu(\sigma)}\}\int_\delta^\sigma e^{\varepsilon x}\lambda_{\nu(x)}\,dx = \infty, \quad \delta>0.$$

(3) $$\limsup e^{-\sigma}\int_\delta^\sigma L(e^x)\lambda_{\nu(x)}\,dx\Big/\int_\sigma^\infty L(e^x)e^{-x}\lambda_{\nu(x)}\,dx = \infty$$

for $L(x)$ continuous and nondecreasing with $x^{-1}L(e^x)\to 0$.

R. P. Boas, Jr. (Evanston, Ill.)

Kamthan, P. K. [Kamthan, Pawan Kumar] 2371

On the mean values of an entire function represented by Dirichlet series.

Acta Math. Acad. Sci. Hungar. **15** (1964), 133–136.

Let $f(s)=\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ ($s=\sigma+it$, $0<\lambda_1<\cdots<\lambda_n<\cdots$, $\limsup_{n\to\infty} n/\lambda_n<\infty$) represent an entire function and put $A(\sigma)=\lim_{T\to\infty}(1/2T)\int_{-T}^{T}|f(s)|^2\,dt$. It is proved that if $\log|a_{n-1}/a_n|/(\lambda_n-\lambda_{n-1})$ is increasing, then

$$\limsup_{\sigma\to\infty}\,[\liminf_{\sigma\to\infty}]\,\frac{1}{\sigma}\log\log A(\sigma) = \rho\ [\lambda],$$

$$\limsup_{\sigma\to\infty}\,[\liminf_{\sigma\to\infty}]\left(\frac{A(\sigma)}{m_k(\sigma)}\right)^{1/\sigma} = e^\rho\ [e^\lambda],$$

where ρ is the (R) order of $f(s)$, λ is the lower order of $f(s)$ and

$$m_k(\sigma) = \lim_{T\to\infty}\frac{1}{2Te^{k\sigma}}\int_0^\sigma\int_{-T}^{T}|f(x+it)|^2 e^{kx}\,dxdt.$$

These results are somewhat analogous to those of S. Mandelbrojt [*Séries lacunaires*, Actualités Sci. Indust., No. 305, Hermann, Paris, 1936] and Q. I. Rahman [Proc. Amer. Math. Soc. **10** (1959), 213–215; MR **23** #A299]. *A. Dvoretzky* (Jerusalem)

Srivastava, R. K. 2372

On the coefficients of an integral function represented by Dirichlet series of finite order.

Gaṇita **13** (1962), 25–35.

Let $f(s)$ be an entire function represented for every s by the Dirichlet series $\sum a_n e^{\lambda_n s}$ (λ_n is positive and strictly increasing, $\lim \lambda_n = \infty$), and suppose that $f(s)$ has finite non-zero Ritt order ρ. The author defines in the obvious way the type T and lower type with respect to $V(\sigma) = e^{\rho\sigma}\sigma^{\alpha_1}(l_1\sigma)^{\alpha_2}\cdots(l_{k-1}\sigma)^{\alpha_k}$, where $l_k\sigma$ is the kth iterate of $\log \sigma$. His results characterize in certain cases the type T and lower type in terms of the coefficients a_n. The result for the type (Theorem 1, part (iii)) is that if, in the usual notation, $\log \mu(\sigma) \sim \log M(\sigma)$ as $\sigma \to \infty$, then

$$T = \limsup \left(\frac{\rho^{\alpha_1-1}\lambda_n}{eu(\lambda_n)} \cdot |a_n|^{\rho/\lambda_n} \right),$$

where

$$u(\lambda_n) = (l_1\lambda_n)^{\alpha_1}\cdots(l_k\lambda_n)^{\alpha_k}.$$

A parallel result is given for the case of zero Ritt order.

M. E. Noble (Canterbury)

Srivastava, R. S. L. 2373

On the maximum term of an integral function defined by Dirichlet series.

Ganita **13** (1962), 75–86.

Further elementary results are obtained concerning the maximum term $\mu(\sigma, f^{(n)})$ and central index $\nu(\sigma, f^{(n)})$ of the derivatives of an entire function $f(s)$ represented by an everywhere absolutely convergent Dirichlet series $\sum a_n e^{\lambda_n s}$ (λ_n is non-negative and strictly increasing, $\lim \lambda_n = \infty$). A typical result is that if the upper and lower Ritt orders are positive and finite, $\varepsilon > 0$ and $\sigma > \sigma_0(\varepsilon)$, then

$$\sigma^{-1} \log \mu(\sigma, f^{(n)}) < (1 - \lambda/\rho + \varepsilon)\frac{\mu'(\sigma, f)}{\mu(\sigma, f)}$$

and

$$\frac{\log \mu(\sigma, f^{(n)})}{\log \lambda_\nu(\sigma, f)} < (1/\lambda - 1/\rho + \varepsilon)\frac{\mu'(\sigma, f)}{\mu(\sigma, f)}.$$

M. E. Noble (Canterbury)

Ionin, V. K. 2374

On a disk embedded in a multiply connected domain. (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. **169** (1963), 8–12.

Let K be a multiply connected domain enclosing an area equal to A, and let its boundary be composed of n non-intersecting closed simple Jordan curves, the sum of whose lengths is equal to L. It is shown that there is a circle with radius $R \geq 2F[L + (L^2 + 4\pi(n-2)F)^{1/2}]$ contained in K. The major part of the paper is devoted to the proof of this result. A second result is stated. If in the unit circle there are chosen n fixed points, then there is a circle of radius $R \geq 1/(1+\sqrt{n})$ belonging to the unit circle and having none of the chosen points in its interior.

W. C. Royster (Lexington, Ky.)

Mitjuk, I. P. 2375

The symmetrization principle for multiply connected domains. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 268–270.

The author proves the following theorem. Let $w = f(z) = w_0 + a_p(z - z_0)^p + a_{p+1}(z - z_0)^{p+1} + \cdots$ be regular in a domain G of the z-plane having inner radius r at z_0, and let the roots of the equation $f(z) = w_0$ in G be z_0 and other points z_k, $k = 1, 2, \cdots$, with multiplicities p_k. Let $g_G(z, \xi)$ be the Green's function of G and let $R = r(G_f, w_0)$ be the inner radius of the image domain G_f at w_0 or of a domain G_f^* obtained from G_f by symmetrizing with respect to a ray or line through w_0. Then

$$R \geqq |a_p| r^p \exp \sum p_k g_G(z_0, z_k).$$

Equality is possible only if $f(z)$ gives an $[m, 1]$ map of G onto G_f. The special case when G is the unit disk, $p = 1$, and there are no points z_k, is treated by the reviewer [e.g., *Multivalent functions*, Chapter 4, Cambridge Univ. Press, Cambridge, England, 1958; MR **21** #7302].

W. K. Hayman (London)

Mitjuk, I. P. 2376

A generalized reduced module and some of its applications. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 2 (39), 110–119.

For a definition of module and reduced module see Jenkins [*Univalent functions and conformal mapping*, Springer, Berlin, 1958; MR **20** #3288]. Let Γ be a non-degenerate boundary component of the multiply connected domain G_z. Consider the class $M(G_z)$ of functions $f(z)$, regular and univalent in G_z, satisfying the conditions $|f(z)| < 1$, $z \in G_z$; $f(z_0) = 0$, $z_0 \in G_z$; $f'(z_0) > 0$; $|f(z)| = 1$ on the outer boundary component of G_z. It has been shown earlier by the author [Dopovidi Akad. Nauk Ukraïn. RSR **1961**, 1115–1118; MR **24** #A2660] that there is a unique extremal function f_0 in $M(G_z)$ maximizing $|f(z_0)|$. The map of G_z under such an extremal function is called an extremal domain of type K. Particular extremal domains of type K are considered. Several inequalities concerning the reduced modules of these domains are obtained. Finally, the author applies these results to univalent functions in G_z, obtaining bounds on the product of the derivatives at a point in G_z of certain functions univalent in G_z when G_z is of type K. These results are similar to results of Alenicyn [Mat. Sb. (N.S.) **39 (81)** (1956), 315–336; MR **18**, 292].

W. C. Royster (Lexington, Ky.)

Nosenko, O. S. 2377

On the range of certain functionals. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1303–1307.

Let $S^0(\alpha)$ be the class of normalized univalent functions in $|z| < 1$, and convex of order $1 - \alpha$ as defined by Robertson [Ann. of Math. (2) **37** (1936), 374–408]. The author proves that for fixed z the domain of values for functions in the class $S^0(\alpha)$ is the closure of the region bounded by the curve $\zeta = z/(1 - ze^{i\tau})^{2\alpha}$, $-\pi \leqq \tau \leqq \pi$. A similar result is obtained for the first derivatives of these functions.

A. W. Goodman (Tampa, Fla.)

Nosenko, O. S. 2378

On the domain of values of the derivative of a function convex and univalent outside the unit circle. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1001–1005.

If $\Phi(z)$ satisfies the conditions set forth in the title, then $|1-\Phi'(z)| \leq 1/|z|^2$, and this result is sharp.
A. W. Goodman (Tampa, Fla.)

Ėzrohi, T. G. 2379
On certain classes of p-valent functions. (Russian)
Ukrain. Mat. Ž. **16** (1964), 558–568.
The author continues his study of the classes L_p, R_p^0, and Q_p which were introduced in an earlier paper [Ann. Univ. Mariae Curie-Skłodowska Sect. A **16** (1962), 137–149; MR **29** #1326]. *A. W. Goodman* (Tampa, Fla.)

Krzyż, Jan 2380
Some remarks concerning my paper: "On univalent functions with two preassigned values". (Polish and Russian summaries)
Ann. Univ. Mariae Curie-Skłodowska Sect. A **16** (1962), 129–136 (1964).
In this note the author corrects an oversight that has appeared in an earlier note [same Ann. **15** (1961), 57–77; MR **25** #2183]; this oversight does not vitiate the main result contained in the cited paper.
M. Reade (Ann Arbor, Mich.)

Privalova, G. K.; Saran, L. A. 2381
Some extremal problems for a class of functions which is star-shaped in the direction of the real axis. (Russian)
Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 4 (41), 126–130.
The class S_R of functions starlike in the direction of the real axis was studied first by Robertson [Amer. J. Math. 58 (1936), 465–472]. Using the known structural formula for functions in S_R, the authors find sharp bounds for $\operatorname{Re} f(z)$, $\operatorname{Im} f(z)$ and $|f(z)|$ when $f\in S_R$. The details and formulas are too involved to exhibit in this review.
W. C. Royster (Lexington, Ky.)

Marcus, Moshe 2382
Transformations of domains in the plane and applications in the theory of functions.
Pacific J. Math. **14** (1964), 613–626.
A class of methods of symmetrization applicable to star-shaped domains was introduced by Szegö [Ann. Mat. Pura Appl. (4) **40** (1955), 113–119; MR **17**, 1074], and the effects on capacity were studied. To each integer $n\geq 2$ there is a method in the class which yields a star-shaped domain with n-fold symmetry. The author shows how these methods of symmetrization can be applied to general domains, and to condensers as defined by Hayman [*Multivalent functions*, Cambridge Univ. Press, Cambridge, England, 1958; MR **21** #7302], and then solves several extremal problems. The essential difficulties which arise in the generalization are clearly pointed out and dealt with in a precise way. A typical result is as follows. Let n be an integer exceeding 1. If $f(z)$ is analytic for $|z|<1$ and if D is the set of values assumed by $f(z)$, then $|f'(0)|$ has as an upper bound a quantity associated with the symmetrized domain having n-fold symmetry, and the case of equality can be completely discussed.
J. L. Ullman (Ann Arbor, Mich.)

Dinghas, Alexander 2383
Zur Abschätzung von Dirichlet-Normen bei Randwertaufgaben in Riemannschen Räumen.
Norske Vid. Selsk. Skr. (Trondheim) **1963**, *no.* 3, 11 *pp.*
Es sei C^{n-1} ein Riemann'scher Raum von konstanter Krümmung K der Dimension $n-1$, A ein "regulärer" Körper in C^{n-1} und S_A die Klasse der auf A zweimal stetig differenzierbaren und auf dem Rande verschwindenden Funktionen, ohne die Konstante 0. Es wird für den Quotienten $D_\alpha(v)/J_\alpha(v)$, $v\in S_A$ eine untere Schranke angegeben. Dabei ist

$$D_\alpha(v)=\int_A |\nabla v|^\alpha\,dS,\qquad J_\alpha(v)=\int_A |v|^\alpha\,dS,$$

$\alpha\geq 1$ und dS das Volumenelement. Ein wesentlicher Punkt der Methode besteht im Beweis des folgenden Resultats: Ist dx das Längenelement auf einer Achse X, $d\sigma$ das Längenelement in C^{n-1} und Z^n das kartesische Produkt $C^{n-1}\times X$ mit der Metrik $ds^2=d\sigma^2+dx^2$, so wird bei Schwarz'scher Symmetriesierung (Abrundung) (wie es für $K=0$ bekannt ist) die Oberfläche verkleinert. Einige der Betrachtungen werden auch für Riemann'sche Räume nicht-konstanter Krümmung durchgeführt. *A. Pfluger* (Zürich)

Mennicke, Jens 2384
Eine Bemerkung zu J. Nielsens Theorie der topologischen Abbildungen geschlossener orientierbarer Flächen.
Math. Z. **83** (1964), 37–40.
The author considers the Klein model H of the hyperbolic plane based on the conic $x^2-3y^2-3z^2$ (x, y, z being homogeneous coordinates), and shows that a closed orientable surface of genus $h\geq 2$ can be realized as H/G_h, where G_h is a group of non-Euclidean translations such that the set of axes of these translations coincides with the set of rational straight lines. The construction for $h=2$ is explicit (and uses the results of the author's previous paper [Proc. Glasgow Math. Assoc. **5** (1961), 49–66; MR **28** #5114]). For $h>2$, G_h is obtained as a normal subgroup of G_2 such that $G_h/G_2\approx\pi_{h-1}$. *L. Bers* (New York)

Earle, Clifford J. 2385
Teichmüller spaces of groups of the second kind.
Acta Math. **112** (1964), 91–97.
A normalized Fuchsian group G is a discontinuous group of conformal self-mappings of U (upper half-plane) with limit points at 0, 1, and ∞. G is of the first [second] kind if its limit set is dense [nowhere dense] on the real axis. Let G be a normalized Fuchsian group of the second kind. If D is the region of discontinuity of G, D/G can be considered as a Riemann surface and can be represented by U/H, where H is a group of the first kind. The symmetry of D induces a symmetry in H. This symmetry induces an involution of the Teichmüller space $T(H)$ of H. The author shows that the part of $T(H)$ invariant under the involution has a natural real-analytic structure and may be interpreted as the space of quasiconformal deformations of G. *L. Keen* (Princeton, N.J.)

Sario, Leo 2386
An integral equation and a general existence theorem for harmonic functions.
Comment. Math. Helv. **38** (1964), 284–292.

The author shows that on an arbitrary Riemann surface a Fredholm integral equation can be used to construct a harmonic function, p, that has assigned singularities and that imitates the behavior of a given harmonic function, s, in a neighborhood of the ideal boundary. Upper and lower bounds are given for the deviation of p from s, bounds which depend only on the behavior of s on a compact set. These bounds prove useful in general value-distribution theory as developed by the author in articles listed in the bibliography.

R. Accola (Providence, R.I.)

Kruškal', S. L. 2387

The method of variations in the theory of quasiconformal mappings of closed Riemann surfaces. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 781–783.

This is an extension of Belinskiĭ's work [Sibirsk. Mat. Ž. **1** (1960), 303–330; MR **28** #212] to quasiconformal mappings of a closed Riemann surface S of genus $g \geqq 2$ onto another S_1. S is represented as U/Γ, where $U = [|z| < 1]$ and Γ is a hyperbolic Fuchsian group. The mapping is defined by a quasiconformal self-mapping $f(z)$ of U, normalized by $f(0) = 0$, $f(1) = 1$, with complex dilatation $h = f_{\bar{z}}/f_z$ satisfying $|h| \leq$ const (> 1) and

$$h(Az)A'(z)^*/A'(z) = h(z)$$

for $A \in \Gamma$ (here $*$ denotes complex conjugation). Thus $S_1 = U/\Gamma_1$, $\Gamma_1 = f\Gamma f^{-1}$. The essential content of Theorem 1 can be rephrased as follows. If $h(z) = th_0(z)$, $0 \leqq t \leqq 1$, then at $t = 0$

$$(1)\quad \frac{\partial f}{\partial t} = \frac{z(1-z)}{\pi} \iint_{U/\Gamma} \left\{ \sum_{A\in\Gamma} \frac{h(\zeta)A'(\zeta)^2}{A(\zeta)(1-A\zeta)(z-A\zeta)} + \left(\frac{h(\zeta)A'(\zeta)^2}{A\zeta(1-A\zeta)(1-z^*A\zeta)} \right)^* \right\} d\xi d\eta.$$

Actually the author first gives a special case of (1), and Theorem 2 is a formula for $\partial f/\partial t$ for $t > 0$, resulting from (1). {The reviewer notes that (1) also follows easily from the results in Ahlfors and the reviewer [Ann. of Math. (2) **72** (1960), 385–404; MR **22** #5813].}

Now let f be subject to the condition $|h| \leqq k < 1$, let $z_1, \cdots, z_k$ be fixed points and Φ a real-valued function of $2k$ complex variables, and set

$$\Phi[f] = \Phi(z_1, \cdots, z_k, f(z_1), \cdots, f(z_k)).$$

Theorem 2 asserts that the problem $\Phi[f] = \min$ has a solution f such that $|h(z)| = \text{const} = k$, and $\arg(f_{\bar{z}}^{-1}/f_z^{-1})$ is given by an explicit formula involving the (unknown) group $\Gamma_1 = f\Gamma f^{-1}$. Theorem 3 deals with a simple generalization of this problem.

It is stated that one can find explicitly a complex Beltrami basis on S (in the sense of the reviewer [Proc. Internat. Congr. Math. 1958, pp. 349–361, Cambridge Univ. Press, New York, 1960; MR **23** #A1796]) such that, in the first approximation, each basis element corresponds to varying precisely one of the $3g-3$ moduli of S (in the sense of Rauch [Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 42–49; MR **17**, 251] and Ahlfors [*Analytic functions*, pp. 45–66, Princeton Univ. Press, Princeton, N.J., 1960; MR **23** #A1798]), and that this, together with the formulas in Ahlfors [J. Analyse Math. **9** (1961/62), 161–176; MR **25** #192] leads to explicit expressions for quantities characterizing the curvature properties of the Teichmüller space T_g. No details or proofs are given.

L. Bers (New York)

Ikegami, Teruo 2388

On the theorems of Constantinescu-Cornea.

Proc. Japan Acad. **40** (1964), 196–199.

The author points out that the fine cluster set [Naïm, Ann. Inst. Fourier (Grenoble) **7** (1957), 183–281; MR **20** #6608] at a minimal Martin boundary point of an analytic function from a hyperbolic Riemann surface to a second Riemann surface is the same as a certain cluster set defined by Constantinescu and Cornea [Nagoya Math. J. **17** (1960), 1–87; MR **23** #A1025]. This fact has also been pointed out by the latter authors [*Ideale Ränder Riemannscher Flächen*, Springer, Berlin, 1963; MR **28** #3151]. The author then discusses some properties of the above described analytic functions tied closely to the properties of the functions near the boundaries of their domains.

J. L. Doob (Urbana, Ill.)

Accola, Robert D. M. 2389

On semi-parabolic Riemann surfaces.

Trans. Amer. Math. Soc. **108** (1963), 437–448.

A bordered Riemann surface $\overline{W}$ is semi-parabolic, denoted $\overline{W} \in SO_g$, if its double is parabolic. This paper treats two aspects of semi-parabolic surfaces.

The first is concerned with the behavior of exact harmonic differentials $du \in \Gamma_{ho}(W)$. It turns out that for such W and u the level set $\{u = t\}$ consists of closed curves for almost all values t. The orthogonal trajectories also behave nicely. Let u be harmonic on W, $\overline{W} \in SO_g$, and γ a compact connected subset of $\partial\overline{W}$. Assume u is harmonic on $W \cup \gamma$, positive on W, and $u = 0$ on γ. Parametrize γ by $[0, a]$ corresponding to a branch of u^*. For each $s \in [0, a]$ let $l(s)$ be the orthogonal trajectory of u issuing from the point on γ corresponding to s. Then $l(s)$ is relatively compact on $\overline{W}$ for almost all s provided u either is bounded or Dirichlet bounded. In case u is a generalized harmonic measure in the sense of Heins, stronger conclusions are obtained. The methods used here also yield results for arbitrary Riemann surfaces. For example, it is shown that if g is a Green's function with fixed pole on an arbitrary surface, then the components of the level set $\{g = \lambda\}$ are compact for almost all λ.

On a finite Riemann surface the differential of a harmonic measure is the period reproducer for a dividing cycle. How this fact generalizes to semi-parabolic surfaces is the second topic of the paper. Some useful integration formulas are established in this connection. Sufficient conditions are found for a differential in Γ_h to be a weak limit of period reproducers. The author has also included some open questions and conjectures.

B. Rodin (La Jolla, Calif.)

Smart, John Roderick 2390

Parametrization of automorphic forms for the Hecke groups $G(\sqrt{2})$ and $G(\sqrt{3})$.

Duke Math. J. **31** (1964), 395–403.

In this paper the author considers automorphic forms for the Hecke groups $G(\sqrt{n})$ for $n = 2$ and 3. ($G(\sqrt{n})$ is the discontinuous group of linear fractional transformations

on the upper half-plane generated by the two transformations $z \to -1/z$ and $z \to z+\sqrt{n}$ [Hecke, Math. Ann. **112** (1936), 664–699].) For $n=2$ and 3 the Riemann surfaces associated to these groups are compact surfaces of genus zero, so the situation is rather similar to that arising in the case of the ordinary modular group $G(1)$. The multiplier systems for these two groups are determined explicitly, using the description of the characters for these two groups given by M. I. Knopp [Duke Math. J. **27** (1960), 497–506; MR **22** #5614], and explicit expansions are derived for all the automorphic forms for these two groups in terms of simple canonical forms.

R. C. Gunning (Princeton, N.J.)

Siegel, Carl Ludwig 2391

Bestimmung der elliptischen Modulfunktion durch eine Transformationsgleichung.

Abh. Math. Sem. Univ. Hamburg **27** (1964), 32–38.

If $Tz=(az+b)/(cz+d)$ is a linear fractional transformation, where a, b, c, d are coprime integers and $ad-bc=n>1$, then the elliptic modular function $j(z)$ and its transform $j(Tz)$ are related by an irreducible algebraic equation over the rationals, $\phi_n(j(Tz),j(z))=0$. In the present paper the author shows that, conversely, this transformation equation determines the elliptic modular function in the following sense. If the transformation T as above is a non-periodic elliptic transformation for which the integer n is square-free, and if $f(z)$ is a non-constant analytic function in an open neighborhood of the fixed point ζ of T in the upper half-plane such that

$$\phi_n(f(Tz),f(z)) = 0 \quad \text{and} \quad f(\zeta) = j(\zeta),$$

then $f(z)=j(Mz)$, where M is a linear fractional transformation with the same fixed points as T. In addition, the author investigates the extent to which the Taylor series coefficients of any such function $f(z)$ are determined recursively by the transformation equation ϕ_n itself.

R. C. Gunning (Princeton, N.J.)

Iwasaki, Koziro 2392

Note on the modular forms.

Proc. Japan Acad. **39** (1963), 333–337.

If $\sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ is a cusp form of weight k for the modular group, then it is known that the series

$$(*) \qquad \sum_{n=1}^{\infty} a_n(x/n)^{(k+r)/2} J_{k+r}(4\pi\sqrt{(xn)})$$

is Abel summable to $C(k,r)\sum_{0<n<x} a_n(x-n)^r$ for non-integral values $x>0$, where J_{k+r} is a Bessel function and $C(k,r)$ a suitable constant; general results of this form can be found in Ferrar [Compositio Math. **1** (1934), 344–360; ibid. **4** (1937), 394–405] and in Bochner [Ann. of Math. (2) **53** (1951), 332–363; MR **13**, 920]. In the present paper the author investigates the actual convergence of this series (*) in the cases $r>\frac{1}{2}$ and $r=0$.

R. C. Gunning (Princeton, N.J.)

Randol, Burton 2393

Generalized zeta-functions.

Ark. Mat. **5**, 101–111 (1964).

Let B be a (not necessarily commutative) Banach algebra with unit. Let $\beta(x)$ be a B-valued function of $x=(x_1,\cdots,x_k)\in E^k-\{0\}$ such that: (1) There exists a positive number N with $\beta(rx)=r^N\beta(x)$ for $r>0$ and $x\neq 0$; (2) $\beta(x)$ is C^∞ in $E^k-\{0\}$, i.e., all mixed derivatives exist in the strong topology; (3) for every $x\neq 0$, the spectrum of $\beta(x)$ is contained in $\mathbf{C}^1-\mathbf{R}_-$. These conditions for $\beta(x)$ extend well-known properties of the usual Riemann zeta-function and its generalizations as they were given by P. Epstein [Math. Ann. **56** (1903), 615–644] and S. Bochner [Ann. of Math. (2) **57** (1953), 32–56; MR **14**, 986]. The author proves further that several properties of these zeta-functions still hold for the more general case considered, also under suitable further assumptions for the Banach algebra B, as, e.g., commutativity and semi-simplicity of it.

A. Mallios (Athens)

Gol'dberg, A. A.; Levin, B. Ja. 2394

Entire functions which are bounded on the real axis. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 19–21.

Let $f(z)$ be an entire function of exponential type σ and bounded by A, B on the negative and positive axes, respectively. The authors establish the bound

$$|f(re^{i\phi})| \leqq B(A/B)^{|\phi|/\pi} e^{\sigma r \sin|\phi|}, \qquad |\phi| < \pi.$$

They also construct for each set of three positive numbers σ, A, B an entire function $f(z)$ such that

$$|f(re^{i\phi})| \sim B(A/B)^{\phi/\pi} e^{\sigma r \sin\phi}$$

as $r\to\infty$ uniformly for $0\leqq\phi\leqq\pi$, and as $r\to\infty$ outside a set of finite logarithmic measure for $-\pi<\phi<0$. The question of the exact bound for $f(z)$ with the above hypotheses was raised as one of the Cornell problems [Bull. Amer. Math. Soc. **68** (1962), 21–24, Problem 4].

W. K. Hayman (London)

Gol'dberg, A. A. 2395

Three examples of entire functions. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 443–446.

The author constructs entire functions f_1, f_2, f_3 of finite positive order ρ such that $\lim r^{-\rho}\log M(r)$ exists for f_1, but $\lim r^{-\rho}T(r)$ does not; for f_2, the opposite is the case; for f_3, both limits exist and are equal. The functions are as follows.

$$f_1(z) = \sum \eta_n z^n/\Gamma(\rho^{-1}n+1),$$

where η_n are defined by taking $r_1=1$, $r_{k+1}=6(\rho^2+1)r_k^{2/\rho}$; $\eta_{2k}=0$; $\eta_{2k+1}=1$ for $r_{2n-1}\leqq 2k+1<r_{2n}$; $\eta_{2k+1}=0$ for $r_{2n}\leqq 2k+1<r_{2n+1}$.

$$f_2(z) = F_1(z)+F_2(-z),$$

where

$$F_j(z) = (2\pi i)^{-1}\int_{L_r}\frac{\exp(t^\rho l_j(t))}{t-z}\,dt, \qquad r>|z|,$$

$$l_j(z) = z\int_0^\infty (t+z)^{-2}\lambda_j(t)\,dt,$$

$$\lambda_1(r) = 1+\eta+\sin\log\log(r+e) \qquad (\eta>0),$$

$$\lambda_2(r) = 2(1+\eta)-\lambda_1(r),$$

and L_r is the boundary of $\{|\arg z| < \pi/2\rho_1, \ |z| > r\}$, $\rho/3 < \rho_1 < \rho$.

$$f_3(z) = \prod (1 + (z/r_k)^{n_k}),$$

where $0 < q < \min(1, \rho)$, $r_k = k^{1/q}$, $n_k = [k^{p/q}] - [(k-1)^{p/q}]$. The necessary estimates for f_1 and f_2 are available from Evgrafov [*Asymptotic estimates and entire functions* (Russian), second revised edition, Fizmatgiz, Moscow, 1962; MR **27** #4924]. *R. P. Boas, Jr.* (Evanston, Ill.)

Haruki, Hiroshi 2396

On a certain family of meromorphic functions.

Proc. Japan Acad. **40** (1964), 88–93.

Let M be the class of functions $f(z)$ meromorphic in the plane for which $\phi(x, y) = f(z)$ satisfies one of the conditions C1: there exists a neighbourhood V where $\phi \neq 0$, $\phi_y \neq 0$ and $\phi_x/\phi_y = A(x)B(y)$, or C2: there exists a neighbourhood V where $\phi \neq 0$, $\phi_x \neq 0$, $\phi_x/\phi_y = A(x)B(y)$. A necessary and sufficient condition is given for $f(z)$ to belong to M, and the members of M are enumerated. Applications to various functional equations are given.

I. N. Baker (London)

Petrenko, V. P. 2397

Some estimates for the logarithmic derivative of a meromorphic function. (Russian. Armenian summary)

Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk **17** (1964), *no.* 1, 23–37.

The author proves: Let $g(z)$ be a meromorphic function of order ρ ($\leqq \infty$) and lower order λ, $\lambda < \rho$. Then, with the usual notations of the Nevanlinna theory,

$$\liminf r\{N(r, g) + N(r, 1/g)\}^{-1} \int_0^{2\pi} |g'(re^{i\theta})/g(re^{i\theta})| \, d\theta \leqq A \min_{\lambda \leqq x \leqq \rho} x^2 |\operatorname{cosec} \pi x|.$$

Here (and below) A denotes a positive number independent of ρ and λ. The result remains true for $\lambda = \rho$, ρ not an integer. (For the case $\lambda \leqq \rho < 1$ the result was proved by I. V. Ostrovskiĭ and I. V. Kazakova [Zap. Meh.-Mat. Fak. Har'kov. Mat. Obšč. **30** (1963), 73–78].) Following a known method [the reviewer, Ann. of Math. (2) **68** (1958), 203–209; MR **20** #7106], the author derives $\sum_a (\delta(a, g))^{1/2} < A \min_{\lambda \leq x \leq \rho} x |\operatorname{cosec} \pi x|$. Also

$$\sum \delta(a, g) < A \min_{\lambda \leqq x \leqq \rho} x^2 |\operatorname{cosec} \pi x|$$

if g has at least two deficient values.

W. H. J. Fuchs (Ithaca, N.Y.)

Selberg, Henrik L. 2398

Über einen Satz von Picard.

Norske Vid. Selsk. Forh. (Trondheim) **36** (1963), 145–149.

Hier handelt es sich um den Satz, dass zwischen zwei meromorphen Funktionen keine algebraische Gleichung von Geschlecht > 1 bestehen kann. Dies folgt ziemlich rasch aus der Nevanlinna'schen Wertverteilungslehre. Verfasser hat früher [Avh. Norske Vid.-Akad. Oslo I Mat.-Naturvid. Kl. **1934**, no. 8] jedoch bemerkt, dass man nur nachzuweisen braucht, dass noch ein erster Hauptsatz bestehen bleibt, wenn man die in Frage stehenden meromorphen Funktionen f mit einer algebraischen Funktion $w \to W$ zusammensetzt, also $F(z) = W(f(z))$ betrachtet.

Dieser Nachweis wird hier auf völlig elementare Weise gegeben. *A. Pfluger* (Zürich)

Tanaka, Chuji 2399

On Blaschke products in the unit-circle.

Mem. School Sci. Engrg. Waseda Univ. Tokyo No. 27 (1963), 67–77.

The author establishes a number of results concerning Blaschke products and functions of class U. Some of the results are new, but proofs of previously announced theorems [the author, Proc. Japan Acad. **39** (1963), 141; MR **27** #2632] and of previously proved theorems [ibid. **39** (1963), 410–412; MR **28** #1308] are included in the paper.

Theorem 1 gives three necessary and sufficient conditions for the zeros a_n ($n = 1, 2, \cdots$) of a Blaschke product B to satisfy (*) $\sum_{n=1}^{\infty} (1 - |a_n|)/|e^{i\theta} - a_n| < \infty$, the most interesting of which is that B is absolutely convergent at $e^{i\theta}$. The author then uses a theorem of O. Frostman [Kungl. Fysiogr. Sällsk. i Lund Förh. **12** (1942), no. 15, 169–182; MR **6**, 262], which asserts that (*) is a necessary and sufficient condition for B and all of its subproducts to have radial limits of modulus one at $e^{i\theta}$, to show (by essentially repeating Frostman's proof) that (Theorem 3) if B is absolutely convergent at $e^{i\theta}$, then B has a radial limit at $e^{i\theta}$ which is equal to the value of the product at that point. Theorem 2 states that if $e^{i\theta}$ is not a limiting point of $\{a_n\}$, then B converges absolutely and uniformly to a regular function in some neighborhood of $e^{i\theta}$. {This result is well known; see, e.g., K. Hoffman [*Banach spaces of analytic functions*, p. 68, Prentice-Hall, Englewood Cliffs, N.J., 1962; MR **24** #A2844].} Using a modification of Theorem 2 in conjunction with theorems of Iversen and Lindelöf, the author proves that (Theorem 5) if $e^{i\theta}$ is an isolated limiting point of $\{a_n\}$ and if infinitely many zeros are contained in some Stolz angle at $e^{i\theta}$, then B assumes every complex value infinitely many times in each neighborhood of $e^{i\theta}$. Theorem 6, which involves an analogue of Milloux's "cercles de remplissage", is based on an existence theorem due to the reviewer. Theorem 7, which follows from a theorem of F. Riesz, asserts that B is radially and continuously prolongable into $|z| > 1$ almost everywhere on $|z| = 1$ (in the sense that $\lim_{r \to 1-} B(re^{i\theta}) = \lim_{s \to 1+} B(se^{i\theta})$).

The foregoing theorems are translated into assertions about functions of class U by means of a theorem of O. Frostman which states that if $f \in U$, then, for each α in the unit disk except for a set of exceptional values of capacity zero, $(f(z) - \alpha)/(1 - \bar{\alpha} f(z))$ is equal to the Blaschke product extended over the α-points of f. The paper closes by sketching the proofs of four theorems concerning the class $U(\theta_1, \theta_2)$.

{Reviewer's notes: In connection with Theorems 1, 5, and 9, there are a number of papers in the literature to which the reader might have been referred, e.g., W. Seidel [Trans. Amer. Math. Soc. **36** (1934), 201–226], M. Ohtsuka [J. Math. Soc. Japan **2** (1950), 1–15; MR **13**, 336], A. J. Lohwater [Ann. Acad. Sci. Fenn. Ser. A I No. 229 (1956); MR **18**, 648], and the reviewer [Duke Math. J. **30** (1963), 143–149; MR **26** #2616]. Theorem 9 apparently was first proved by M. Ohtsuka [loc. cit.]. The

reviewer was unable to follow the proofs of Theorem 4 and the last part of Theorem 1, where evidently it is assumed that if $\{a_n\}$ is a sequence of points in the open unit disk and $e^{i\theta}$ is a point on the unit circle, then $\{a_n\}$ can be decomposed into two complementary subsequences in such a way that one subsequence converges to $e^{i\theta}$ and the other does not have $e^{i\theta}$ as a limiting point. Nevertheless, these theorems are true; in fact, Theorem 4 is essentially Lemma 2 of the reviewer's paper [J. Math. Anal. Appl. **5** (1962), 1–16; MR **25** #5182].}

G. T. Cargo (Syracuse, N.Y.)

Arakeljan, N. U. 2400

Uniform and asymptotic approximation by entire functions on unbounded closed sets. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 9–11.

Appelons un ensemble E fermé dans le plan un K-ensemble s'il existe une fonction non négative $r_E(t)\nearrow\infty$ lorsque $t\nearrow\infty$ telle que chaque point $z\in \complement E$ peut être joint à l'infini par une courbe de Jordan située dans $\complement E$ et en dehors du disque $|\zeta|\leqq r_E(|z|)$. Le théorème 1 de cette note dit que toute fonction continue en chaque point de E (hormis $z=\infty$) et holomorphe à l'intérieur E^0 de E est limite uniforme sur E d'une suite de fonctions entières si et seulement si E est un K-ensemble (ou coïncide avec le plan).

Appelons un ensemble fermé E un ensemble de Carleman si toute fonction $f(z)$ continue en chaque point $z\neq\infty$ de E permet une approximation

$$(1)\qquad |f(z)-G(z)| < \varepsilon(|z|),\qquad z\in E,$$

$G(z)$ étant entière et $\varepsilon(t)$ étant assujétie seulement à la condition d'avoir une borne inférieure positive sur chaque intervalle fini. Le théorème 2 dit qu'un fermé E est un ensemble de Carleman si et seulement si (1°) E est maigre, (2°) E est un K-ensemble.

Dans le cas où E comporte des points intérieurs on a le théorème 3: Une fonction $f(z)$ continue sur E et holomorphe sur E^0 permet l'approximation (1) avec $G(z)$ entière et $\int_1^\infty \log\varepsilon(t)/t^{3/2}\,dt>-\infty$ si et seulement si E est un K-ensemble (ou coïncide avec le plan). Si

$$\int_1^\infty \log\varepsilon(t)/t^{3/2}\,dt = -\infty,$$

alors il existe un K-ensemble E et une fonction $f(z)$ holomorphe sur E telle que (1) est en défaut pour chaque fonction entière $G(z)$.

E. J. Akutowicz (Bologna)

Sabitov, I. H. 2401

A boundary-value problem with linear conjugation. (Russian)

Mat. Sb. (N.S.) **64 (106)** (1964), 262–274.

Let Γ denote the unit circle $|z|=1$, D^- its exterior, D^+ its interior. In an earlier paper [Sibirsk. Mat. Ž. **5** (1964), 124–129; MR **28** #3164] the author considered the problem of determining functions $\varphi^+(z)$, $\varphi^-(z)$, analytic respectively in D^+, D^-, with boundary values $\varphi^+(t)$, $\varphi^-(t)$, satisfying on Γ a relation of the form

$$(1)\quad \varphi^+(t) = a(t)\varphi^-(t)+b(t)\overline{\varphi^-(t)}+c(t),\qquad \varphi^-(\infty)=0,$$

where a, b, c are given and Hölder-continuous on Γ, $a\neq 0$. In the present paper the left member of (1) is replaced by $\varphi^+[\alpha(t)]$, where $\alpha'(t)$ is a non-vanishing Hölder-continuous function and $\alpha(t)$ maps Γ onto itself in a one-to-one, continuous, direction-preserving manner. The coefficients a, b are again taken Hölder-continuous; $c(t)\in L_p(\Gamma)$, $p>1$, $a(t)\neq 0$. The author derives estimates for the number of solutions of the corresponding homogeneous problem when $\operatorname{ind}_\Gamma a(t)$ lies in specified intervals.

J. F. Heyda (King of Prussia, Pa.)

Paatašvili, V. A. 2402

On a discontinuous problem in linear conjugation with summable coefficients. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **34** (1964), 519–524.

The author considers the Riemann boundary problem (*) $\phi^+(t)=G(t)\phi^-(t)+g(t)$, $t\in\Gamma$, where Γ is a simple, smooth, closed curve, the angular orientation of whose tangent line is a Hölder-continuous function of arc length. The coefficient $G(t)$ is assumed to belong to $L_r(\Gamma)$, $r>1$, and $[G(t)]^{-1}\in L_\infty(\Gamma)$; the function $g(t)\in L_{p+\varepsilon}(\Gamma)$, $p>1$, where ε is an arbitrary positive number. In addition, there is assumed to be a $\delta>0$ such that each $t\in\Gamma$ is included in a neighborhood of Γ for which the corresponding $G(t)$ points lie in a sector of opening angle

$$\beta = \min\{(2\pi-\delta)/\max(p,p'),\ (2\pi-\delta)/\max(p_1,p_1')\},$$

$p_1=2rp(p+r)^{-1}$, $p'=p/(p-1)$, $p_1'=p_1/(p_1-1)$. Under these conditions he then states and proves a theorem concerning the number of linearly independent solutions of (*) when the index of $G(t)$ is positive, negative or zero, respectively.

J. F. Heyda (King of Prussia, Pa.)

Pyhteev, G. N. 2403

Solution of a mixed boundary-value problem involving a shift, and inversion formulae for certain singular integral equations of the first kind. (Russian)

Sibirsk. Mat. Ž. **4** (1963), 845–861.

The author considers the following generalization of the Riemann-Hilbert problem. Let $a<d<c<b$, and let $\varphi(x)$ be a one-to-one mapping of $[c,b]$ onto $[a,d]$. The function $\Phi(x)$ is given in the intervals $(-\infty,a]$ and $[b,\infty)$, the function $\Psi(x)$ is given in the interval $[d,c]$, and γ is a given real constant. It is desired to find a function $F(z)$, analytic in the half-plane $\operatorname{Im} z>0$, continuous in $\operatorname{Im} z\geqq 0$, and bounded at infinity, such that

$$\begin{aligned} \operatorname{Re} F(x) &= \Phi(x) && (x\leqq a,\ x\geqq b),\\ \operatorname{Im} F(x) &= \Psi(x) && (d\leqq x\leqq c),\\ F[\varphi(x)] &= F(x)+2\gamma && (c\leqq x\leqq b). \end{aligned}$$

It is shown that, under appropriate conditions on the given functions, the problem can be reduced to the solution of a singular integral equation of the first kind for the function

$$\nu(x) = \operatorname{Re} F(x)-\Phi(b)\quad (c\leqq x\leqq b).$$

Regularization of this equation reduces it to a Fredholm equation possessing a unique solution. The details are too complicated to be described here.

The special cases where $\varphi(x)$ is a linear function or a quotient of two linear functions lead to elegant explicit inversion formulae for the solution of the singular integral equation.

F. Smithies (Cambridge, England)

Wend, David V. V. 2404
Branched regular curve families and finite asymptotic values.
Amer. J. Math. **86** (1964), 493–520.
The principal results of this paper rest upon the concept of b-curves belonging to a family of curves F, where F is a branched regular curve family filling the complex plane minus a set of branch points. Briefly, a curve b of F is a b-curve if (i) b is a regular curve or a maximal chain of F, and (ii) if there exists a regular curve or maximal chain c of F such that b and c form the complete boundary of a simply connected domain, say V, and such that no curve in the complete boundary of V has an endpoint on b or c and no regular curve or infinite chain separates b and c. Note that c is also a b-curve. This concept leads to (1) sufficient conditions that any analytic function with real or imaginary part contoured by a given family of curves have a certain number of distinct finite asymptotic values, (2) sufficient conditions that there exists an analytic function with no finite asymptotic values with real or imaginary part contoured by a given family of curves, and (3) elegant characterizations of the level curve families of harmonic polynomials.
P. Porcelli (Bloomington, Ind.)

Li, Zhong [Li, Chung]; Wen, Guo-chun [Wen, Kuo-chun] 2405
On the Cauchy formula for elliptic systems of linear partial differential equations of the first order.
Acta Math. Sinica **14** (1964), 23–32 (*Chinese*); *translated as Chinese Math.* **5** (1964), 25–35.
Relations generalizing the Cauchy integral formula to elliptic systems have been obtained by the reviewer [*Theory of pseudo-analytic functions*, Inst. Math. Mech., New York Univ., New York, 1953; MR **15**, 211; Comm. Pure Appl. Math. **9** (1956), 693–711; MR **19**, 861] and by Vekua [*Generalized analytic functions*, English transl., Pergamon, London, 1962; MR **27** #321]. The authors extend these results to the most general linear uniformly elliptic system which may be written, in complex notation, as

$$(1)\qquad w_{\bar z}-\mu w_z-\nu\bar w_z = aw+b\bar w+c.$$

They assume that μ and ν are measurable functions which are Hölder continuous near the boundary of the domain D considered, and satisfy $|\mu|+|\nu|\leqq \text{const}<1$, while a, b, c lie in $L_p(D)$ for some $p>2$. In order to obtain the desired Cauchy formulas, (1) is first transformed into an equation with $\nu\equiv 0$ (by a quasiconformal change of the independent variables) and then into one with $a\equiv b\equiv 0$ (by a change of the dependent variables corresponding to the transition from pseudo-analytic functions of the first kind to those of the second kind in the reviewer's theory).
L. Bers (New York)

Peschl, Ernst 2406
Über die Verwendung von Differentialinvarianten bei gewissen Funktionenfamilien und die Übertragung einer darauf gegründeten Methode auf partielle Differentialgleichungen vom elliptischen Typus.
Ann. Acad. Sci. Fenn. Ser. A I No. 336/6 (1963), 22 *pp.*
The author starts out with a type of "Verzerrung" theorem for conformal mappings of the open unit disc which implies the Bloch-Landau-Ahlfors theorem: the image of a conformal mapping $w=z+\cdots$ in $z\bar z<1$ contains a closed disc of radius $\frac{1}{2}-\varepsilon$ with arbitrarily small $\varepsilon>0$. The Beltrami operators with respect to the hyperbolic metric in $z\bar z<1$ are used in the proof. This leads to a differential equation $D_0[f]=0$,

$$D_0[f] \equiv -f(2+f'')+(f'+1)(f'+2),$$

$f=f(\alpha)$, $\alpha\in\mathbf{R}$, and then a modified maximum principle yields a crucial inequality involving a family of solutions for $D_0[f]=0$, finishing the proof of the theorem mentioned. The ideas and methods of this proof are carried over to several more general cases: (1) similar considerations are possible if the hyperbolic metric in $z\bar z<1$ and the euclidean metric in the w-plane are replaced by metrics $e^{2a}dzd\bar z$ and $e^{2b}dwd\bar w$; (2) it is proposed to replace $D_0[f]=0$ and related differential equations in the original case by more general (partial) differential equations of elliptic type; (3) computations are made in higher dimensions for Kähler-Einstein metrics (complex case) and for Einstein metrics (real case) which suggest analogous results. Not all of these generalizations are followed to the end, and the paper has definitely an expository and often summarizing character. *A. Aeppli* (Minneapolis, Minn.)

Alpár, László 2407
Sur une forme symétrique des équations de Beltrami. (Russian summary)
Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 141–150.
Formal manipulations. Example: If $W(z,y)$ and $Z(x,y)$ are complex-valued functions, then, under suitable hypotheses, $\partial W/\partial\bar Z=0$ may be written as $\partial(W,Z)/\partial(x,y)=0$. *L. Bers* (New York)

Virčenko, N. O. [Virčenko, N. A.] 2408
Some boundary-value problems for x-analytic functions. (Ukrainian. Russian and English summaries)
Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1577–1581.
Author's summary: "The author determines the connection between x-analytic and analytic functions of a complex variable for a half-strip. This makes it possible to solve in a closed form some mixed boundary problems of potential theory. As an example, the problem of definition of an x-analytic function in a half-strip $x>0$, $0<y<h$ is considered when the imaginary part of the x-analytic function is known on the lower part of boundary $y=0$ for $x<a$, and the real part of the unknown x-analytic function is known for $x>a$, the imaginary part being known also on the upper part of boundary $y=h$. The possibility of extending the results obtained to x^k-analytic functions (k is an odd number) is indicated."

Virčenko, N. O. [Virčenko, N. A.] 2409
An integral relation in the class of x^k-analytic functions. (Ukrainian. Russian and English summaries)
Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 734–736.
The author derives an integral relation connecting three x^k-analytic functions defined in a half-strip, and points out that this relation may be used for solving certain boundary-value problems in the class of x^k-analytic functions. A number of corrections are made to his earlier paper reviewed above [#2408].

Polozii, G. N. 2410

A fundamental integral representation for p-analytic functions with characteristic $p=x^k$. (Russian)

Ukrain. Mat. Ž. **16** (1964), 254–259.

The paper, a generalization of previous work by the author [for instance, same Ž. **15** (1963), 25–45; MR **28** #2670] contains various integral formulas representing x^k-analytic functions, $\tilde f(z)=\tilde u+i\tilde v$, i.e., solutions of the system $\tilde u_x=x^k\tilde v_y$, $\tilde u_y=-x^k\tilde v_x$, $k>0$, in terms of analytic functions $f(z)$. A typical result is Theorem 1: If G is a domain, the boundary of which contains a segment L of the imaginary axis, G^* the domain obtained by reflecting G about L, Γ an arc in $G\cup L\cup G^*$ joining a point a with $-\bar a$ and intersecting L, and if $f(z)$ is analytic in G and real on L, then

$$\tilde f(z)=\frac{1}{2}\int_\Gamma f(\zeta)\left[\left(\frac{z+\bar z}{2}\right)^{1-k}+\zeta-\frac{z-\bar z}{2}\right][(z-\zeta)(\bar z+\zeta)]^{(k-2)/2}\,d\zeta$$

is x^k-analytic in G and real on L. The correspondence between f and $\tilde f$ is one-to-one. *L. Bers* (New York)

Položiĭ, G. M. [Položiĭ, G. N.] 2411

Remarks on the fundamental integral representation of p-analytic functions with characteristic $p=x^k$. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 839–841.

Minor extensions of the results of the preceding paper [#2410]. *A. J. Lohwater* (Providence, R.I.)

Cullen, C. G. 2412

Analytic primary functions on quaternions.

Mem. School Sci. Engrg. Waseda Univ. Tokyo No. 27 (1963), 103–106.

Let $F(z)=U(x,y)+iV(x,y)$ be a function analytic in some open region of the complex plane and satisfying $F(\bar z)=\overline{F(z)}$, and let $F(\xi)$ be its extension to Q, the algebra of real quaternions.

Let $i_0=1$, i_1, i_2, i_3 be a basis for Q, with $i_1i_2=-i_2i_1=i_3$, $i_1{}^2=i_2{}^2=-1$, and let $\xi=\sum x_ji_j$ be written in the form $\xi=x_0+p\mu$, where $p=(x_1{}^2+x_2{}^2+x_3{}^2)^{1/2}$, and μ is a unit vector quaternion. The reviewer [Duke Math. J. **27** (1960), 1–19; MR **22** #11104] has shown that $f(\xi)=\sum_j f_j(x_0;x_1,x_2,x_3)i_j$ is of the form $U(x_0,p)+V(x_0,p)\mu$ and that $f(\xi)$ is Hausdorff-differentiable in its domain of definition, i.e., the Fréchet differential of $F(\xi)$ exists and can be written in the form (1) $dF(\xi)=\sum_{k=1}^m A_kd\xi B_k$, where the A_k, B_k are independent of $d\xi$ $(=\sum_{j=0}^3 dx_ji_j)$.

The principal result of the present paper is the determination of a simple form of the A_k, B_k; specifically, $dF(\xi)=\frac12(V/p+F'(\xi))+\frac12(V/p-F'(\xi))\mu d\xi\mu$, $F'(\xi)$ being the extension of $F'(z)$ to Q. The proof utilizes the Cauchy-Riemann equations and the algebraic properties of Q.

For general functions from Q to Q with differentiable component functions, Nisigaki [Mem. Fac. Sci. Engrg. Waseda Univ. No. 15 (1951), 6–14; MR **14**, 633; ibid. No. 16 (1952), 122–126; MR **14**, 633; Mem. School Sci. Engrg. Waseda Univ. Tokyo No. 18 (1954), 108–109; MR **16**, 583] has shown that m in (1) need not exceed 4 and has classified such functions into classes C_1, C_2, C_3, C_4 according to the number of terms required in (1). In the present paper the author shows by means of his result that Nisigaki's characterization of C_2 is in apparent error. The nature of the error is not identified.

R. F. Rinehart (Cleveland, Ohio)

Dragilev, M. M. 2413

The regular convergence of basis expansions of analytic functions. (Russian)

Naučn. Dokl. Vysš. Školy Fiz.-Mat. Nauki **1958**, *no.* 4, 27–32.

The reviewer has shown [Math. Ann. **135** (1958), 283–293; MR **20** #7216] that if $\{x_n\}$ is an absolutely convergent basis in a Fréchet space $\mathscr{U}$ and $\{y_n\}$ is a basis in a Fréchet space $\mathscr{V}$ similar to $\{x_n\}$, then $\{y_n\}$ is likewise absolutely convergent. Using different methods, the author establishes the special case of this result in which $\mathscr{U}$ and $\mathscr{V}$ are both the space of all analytic functions on the disc $|z|<R$ and $\{x_n\}$ is the fundamental basis consisting of successive powers of z. *M. G. Arsove* (Seattle, Wash.)

POTENTIAL THEORY

See also 2247, 2365a–c, 2386, 2388, 2517.

Meinhold, Peter 2414

Über ein verallgemeinertes Balayageprinzip.

Wiss. Z. Techn. Univ. Dresden **12** (1963), 1597–1600.

Let K be a compact subset of the compact set E, $M(K)$ and $M(E)$ the corresponding spaces of Radon measures, and let D be a linear subspace of $C(E)$, the continuous functions on E. Following Anger [same Z. **11** (1962), 417–426; MR **27** #332], a measure $\nu\in M(E)$ satisfies the (D,K)-balayage principle if there is a measure $\mu\in M(K)$ such that $\int h\,d\mu=\int h\,d\nu$ for all $h\in D$. The dual system $(C(E),M(E))$ is utilized to study the projection on a subspace of $M(E)$ determined by balayage. Let $J(K)=\{f\in C(E):f=0 \text{ on } K\}$. The projection is weakly continuous if and only if $\bar D+J(K)$ is closed in $C(E)$ relative to the topology of uniform convergence. This property is valid and $\bar D\cap J(K)=\{0\}$ if and only if each $\nu\in M(E)$ satisfies the (D,K)-balayage principle. Other criteria for balayage are given. Special cases include the balayage of potential theory and barycentric representation in a compact convex subset of a locally convex space.

Guy Johnson (Syracuse, N.Y.)

Kishi, Masanori 2415

On the uniqueness of balayaged measures.

Proc. Japan Acad. **39** (1963), 749–752.

L'auteur avait montré [Osaka Math. J. **13** (1961), 41–74; MR **26** #1486] que dans un espace séparé localement compact et localement séparable, pour un noyau continu $G>0$ régulier symétrique satisfaisant au principe du balayage, les principes réunis U (de détermination des masses par les potentiels, c'est-à-dire l'égalité des mesures si les potentiels sont égaux sauf sur un ensemble de mesure nulle pour toute mesure d'energie finie) et BU (unicité du balayage sur un compact) équivalent à ce que le noyau G soit non dégénéré (c'est-à-dire $G(x,y_1)/G(x,y_2)$ non constant pour y_1, y_2 fixés quelconques).

Il établit surtout ici le même théorème en remplaçant la symétrie par la régularité du noyau adjoint.

M. Brelot (Paris)

Constantinescu, C.; Cornea, A. 2416

On the axiomatic of harmonic functions. I, II.

Ann. Inst. Fourier (Grenoble) **13** (1963), *fasc.* 2, 373–388; *ibid.* **13** (1963). *fasc.* 2, 389–394.

Ces deux articles démontrent quelques propriétés intéressantes relatives à l'axiomatique de M. Brelot, en particulier: L'extension à cette axiomatique d'un théorème démontré par M. Brelot dans le cas classique: l'enveloppe inférieure des fonctions hyperharmoniques $\geqq f$ s.c.i. sur un ouvert X, est hyperharmonique, et continue en tout point où f est continue; en outre, si elle est surharmonique et $>f(x)$ au point x, elle est harmonique dans un voisinage de x. On en déduit qu'une fonction surharmonique $\geqq$ une fonction continue est l'enveloppe supérieure de ses minorantes surharmoniques finies continues.

S'il existe sur l'espace X une fonction surharmonique >0 et si X a localement une base dénombrable, alors X a une base dénombrable. Pour toute fonction s surharmonique $\geqq 0$ sur X, et tout ensemble $E\subset X$, $\hat{R}_s^E=s$ sur l'intérieur fin de E. On en déduit: $R_s^G=\hat{R}_s^G$ pour tout ouvert fin G; puis $R_s^E=\inf_{G\supset E} R_s^G$ pour E quelconque; enfin l'additivité de R_s^E et $\hat{R}_s^E$ par rapport à s.

R. M. Hervé (Nancy)

Sjöberg, Boris 2417

Übertragung einiger Sätze aus der Theorie der meromorphen Funktionen endlicher Ordnung auf die Potentialtheorie.

Acta Acad. Abo. Math. Phys. **23**, *no.* 7, 51 *pp.* (1963).

The author succeeds in carrying over to subharmonic and δ-subharmonic functions in n-dimensional space ($n\geqq 3$) the basic concepts and results of the theory of entire and meromorphic functions of finite order. Many of the techniques parallel those given by the reviewer for the case of $n=2$ [Trans. Amer. Math. Soc. **75** (1953), 526–551; MR **15**, 662], but new methods have also had to be devised. (As the author has noted later [#2418 below], the condition that the functions $u-v$ under consideration have the property that u and v are not simultaneously infinite is ill-chosen and, in fact, negates the second half of Theorem 3.) *M. G. Arsove* (Seattle, Wash.)

Sjöberg, Boris 2418

Über n-fach periodische Potentiale im n-dimensionalen Raum. I.

Acta Acad. Abo. Math. Phys. **23**, *no.* 11, 9 *pp.* (1963).

Multiply periodic δ-subharmonic functions are here examined with a view to extending known properties of elliptic functions. Equality of the number of zeros and poles in a period parallelogram, for example, has its analogue in the author's Theorem 3: If w is an n-fold periodic δ-subharmonic function in R^n, then the total mass for w over a period parallelogram must vanish. Moreover, w and its mass distribution are shown to be of mean type, order 2, and genus 3. A basic property in developing the theory is that a periodic δ-subharmonic function defined everywhere on R^n and assumed not constant in each direction cannot have arbitrarily small periods. This property holds, in fact, for any periodic function on R^n which at each point x equals the limit as $r\to 0$ of its average over the sphere of radius r about x.

M. G. Arsove (Seattle, Wash.)

Gilbert, R. P. 2419

Composition formulas in generalized axially symmetric potential theory.

J. Math. Mech. **13** (1964), 577–588.

The author continues his study of solutions of the generalized axially symmetric partial differential equation $\partial^2u/\partial x^2+\partial^2u/\partial y^2+(k/y)\partial u/\partial y=0$, $k>0$, given by the integral operator

$$u(z) = \mathscr{A}_k[f(\sigma)] = \alpha_k\int_{\mathscr{L}} f(\sigma)[\zeta-\zeta^{-1}]^{k-1}\zeta^{-1}\,d\zeta$$

in the neighborhood of an initial point z^0, where α_k is a complex constant, $\sigma=x+\frac{1}{2}iy(\zeta+\zeta^{-1})$, $z=x+iy$, $\mathscr{L}$ the upper half of the unit circle and f an analytic function of one complex variable [see the author, J. Reine Angew. Math. **212** (1963), 158–168; MR **27** #5923; Amer. J. Math. **84** (1962), 475–484; MR **26** #1479]. A quasi-multiplication is introduced by the formula $u_1(z)*u_2(z)=\mathscr{A}_k[f_1\cdot f_2]$, $u_j(z)=\mathscr{A}_k[f_j]$, and is used to obtain a residue calculus for the integral $\mathrm{Re}\int_{\mathscr{J}} w(z)\,dz$, where $w(z)=u(z)+iy^{-k}v(z)$ and $\mathscr{J}$ is a smooth Jordan curve in the z-plane. An analogue of the argument principle in one complex variable is given when the associate of $w(z)$ is $-g(\sigma)f'(\sigma)(\zeta-\zeta^{-1})^2/2f(\sigma)$, and when f has a finite number of poles and zeros and g is regular for z in the interior of $\mathscr{J}$.

J. Mitchell (Madison, Wis.)

Gilbert, R. P. 2420

Bergman's integral operator method in generalized axially symmetric potential theory.

J. Mathematical Phys. **5** (1964), 983–997.

This paper is a lucid survey of the author's recent researches on the generalized axially symmetric potential equation $L_K[u]=\partial^2u/\partial x^2+\partial^2u/\partial y^2+(K/y)\partial u/\partial y=0$, $K>0$, first studied by Weinstein [see the author, Arch. Rational Mech. Anal. **6** (1960), 171–176; MR **22** #11231; J. Reine Angew. Math. **212** (1963), 158–168; MR **27** #5923; Amer. J. Math. **84** (1962), 475–484; MR **26** #1479; Ann. Mat. Pura Appl. (4) **61** (1963), 337–348; MR **27** #5922; see also #2419 above]. Solutions are represented in a sufficiently small neighborhood of a point z^0 by the integral operator $u(z)=\mathscr{A}_K[f]=\alpha_K\int_{\mathscr{L}} f(\sigma)[\zeta-\zeta^{-1}]^{K-1}\zeta^{-1}\,d\zeta$, where α_K is a constant, $z=x+iy$, $\sigma=x+\frac{1}{2}iy(\zeta+\zeta^{-1})$, $\mathscr{L}$ is the upper half of the unit circle and $f(\sigma)$ is an analytic function, regular at the origin. This operator is closely connected with the Bergman-Whittaker operator which transforms analytic functions of two complex variables into solutions of the Laplace equation in three real variables [*Integral operators in the theory of linear partial differential equations*, Springer, Berlin, 1961; MR **25** #5277]. Briefly, the method consists in using properties of the analytic function f to obtain analogous properties of u. Problems discussed are (a) inverse operators $\mathscr{A}_K^{-1}[u]$, (b) the relation of singularities of u to those of f relative to position and number, (c) bounds for u when f is a meromorphic function, (d) properties of entire solutions, (e) a residue calculus for the integral $\int_{\mathscr{J}_z}(u\,dx-y^{-K}v\,dy)$, where $\mathscr{J}_z$ is a smooth curve and v satisfies the Stokes-Beltrami equations $y^K\partial u/\partial x=\partial v/\partial y$, $y^K\partial u/\partial y=-\partial v/\partial x$, (f) solutions of Poisson's equation $L_K[u]=\rho(x,y)$ given by integral operators. It is expected that the methods used can be extended to other singular partial differential equations such as the Euler-Poisson-Darboux equation, the doubly symmetric equation and the symmetric Helmholtz equation. *J. Mitchell* (University Park, Pa.)

Bade, W. G.; Freeman, R. S. 2421
Closed extensions of the Laplace operator determined by a general class of boundary conditions.
Pacific J. Math. **12** (1962), 395–410.

Let G be a bounded domain in $\mathbf{R}^m$ whose boundary ∂G is a $C^{1,1}$ surface. For u defined in G, the boundary values of u and $\partial u/\partial n$ on ∂G (if they exist in an appropriate sense) are denoted by $\tilde{u}$ and $\tilde{u}_n$.

The authors first prove the following results. Let $1 < p < \infty$. If $f \in L_p(\partial G)$ and u is the single-layer potential with charge f, then the linear maps $B: f \to \tilde{u}$ and $T: f \to \tilde{u}_n$ are bounded in $L_p(\partial G)$; conversely, if $k \in L_p(\partial G)$ and $\int_{\partial G} k(\theta)\,d\theta = 0$, then there is a single-layer potential u with charge in $L_p(\partial G)$ such that $\tilde{u}_n = k$.

Now let $D_1(G)$ be the set of functions $u \in L_2(G)$ such that: (i) if G_1 is an open set with $\bar{G}_1 \subset G$, then $u \in W^{2,2}(G)$; (ii) $\Delta u \in L_2(G)$, where Δ is the Laplace operator; (iii) u and $\partial u/\partial n$ have L_2 boundary values $\tilde{u}$ and $\tilde{u}_n$ in $L_2(\partial G)$. It is shown that $\underline{D}_1(G)$ coincides with the set of functions $u = p + h$, where p is the space potential of a charge in $L_2(G)$ and h is the single-layer potential of a charge in $L_2(\partial G)$.

The following results are then proved. In $L_2(G) \oplus L_2(\partial G)$, the operator S is defined by the equation

$$S[u, \tilde{u}] = [-\Delta u, \tilde{u}_n]$$

whenever $u \in \underline{D}_1(G)$; then S is a self-adjoint operator with a compact resolvent. Finally, let L be an arbitrary bounded linear operator in $L_2(\partial G)$ and let T_L be the operator defined by $T_L u = -\Delta u$ for those $u \in \underline{D}_1(G)$ such that $\tilde{u}_n = L\tilde{u}$. Then T_L is closed, its resolvent is compact, $(T_L)^* = T_{L^*}$, and the spectrum of T_L lies inside a parabola $\tau^2 = a\sigma + b$ in the (σ, τ)-plane, where a and b are positive.

F. Smithies (Cambridge, England)

Čigar'ov, V. M. [Čigarev, V. N.] 2422
On a Dirichlet problem. (Ukrainian. Russian and English summaries)
Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 997–1001.

Author's summary: "A solution is given for Dirichlet's problem for an annular region with four symmetrical holes bounded by a section of the radii and arcs of the circumferences on the basis of a harmonic polynomial in the representation of the function sought. The polynomial coefficients are determined from a system of N equations obtained by substituting in the boundary condition of the problem the coordinates r and φ of points taken from the contours of the region. A theorem is proved which permits considering the constants of the boundary condition as equalling zero in the given case. The method permits obtaining both an approximate expression for the function with a finite 'thickness' of the crack Δ and an exact expression, by passing to the limit, with $\Delta = 0$."

Morita, Yoshihito 2423
On a solution of Laplace's differential equation in two dimensions satisfying certain boundary conditions.
Mem. Defense Acad. **3**, *no.* 2, 79–88 (1964).

The solution $u(x, y)$ to the two-dimensional Laplace equation in the infinite strip $G: x > 0,\ -1 < y < 1$ is given for the following boundary-value problem: (a) $u(0, y) = 1$ $(-1 \leqq y \leqq 1)$; (b) $\lim_{x\to\infty} u(x, y) = 0$ $(-1 \leqq y \leqq 1)$; (c) u_x, u_y are continuous for $x > 0$, $-1 \leqq y \leqq 1$, with $u_x = u_y$ on $x > 0$, $y = 1$ and $u_x = -u_y$ on $x > 0$, $y = -1$; (d) $u_x^2 + u_y^2 \neq 0$ on $x > 0$, $-1 < y < 1$. These conditions are cleverly designed to give the following relation between $u(x, y)$ and its conjugate harmonic function $v(x, y)$ on the boundary: $u(x, 1) + v(x, 1) = \text{const}$, $u(x, -1) - v(x, -1) = \text{const}$.

The regular function $\omega = f(z) = u(x, y) + iv(x, y)$ is then explored as a conformal map from the (x,y)-plane to the (u,v)-plane. It is first shown that under the given boundary conditions the boundary of the strip G has an image which is a right triangle enclosing a region G' in the (u,v)-plane. It is then shown that G' is the image of G. The mapping $\omega = f(z)$ is then found by finding (in implicit form) the conformal map which maps the triangle region G' onto the strip region G. The solution u is then the real part of this map. The solution is then shown in the form of a Fourier series.

It is then shown that a similar mapping procedure yields a solution to a slightly generalized version of this boundary-value problem.

M. Wachman (Valley Forge, Pa.)

Ball, R.; Bartlett, D.; Bayer, R.; Partovi, F. 2424
Some new geometries for which Laplace's equation in three dimensions is soluble.
J. Math. and Phys. **43** (1964), 144–157.

Spherical inversion is known to take a solution of Laplace's equation in three dimensions into another solution of the same equation. Specifically, from the solution $\psi(x, y, z)$ one obtains a solution $\psi'(u, v, w)$, where $u = R^{-2}x$, $v = R^{-2}y$, $w = R^{-2}z$, $R^2 = x^2 + y^2 + z^2$, and $\psi'(u, v, w) = S^{-1}\psi(S^{-2}u, S^{-2}v, S^{-2}w)$, $S^2 = u^2 + v^2 + w^2$. The coordinate surfaces of the prolate and oblate spheroidal coordinate systems are transformed by spherical inversion into certain surfaces in (u, v, w)-space. Solutions $\psi'(u, v, w)$ of Laplace's equation which are constant on these inverted surfaces may be obtained by the method of separation of variables applied in (x, y, z)-space. As the authors point out, this idea originates in the treatises on spherical harmonics of F. Neumann and E. Heine.

The authors classify surfaces in (u, v, w)-space obtained in this way for various ranges of the parameters in the spheroidal coordinate systems. It is not indicated that any problems of applied interest actually lead to these as surfaces of constant potential.

I. Marx (Lafayette, Ind.)

Wallin, Hans 2425
Continuous functions and potential theory.
Ark. Mat. **5**, 55–84 (1963).

A famous theorem of G. C. Evans asserts that if F_0 is a compact subset of R^m having capacity zero, then there exists a positive mass distribution on F_0 admitting a potential which is infinite on F_0. Here, the related result is established that for a compact subset F_0 of R^m to have capacity zero it is necessary and sufficient that every continuous function on F_0 be the restriction to F_0 of the potential of some mass distribution having compact support. This theorem is proved, in fact, for more general kernels, including those for the α-potentials. Techniques, of interest in themselves, are also devised to obtain the following theorem. Let F_0 be a closed subset of the boundary of the unit ball in R^m; then for F_0 to have capacity zero it is necessary and sufficient that every

continuous function on F_0 admit an extension, continuous on the closed unit ball, which is harmonic on the open unit ball and has finite Dirichlet integral. As in the case of the previous theorem, this property is derived for a general class of kernels. *M. G. Arsove* (Seattle, Wash.)

SEVERAL COMPLEX VARIABLES

See also 2324, 2671, 2698.

Higuchi, Teiichi 2426

On the distribution theorem of holomorphic mappings in several complex variables.

Sci. Rep. Tokyo Kyoiku Daigaku Sect. A 8, 99–121 (1964).

In this paper the author derives, from formal properties of the integrals, various integral formulas for counting the number of times certain analytic local homeomorphisms between bounded domains in C^n have a given point as image point. These results are then applied to derive conditions that analytic mappings be at most p-valent, be star-like, or be convex-like.

R. C. Gunning (Princeton, N.J.)

Vituškin, A. G. 2427

A proof of the existence of analytic functions of several variables not representable by linear superpositions of continuously differentiable functions of fewer variables. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 1258–1261.

A. N. Kolmogorov bewies die Existenz reellwertiger, stetiger Funktionen $\alpha_m(x_1)$, $\beta_m(x_2)$ $(m=1, 2, \cdots, 5)$ mit der Eigenschaft, daß sich jede reelle, stetige Funktion $f(x_1, x_2)$ darstellen läßt in der Form

$$f(x_1, x_2) = \sum_{m=1}^{5} f_m(\alpha_m(x_1)+\beta_m(x_2)),$$

wobei die $f_m(t)$ geeignete stetige Funktionen sind. In der vorliegenden Arbeit wird als Gegenstück dieser Aussage gezeigt: Zu jeder Menge $D\subset\mathbf{R}^2$ und beliebigen, reellen, stetigen Funktionen $p_m=p_m(x_1, x_2)$ und reellen, stetig differenzierbaren Funktionen $q_m=q_m(x_1, x_2)$ $(m=1, 2, \cdots, N)$ existiert eine Zahl $\rho>0$ und eine in $\{(z_1, z_2)\in\mathbf{C}^2: z_k=x_k+iy_k, |y_k|\leqq\rho\}$ holomorphe, beschränkte Funktion $f(z_1, z_2)$, die auf der Menge $y_1=y_2=0$ reelle Werte annimmt und für kein N-tupel stetiger Funktionen $f_m(t)$ in D darstellbar ist durch $\sum_{m=1}^{N} p_m(x_1, x_2)f_m(q_m(x_1, x_2))$.

J. Spilker (Freiburg)

Siciak, J. 2428

A generalization of Schwarz's lemma and of Hadamard's three circles theorem.

Colloq. Math. **11** (1963/64), 203–207.

Sei $S(z): \mathbf{C}^n\to\mathbf{R}^+$ eine absolut homogene Funktion vom Grade 1 mit $S(z)>0$ für $z\neq0$ und $T(w): \mathbf{C}^m\to\mathbf{R}^+$ eine plurisubharmonische Funktion mit

$$\frac{1}{|\lambda|}\,T(w) \leqq T\left(\frac{1}{\lambda}w\right) \quad \text{für } w\in\mathbf{C}^m,\ |\lambda|\leqq 1.$$

Für $|\lambda|=1$ gelte sogar das Gleichheitszeichen. Der Verfasser beweist folgendes Analogon des Schwarzschen Lemmas: Für nach oben halbstetiges $S(z)$ und jede holomorphe Abbildung $f: D=\{z\in\mathbf{C}^n: S(z)<1\}\to\mathbf{C}^m$, deren sämtliche Komponenten von mindestens ν-ter Ordnung im Nullpunkt verschwinden, gilt

$$T(f(z)) \leqq \sup T(f(D))\cdot S^\nu(z), \qquad z\in D.$$

Mit einem früheren Resultat des Verfassers ergibt sich hieraus ein neuer Beweis des Poincaréschen Satzes über die biholomorphe Inäquivalenz von Kugel und Polyzylinder im $\mathbf{C}^n$ $(n>1)$. Wenn $G<\mathbf{C}^m$ ein Gebiet und $H(w): G\to\mathbf{R}^+$ eine Abbildung sowie $\log H(w)$ plurisubharmonisch ist, dann gilt der verallgemeinerte Hadamardsche Dreikreisesatz: Sei S zusätzlich stetig und

$$f: \{z\in\mathbf{C}^n: r_1\leqq S(z)\leqq r_2\}\to G$$

holomorph, so ist $\log\max H(f(z))$ eine konvexe Funktion bezüglich $\log r$ in $r_1\leqq r\leqq r_2$. Die Beweise sind einfache Ausdehnungen der Beweise für die bekannten Spezialfälle.

J. Spilker (Freiburg)

Michiwaki, Yoshimasa 2429

Note on some coefficients in a star-like function of two complex variables.

Res. Rep. Nagaoka Tech. College **1**, 151–153 (1963).

Let $Z=(z_1, z_2)'$ denote a vector with the sign $'$ indicating transposition. Write the Taylor expansion for a scalar function as $f(Z)=\sum_{m=0}^{\infty} A_mZ^m$. A_m is a row vector and Z^m is defined by the use of the Kronecker product. If M_E is a class of functions $f(Z)$, $f(0)=1$, regular in the bicylinder $\mathbf{E}: (|z_1|<1, |z_2|<1)$, satisfying in $\mathbf{E}$ the conditions

$$\operatorname{Re}\frac{Z}{f(Z)}\frac{df(Z)}{dZ} > -1,$$

then the author proves the following theorem. If the function $f(Z)=\sum_{m=0}^{\infty} A_mZ^m\in M_E$, with $A_m=(a_1, \cdots, a_m)'$, then

$$|A_m| \equiv \left(\sum_{j=1}^{m}|a_j|^2\right)^{1/2} \leqq (m+1)\left(\sum_{j=1}^{m}\frac{1}{{}_mC_j}\right)^{1/2}.$$

This verifies a previous result of I. I. Bavrin on the estimates of the modulus of coefficients of a starlike function of several complex variables [see I. I. Bavrin, Dokl. Akad. Nauk SSSR **143** (1962), 1011–1013; MR **26** #352].

J. A. Cima (Chapel Hill, N.C.)

Stoll, Wilhelm 2430

The growth of the area of a transcendental analytic set of dimension one.

Math. Z. **81** (1963), 76–98.

Von H. Rutishauser und dem Autor war in früheren Arbeiten ein Kriterium für die Transzendenz einer rein $(n-1)$-dimensionalen analytischen Menge M im n-dimensionalen komplexen Vektorraum C^n angegeben worden [vgl. H. Rutishauser, Acta Math. **83** (1950), 249–325; MR **12**, 90; der Verfasser, Math. Z. **60** (1954), 287–304; MR **16**, 463]. Bezeichnet man mit $V(r, M)$ den Flächeninhalt von $M\cap\{\mathfrak{z}, |\mathfrak{z}|<r\}$, so ist M genau dann transzendent, wenn für $r\to\infty$ auch $V(r, M)/r^{2(n-1)}\to\infty$ gilt. Der Autor beweist nun in der vorliegenden Arbeit, indem er sich vor allem auf Ergebnisse von H. Weyl und J. Weyl [H. Weyl, *Meromorphic functions and analytic curves*, Princeton Univ. Press, Princeton, N.J., 1943;

MR 5, 94] stützt, das entsprechende Resultat für rein eindimensionale analytische Mengen M: M ist genau dann transzendent, wenn für $r \to \infty$, $V(r, M)/r^2 \to \infty$ gilt.
R. Epe (Münster)

Stoll, Wilhelm 2431
Normal families of non-negative divisors.
Math. Z. **84** (1964), 154–218.

Let G be a connected complex manifold of complex dimension p, and for each open $U \subset G$ let $H(U)$ be the family of holomorphic functions on U which do not vanish identically on any component of U. Each element f of $H(U)$ determines a function ν_f from U to the non-negative integers Z, called the divisor of f, which associates to each z in U the order $\nu_f(z)$ to which f vanishes at z. If ν is a function from G to Z which can be represented locally as the divisor of a holomorphic function, ν is called a non-negative divisor on G. Thus there is a natural map $\lambda: f \to \nu_f$ of $H(G)$ into the family $\mathfrak{D}^+(G)$ of non-negative divisors on G. $H(G)$ is given the topology of uniform convergence on compact sets and $\mathfrak{D}^+(G)$ is topologized in a natural way so as to make λ continuous.

The paper develops integral formulas, of intrinsic interest but too complicated to describe here, which are used to construct a function f in $H(G)$ having a given divisor in $\mathfrak{D}^+(G)$. This is done in case G is $\mathbf{C}^p$ (giving a generalization of the Weierstrass product construction of entire functions in $\mathbf{C}^1$) and in case G is an open ball in $\mathbf{C}^p$. Thus a useful inverse to the map λ can sometimes be constructed. For instance, if $\mathfrak{N}$ is any normal (i.e., relatively compact) subset of $\mathfrak{D}^+(\mathbf{C}^p)$, consisting of ν for which $\nu(0) = 0$, a continuous map of $\mathfrak{N}$ into $H(\mathbf{C}^p)$ is constructed which is inverse to λ. Results are also obtained for open balls in $\mathbf{C}^p$.

The integral formulas are applied to establish a Montel-type theorem for divisors. The area $Y_\nu(K)$ of a divisor ν in $\mathfrak{D}^+(G)$ on a compact subset K of G is defined, and a set $\mathfrak{N} \subset \mathfrak{D}^+(G)$ is called bounded if $\{Y_\nu(K) : \nu \in \mathfrak{N}\}$ is bounded for each compact $K \subset G$.

It is proved that $\mathfrak{N}$ is normal if and only if it is bounded. A Montel theorem for analytic subsets of G of pure codimension 1 is also obtained by exploiting the relation between such an analytic set and its divisor. The theorem states that a family of such sets is normal if their $(2p-2)$-dimensional volumes are uniformly bounded on each compact subset of G. *E. A. Bishop* (Berkeley, Calif.)

Sorani, G. 2432
Homologie des q-paires de Runge.
Ann. Scuola Norm. Sup. Pisa (3) **17** (1963), 319–332.

Recall that a complex manifold X is called "q-complete" if there is a strongly q-plurisubharmonic function f on X (i.e., a C^∞ real-valued function for which the hermitian form $\sum_{ij} (\partial^2 f/\partial z_i \partial \bar{z}_j)\zeta_i \bar{\zeta}_j$ has $n-q+1$ positive eigenvalues at each point of X) such that the set $\{x \in X \mid f(x) < c\}$ is relatively compact in X for any real number c [cf. Andreotti and Grauert, Bull. Soc. Math. France **90** (1962), 193–259; MR **27** #343]. Let $Z^{k,r}(X)$ be the space of C^∞ complex-valued differential forms of bi-degree (k, r) on a complex manifold X and $Y \subset X$ be an open subset; then (X, Y) is called a "q-pair of Runge" if for each index k, $0 \leq k \leq n$, the image of the restriction map $r: Z^{k,q-1}(X) \to Z^{k,q-1}(Y)$ is dense in $Z^{k,q-1}(Y)$, in the topology of compact convergence of the coefficients of the differential forms and of all their derivatives. The author shows that if X, Y are Stein manifolds, then a 1-pair of Runge is a Runge pair in the usual sense [Docquier and Grauert, Math. Ann. **140** (1960), 94–123; MR **26** #6435]; and if X is q-complete with respect to a function f, then for any real number c the pair $(X, \{x \in X \mid f(x) < c\})$ is a q-pair of Runge, generalizing the corresponding assertion for $q = 1$ [Docquier and Grauert, loc. cit.]. Further, extending a result of Andreotti and Narasimhan [Ann. of Math. (2) **76** (1962), 499–509; MR **25** #4128], the author shows that if X and Y are q-complete manifolds and (X, Y) is a q-pair of Runge, then $H_i(X \bmod Y, \mathbf{Z}) = 0$ for $i > n+q-1$.
R. C. Gunning (Princeton, N.J.)

Kajiwara, Joji 2433
Note on the Levi problem.
Sci. Rep. Kanazawa Univ. **8**, 259–270 (1963).

This paper contains various examples of Cartan pseudoconvex domains of complex manifolds which are not holomorphically convex [cf. G. Scheja, Math. Ann. **140** (1960), 33–50; MR **22** #8535]. *R. C. Gunning* (Princeton, N.J.)

Klingen, Helmut 2434
Über einen Zusammenhang zwischen Siegelschen und Hermiteschen Modulfunktionen.
Abh. Math. Sem. Univ. Hamburg **27** (1964), 1–12.

The Siegel upper half-plane $\mathfrak{S}_n$ is the space of symmetric $n \times n$ complex matrices Z such that $\operatorname{Im} Z = (2i)^{-1}(Z - \bar{Z})$ is positive definite, and the Siegel modular functions are those meromorphic functions on $\mathfrak{S}_n$ which are invariant under the action of the symplectic modular group Γ_n [Siegel, Math. Ann. **116** (1939), 617–657; MR **1**, 203]. The Hermitian upper half-plane $\mathfrak{H}_n$ is the space of $n \times n$ complex matrices Z such that $(2i)^{-1}(Z - \bar{Z}')$ is positive definite Hermitian, and correspondingly, Hermitian modular functions are meromorphic functions on $\mathfrak{H}_n$ which are invariant under the action of Hermitian modular groups Φ_n [Braun, Ann. of Math. (2) **50** (1949), 827–855; MR **11**, 333]. The space $\mathfrak{S}_n$ is embedded in $\mathfrak{H}_n$ as an analytic submanifold in a natural manner, namely,

$$\mathfrak{S}_n = \{Z \in \mathfrak{H}_n \mid Z = Z'\};$$

and Γ_n is then the subgroup of Φ_n consisting of those transformations which preserve the submanifold $\mathfrak{S}_n$ [Klingen, Math. Ann. **129** (1955), 345–369; MR **18**, 299]. The purpose of the present paper is to show that the Siegel modular functions consist precisely of the restrictions to $\mathfrak{S}_n$ of those Hermitian modular functions which are not everywhere singular on $\mathfrak{S}_n \subset \mathfrak{H}_n$.
R. C. Gunning (Princeton, N.J.)

SPECIAL FUNCTIONS

See also 2240, 2605, 2606, 2607, 2941.

Karpov, K. A. [Карпов, К. А.]; 2435
Čistova, È. A. [Чистова, Э. А.]
★Tables of the Weber functions. Vol. II [Таблицы функций Вебера. Том II].
Vyčisl. Centr Akad. Nauk SSSR, Moscow, 1964. xx + 341 *pp*. (2 *inserts*) 3.67 *r*.

This is a companion to Vol. I (by Kireeva and Karpov) [1959; MR **27** #2658] dealing in the same way with the range $-2 \leq p \leq 0$ as the earlier volume handles the range $0 \leq p \leq 2$. One of the inserts corrects errors in the constant terms of the expression for A_1 and A_3 at the bottom of p. xix of Vol. I. The correct values are $A_1 = 2^{p/2}$ and $A_3 = \sqrt{\pi}/2^{p/2}\Gamma(-p)$ and 10D tables for these quantities for $p = .05(.05)1.95$ are given. *John Todd* (Pasadena, Calif.)

Vogel, Alfred 2436

★Vierstellige Funktionentafeln.

Dritte Auflage. Mit ungebundenem Anhang: "Mathematische Formelsammlung".

Verlag Konrad Wittwer, Stuttgart, 1963. viii + 158 *pp.* + 24 *pp. Anhang. DM* 7.60.

This is a well-printed collection. It includes some physical, chemical, astronomical and geographical tables in addition to the usual mathematical ones. It is notable for elaborate devices to get the most out of four figures: in addition to the upper and lower dots we find the following: among consecutive entries which are identical to four places, there is one which is in italics—the corresponding argument is the appropriate value of the inverse function. The interval of tabulation is chosen to make linear interpolation generally applicable; where it is not, the entries are marked with a danger signal.

John Todd (Pasadena, Calif.)

Artin, Emil 2437

★The gamma function.

Translated by Michael Butler. Athena Series: Selected Topics in Mathematics.

Holt, Rinehart and Winston, New York-Toronto-London, 1964. vii + 39 *pp.* $2.25.

From the Editor's Preface, by Edwin Hewitt: "A generation has passed since the late Emil Artin's little classic on the gamma function appeared in the Hamburger Mathematische Einzelschriften, Heft 11 [*Einführung in die Theorie der Gammafunktion*, Teubner, Leipzig, 1931]. Since that time, it has been read with joy and fascination by many thousands of mathematicians and students of mathematics. In the United States (and presumably elsewhere as well), it has for many years been hard to find, and dog-eared copies and crude photocopies have been passed from hand to hand. Professor Artin's monograph has given many a student his first look at genuine analysis —the delicacy of its arguments, the precision of its results. Artin had a deep feeling for these aspects of analysis, and he treated them with a master's hand. His undergraduate lectures in the calculus, for example, were filled with elegant constructions and theorems which, alas, Artin never had time to put into printed form. We may be all the more grateful for this beautiful essay, and for its appearance in a new English edition. Various changes made by Artin himself have been incorporated in the present edition. In particular, a small error following formula (59) (this edition) was corrected on the basis of a suggestion by Professor Børge Jessen."

P. J. Davis (Providence, R.I.)

Grinšpun, Z. S. 2438

The evaluation of a class of elliptic integrals. (Russian. Kazak summary)

Izv. Akad. Nauk Kazah. SSR Ser. Fiz.-Mat. Nauk **1963**, *no.* 3, 107–112.

Der Verfasser betrachtet eine Klasse uneigentlicher elliptischer Integralen

$$(1)\qquad I(\alpha,\beta,\gamma,\delta) = \int_\beta^\gamma \sqrt{\frac{(\gamma-x)(x-\alpha)}{(x-\beta)(\delta-x)}}\,dx$$

$$(\min(\alpha,\delta) < \beta < \gamma < \max(\alpha,\delta))$$

und zeigt dass sich ein solches Integral in der Form

$$(2)\qquad I(\alpha,\beta,\gamma,\delta) = \frac{\pi}{2}\sqrt{\frac{\beta-\alpha}{\delta-\beta}}\sum_{n=0}^\infty C_n(\gamma-\beta)^{n+1}\frac{(2n)!}{n!(n+1)!2^{2n}}$$

darstellen lässt. Dabei ist

$$C_n = \frac{1}{2^n n!}\sum_{k=1}^n \sigma_{n-k}\tau_k T_{k+1}^{(n)},$$

$$\sigma_k = 1 \quad (k=0,1),$$
$$= (-1)^{k-1}(2k-3)!! \quad (k=2,3,\cdots,n);$$
$$\tau_k = 1 \quad (k=0),$$
$$= (2k-1)!! \quad (k=1,2,\cdots,n),$$

und $T_{k+1}^{(n)}$ sind Glieder der Binomialentwicklung $[1/(\beta-\alpha)+1/(\delta-\beta)]$, d.h.,

$$T_{k+1}^{(n)} = \frac{1}{(\beta-\alpha)^{n-k}(\delta-\beta)^k}\binom{n}{k}.$$

Die Reihe (2), also auch das Integral (1), konvergiert wenn $\gamma-\beta < R$ ($R = \min(\delta-\beta, \beta-\alpha)$). Fur das Restglied r_m der Reihe (2) hat man

$$(3)\qquad |r_m| < Aq^{m+1}[(2m+3)(1-q)]^{-1/2} \quad (m=0,1,2,\cdots),$$

wo

$$A = \frac{\pi(4|\beta-\alpha|+\gamma-\beta+R)}{4(2|\delta-\beta|-\gamma+\beta-R)};\qquad q = \frac{2(\gamma-\beta)}{\gamma-\beta+R} < 1.$$

Die Ungleichung (3) ermöglicht das Integral (1) zu berechnen mit einer beliebigen, in vorhinein gegebenen Genauigkeit. *S. Fempl* (Belgrade)

Weiss, George H. 2439

A note on a generalized elliptic integral.

J. Res. Nat. Bur. Standards Sect. B **68B** (1964), 1–2.

Für das Integral

$$\Omega_j(k) = \int_0^\pi \frac{d\theta}{(1-k^2\cos\theta)^{j+1/2}}$$

$$(j = 0,1,2,\cdots, 0 \leq k^2 \leq 1)$$

gibt der Verfasser eine asymptotische Entwicklung

$$\Omega_j(k) \sim \sqrt{\frac{\pi}{2}}\frac{1}{k^{2j+1}\Gamma(j+\frac12)}$$
$$\times\left\{\sum_{n=0}^{j-1}\frac{(-1)^n\Gamma(\frac12+n)}{2^n\Gamma(\frac12-n)}\frac{(j-n-1)!}{n!}\left(\frac{k^2}{1-k^2}\right)^{j-n}\right.$$
$$\left.+\sum_{n=j}^\infty\frac{(-1)^n}{2^n}\frac{\Gamma(\frac12+n)}{\Gamma(\frac12-n)}\frac{1}{(n-j)!}\left(\frac{1-k^2}{k^2}\right)^{n-j}\ln\left(\frac{1-k^2}{k^2}\right)\right\},$$

die zweckmässig für Werte von k die der Einheit nahe kommen ist. *S. Fempl* (Belgrade)

Soudack, A. C. 2440

Some nonlinear differential equations satisfied by the Jacobian elliptic functions.

Math. Mag. **37** (1964), 138–147.

Der Verfasser behandelt die Differentialgleichung

$$\frac{d^2x}{dt^2}+ax+bx^3 = 0$$

die für verschiedene Werte von a und b mit elliptischen Funktionen sn, cn, dn und tn von t befriedigt wird.

S. Fempl (Belgrade)

Nörlund, N. E. 2441

The logarithmic solutions of the hypergeometric equation.

Mat.-Fys. Skr. Danske Vid. Selsk. **2**, *no.* 5, 58 *pp.* (1963).

In this important paper the author discusses in a clear and detailed way the complete logarithmic solutions of the hypergeometric differential equation satisfied by the Gauss function $F[a;b;c;z]$, that is to say, those solutions which arise when one or more of the quantities $a-b$, $c-a-b$, c, are integers.

Complete tables are given of the linear and quadratic relations which hold between the various solutions in every possible special case. Tables are also given for the continuation formulae which hold between the logarithmic and other cases of Riemann's P-function, and the paper concludes with a very clear statement of the logarithmic solutions of the confluent hypergeometric equation satisfied by Kummer's function $F[a;b;z]$.

L. J. Slater (Cambridge, England)

Schmidt, Paul W. 2442

Asymptotic expansion of certain integrals containing the Bessel function $J_0(x)$.

J. Mathematical Phys. 5 (1964), 1183–1184.

Author's summary: "An asymptotic expansion is calculated for the integral

$$\frac{2}{\pi}\int_0^1 \frac{dxf(x)J_0(kx)}{(1-x^2)^{1/2}},$$

where $J_0(x)$ is the zero-order Bessel function of the first kind, and the function $f(x)$ and all its derivatives are continuous for $0 \leqq x \leqq 1$. To find the asymptotic expansion, the integral is converted to a Fourier integral, to which asymptotic expansion techniques are applied. The method can be generalized to apply to related integrals."

Van Tuyl, A. H. 2443

The evaluation of some definite integrals involving Bessel functions which occur in hydrodynamics and elasticity.

Math. Comp. **18** (1964), 421–432.

Consider

$$I_1(m,\nu) = \int_0^\infty e^{-zt}J_\nu(yt)J_\nu(bt)t^m\,dt,$$

$$I_2(m,\nu) = \int_0^\infty e^{-zt}J_{\nu-1}(yt)J_\nu(bt)t^{m+1}\,dt$$

and the more general integral

$$I = \int_0^\infty e^{-ct}J_\mu(at)J_\nu(bt)t^\lambda\,dt.$$

It is shown that I_1 may be expressed in terms of associated Legendre functions for general ν when $m \geqq 0$, and for ν an integer when $m<0$. Recurrence equations are derived for both I_1 and I_2. The integral I has been studied by Eason, Noble, and Sneddon [Philos. Trans. Roy. Soc. London Ser. A **247** (1955), 529–551; MR **16**, 1107], and the results have been summarized by the reviewer in his book [*Integrals of Bessel functions*, McGraw-Hill, New York, 1962; MR **25** #5198]. The recurrence equations of the present author are not special cases of those derived by Eason, Noble and Sneddon.

Y. L. Luke (Kansas City, Mo.)

Kuipers, L. 2444

Some remarks on properties of Jacobi polynomials and generalized Legendre functions.

Bull. Calcutta Math. Soc. **53** (1961), 181–184.

The author shows that certain results on Jacobi polynomials obtained by Bhonsle [same Bull. **50** (1958), 159–164; MR **21** #2759] may be used to obtain similar results for generalized Legendre functions [Kuipers and Meulenbeld, Nederl. Akad. Wetensch. Proc. Ser. A **60** (1957), 436–450; MR **20** #5306]. He also notes some errors in Section 16.4 of *Higher transcendental functions*, Vol. III [McGraw-Hill, New York, 1955; MR **16**, 586] by Erdélyi, Magnus, Oberhettinger, and Tricomi.

A. Erdélyi (Edinburgh)

Rangarajan, S. K. 2445

Series involving products of Laguerre polynomials.

Proc. Indian Acad. Sci. Sect. A **58** (1963), 362–367.

The author evaluates

$$\sum n!\,[\Gamma(\alpha+n+1)]^{-1}L_n^{\alpha}(x)L_n^{\beta}(y)z^n,$$

and also a similar sum in which β has been replaced by $\beta-n$, in closed form (in terms of a confluent hypergeometric function of two variables).

A. Erdélyi (Edinburgh)

Remez, E. Ja. 2446

The Chebyshev "trigonometrical" polynomials are also hyperbolic. (Russian)

Ukrain. Mat. Ž. **16** (1964), 406–408.

The author remarks that the Chebyshev polynomial $T_n(x)=\cos(n \arccos x)$ can also be written in terms of hyperbolic functions as $\operatorname{ch}(n \operatorname{ar\,ch} x)$, and that this representation is convenient when considering problems involving the $T_n(x)$ outside the interval $[-1,+1]$. In particular, he gives a short proof of an inequality of Chebyshev (not explicitly involving the $T_n(x)$) on the basis of a classical extremal property of the $T_n(x)$ in the interval $[1,\infty)$. *J. H. Williamson* (Cambridge, England)

Gloden, R. F. [Gloden, Raoul-François] 2447

★Propriétés des polynômes associés aux fonctions de Laguerre de seconde espèce généralisées et questions connexes. (English summary)

EUR 207.f.

Communauté Européenne de l'Énergie Atomique—EURATOM, Centre Commun de Recherche Nucléaire, Établissement d'Ispra—Italie, Centre de Traitement des Informations Scientifiques—CETIS, ***Brussels*, 1963.** 5 *pp.* *FB* 25.00.

The denominators of the approximants of the continued fraction

$$\frac{1}{|x-(\beta+1)}-\frac{1(\beta+1)|}{|x-(\beta+3)}-\cdots-\frac{m(\beta+n)|}{|x-(\beta+2n+1)}-\cdots$$

are the Laguerre polynomials $L_n^\beta(x)$ with leading coefficients one; the numerators are called the associated Laguerre polynomials of the second kind and are denoted by $K_n^\beta(x)$. A recurrence relation and a representation of $K_n^\beta(x)$ as a determinant are obtained. The zeros of $K_n^\beta(x)$ are real and distinct, and the zeros of $K_{n-1}^\beta(x)$ separate those of $K_n^\beta(x)$.

The representation

$$K_{n-1}^\beta(x) = [L_n^\beta(x)l_0^\beta(x)-l_n^\beta(x)]e^{-x}x^\beta,$$

where $l_n^\beta(x)$ is the Laguerre function of the second kind, is obtained, and $K_{2m-1}^\beta(x)$ and $K_{2m}^\beta(x)$ are expressed as sums involving the derivatives of the Laguerre polynomials. A second-order and a fourth-order differential equation satisfied by $K_n^\beta(x)$ are derived. The general solution of the latter is obtained.

The methods of proof are indicated without any details.

A. E. Danese (Buffalo, N.Y.)

ORDINARY DIFFERENTIAL EQUATIONS

See also 2348, 2440, 2532, 2673, 2677, 2678, 2680, 2682, 2818a-b, 2819, 2974, 2975, 2977, 2981, 3053, 3143, 3240, 3301, 3305.

Crowson, Henry L. 2448

An analysis of a second order linear ordinary differential equation with five regular singular points.

J. Math. and Phys. **43** (1964), 38–44.

The five regular singular points are $z=0$, 1, a, b and c, the respective exponents at these singularities being $\alpha_r=0$ ($r=1,2,3,4$) and $\alpha_5=\alpha$ and $\beta_1=1-\gamma$, $\beta_2=1-\delta$, $\beta_3=1-\varepsilon$, $\beta_4=1-\eta$ and $\beta_5=\beta$. Thus the differential equation is

$$\frac{d^2u}{dz^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\varepsilon}{z-a}+\frac{\eta}{z-b}+\frac{1-\alpha-\beta}{z-c}\right)\frac{du}{dz}$$
$$+\left(\frac{\alpha\beta}{(z-c)^2}+\frac{D_1}{z}+\frac{D_2}{z-1}+\frac{D_3}{z-a}+\frac{D_4}{z-b}+\frac{D_5}{z-c}\right)u = 0,$$

where

$$D_1+D_2+D_3+D_4+D_5 = 0,$$
$$\alpha\beta+D_2+aD_3+bD_4+cD_5 = 0,$$
$$2c\alpha\beta+D_2+a^2D_3+b^2D_4+c^2D_5 = 0,$$
$$\gamma+\delta+\varepsilon+\eta-\alpha-\beta = 1.$$

For the singular point c the value ∞ is chosen. In the neighborhood of $z=0$ the method of Frobenius leads to a four-term recursion formula for the coefficients of the power series representing the solution. For a proof of the solution obtained, Sheffé's lemma is applied. An exhaustive discussion of the cases arising from the application completes the proof.

M. J. O. Strutt (Zürich)

Mirak'jan, M. G. 2449

On the existence of a regular solution of a system of first-order differential equations near a singularity. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 454–457.

Author's summary: "This article deals with the question of analytical solutions of nonlinear differential equations

$$z^p\frac{dw_k}{dz} = za_{00}^{(k)}(z)+a_{11}^{(k)}(z)w_1+\cdots+a_{1n}^{(k)}(z)w_n + \sum_{j=r_1+\cdots+r_l=2}^{\infty} a_{ij}^{(k)}(z)w_{r_1}w_{r_2}\cdots w_{r_l}$$

in the neighborhood of a singular point $z=0$. In the case when $p=2$ a connection is found between the analytical solutions of this system and the formal solutions in the form of power series."

Aliev, R. G.; Ostroumov, V. V.; Pak, S. A. 2450

Some properties of the Cauchy function. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 4 (41), 9–11.

Let $u_i(t)$ ($i=0,1,\cdots,n-1$) be a fundamental system of solutions (f.d.s.) of the equation

$$L(y) \equiv y^{(n)}-\sum_{k=0}^{n-1}g_k(t)y^{(k)} = 0$$

where $g_k(t)$ are summable in $[a,b]$ and let $D(t)$ be the Wronskian of the system. Then

$$K(t,s) = \frac{1}{D(t)}\begin{vmatrix} u_0(s) & \cdot\ \cdot\ \cdot & u_{n-1}(s) \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ u_0^{(n-2)}(s) & \cdot\ \cdot\ \cdot & u_{n-1}^{(n-2)}(s) \\ u_0(t) & \cdot\ \cdot\ \cdot & u_{n-1}(t) \end{vmatrix}$$

is called the Cauchy function of $L(y)=0$. The authors obtain the following results. There exist points $s_i\in[a,b]$ such that the functions $y_i(t)=K(t,s_i)$ constitute an f.d.s. of $L(y)=0$. Let $[a,a+h]$ be the interval of non-oscillation for $L(y)=0$, and let $g_k(t)$ be k times continuously differentiable. Then for arbitrary distinct points $s_i\in[a,a+h]$, the functions $y_i(t)=K(t,s_i)$ represent an f.d.s. of $L(y)=0$. The authors also give necessary and sufficient conditions for the inequalities $K(x,s)>0$ for $a<s<x<b$, $(-1)^{n-1}K(x,s)>0$ for $a<x<s<b$ to hold.

V. Lakshmikantham (Aurangabad)

Perčinkova, Danica 2451

★Contribution to the study of homogeneous eigenvalue problems for linear ordinary differential equations. (Macedonian. French summary)

Faculté des Sciences de l'Université de Skopje Éditions Spéciales, Livre 13.

Université de Skopje, Skopje, 1963. 95 *pp.*

Le travail est consacré à l'étude des problèmes aux limites (E) $y^{(p+1)}+\lambda y=0$, (L) $\sum_{\nu=0}^{p}[\alpha_{\mu\nu}y^{(\nu)}(a)+\beta_{\mu\nu}y^{(\nu)}(b)]=0$, $\mu=1,2,\cdots,p+1$, pour $p=1,2,3,4$. On obtient chaque fois l'équation caractéristique pour les valeurs propres du problème et l'on donne l'expression des fonctions propres. Les constantes $\alpha_{\mu\nu}$ et $\beta_{\mu\nu}$ sont telles que la matrice $\|\alpha_{\mu\nu};\beta_{\mu\nu}\|$ ait le rang $p+1$. Pour des valeurs particulières des constantes $\alpha_{\mu\nu}$ et $\beta_{\mu\nu}$ on donne les valeurs approximatives des valeurs caractéristiques, on fait la liste des

fonctions propres, etc. L'auteur remarque que la résolution des équations caractéristiques pour $p > 4$ est un problème assez difficile au point de vue technique.

C. Corduneanu (Iaşi)

Buravs'kiĭ, È. S.; Jaremčuk, F. P. 2452

On the solution of boundary-value problems for a non-homogeneous ordinary differential equation of second order. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1411–1415.

Authors' summary: "A method is proposed for the approximate solution of a boundary-value problem for an inhomogeneous ordinary linear differential equation of the second order. The boundary-value problem connected with the study of the flow of liquid in canals, for instance, reduces to a problem of this form. It is shown that the proposed method may be applied to such problems."

Arscott, F. M. 2453

Two-parameter eigenvalue problems in differential equations.

Proc. London Math. Soc. (3) **14** (1964), 459–470.

If w is a solution of the ordinary differential equation

$$d^2w/dz^2 + [\lambda + \mu f(z) + g(z)]w = 0,$$

then $W(\alpha, \beta) = w(\alpha)w(\beta)$ is a solution of the partial differential equation

$$\partial^2 W/\partial\alpha^2 - \partial^2 W/\partial\beta^2 + \{\mu[f(\alpha) - f(\beta)] + g(\alpha) - g(\beta)\}W = 0.$$

From this fact the author deduces, chiefly by integration by parts in the complex plane, certain orthogonality relations, integral equations, and integral representations for solutions w. He derives conditions for reality of the characteristic values λ, μ and properties of expansions in characteristic functions; however, the approach is purely formal, and no attempt is made to prove existence of characteristic values or of expansions in characteristic functions.

I. Marx (Lafayette, Ind.)

Gredžuk, I. F. 2454

A method of solving a boundary-value problem for a linear second-order differential equation. (Russian)

Ukraïn. Mat. Ž. **16** (1964), 375–382.

The author presents a method for finding upper and lower bounding functions for the solution $y(x)$ of $y'' - Q(x)y = 0$ subject to $y(0) = y_0$, $y(1) = y_1$, where Q is continuous and positive on [0, 1] and the conditions are such that $y(x) > 0$ on [0, 1]. With $n \geqq 4$ let $z(x) = \sum_{i=0}^{n} a_i x^{n-i}$ and let $\chi(x) = z''(x)/z(x)$, so that $z'' - \chi(x)z = 0$. The conditions $z(0) = y_0$, $z(1) = y_1$, $\chi(0) = Q(0)$ and $\chi(1) = Q(1)$ define four of the a_i in terms of the remaining ones. Let α denote a vector having the $n-4$ free parameters as components, and let $E_1 = \{\alpha | \chi(x) \leqq Q(x) \text{ on } [0, 1]\}$ and $E_2 = \{\alpha | \chi(x) \geqq Q(x) \text{ on } [0, 1]\}$. By a comparison theorem, $\alpha \in E_1$ implies $z(x) \geqq y(x)$ on [0, 1] while $\alpha \in E_2$ implies $z(x) \leqq y(x)$ on [0, 1]. If $\alpha \in E_1 \cap E_2$, then $z(x) \equiv y(x)$. If $E_1 \cap E_2$ is void, an approximate method is described for obtaining $\alpha_1 \in E_1$ and $\alpha_2 \in E_2$ such that the euclidean distance between α_1 and α_2 is a minimum. The inequalities $\chi(x) \leqq Q(x)$ and $\chi(x) \geqq Q(x)$ are equivalent to ones which are linear in the components of α, with coefficients being functions of x. These inequalities, to hold on [0, 1], are approximated by the systems obtained therefrom by evaluating the coefficients at $x = j/k$, $j = 0, 1, \cdots, k$. The optimal α_1 and α_2 from the resulting systems provide approximations to the desired $\alpha_1 \in E_1$ and $\alpha_2 \in E_2$.

C. E. Langenhop (Carbondale, Ill.)

Gehtman, M. M. 2455

On the spectrum of the non-selfadjoint Sturm-Liouville operator. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, *no.* 1, 29–36.

English summary: "Let L be an operator on $L_2(0, \infty)$, $L(y) \equiv -y'' + (q + ir)y$, $y'(0) - \theta y(0) = 0$. Here $q(x)$ and $r(x)$ are real functions, $\theta \neq \infty$ is also real. The following theorem is proved in the paper: If $r(x) \to 0$ as $x \to \infty$, then the spectrum of the operator L lies in the complex λ-plane $(\lambda = \sigma + i\tau)$ in the domain $|\tau| \leqq K(\sigma)$, where $K(\sigma)$ is a non-negative function satisfying the condition $K(\sigma) \to 0$ as $\sigma \to \infty$."

G. K. Kalisch (Minneapolis, Minn.)

Leptin, H. 2456

Kamkesche Eigenwertprobleme und Hilbert-Schmidt-Operatoren.

Math. Z. **82** (1963), 133–151.

The author discusses eigenvalue problems of the form $My = \lambda Ny$, where M and N are formally self-adjoint differential operators of orders m and n, with $m \geqq n + 2$, in an interval $[\alpha, \beta]$, under self-adjoint homogeneous boundary conditions involving the values of y and its derivatives up to order $m - 1$ at the points α and β. The formal structure of the method he uses is as follows: Let Γ be the Green's function operator corresponding to M and the boundary conditions, supposed positive definite, and let $T = \Gamma^{1/2}$, $H = TNT$. Then there is a (1, 1) correspondence between solutions of the equation $y = \lambda Hy$ and those of $Mz = \lambda Nz$, given by $z = Ty$. This argument is made precise, under appropriate restrictions on the coefficients of the differential operators and boundary conditions.

In the last part of the paper the author considers problems of the more general form, $My = N(\lambda)y$, where N depends, in a certain sense, continuously on λ. Using perturbation methods in Hilbert space and some other devices, the author establishes some sufficient conditions for the existence of eigenvalues.

The author does not seem to be aware of the related work of D. F. Harazov, contained in a long series of papers published since about 1954 [see, e.g., Mat. Sb. (N.S.) **42** (84) (1957), 129–178; MR **19**, 666].

F. Smithies (Cambridge, England)

Džabrailova, È. M. 2457

The completeness of finite-dimensional invariant subspaces of a non-selfadjoint singular differential operator of second order relative to a real weight function. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 774–776.

Soient $L[y] = -y'' + p(x)y$, $Q[y] = q(x)y$ deux opérateurs différentiels. On considère le problème aux limites $L[y] = \lambda Q[y]$, $y(x) \in L^2(-\infty, +\infty)$. Posons $p(x) = p_1(x) + ip_2(x)$.

On admet que les fonctions $p_1(x)$, $p_2(x)$ et $q(x)$ sont sommables sur tout intervalle compact et que $q(x)$ est essentiellement bornée sur l'axe réel, pouvant s'annuler seulement sur un ensemble de mesure nulle. Le sous-espace R de $L^2(-\infty, +\infty)$ s'appelle invariant si l'on a $QR \subset LR$. Pour assurer la validité de la propriété mentionnée dans le titre, il suffit: $\liminf_{|x|\to\infty} (p_1(x)/|x|^\alpha) = C > 0$, $|p_2(x)| < p_1(x)$, $\lim_{|x|\to\infty} (|p_2(x)|/p_1(x)) = 0$.

C. Corduneanu (Iaşi)

Džabrailova, È. M. 2458

On the spectrum of a singular non-selfadjoint second-order differential equation with respect to a complex weight function. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 1021–1023.

The author considers the equation

$$(1)\qquad -y'' + p(x)y - \lambda q(x)y = 0$$

in $L_2(0, \infty)$, where $p(x)$ and $q(x)$ are locally integrable complex-valued functions. Under various growth conditions on $p(x)$ and $q(x)$ he shows that (1) has only one solution in $L_2(0, \infty)$, and by imposing the boundary condition $y'(0) - \theta y(0) = 0$ (θ complex) shows that the corresponding operator equation, in an appropriate domain, has a spectrum consisting of eigenvalues with no finite limit point.

The author next considers two sets of conditions on $p(x)$ and $q(x)$ giving two solutions of (1) in $L_2(0, \infty)$ and accordingly imposes two boundary conditions. In one case he shows that the spectrum is again discrete and in the other, that it is the whole real axis plus a discrete part.

R. E. L. Turner (Madison, Wis.)

Chaudhuri, J. 2459

On the convergence of the eigenfunction expansion associated with a fourth-order differential equation.

Quart. J. Math. Oxford Ser. (2) **15** (1964), 258–274.

This paper provides the generalization of the results of E. C. Titchmarsh [*Eigenfunction expansions associated with second-order differential equations*, Chapter 3, Theorems 3.2–3.4, Clarendon Press, Oxford, 1946; MR 8, 458] on the convergence of the eigenfunction expansion associated with a second-order self-adjoint problem on $[0, \infty)$, to even-order self-adjoint problems in general, and to the fourth-order case in particular.

Specifically, given

$$y^{(4)} + q(x)y = \lambda y$$

with $q(x) \in L[0, \infty)$, such that

$$\sum_1^4 a_r y^{(r-1)}(0) = 0, \qquad \sum_1^4 b_r y^{(r-1)}(0) = 0,$$

$$a_1b_4 - a_4b_1 + a_3b_2 - a_2b_3 = 0 \text{ (self-adjointness condition)},$$

then for an $f(x)$ which is both $L[0, \infty)$ and $L^2[0, \infty)$, the eigenexpansion of $f(x)$ converges under the same conditions as the corresponding expansion for the case $q(x) \equiv 0$.

The techniques used are direct generalizations of those found in Titchmarsh's book [op. cit.].

G. E. Latta (Stanford, Calif.)

Kearsley, Elliot A. 2460

Solutions of the equation

$$\Psi_{xx} + \frac{1}{x}\Psi_x + Kx^n e^\Psi = 0.$$

J. Res. Nat. Bur. Standards Sect. B **67B** (1963), 245–247.

The non-linear second-order differential equation cited in the title arises for a class of steady viscous flows in which the viscosity depends exponentially on temperature. The same equation may also arise in calculating the temperature distribution in a dielectric in an alternating field. With K and n arbitrary real constants, the general solution of this strongly non-linear equation is exhibited in terms of simple tabulated functions. The existence and uniqueness of the boundary-value problem in which Ψ is prescribed at two fixed values of x, and is a smooth bounded function in the interval, is determined as a function of the parameter K.

K. B. Ranger (Toronto, Ont.)

Sansone, Giovanni 2461

L'equazione $\ddot\theta + f(\theta, \alpha)h(\dot\theta) = g(\theta) + p(t)$.

Univ. e Politec. Torino Rend. Sem. Mat. **22** (1962/63), 81–89 (1964).

This is a summary of results obtained by the author in a paper which will appear in J. Math. Pures Appl. (9). They deal with the continuation of solutions and the existence of bounded or periodic solutions for the equation in the title.

W. A. Coppel (Canberra)

Ráb, Miloš 2462

Asymptotische Formeln für die Lösungen der Differentialgleichung $y'' + q(x)y = 0$. (Russian summary)

Czechoslovak Math. J. **14** (**89**) (1964), 203–221.

Let the two differential equations (1) $y'' + q(x)y = 0$ and (2) $\ddot Y + Q(X)Y = 0$, where q, Q are continuous on $[a, \infty)$, be simultaneously oscillatory or non-oscillatory. There are obtained certain asymptotic formulas for solutions of (1) in terms of solutions of (2). Among other things it is shown that if U, V are independent solutions of (2), then there exist some $\xi \geq a$ and functions f, F such that (3) $f(x)U[F(x)]$, $f(x)V[F(x)]$ are independent solutions of (1) on $[\xi, \infty)$. The functions f, F satisfy the non-linear equations (4) $fF'' + 2f'F' = 0$, $f'' + [q - F'^2Q(F)]f = 0$. There is then investigated the problem of the connection between the solutions of (1) and the functions corresponding to (3), but with f and F replaced by ϕ and Φ, where ϕ and Φ are in a certain sense approximate solutions of (4). The equation $(p(x)y')' + q(x)y = 0$, where $p > 0$ and p', q are continuous, is also considered.

C. R. Putnam (Lafayette, Ind.)

Kukles, I. S.; Nurov, T. 2463

On distinguishing center and focus. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1963**, no. 6 (37), 98–108.

On sait [voir H. Poincaré, J. Math. Pures Appl. (3) **7** (1881), 375–422; ibid. (3) **8** (1882), 251–296; ibid. (4) **1** (1885), 167–244; ibid. (4) **2** (1886), 151–217; A. M. Ljapunov, Mat. Sb. **17** (1893), no. 2, 253–333] q'une con-

dition nécessaire et suffisante pour que le point $x=y=0$ soit un centre pour l'équation

$$(1)\qquad \frac{dy}{dx} = -\frac{x+p_2(x,y)+\cdots+p_n(x,y)+\cdots}{y+q_2(x,y)+\cdots+q_n(x,y)+\cdots},$$

où $p_i(x,y)=\sum_{k=0}^{i} a_{i-k,k}x^{i-k}y^k$, $q_i(x,y)=\sum_{k=0}^{i} b_{i-k,k}x^{i-k}y^k$ ($a_{i-k,k}$, $b_{i-k,k}$ sont des constantes réelles), est qu'il existe pour l'équation (1) une intégrale réelle $V(x,y)=C$, holomorphe dans un voisinage de $x=y=0$. En substituant dans (1) une telle intégrale $V(x,y)=\sum_{i=2}^{\infty}\omega_i(x,y)=C$, où

$$\omega_2(x,y) = \tfrac{1}{2}(x^2-y^2),$$

$$\omega_i(x,y) = \sum_{k=0}^{i}\lambda_{i-k,k}x^{i-k}y^k, \qquad i>2$$

(les constantes réelles $\lambda_{i,k}$ n'étant pas déterminées), les auteurs trouvent des conditions nécessaires et suffisantes pour le centre, écrites sous la forme d'une suite infinie d'équations $D_i=0$, où les D_i sont certains déterminants. On donne aussi une règle générale pour construire ces déterminants sous une forme explicite. On considère aussi l'équation

$$\frac{dy}{dx} = -\frac{x^{2n+1}+P(x,y)}{y^{2k+1}+Q(x,y)},$$

où P, Q sont holomorphes dans un voisinage de $x=y=0$, et quelques exemples. *M. Reghiș* (Timișoara)

Halanay, A. 2464

★Teoria calitativă a ecuaţiilor diferenţiale. Stabilitatea după liapunov. Oscilaţii. Sisteme cu argument întîrziat [Qualitative theory of differential equations. Liapunov stability. Oscillations. Systems with retarded argument].

Editura Academiei Republicii Populare Romîne, Bucharest, 1963. 482 *pp.* *Lei* 21.50.

This book consists of four parts: Lyapunov stability theory, the theory of nonlinear feedback controls, the theory of nonlinear oscillations, and differential equations with retarded arguments, sometimes called differential-difference equations or functional-differential equations.

The part on the theory of Lyapunov stability has the same flavor as the books of Hahn [*Theory and application of Liapunov's direct method*, Prentice Hall, Englewood Cliffs, N.J., 1963; MR **26** #5230] and Krasovskiĭ [*Stability of motion*, Stanford Univ. Press, Stanford, Calif., 1963; MR **26** #5258].

The part on feedback control presents for the first time in a book the important and far-reaching recent results of Popov [Avtomat. i Telemeh. **22** (1961), 961–979; MR **24** #A3394; ibid. **23** (1962), 3–24; MR **28** #3212] on the stability of an automatic regulator. This chapter also contains the recent work of Lefschetz [Bol. Soc. Mat. Mexicana (2) 5 (1960), 139–143; MR **24** #A2730] and Jakubovič [Dokl. Akad. Nauk SSSR **143** (1962), 1304–1307; MR **24** #A3167].

The part on nonlinear oscillations touches most of the topics usually considered in this field: periodic and almost periodic solutions by methods of successive approximations, periodic perturbations of nonlinear autonomous systems (integral manifolds) and fixed-point methods for periodic solutions.

The last part of the book treats all of the above problems for differential-difference equations. At present, this theory is in a state of intense development and unification, which naturally causes some sections of this part of the book to be already outdated, especially much of the part dealing with equations with a small parameter. A careful reading of the chapter on differential-difference equations is most rewarding, for it will stimulate the reader to many new areas of research.

This book will serve as a valuable reference since it presents in one place so many of the aspects of differential equations. There are no examples and, for this reason, the book will be more beneficial to the specialist than the student or someone beginning to study this area for the first time.

{The reviewer has just been informed that this book will be published in English by Academic Press. Also, the chapter on differential-difference equations will be completely revised.} *J. K. Hale* (Providence, R.I.)

Nehari, Zeev 2465

On a nonlinear differential equation arising in nuclear physics.

Proc. Roy. Irish Acad. Sect. A **62**, 117–135 (1963).

The author studies the differential equation

$$\ddot{x}+\frac{2\dot{x}}{t}-x-x^2 = 0$$

previously considered by Synge [same Proc. **62** (1961), 17–41; MR **25** #2333] and Takahashi [Nuclear Phys. **26** (1961), 658–669; MR **29** #4509] and gives a proof of the existence of a continuous solution for $0\leq t<\infty$ that tends to zero as $t\to\infty$. The proof also applies to the equation

$$\ddot{x}+\frac{2\dot{x}}{t}-x+x^k = 0,$$

where $1<k\leqq 4$. The method used in the proof is related to that considered in the author's paper [Trans. Amer. Math. Soc. **95** (1960), 101–123; MR **22** #2756] and has been extended to cover this singular case. The author also proposes two methods for the construction of successive convergents to the solution of (1). By these methods he confirms a numerical result already obtained by Synge.

The proof of the uniqueness of the solution remains an open problem. *E. L. Ortiz* (London)

Miron, S. I. 2466

A global study of the solutions of a class of non-linear differential equations. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 169–173.

Author's summary: "The problem of the integral representation of the solution

$$W = \int_a^b \frac{d\sigma(\tau,z)}{\tau+\dfrac{1}{\omega}}$$

is stated for the equation $dW/dz=Wf(W,z)$. The problem is solved for the auxiliary equation $dW/dx=Wf(W)$."

Šarkovs'kiĭ, O. M. [Šarkovskiĭ, A. N.] 2467

Fixed points and the center of a continuous mapping of the line into itself. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 865–868.

Author's summary: "It is shown in this note that the centre and minimal centre of attraction of a continuous mapping of a segment of the line is the closure of the set of points of the cycles."

Kluczny, Czeslaw 2468

Sur certaines familles de courbes en relation avec la théorie des équations différentielles ordinaires. II. (Polish and Russian summaries)

Ann. Univ. Mariae Curie-Skłodowska Sect. A **16** (1962), 5–18 (1964).

This paper is a continuation of Part I [same Ann. **15** (1961), 13–40; MR **25** #2284]. Again the curve families arise as solutions of the vector differential equation $x'=f(t,x)$, defined and continuous in a region of E^n. Uniqueness of solutions is not assumed. Consequences of non-asymptoticity are secured.

W. R. Utz (Columbia, Mo.)

Hale, J. K.; Perelló, C. 2469

The neighborhood of a singular point of functional differential equations.

Contributions to Differential Equations **3** (1964), 351–375.

The authors continue the investigation of the geometry of solutions of functional differential equations, begun by one of them [J. K. Hale, Contributions to Differential Equations **2** (1963), 291–317; MR **27** #2700]. The setting for this geometry is the phase space C of continuous functions with fixed domain $[-r, 0]$ and range E^n. The object of the present paper is to demonstrate that the set of initial values of those solutions of a nonlinear equation which tend to zero as $t\to\infty$ (or $t\to-\infty$) is homeomorphic to the corresponding set for the linear approximating equation. More precisely, consider the linear autonomous equation (notation in previous paper) (1) $\dot{u}(t)=L(u_t)$ and the nonlinear equation (2) $\dot{x}(t)=L(x_t)+N(x_t,t)$. Here L is a continuous linear function on C into E^n and N is a nonlinear function satisfying $N(0,t)=0$ $(-\infty<t<\infty)$ and $|N(\varphi,t)-N(\psi,t)|\leqq\eta(\delta)\|\varphi-\psi\|$ for $\|\varphi\|$, $\|\psi\|\leqq\delta$, where $\eta(0)=0$. Let γ be any real number and let $\Lambda(\gamma)$ be the (finite) set of characteristic roots of (1) with real part $\geqq\gamma$. The space C can be decomposed (see the paper cited above) into the sum of spaces $R=R(\Lambda)$ and $S=S(\Lambda)$, where R is the generalized eigenspace of the linear equation (1) corresponding to Λ. Thus every φ in C has a unique representation $\varphi=\varphi^R+\varphi^S$, $\varphi^R\in R$, $\varphi^S\in S$. Define projection operators p_R and p_S by $p_R\varphi=\varphi^R$, $p_S\varphi=\varphi^S$, and let C_H denote the subset of C for which $\|\varphi\|<H$. Let $x(\sigma,\varphi)$ denote the solution of (2) with initial value φ at σ. The principal result is Theorem 1: If $\gamma\leqq 0$ and no characteristic root has real part γ, then there exist constants $K\geqq 1$, $\delta>0$ such that, for any σ in $(-\infty,\infty)$, p_S is a homeomorphism from the set $\mathscr{S}(\gamma,\sigma,\delta)=\{\varphi\in C:\|\varphi^S\|\leqq\delta/2K$, $\|x_t(\sigma,\varphi)\|\leqq\delta$ for $t\geqq\sigma$, $\|x_t(\sigma,\varphi)\|=O(e^{\gamma(t-\sigma)})$ as $t\to\infty\}$ onto $S\cap C_{\delta/2K}$, and $\mathscr{S}(\gamma,\sigma,\delta)$ is tangent to S at 0. Furthermore, for φ in $\mathscr{S}(\gamma,\sigma,\delta)$, $x_t(\sigma,\varphi)$ tends exponentially to zero as $t\to\infty$.

Also taking $\gamma=0$, but allowing characteristic roots on the imaginary axis, the authors establish a homeomorphism between the initial values of solutions tending exponentially to zero. Considering $t\to-\infty$, they show that p_R is a homeomorphism between $R\cap C_{\delta/2K}$ and a certain manifold $\mathscr{R}(\gamma,\sigma,\delta)$. Finally, descriptive results are obtained for the solutions of (2) starting off the manifolds $\mathscr{R}$ and $\mathscr{S}$.

K. L. Cooke (Claremont, Calif.)

Sun, Shun-hua 2470

The equations of the closed trajectories in rotated vector fields.

Acta Math. Sinica **14** (1964), 50–57 (*Chinese*); *translated as Chinese Math.* **5** (1964), 54–62.

The concept of rotated vector fields was introduced by Duff [Ann. of Math. (2) **57** (1953), 15–31; MR **14**, 751] and later extended by Seifert [*Contributions to the theory of nonlinear oscillations*, Vol. IV, pp. 125–139, Princeton Univ. Press, Princeton, N.J., 1958; MR **21** #4278]. The purpose of the present paper is to establish the equations of the closed trajectories of rotated vector fields. The author illustrates the application of the results to the calculation of the limit cycle of van der Pol's equation with a small parameter.

J. K. Hale (Providence, R.I.)

Wexler, Dinu 2471

Solutions périodiques et presque périodiques des systèmes d'équations différentielles aux impulsions.

C. R. Acad. Sci. Paris **259** (1964), 287–289.

L'auteur étudie le problème de l'existence des solutions périodiques et presque périodiques du système (S) $\dot{x}=A(t)x+\sum s_i\delta_i$, dans lequel $A(t)$ est une matrice continue sur l'axe réel, s_i sont des vecteurs et δ_i est la mesure de Dirac placée au point t_i. On admet que $t_i\to\infty$ pour $i\to\infty$ et $t_i\to-\infty$ pour $i\to-\infty$, et aussi $t_i<t_{i+1}$. On cherche les solutions du système (S) parmi les distributions et l'on obtient des résultats qui rapellent les résultats connus de la théorie des oscillations linéaires. Par exemple, le système (S) admet une solution périodique de période ω, quelle que soit la distribution qui figure dans le second membre, si et seulement si le système homogène associé n'admet aucune solution périodique non nulle (on a supposée $A(t)$ de période ω). Si $A(t)$ et la suite (s_i) sont presque périodiques et si la solution nulle du système homogène est uniformément asymptotiquement stable, alors le système (S) admet une solution presque périodique unique (on a imposé d'autres conditions à la suite t_i). L'auteur ne donne pas de démonstrations.

C. Corduneanu (Iaşi)

Lochak, Georges 2472

Sur les perturbations rapidement oscillantes d'un système dynamique à stabilité asymptotique.

C. R. Acad. Sci. Paris **258** (1964), 3172–3175.

Given $dx_i/dt=X_i(x,t)$, a dynamical system with $x=0$ being asymptotically stable, and perturbation terms $R_i(x,t)$ such that $|R_i(x,t)|$ is bounded for bounded $|x|$, and all t; $|R_i(x,t)-R_i(x',t)|\leqq\varphi(|x-x'|)$ for some continuous non-negative φ, vanishing for zero argument; each $\int_{t'}^{t}R_i(x,\theta)\,d\theta$ vanishes at least once in every interval of length τ; then the author proves, by means of a Liapounov function, the following theorem: Given ε, sufficiently small, one can find numbers δ and τ such that for every solution of

$$\frac{dx_i}{dt}=X_i(x,t)+R_i(x,t),$$

one has $|x|<\varepsilon$, provided $|x_0|<\delta$ and τ is small enough; here, x_0 is the initial value, and the bounds on the

$|R_i(x, t)|$ are unimportant, as long as they are bounded. (In the proof, it is necessary to choose τ "sufficiently small", so that it appears as if the $R_i(x, t)$ must be "sufficiently rapidly oscillatory" to be able to do so.)

G. Latta (Stanford, Calif.)

Böning, W. 2473

Die Differentialgleichung der verlustbehafteten Spule mit gekrümmter magnetischer Kennlinie.

Z. Angew. Math. Mech. **43** (1963), 564–565.

The nonlinear problem $dx/dt + \alpha \sinh x = \beta \sin t$, $x(0) = 0$, is solved in terms of integrals involving a certain periodic solution $x_p(t)$ of the differential equation, where x_p can be approximated by a trigonometric series.

R. V. Churchill (Ann Arbor, Mich.)

Senik, P. M. 2474

Program quasi-linear oscillatory processes. (Russian)

Ukrain. Mat. Ž. **16** (1964), 268–273.

In this note the author considers the problem of finding the control parameters b_r $(r = 1, \cdots, m)$ so that the oscillatory process described by the system of differential equations

$$\dot{y} + \omega x = \sum_{r=1}^{m} F_r(x, y) b_r, \qquad \dot{x} - \omega y = 0$$

on the interval $t_0 \leqq t < t_N < \infty$ has given amplitude and frequency $a = \gamma(\tau)$, $\Omega = \omega + \varepsilon\lambda(\tau)$, and the integral

$$I = \int_{\tau_0}^{\tau_N} \{[x - au]^2 + [y - av]^2\} \, d\tau$$

has small values.

L. Pintér (Szeged)

Neĭmark, Ju. I. 2475

The method of point transformations in the theory of non-linear oscillations. (Russian. English summary)

Qualitative methods in the theory of non-linear vibrations (Proc. Internat. Sympos. Non-linear Vibrations, Vol. II, 1961), *pp.* 268–307. *Izdat. Akad. Nauk Ukrain. SSR, Kiev,* 1963.

Generally speaking, the method of point transformations consists in reducing the investigation of a continuous flow to that of a discrete one. In particular, if one has a piecewise linear differential system, it is natural to consider the transformation which replaces every switching point by the following one on the same orbit. (A switching point is a point where an orbit reaches the boundary of a linearity region.) In this way, e.g., the problem of stability of a periodic solution can be reduced to deciding the stability of the corresponding fixed point of the point transformation. The main part of the present survey paper covers the following subjects: Stability criteria for fixed points; connection between fixed points and periodic motions; dependence of fixed points and periodic motions on a parameter; safe and dangerous parts of the boundary of the region of stability (the theory of Bautin); connection with the method of small parameters; stability of equilibria on the surfaces of discontinuity (switching surfaces). The method of point transformations was initiated, essentially, by Andronov, Bautin, and Mayer beginning in 1944, and was developed further by the author and other researchers in Gorki. Of the 142 papers quoted in the report we can mention here only the following: the author and Šil'nikov, Izv. Akad. Nauk SSSR Otd. Tehn. Nauk Meh. Mašinostr. **1959**, no. 6, 51–59 [MR **25** #2295]; the author, Izv. Vysš. Učebn. Zaved. Radiofizika **1** (1958), no. 1, 41–66; ibid. **1** (1958), no. 2, 95–117; ibid. **1** (1958), no. 5–6, 146–165; ibid. **2** (1959), no. 3, 507–508; ibid. **3** (1960), 342–343; Haag, Bull. Sci. Math. (2) **73** (1949), 123–134 [MR **11**, 514]; the author, Dokl. Akad. Nauk SSSR **129** (1959), 736–739 [MR **24** #A2101]; Leonov, Izv. Vysš. Učebn. Zaved. Radiofizika **2** (1959), no. 6, 942–956.

P. Seibert (Mexico City)

Mel'nikov, G. I. 2476

On the theory of non-linear vibrations. (Russian. English summary)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **19** (1964), *no.* 1, 88–98.

The author studies solutions in the neighborhood of the origin of the autonomous system

$$(1) \quad \frac{dy_s}{dt} = \lambda_s y_s + P_s(y_1, \cdots, y_n) \qquad (s = 1, 2, \cdots, n),$$

where the y's are complex and the P's are either finite sums or absolutely convergent power series with bounded coefficients in some sphere about the origin. The P_s begin with terms of order ν.

The case studied by the author is that in which λ_1 and λ_2 are conjugate complex, $\lambda_{1,2} = \alpha \pm i\beta$, $|\alpha| \ll \beta$, $|\alpha| \ll |\mathrm{Re}\, \lambda_s|$, $s = 3, \cdots, n$. It is also assumed that the real parts of $\lambda_3, \cdots, \lambda_n$ are negative. It is also assumed that (2) for $k = \nu, \nu + 1, \cdots, m$, and $\nu_1, \nu_2, \cdots, \nu_n$ non-negative integers with sum k, the relations $\lambda_s = \nu_1\lambda_1 + \cdots + \nu_n\lambda_n$ are impossible except for inevitable cases with odd k which arise as a result of λ_1 and λ_2 having very small or zero real parts.

The author's method is to transform (1) to the form

$$(3) \quad \frac{dz_s}{dt} = \left(\lambda_s + \sum_{\substack{k=\nu \\ k \text{ odd}}}^{m} a_s^{(k)} r^{k-1}\right) z_s + Z_s^{(m+1)} \qquad (s = 1, \cdots, n)$$

by means of a polynomial transformation

$$(4) \quad z_s = y_s + \sum_{\nu_1 + \cdots + \nu_n = \nu}^{m} A_s^{(\nu_1, \nu_2, \cdots, \nu_n)} y_1^{\nu_1} y_2^{\nu_2} \cdots y_n^{\nu_n}.$$

In (3) the $a_s^{(k)}$ occur because of the exceptional cases in (2), $r = |z_1| = \sqrt{(z_1 z_2)}$, and the $Z_s^{(m+1)}$ contain terms of order $m+1$ or higher. This transformation is possible because of assumption (2). The system (3) is solved when the terms $Z_s^{(m+1)}$ are neglected, and the result, transformed back to the y's by the inverse of (4), is used as an approximate solution for (1).

The author investigates the stability of the solutions of (3) near the origin by means of Ljapunov functions. He obtains error estimates on the interval $0 \leqq t < \infty$. The ideas are illustrated with van der Pol's equation:

$$\frac{dx}{dt} = \varepsilon x - y - \varepsilon x y^2, \qquad \frac{dy}{dt} = x,$$

for small ε.

W. S. Loud (Darmstadt)

Kozinec, B. N. 2477

A stability criterion for Hill's equation. (Russian. English summary)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **19** (1964), *no.* 3, 18–24.

Author's summary: "Hill's differential equation

$$(1) \qquad y''+p(t)y = 0$$

is considered. The function $p(t)$ is said to belong to class $\mathbf{M}_{\alpha\beta}$ if the conditions

$$\frac{1}{T}\int_0^T p(t)\,dt = \alpha, \qquad \frac{1}{T}\int_0^T |p(t)-\alpha|\,dt = \beta$$

are fulfilled. The point (α_0, β_0) is said to be a point of stability if there exists a neighbourhood δ of (α_0, β_0) such that any equation (1) with the function $p(t)\in\mathbf{M}_{\alpha\beta}$ $((\alpha,\beta)\in\delta)$ possesses only bounded solutions. G. Borg [Ark. Mat. Astronom. Fys. **31A** (1944), no. 1; MR **8**, 70] obtained some sufficient conditions of stability of equation (1). In this note it is shown that these conditions are necessary."

Hochstadt, Harry 2478

Results, old and new, in the theory of Hill's equation.

Trans. New York Acad. Sci. (2) **26** (1963/64), 887–901.

This is a summary of some recent results on the spectrum of Hill's equation: G. Borg [Acta Math. **78** (1946), 1–96; MR **7**, 382], W. Magnus and A. Shenitzer [Div. Electromag. Res. Inst. Math. Sci. New York Univ. Res. Rep. No. BR-22 (1957); MR **19**, 856], the author [Comm. Pure Appl. Math. **14** (1961), 749–764; MR **24** #A2699; Amer. Math. Monthly **70** (1963), 18–26; MR **26** #2680; Arch. Math. **14** (1963), 34–38; MR **26** #5207; Math. Z. **82** (1963), 237–242; MR **27** #5955; Proc. Amer. Math. Soc. **14** (1963), 930–932; MR **27** #5956; Comm. Pure Appl. Math. **17** (1964), 251–255; MR **28** #2310], and D. M. Levy and J. B. Keller [ibid. **16** (1963), 469–476; MR **27** #3875].

N. P. Bhatia (Cleveland, Ohio)

Vu, Tuan 2479

On the theory of nth-order linear differential equations with periodic coefficients. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 728–732.

Author's summary: "The solution of a linear differential equation of the nth order of the form

$$\frac{d^nx}{dt^n}+a_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}}+\cdots+a_1(t)\frac{dx}{dt}+a_0(t) = e^{pt},$$

where

$$a_k(t) = a_k{}^0+\varepsilon\sum_\nu A_{k,\nu}e^{i\nu t},$$

is given in this paper. The author shows the existence for this equation of a unique solution $x=\sum_\nu C_\nu e^{(p+i\nu)t}$, where C_ν, values independent of t, are analytical functions with respect to the parameters p and ε."

Vu, Tuan 2480

On the stability of solutions of nth-order linear differential equations with variable coefficients. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1431–1433.

Author's summary: "A theorem is proved on the strong stability of a linear inhomogeneous differential equation of the nth order."

Onuchic, Nelson 2481

Relationships among the solutions of two systems of ordinary differential equations.

Michigan Math. J. **10** (1963), 129–139.

The stability of the system (1) $\dot{x}=A(t)x+g(t,x)$ with respect to the system (2) $\dot{y}=A(t)y$ as $t\to\infty$ is studied by means of Ważewski's topological methods [see L. Cesari's book on stability, *Asymptotic behavior and stability problems in ordinary differential equations*, second edition, Academic Press, New York, 1963; MR **27** #1661].

The results include sufficiency conditions for the existence of $x(t)-y(t)\to 0$ as $t\to\infty$, e.g., $A(t)=(f_{ij}(t))$ and $|g_j(t,x)|\leq h_j(t)$ such that

$$\int_t^\infty h_k(v)\left[\exp\int_v^t R(f_{kk}(s))\,ds\right]dv\to 0,$$

$$\int_t^\infty |f_{ij}(v)|\left[\exp\int_v^t R(f_{ii}(s))\,ds\right]dv+0, \qquad i\neq j,\ t\to\infty.$$

G. Latta (Stanford, Calif.)

Anitova, E. S. 2482

On the boundedness of the solutions of a system of differential equations of third order. (Russian. English summary)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **19** (1964), *no.* 1, 5–14.

A system of differential equations $w'=f(t,w)$ is said to be dissipative if there exists a constant $C>0$ such that for every solution $w(t)$ of this system $\limsup_{t\to\infty}|w(t)|<C$. The author considers the system

$$x' = a_{11}x+a_{12}y+a_{13}z+p_1(t,x,y,z),$$
$$y' = f(x)+a_{22}y+a_{23}z+p_2(t,x,y,z),$$
$$z' = a_{31}x+a_{32}y+a_{33}z+p_3(t,x,y,z),$$

where the functions p_i are bounded, and shows that it is dissipative if $f(x)$ satisfies inequalities of Hurwitz type. Hence if the functions p_i have period T, there exists a solution of period T. The paper is related to previous ones by Pliss and Ezeilo for scalar third-order equations. {To the references given in the paper, add J. O. C. Ezeilo [Proc. London Math. Soc. (3) **13** (1963), 99–124; MR **26** #418].}

W. A. Coppel (Canberra)

Huaux, Aimé 2483

Sur l'instabilité des solutions constantes, pour n impair, de l'équation différentielle $(d^nx/dt^n)+\varphi(x)=0$.

C. R. Acad. Sci. Paris **258** (1964), 5321–5323.

Author's summary: "Par des calculs élémentaires et la deuxième méthode de Ljapounov, on établit que toutes les solutions constantes réelles de l'équation différentielle ci-dessus sont instables dès que n est impair. Ces solutions constantes sont les zéros réels de $\varphi(x)$; on les suppose isolés."

To be more explicit, the author proves the following theorem for the differential equation mentioned: If (1) conditions for existence, unicity and continuity with respect to initial conditions of the solution of the differential equation are satisfied, (2) $\varphi(x)$ is continuous and the real zeros of $\varphi(x)$ (which are the "solutions constantes") are isolated, and (3) $n\geq 3$ is uneven, then any "solution

constante" is unstable. The proof makes use of Četaev's theorem. Unfortunately, this theorem is not explicitly stated. *J. Kestens* (Brussels)

Huaux, Aimé 2484

Sur l'instabilité des solutions constantes, pour n impair, du système différentiel $(d^n x_i/dt^n)+\varphi_i(x_1, x_2, \cdots, x_p)=0$.
C. R. Acad. Sci. Paris **258** (1964), 5573–5576.

The author considers constant solutions of the system

$$\frac{d^n x_i}{dt^n}+\varphi_i(x_1, x_2, \cdots, x_p) = 0 \qquad (i = 1, 2, \cdots, p)$$

which are assumed to be isolated. He proves that these solutions are unstable under the conditions: (1) the general solution exists, is unique and depends continuously on the initial values; (2) $\varphi_i(x_1, \cdots, x_p)$, $i=1, \cdots, p$, are continuous in a certain domain containing the considered solution; (3) n is odd and $\geqq 3$. The proof uses the second method of Lyapunov and a theorem of Četaev. It is shown that the condition $n \geqq 3$ must not be omitted.
H. Ehrmann (Clausthal-Zellerfeld)

Akmagambetov, B. A. 2485

A three-dimensional turbulent hereditary dynamical system. (Russian. Kazak summary)
Izv. Akad. Nauk Kazah. SSR Ser. Fiz.-Mat. Nauk **1963**, *no.* 3, 53–58.

Under certain restrictions, the author determines a region of stability for the system of equations

$$\ddot{x}+a(t)\dot{x}+b(t)x = p(t)y,$$

$$\dot{y} = q(t)x-\operatorname{Re} p\{y[t-d(t)]\},$$

where

$$\operatorname{Re} p(z) = +1, \quad z \geqq \xi_0 > 0,$$

$$= -1, \quad z \leqq -\xi_0,$$

a control with an open region of insensitivity, and also establishes the existence of a region of asymptotic stability. *M. A. Geraghty* (Iowa City, Iowa)

Hohrjakov, A. Ja. 2486

On a periodic boundary-value problem for a differential equation of third order. (Russian)
Mat. Sb. (N.S.) **63 (105)** (1964), 639–645.

In the first part of this article the author considers the boundary-value problem $K(a, f, y)$:

(1) $$L[y] \equiv y''' + a(x)y = f(x),$$

(2) $$y(\alpha)-y(\beta) = 0, \quad y'(\alpha)-y'(\beta) = 0, \quad y''(\alpha)-y''(\beta) = 0,$$

where the functions $a(x)$ ($\not\equiv 0$) and $f(x)$ are continuous and $a(x)$ does change its sign. Let $[\alpha, r(\alpha))$ be the maximal non-oscillatory interval of $L[y]=0$ and $\beta \in [\alpha, r(\alpha))$. It is proved that the problem $K(a, f, y)$ has a uniquely determined solution for every continuous function $f(x)$. Further, assuming that $a(x)$ and $f(x)$ are continuous periodic functions with period $\omega=\beta-\alpha$, the equation (1) has a uniquely determined periodic solution with period ω. A comparison theorem is also proved.

In the second part the nonlinear boundary-value problem: (3) $N[y]\equiv y'''+f(x, y)=0$, $y(\alpha)-y(\beta)=0$, $y'(\alpha)-y'(\beta)=0$, $y''(\alpha)-y''(\beta)=0$ is considered. The function $f(x, y)$ is defined in the convex domain G and is continuous there. According to N. V. Azbelev, the function $f(x)$ has the property L_1 [respectively, L_2] in G, if for arbitrary continuous functions $u(x)$, $v(x)$ with $u(x)\geqq v(x)$ and with graphs in G, there exists a continuous function $p(x)$ [$q(x)$] such that $f(x, u)-f(x, v)\geqq p(x)(u-v)$ [$f(x, u)-f(x, v)\leqq q(x)(u-v)$]. A typical result is the following: The function $f(x, y)$ satisfies L_1 and L_2 in G with the functions $p(x)$ and $q(x)$ ($p(x)$ and $q(x)$ are of constant sign). Suppose that for the functions $z_1(x)$ and $z_2(x)$ (both in G) with the property (2), the inequalities $N[z_1]\leqq 0$, $N[z_2]\geqq 0$ are valid. Then the problem (3) has a uniquely determined solution $y(x)$, with $z_1(x)\leqq y(x)\leqq z_2(x)$ if $p(x)\geqq 0$ and $z_1(x)\geqq y(x)\geqq z_2(x)$ if $q(x)\leqq 0$.

In the proofs the Green's functions play an important role. *L. Pintér* (Szeged)

Zadiraka, K. V. 2487

On periodic solutions of an irregularly perturbed equation which is weakly non-linear. (Russian. English summary)
Ukrain. Mat. Ž. **16** (1964), 292–299.

Consider the system

(1) $$\dot{x} = f(x, z, \mu), \qquad \dot{z} = F(x, z, \varepsilon, \mu),$$

where ε, μ are small parameters of the same order of smallness, and the corresponding degenerate system

(2) $$\dot{x} = f(z, x, 0), \qquad 0 = F(x, z, \varepsilon, 0).$$

Notice that ε is not zero in (2). Assuming a periodic solution of (2) satisfying certain properties, the author applies the method of integral manifolds [see Bogoljubov and Mitropol'skiĭ, *Asymptotic methods in the theory of nonlinear oscillations* (Russian), second edition, Fizmatgiz, Moscow, 1958; MR **20** #6812] to obtain the existence of a limit cycle of (1) for ε and μ small. Conditions are also given for the orbital stability of this limit cycle. The paper terminates with an interesting application to a second-order equation perturbed by terms in μ in the higher derivatives. *J. K. Hale* (Providence, R.I.)

Inaba, Mituo 2488

Correction: "On the theory of differential equations in coordinated spaces".
Kumamoto J. Sci. Ser. A **6**, 48–49 (1963).

The author amends and corrects the proof of a theorem of his earlier paper [same J. **5** (1961), 119–136; MR **25** #5251].

Vasil'eva, A. B.; Zimin, A. B. 2489

Asymptotic behaviour of certain classes of differential equations with a small parameter multiplying the highest derivative. (Russian. English summary)
Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, *no.* 4, 21–29.

Previously [Mat. Sb. (N.S.) **50 (92)** (1960), 43–58; MR **22** #2772] the first author treated the problem of representing

a solution of the singular perturbation problem, $\mu\, dz/dt = F(z, t)$, $z(0) = z^0$, asymptotically in μ under the conditions that $dF(z, t)/dz < 0$ when $z = \varphi(t)$, where $F(\varphi(t), t) = 0$ and $\varphi(0) = z^0$, while every stationary point of $dz/d\tau = F(z, t^0)$, $\tau = t\mu^{-1}$, $0 \leq t_0 \leq T$, is asymptotically stable. In this paper the condition on dF/dz is replaced by $d^kF/dz^k = 0$ for $k < 2m+1$, $d^{2m+1}F/dz^{2m+1} < 0$. It is also assumed that $d\varphi/dt \neq 0$, $0 \leq t \leq T$. There results an asymptotic expansion of the solution of the singular perturbation in terms of $\mu^{1/(2m+1)}$ valid on an interval $[t_0, T]$, where t_0 is as small as desired. *C. S. Coleman* (Claremont, Calif.)

Lin', Czun-Či [Lin, Chün Chi] 2490

Asymptotic behaviour of solutions of the Cauchy problem when the limiting equation has a singularity. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 522–525.

Soit l'équation (E) $\varepsilon^2 y'' + a(x)y' + b(x)y = f(x)$, avec les conditions initiales (I) $y(0) = c_0$, $y'(0) = 0$, dans laquelle ε est un petit paramètre positif et $a(x)$, $b(x)$, $f(x)$ sont des fonctions définies pour $x > 0$ à l'aide des formules $a(x) = x + \sum_2^\infty a_k x^k$, $b(x) = \sum_0^\infty b_k x^k$, $f(x) = \sum_0^\infty f_k x^k$. Pour $\varepsilon = 0$ on obtient l'équation (E_0) $a(x)y' + b(x)y = f(x)$ qui admet une singularité au point $x = 0$. L'auteur cherche la solution $y_\varepsilon(x)$ de l'équation (E), satisfaisant aux conditions (I), sous la forme $y_\varepsilon(x) = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$. On obtient deux expressions différentes, la première étant valable pour $0 < x < \varepsilon^{1/2}$ et la seconde pour $\varepsilon^{1/2} < x < N$, où N est un nombre qu'on peut déterminer et qui ne dépend pas de ε. Le travail contient la méthode qui permet de trouver les fonctions $y_k(x)$ et quelques renseignements sur ces fonctions. *C. Corduneanu* (Iași)

Bunke, H. 2491

Zur Bestimmung der Stabilitätswahrscheinlichkeit stochastischer Systeme.

Monatsb. Deutsch. Akad. Wiss. Berlin **5** (1963), 277–280.

Given a stochastic process $\xi = \xi_t$ and a reasonable function $f = f(\xi, \eta, t)$ with $f(\xi, 0, t) \equiv 0$, the problem is posed of computing the probability that the null solution $\eta = \eta_t \equiv 0$ of $\eta^\bullet = f(\xi, \eta, t)$ should be stable in the sense of Liapunov. A brief discussion follows.

H. P. McKean, Jr. (Cambridge, Mass.)

Salvadori, Luigi 2492

Ancora sulla ricerca di una funzione di Liapounoff sotto forma polinomiale per un sistema differenziale interessante la Meccanica.

Ricerche Mat. **13** (1964), 92–96.

On considère la stabilité asymptotique du système autonome

$$\frac{dx_i}{dt} = -\lambda_i y_i + X_i, \quad \frac{dy_i}{dt} = \lambda_i x_i + Y_i \quad (i = 1, 2, \cdots, n).$$

L'existence d'un polynôme V réduit la question de la stabilité asymptotique à l'examen qualitatif de V. Deux classes I, I' de polynômes sont convenables pour la recherche de V. Cette note, contenant une condition nécessaire et suffisante, confirme que cette direction de recherches n'implique aucune restriction essentielle pour l'application concrète. Le rôle central appartient à la fonction

$$\sum_1^n \left[\frac{\partial f}{\partial x_i}(-\lambda y_i + X_i) + \frac{\partial f}{\partial y_i}(\lambda_i x_i + Y_i)\right]$$

pour $f = V$, ainsi qu'aux formes

$$R_m = \sum G_{p_1 p_2 \cdots p_n}(x_1^{2p_1} x_2^{2p_2} \cdots x_n^{2p_n} + y_1^{2p_1} y_2^{2p_2} \cdots y_n^{2p_n}),$$

$$R_m' = \sum G'_{p_1 p_2 \cdots p_n}(x_1^2 + y_1^2)^{p_1}(x_2^2 + y_2^2)^{p_2} \cdots (x_n^2 + y_n^2)^{p_n}$$

(où $p_1 + p_2 + \cdots + p_n = p$) correspondant aux classes I, I'.

M. Bertolino (Belgrade)

Bykov, Ja. V. 2493

Conditions for the existence of a limiting periodic regime for non-linear integro-differential equations. (Russian. English summary)

Qualitative methods in the theory of non-linear vibrations (*Proc. Internat. Sympos. Non-linear Vibrations, Vol. II*, 1961), *pp.* 98–99. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, 1963.

The author states without proof conditions under which there exists a limiting periodic (or almost periodic) regime for the integro-differential equation

$$(1)\quad \frac{du}{dt} = B(t)u + \int_a^t K(t-s)u(s)\,ds + \psi[t, u(t)] + \int_a^t \Psi[t, s, u(s)]\,ds.$$

A limiting regime is a vector function $y(t)$ such that as $t \to +\infty$ or $-\infty$, all solutions $u(t)$ of (1) converge to $y(t)$ (i.e., $\|u(t) - y(t)\| \to 0$). It is assumed that $f(t) \equiv \psi(t, 0) + \int_a^t \Psi(t, s, 0)\,ds$ is periodic (or almost periodic) and that the roots of a certain characteristic equation have negative real parts. The complete translation of the paper into English is given. *W. S. Loud* (Darmstadt)

Norkin, S. B. 2494

On the solutions of a linear, non-homogeneous, second-order differential equation with retarded argument. (Russian)

Uspehi Mat. Nauk **16** (1961), *no.* 2 (98), 143–148.

The author considers the equation

$$(1)\quad y''(x) + N(x)y(x) + M(x)y(x - \Delta(x)) = f(x),$$

where $M(x)$, $N(x)$, $f(x)$, and $\Delta(x) \geq 0$ are continuous functions on $[A, B)$, with the initial conditions $y(x) = \bar\varphi(x)$, a given continuous function, on the initial set E_A and $y'(A+0) = y_A'$.

Assume that the non-trivial solutions of the homogeneous equation $y''(x) + N(x)y(x) + M(x)y(x - \Delta(x)) = 0$, with an appropriate class of initial conditions, do not have multiple zeros. {The author has obtained conditions which guarantee this in earlier papers [e.g., Uspehi Mat. Nauk **14** (1959), no. 1 (85), 199–206; MR **21** #5052].} Then the solution of (1) can be represented in terms of the solution of a linear Volterra integral equation, the kernel of which is expressed in terms of a "fundamental system" of two solutions of the homogeneous equation. Boundedness of the kernel is proved for two examples.

R. D. Driver (Albuquerque, N.M.)

Norkin, S. B. 2495
On the coalescence of solutions of a linear, homogeneous, second-order differential equation with retarded argument. (Russian)
Approximate methods of solving differential equations, pp. 82–89. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, 1963.

Continuing his study of the equation $y''(x)+N(x)y(x)+M(x)y(x-\Delta(x))=0$ on $[A, B)$ [Uspehi Mat. Nauk **14** (1959), no. 1 (85), 199–206; MR **21** #5052], the author discusses conditions under which two solutions can (or cannot) coincide on certain intervals without being identical on $[A, B)$. *R. D. Driver* (Albuquerque, N.M.)

Norkin, S. B. 2496
Periodic motions of a class of oscillatory systems with time lag. (Russian. English summary)
Qualitative methods in the theory of non-linear vibrations (*Proc. Internat. Sympos. Non-linear Vibrations, Vol. II*, 1961), pp. 315–321. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, 1963.

The author considers the motion of a mass suspended on a weightless spring with a Stokes friction. The forcing function is a product of a periodic function and the displacement of the mass at a certain previous instant. The amount of delay of the argument in this displacement depends on time and the displacement as well. The author derives the conditions that must be satisfied by the friction coefficient, by the spring constant, and by the amount of delay, such that a strictly periodic motion results. *Č. Masaitis* (Aberdeen, Md.)

Fodčuk, V. I. 2497
On the continuous dependence on a parameter of solutions of differential equations with retarded argument. (Russian)
Ukrain. Mat. Ž. **16** (1964), 273–279.

Consider a difference-differential system

(1) $dx(t)/dt = f[t, x(t), x(t-\Delta), \lambda]$ for $0 < t < T$,

with (2) $x(t, \lambda)=\varphi(t, \lambda)$ for $-\Delta \leqq t \leqq 0$, where $f(t, x, y, \lambda)$ satisfies conditions slightly stricter than Carathéodory's for ordinary differential equations, $\varphi(t, \lambda)$ is bounded and continuous with respect to t, and the solution of (1), (2) is assumed unique when $\lambda=\lambda_0$. The author then shows that the solution depends continuously on λ at λ_0 provided

$$\lim_{\lambda\to\lambda_0} \int_0^t f(\tau, x, y, \lambda)\, d\tau = \int_0^t f(\tau, x, y, \lambda_0)\, d\tau$$

and

$$\lim_{\lambda\to\lambda_0} \varphi(t, \lambda) = \varphi(t, \lambda_0)$$

uniformly with respect to t, x, y, and t, respectively. The proof makes use of methods of Kurzweil and Vorel [Czechoslovak Math. J. **7 (82)** (1957), 568–583; MR **22** #2734]. The above result is then applied to obtain a theorem on the method of averaging.

For the existence theorem under Carathéodory-type hypotheses the reader is referred to Fodčuk [Dopovidi Akad. Nauk Ukraïn. RSR **1962**, 1541–1545; MR **26** #4005]. A similar result was also given by Sansone [Ann. Mat. Pura Appl. (4) **39** (1955), 65–67; MR **17**, 1084]. *R. D. Driver* (Albuquerque, N.M.)

PARTIAL DIFFERENTIAL EQUATIONS

See also 2219, 2405, 2406, 2407, 2419, 2420, 2421, 2424, 2615, 2617, 2624, 2627, 2639, 2672, 2674, 2675, 2676, 2682, 2684, 2820, 2858, 2982, 2983, 3067.

Tikhonov, A. N. [Tihonov, A. N.]; Samarskii, A. A. [Samarskiĭ, A. A.] 2498
★Equations of mathematical physics.
Translated by A. R. M. Robson and P. Basu; translation edited by D. M. Brink. A Pergamon Press Book.
The Macmillan Co., New York, 1963. xvi+765 pp. $17.50.

This is a translation of the second Russian edition [GITTL, Moscow, 1953; MR **16**, 364]; the first edition (1951) was reviewed earlier [MR **15**, 430].

For anyone reading the English version it might have been helpful to replace some of the references to "well-known" facts [e.g., p. 5, first footnote] by references not in Russian, and to replace references to Russian translations of Western classics by the original Western references.

Palamodov, V. P. 2499
Underdetermined and overdetermined systems of differential equations with constant coefficients. (Russian)
Dokl. Akad. Nauk SSSR **156** (1964), 1288–1291.

Let $pu=w$ be a system of partial differential equations, where p is a given $m\times l$ matrix of differential operators with constant coefficients, w [respectively, u] being m-[l-]tuples of functions or distributions. The author assumes mainly that the system is under- [over-] determined, i.e., that the rank of p [of its transpose p'] is less than l. The subject of the paper can be roughly described as follows. One considers some properties of p given by relations between the space of solutions in a domain G and the space of solutions in a neighborhood of the boundary of G. The author states several interesting results on the relationship between these properties and the properties of some chains of syzygies derived from p. The announcements are too complicated to be formulated here in detail. *S. Łojasiewicz* (Kraków)

Cazzani Nieri, Maria Grazia 2500
Un teorema di esistenza per un sistema di equazioni a derivate parziali del primo ordine.
Ist. Lombardo Accad. Sci. Lett. Rend. A **97** (1963), 455–481.

Uniqueness of the solution of Darboux's problem for a system of partial differential equations of the first order has been discussed in a generalized sense in an earlier paper by the same author [same Rend. **96** (1962), 334–342; MR **27** #2708]. In the present paper existence of the solution is discussed. For this discussion, instead of using the unknown functions $z_j(x, y)$ $(j=1, \cdots, m)$, new functions $u_j(x, y)$ $(j=1, \cdots, m)$ are introduced. Relations between these two sets of functions are given in the following form:

$$\sum_{j=1}^{m} a_{ij}(0, 0; 0, \cdots, 0)z_j(x, y) = u_i(x, y) \qquad (i = 1, \cdots, m).$$

Proving that $u_j(x, y)$ exist as solutions, existence of solutions $z_j(x, y)$ can be shown by using the above relations. *O. Gürel* (Ankara)

Kušnirčuk, Ĭ. F. 2501

The Cauchy problem for a class of partial differential equations of higher order. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 720–725.

Author's summary: "In this note the Cauchy problem for a partial differential equation is solved by the method of increasing the number of independent variables, under the condition that the characteristic differential equations in the complex-real space are linear in the complex variable."

Knowles, J. K.; Messick, R. E. 2502

On a class of singular perturbation problems.

J. Math. Anal. Appl. **9** (1964), 42–58.

This paper presents a detailed "boundary layer" and "corner layer" analysis of a boundary-value problem for $\Delta u - \lambda u_y = 0$ in a rectangular semi-infinite strip $0 < x < \infty$, $0 < y < 1$, with $\lambda \gg 1$. The edge $x = 0$ of the strip is a characteristic line for the reduced equation, $u_y = 0$, and the authors point out that in this case, the boundary-layer approximations involve partial differential equations, in contrast to the case of a non-characteristic boundary, in which only ordinary differential equations arise.

The formal results are then compared with the exact solution, verifying the approximations.

G. E. Latta (Stanford, Calif.)

Ranger, K. B. 2503

On a class of boundary value problems for the biharmonic equation and the Stokes repeated operator equation.

J. Math. and Phys. **43** (1964), 75–82.

The author gives integral representation formulae which transform solutions to n-dimensional repeated operator equations of the Stokes type into solutions of the $(n+1)$-dimensional operator equation, and vice versa. One particular application which the author has in mind is the reduction of Stokes's flow problems in three dimensions to boundary-value problems for the biharmonic operator, these latter problems having been exhaustively treated in the literature. Integral transform formulae in various coordinate systems are given.

L. E. Payne (College Park, Md.)

Mullikin, A. L.; Smith, K. T. 2504

Elliptic differential problems with high order boundary conditions.

Math. Scand. **12** (1963), 178–188.

Les auteurs considèrent A, A_i, B, les opérateurs différentiels à coefficients constants de l'ordre $2m$, m_i, $i = 1, 2, \cdots, m$, $s \leqq 2m$, respectivement. On désigne par x, ξ le point générique de R^n, C^n respectivement, $x = (x', x_n)$. Le système A, A_i satisfait aux conditions usuelles [cf. S. Agmon, A. Douglis, et L. Nirenberg, Comm. Pure Appl. Math. **12** (1959), 623–727; MR **23** #A2610]: pour chaque $\xi' \in R^{n-1}$, $\xi' \neq 0$, le polynôme $A(\xi', \xi_n)$ a précisément m racines à partie imaginaire positive, $\tau_1, \tau_2, \cdots, \tau_m$, et m à partie imaginaire négative. Les polynômes $A_i(\xi', \xi_n)$ sont linéairement indépendants modulo $(\xi_n - \tau_1), \cdots, (\xi_n - \tau_m)$.

On désigne encore par $C_0^\infty(R_+^n)$ l'espace des fonctions indéfiniment différentiables dans le demi-espace R_+^n, à support compact, et par

$$|u|_{k,L^p(\Omega)} = \left(\sum_{|\alpha|=k} \int_\Omega |D_\alpha u|^p \, dx\right)^{1/p},$$

$$\|u\|_{k,L^p(\Omega)} = \left(\sum_{|\alpha|\leqq k} |D_\alpha u|^p \, dx\right)^{1/p}.$$

Théorème: Pour chaque $v \in C_0^\infty(R_+^n)$, il existe $u \in C^\infty(R_+^n)$ telle que $Au = Bv$ dans R_+^n, $A_i u = 0$ pour $x_n = 0$, u est de croissance polynomiale dans ∞ et

$$(*) \qquad |u|_{2m-s+k,L^p} \leqq \text{const}\, |v|_{k,L^p},$$

$$1 < p < \infty, \quad k \geqq 0, \text{ un entier.}$$

Les auteurs définissent ensuite des autres opérateurs différentiels, soit B_i, $i = 1, 2, \cdots, m$, et démontrent encore le même énoncé avec $v \in C_0^\infty(R^n)$, $B_i v = 0$ pour $x_n = 0$. L'appartenance de v à cette classe est aussi une condition nécessaire pour la validité de (*) pour les fonctions de croissance polynomiale dans ∞. La fonction u de (*) est déterminée uniquement sauf un polynôme.

On désigne par $L_m^p(\Omega)$ l'espace des fonctions de $p^{\text{ième}}$ puissance sommable dans Ω avec toutes les dérivées de l'ordre $\leqq m$.

Comme une application du théorème mentionné, il est démontré la représentabilité de chaque fonctionnelle $\varphi(u)$ sur $L_m^p(\Omega)$, pour Ω de C^m, sous la forme

$$\int_\Omega \sum_j P_j u P_j \bar{v} \, dx = \varphi(u)$$

avec $v \in L_m^{p'}(\Omega)$. Ici P_j sont des opérateurs différentiels à coefficients assez réguliers dans $\bar{\Omega}$, de l'ordre $\leqq m$, p_j leurs parties principales. On suppose: (a) pour $x \in \Omega$, $p_j(x, \xi)$ n'ont pas dans R^n une racine commune $\xi \neq 0$, (b) pour $x \in \partial\Omega$, $p_j(x, \xi)$ n'ont pas dans C^n une racine commune $\xi \neq 0$ telle que Im ξ soit orthogonal à $\partial\Omega$ dans x. Les auteurs ont ainsi obtenu une généralisation du résultat de J. L. Lions et E. Magenes [Ann. Scuola Norm. Sup. Pisa (3) **15** (1961), 41–103; MR **26** #4048]; cf. encore M. Schechter [Amer. J. Math. **85** (1963), 1–13].

J. Nečas (Prague)

Jenkins, Howard 2505

Super-solutions for quasi-linear elliptic equations.

Arch. Rational Mech. Anal. **16** (1964), 402–410.

The author studies elliptic equations

$$(1) \quad L[\varphi] = A(x, y, \varphi, \varphi_x, \varphi_y)\varphi_{xx} + 2B(x, y, \varphi, \varphi_x, \varphi_y)\varphi_{xy} + C(x, y, \varphi, \varphi_x, \varphi_y)\varphi_{yy} = 0$$

which are of "minimal surface type" in a sense defined by the reviewer [Ann. of Math. (2) **60** (1954), 397–416; MR **16**, 592]. This condition imposes a bound on the ratio of principal curvatures at each point of any solution surface. The author observes that, conversely, any surface $\varphi(x, y)$ for which the ratio of curvatures lies everywhere on one side of this bound satisfies an inequality $L[\varphi] < 0$, and hence exceeds, in any domain, any solution of (1) which achieves the same boundary data. By exploiting this geometrical property of the solutions, he is able to construct explicitly two families of "super-solutions" of (1). The first of these consists of functions defined in a domain bounded by a parabolic arc and by a straight segment; each function $\to 0$ on the arc and $\to +\infty$ on the segment. This result

implies for all solutions of equations of the indicated type a remarkable a priori bound (Theorem 1), of a kind first indicated by the author and Serrin [same Arch. **12** (1963), 185–212; Lemma 6; MR **26** #2729] and later announced in improved form as a general principle by the reviewer [ibid. **14** (1963), 337–375; Theorem VIII. 6; MR **28** #336; see also J. Analyse Math. (to appear), esp. Theorems 6 and 9]. The essential feature of the result is that in a domain bounded in part by a straight segment Γ, there is a bound on the solution (and, at least in some cases, of its derivatives) which depends only on the boundary data on the complement of Γ. The author's result enlarges in an important way the domain of validity of this principle.

The second family of majorants consists of functions with circumferential symmetry in an annulus, and are obtained under an assumption significantly weaker than that of "minimal surface type". The author uses them to derive local bounds on the solution. They could also have served, as the reviewer has remarked [Theorem 6, J. Analyse Math. (to appear)], to obtain the author's Theorem 1 in a much more general case than he has indicated.

It would be interesting to determine whether interior bounds on the derivatives of the solution can be obtained under the full generality of the author's hypotheses. Such bounds would lead to a general existence theory paralleling but differing in significant ways from the established theory for uniformly elliptic equations.

An underlying idea similar to that of the author has also been developed by the reviewer, although along somewhat different lines, leading to a different but closely related set of results [J. Analyse Math. (to appear)].

R. Finn (Stanford, Calif.)

Burčuladze, T. V. 2506

Some boundary-value problems for a class of elliptic systems. (Georgian. Russian summary)

Soobšč. Akad. Nauk Gruzin. SSR **31** (1963), 513–520.

In an earlier paper [same Soobšč. **30** (1963), 697–704; MR 28 #3241] the author generalized the method of potentials for certain elliptic systems. In the present paper he gives an application in the particular case that $m=2$ and n is an integer. In particular, an existence and uniqueness theorem is proved for an equation of the form $A(\partial/\partial x)u(x)=\omega(x)$, where the operator A is given by

$$A\left(\frac{\partial}{\partial x}\right) = \sum_{i_1,i_2=1}^{2} A^{(i_1,i_2)}\frac{\partial^m}{\partial x_{i_1}\partial x_{i_2}},$$

and the $A^{(i_1,i_2)}$ are $n\times n$ matrices satisfying a certain condition.

Skorobogat'ko, V. Ja.; Bobik, O. I. 2507

New uniqueness criteria for the solution of the first boundary-value problem for an equation of elliptic type and for the system of equations of an atomic reactor. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 703–706.

The equation in question is

$$\sum_{k,l=1}^{m} a_{kl}(x)\frac{\partial^2 u}{\partial x_k \partial x_l} + \sum_{l=1}^{m} b_l(x)\frac{\partial u}{\partial x_l} + c(x)u = 0,$$

where $x=(x_1,\cdots,x_m)$, and where the domain D of definition has a piecewise smooth boundary S. The coefficients $b_l(x)$ are continuously differentiable in D, the $a_{kl}(x)$ are twice continuously differentiable in D, while $c(x)$ is piecewise continuous in D.

Ikebe, Nobunori; Ono, Akira 2508

On the Cauchy problem of elliptic partial differential equations.

Mem. Fac. Sci. Kyushu Univ. Ser. A **17** (1963), 153–163.

The Cauchy problem in a half-space is considered for homogeneous, constant-coefficient elliptic operators. If

$$L(\xi_1,\cdots,\xi_n,\tau) = \sum_{|k|=0}^{2m} a_{k_1\cdots k_n,2m-|k|}\xi_1^{k_1}\cdots\xi_n^{k_n}\tau^{2m-|k|},\qquad |k| = k_1+\cdots+k_n,$$

the solution of

$$L\left(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_n},\frac{\partial}{\partial t}\right)u = f(x,t),\qquad t>0,$$

$$\left.\frac{\partial^j u}{\partial t^j}\right|_{t=0} = \phi_j(x_1,\cdots,x_n),\qquad j=0,1,\cdots,2m-1,$$

is constructed by means of kernels following Agmon, Douglis and Nirenberg [Comm. Pure Appl. Math. **12** (1959), 623–727; MR **23** #A2610]. Schauder-type estimates are then derived from the representation formulas thus obtained. *M. Schechter* (New York)

Parasjuk, L. S. 2509

Fundamental solutions of elliptic systems of differential equations with discontinuous coefficients. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 986–989.

Author's summary: "An interesting fact is established for elliptical systems of differential equations, the coefficients of which have a weak polar capacity with lower-order derivatives: the estimate of the fundamental solution and some of its derivatives proved to be the same as in the case of smooth coefficients."

Parasjuk, L. S. 2510

Behaviour of solutions of elliptic systems of differential equations with discontinuous coefficients. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1127–1129.

The author studies the behaviour of the solutions of elliptic equations in the case that the coefficients of lower-order derivatives are permitted to have singularities like poles. A typical result is the following (Theorem 2). Let $U(x)$ be a solution of the equation $A(x,\partial/\partial x)U(x)=0$ which is regular in a region D, except at $x=0$, with $|x|^\beta U(x)$ regular. Then the solution $U(x)$ can be written in the form

$$U(x) = \sum_{|k|\leq s-1}\left(\frac{\partial}{\partial y}\right)^k \bar{\omega}(x,y)|_{y=0}C_k + U_0(x),$$

where $\bar{\omega}(x,y)$ is a normal fundamental solution, the C_k are certain constant matrices, and $U_0(x)$ is a regular solution of the equation $A(x,\partial/\partial x)U(x)=0$ in D.

Roĭtberg, Ja. A. 2511

Elliptic problems with non-homogeneous boundary conditions and local increase of smoothness of generalized solutions up to the boundary. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 798–801.

The author states results on regularity and solvability of elliptic boundary problems, concerning maps $A:u \to (Lu, B_1u, \cdots, B_mu)$, where L is an elliptic operator of order $2m$, and the B_j form a "normal covering" set of boundary operators on the boundary of a smooth bounded region G. The main object is to obtain results when u is a distribution in a class $\tilde{W}_2^s$ (s an arbitrary integer), taken as the closure of the smooth functions on G in the norm

$$|||u|||_s = \|u\|_{W_2^s(G)} + \sum_{k=1}^{2m} \|\partial^{k-1}u/\partial\nu^{k-1}\|_{W_2^{s-k+1/2}(\partial G)},$$

where ν is a normal to ∂G, and $W_2^s(G)$ for $s<0$ is defined as in Berezanskiĭ [Uspehi Mat. Nauk **18** (1963), no. 1 (109), 63–96; MR **29** #1553]. In this setting, A has closed range and finite-dimensional kernel and cokernel, as an operator from $\tilde{W}_2^s(G)$ to $W_2^{s-2m}(G) \oplus \sum W_2^{s-m_j-1/2}(\partial G)$. The equivalence of weak and strong solutions to such problems is stated, as well as a mixed global-local regularity theorem: Let χ be a C^∞ function with vanishing normal derivative on ∂G; if $u \in W_2^s(G)$ and $\chi Au \in W_2^{s-2m+1}(G) \oplus \sum_1^m W_2^{s-m_j+1/2}(\partial G)$, then $\chi u \in W_2^{s+1}(G)$.

For closely related results see Lions and Magenes [Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **32** (1962), 873–883; MR **26** #6565] and Berezanskiĭ, Kreĭn, and the author [Dokl. Akad. Nauk SSSR **148** (1963), 745–748; MR **26** #4030]. The present article appears to extend each of these in some respects.

R. T. Seeley (Waltham, Mass.)

Marcinkowska, H. 2512

Differentiability theorems for differential equations, which are elliptic with respect to some of the independent variables.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 737–743.

The author considers linear partial differential operators of the form $L(x, y, D_x, D_y) = L_1(x, y, D_x) + L_2(x, y, D_x, D_y)$, where L_1 is of order $2m$ in D_x and L_2 is of order k in D_x and n in D_y, with $k \leqq 2m-1$. The coefficients of the operators and the regions under consideration are subject to conditions similar to those of his earlier papers [Studia Math. **22** (1962/63), 91–107; MR **26** #492; ibid. **23** (1963), 85–103; MR **29** #1423], so that L_1 is uniformly elliptic. Using the methods and results of the aforementioned papers, he discusses interior regularity and regularity of the periodic solutions of these equations.

R. S. Freeman (College Park, Md.)

Bers, Lipman; Schechter, Martin 2513

Elliptic equations.

Partial Differential Equations (Proc. Summer Seminar, Boulder, Col., 1957), *pp.* 131–299. *Interscience, New York,* 1964.

The first chapter is an interesting survey of some of the properties of solutions of elliptic equations, and a number of important theorems are stated without proofs. Weak and strong solutions are defined and the strong nature of weak solutions is noted. Some discussion is given on analyticity, Hölder continuity, unique continuation, the Runge property, normal boundary conditions, and strongly and properly elliptic operators. Chapter 2 is on the maximum principle for second-order elliptic operators. Various forms of this are proved and applied to give uniqueness theorems for the Dirichlet and Neumann problems. Next, the maximum principle is applied to the effective numerical computation of a solution of a Dirichlet problem by the method of finite differences. Then a maximum principle for gradients is proved and used to prove Carleman's unique continuation theorem.

Chapter 3 begins the study of Hilbert-space methods with the particular case of periodic solutions of an elliptic equation which lends itself well to illustrative treatment. For $u = \sum \alpha_l \exp(il\cdot x)$, $v = \sum \beta_l \exp(il\cdot x)$, one defines $(u, v)_t = (2\pi)^n \sum (1 + l\cdot l)^t \alpha_l \beta_{-l}$ $(\beta_{-l} = \bar{\beta}_l)$ and denotes by H_t the completion of the space of trigonometric polynomials in the t-norm obtained from this scalar product. The scale of spaces H_t with $H_\infty = \bigcap H_t$, $H_{-\infty} = \bigcup H_t$, has many interesting properties which are developed (such scales, in a more general context, play an increasingly important role in modern work). Gårding's inequality is proved for the elliptic operator $L = \sum_{|p| \leq m} a_p(x) D^p$, and some basic a priori estimates are established. It is then proved that L and L^* form a Fredholm pair. In two appendices there are proved some facts about projections and continuous linear functionals on a Hilbert space, the Lax-Milgram theorem, the Banach-Saks theorem, and some results of the Fredholm-Riesz-Schauder theory.

Chapter 4 deals first with interior regularity and then a general Dirichlet problem for a bounded region is studied by Hilbert space methods. A scale of spaces H^t is constructed with $(u, v)_t = \sum_{|p| \leq t} \int D^p u D^p v$, and H_0^t is the closure in H^t of C^∞ functions with compact support. The existence of a unique weak solution $u \in H_0^{m/2}$ of the Dirichlet problem $(L+\lambda)u = f$, $f \in H^0$, $\lambda \geqq \lambda_0$, is obtained by means of a Gårding inequality and the Lax-Milgram theorem. Then regularity at the boundary is studied and $f \in C^\infty$ is shown to imply $u \in C^\infty$ if $u \in H_0^{m/2}$ is a solution of the Dirichlet problem $Lu = f$ (hence u is a classical solution); L is assumed to be of even order m. In an appendix it is proved by Hilbert-space methods that if f is real-analytic and $Lu = f$ with the coefficients of the elliptic operator L real-analytic, then u is real-analytic.

Chapter 5 describes the potential-theoretic approach and the construction of a fundamental solution. Then, after a series of lemmas (involving the Sobolev-Kondrašov theorem, the Hölder-Korn-Lichtenstein-Giraud inequality, and a Calderón-Zygmund inequality), a local existence theorem is proved to the effect that if L is elliptic of order m with Hölder-continuous coefficients and if f is Hölder-continuous, then locally there exists a solution u of $Lu = f$ with Hölder-continuous derivatives of order m, and u with its derivatives of order less than or equal to $m-1$ may have prescribed values at a fixed point. Variations on this for $f \in L^q$, $q > 1$, are given. Then two theorems are proved giving interior Schauder-type estimates of the form, for example, $\|u\|_{C_{m+\alpha}(\Omega_0)} \leqq c(\|Lu\|_{C_\alpha(\Omega)} + \|u\|_{C_0(\Omega)})$, where $\Omega_0 \subset \Omega$ is compact. Also a priori estimates up to the boundary are given for second-order equations. These estimates are then applied to the solution of the Dirichlet problem for a second-order equation by the continuity method in a suitable domain. Finally, a theorem on the smoothness of strong solutions is proved and some appendices give the proofs of various fundamental inequalities.

Chapter 6 deals with function-theoretic methods and some applications to elliptic problems. First a linear uniformly elliptic system of two first-order equations in two independent variables is reduced to complex form and

some properties of the Beltrami equation are discussed (with proofs in an appendix). Then a representation theorem of Bers and Nirenberg for solutions of the complex equation is proved, which leads to the result that the zeros of the solution are isolated, and to a unique continuation theorem. Uniformly elliptic second-order equations in two independent variables of a special kind (for simplicity), related to the above elliptic systems, are then studied. A general maximum principle is obtained, and the Dirichlet and Neumann problems are treated using the representation theorem and a theorem of Privalov in function theory (the latter is proved in an appendix). Chapter 7 treats briefly some quasi-linear equations in two independent variables, and some methods for proving the existence of fixed points are discussed (successive approximations, continuity method, Leray-Schauder method, and Birkhoff-Kellogg-Schauder method). Applications to the Dirichlet and Neumann problems are given.

The presentation has a modern flavor and would be well worth consulting for anyone interested in having a readable account of the main ideas concerning elliptic equations. One might have hoped for more information on non-linear and quasi-linear equations, but this is hardly a criticism in view of the clarity and completeness already achieved in 167 pages and the fact that a great deal of work in the non-linear case is too recent to have been included in these lectures. In fact, these lectures could be of invaluable aid in orienting and informing a reader in a field where so much has been happening.

R. Carroll (Urbana, Ill.)

Nečas, Jindřich 2514

Sur l'existence de la solution classique du problème de Poisson pour les domaines plans.

Ann. Scuola Norm. Sup. Pisa (3) **16** (1962), 285–296.

Sia Ω un dominio piano limitato tale che ciascun punto della sua frontiera Γ è vertice di un triangolo, lo stesso per tutti i punti di Γ, formato di punti esterni a Ω. Sia $Au=\sum_{|\alpha|,|\beta|\le k}(-1)^{|\alpha|}D^\alpha(a_{\alpha\beta}D^\beta u)$ un operatore differenziale lineare d'ordine $2k$ a coefficienti $a_{\alpha\beta}$ misurabili e limitati in Ω ed ellittico rispetto al problema di Dirichlet-Poisson

$$(1)\quad Au=f \quad \text{in } \Omega, \qquad u=\frac{\partial u}{\partial \nu}=\cdots=\frac{\partial^{k-1}u}{\partial \nu^{k-1}}=0 \quad \text{su } \Gamma$$

(ν normale esterna a Γ), cioè tale che

$$\int_\Omega \sum_{|\alpha|,|\beta|\le k} \overline{a_{\alpha\beta}D^\alpha\phi}\,\overline{D^\beta\phi}\,dx \ge c\sum_{|\alpha|\le k}\int_\Omega |D^\alpha\phi|^2\,dx, \qquad \phi\in\mathscr{D}(\Omega),$$

dove c è una costante positiva e $\mathscr{D}(\Omega)$ è lo spazio delle funzioni infinitamente differenziabili e a supporto compatto in Ω. Il principale risultato di questo lavoro è il seguente interessante teorema di regolarizzazione: Se f appartiene al duale dello spazio $H_0^{k-1}(\Omega)$ (chiusura di $\mathscr{D}(\Omega)$ nello spazio di Sobolev $H^k(\Omega)$ delle funzioni u tali che $u\in L^2(\Omega)$ e $D^\alpha u\in L^2(\Omega)$, $|\alpha|\le k$), allora la soluzione debole del problema (1) è continua in Ω insieme alle sue derivate fino all'ordine $k-1$.

La dimostrazione si fonda su un metodo analogo a quello utilizzato per l'operatore di Laplace da R. Courant e D. Hilbert [*Methoden der mathematischen Physik*, Springer, Berlin, 1931]. *E. Magenes* (Pavia)

Višik, M. I.; Èskin, G. I. 2515

Singular elliptic equations and systems of variable order. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 243–246.

This is an extension of the results of a previous note [same Dokl. **155** (1964), 24–27; MR **28** #4319]. On a closed manifold M^n the authors consider the equation

$$(1)\qquad L_{\alpha(x)}u \equiv K_{\alpha(x)}u+Tu=f,$$

where, in a coordinate neighborhood $V\subset M^n$, $K_{\alpha(x)}$ is of the form

$$K_{\alpha(x)}(\varphi u)=\int_{R^n}K_{\alpha(x)}(x,x-y)\varphi(y)u(y)\,dy+T_1(\varphi u), \qquad \operatorname{supp}\varphi\subset V,\ \varphi\in C^\infty.$$

It is assumed that $\hat{K}_{\alpha(x)}(x,\zeta)$, the Fourier transform of $K_{\alpha(x)}(x,z)$ with respect to z (the "symbol" of $K_{\alpha(x)}$) is homogeneous of degree $\alpha(x)$ in ζ and does not vanish if $\zeta\ne0$. Further, there are introduced Hilbert spaces $H^{\beta(x)}$ generalizing the well-known spaces of functions whose derivatives up to a certain fixed order are in L_2. The following inequality holds:

$$\|u\|_{\beta(x)} \le C(\|f\|_{\beta(x)-\alpha(x)}+\|u\|_{\beta_0}),$$

where C is a constant provided T is compact as an operator from $H^{\beta(x)}$ to $H^{\beta(x)-\alpha(x)}$. Next, the same equation (1) is considered in a bounded domain $G\subset R^n$ with smooth boundary Γ, approximate boundary conditions being imposed on Γ. A similar inequality holds involving now also boundary data. Also a sort of dual boundary problem is considered. Finally the case of systems is discussed.

J. Peetre (Lund)

Wigley, Neil M. 2516

Asymptotic expansions at a corner of solutions of mixed boundary value problems.

J. Math. Mech. **13** (1964), 549–576.

Let D be a region of the $z=x+iy$ plane whose boundary contains two analytic arcs Γ_1, Γ_2 that meet at the origin and form there an angle $\pi\alpha>0$. R. S. Lehman [same J. **8** (1959), 727–760; MR **21** #4291] has derived asymptotic series, as $z\to0$, for the solutions of Dirichlet boundary-value problems in D for certain analytic elliptic second-order differential equations. The present paper extends these results in that it admits non-analytic data, as well as boundary conditions involving first derivatives. More precisely, the differential equation is

$$\Delta U+KU=F$$

and the boundary condition is of the form

$$\delta_i\frac{\partial U}{\partial n}+A_iU=B_i \quad \text{on } \Gamma_i, \qquad i=1,2,$$

where the δ_i are either 1 or 0. The functions K, F are of class C^N ($N\geqq2$) in $D\cup\Gamma_1\cup\Gamma_2\cup\{0\}$, while the functions A_i, B_i are of class C^N in $\Gamma_i\cup\{0\}$. The asymptotic expressions found for U are, of course, approximations of finite order only, not infinite series. For instance, if $\delta_1=1$, $\delta_2=0$, it is proved that $U(z)=P+O(z^{N-\varepsilon})$ ($\varepsilon>0$, arbitrary), where P is a polynomial in z, $\bar z$, $z^{1/2\alpha}$ and $\bar z^{1/2\alpha}$ if α is irrational. If $\alpha=p/q$ (p, q relatively prime), P is a polynomial in the above quantities as well as in $z^{q/2}\log z$ and $\bar z^{q/2}\log\bar z$, or in $z^q\log z$, $\bar z^q\log z$, depending on whether q is even or odd. The approximation may be formally differentiated $N-1$

times. In addition to the hypotheses mentioned before, the author assumes that $U=o(z^{\mu})$ as $z\to 0$, where $\mu>\max(-1,-1/2\alpha)$ for the particular problem quoted here. The method is based on an integral representation for U in terms of Green's function for a sector. The asymptotic expansion of that integral requires extensive calculations.

W. Wasow (Madison, Wis.)

Eichhorn, Wolfgang 2517

Verallgemeinerte Cauchy-Riemannsche und Laplacesche Differentialgleichungen.

Math. Z. **84** (1964), 279–293.

Let A be an algebra of order n with identity e_1 over K (the real or complex field), with multiplication constants α_{ijk} relative to the basis $e_1,\cdots,e_n$, and let w be a function with domain and range in A.

If A is assumed associative and commutative, then the assumption that w is Hausdorff differentiable (there exists a w' such that $dw=w'd\xi$, $d\xi=d\xi_1 e_1+\cdots+d\xi_m e_m$) leads to a system of partial differential equations (I) $\partial\varphi_k/\partial\xi_j=\sum_{i=1}^n(\partial\varphi_i/\partial\xi_1)\alpha_{ijk}$, $j=2,3,\cdots,m\leqq n$, where the $\varphi_i(\xi_1,\cdots,\xi_m)$ are the components of w. Such a system is called a generalized Cauchy-Riemann system of Type I.

If A is assumed associative, then the so-called regularity conditions on w lead to a system of partial differential equations

$$\text{(II)}\qquad \Delta w=\left(\sum_{j=1}^m \gamma_{ijk}\,\partial/\partial\xi_j\right)(\varphi_1,\cdots,\varphi_n)^t=0,$$

$$\gamma_{i1k}=-\delta_{ik},\quad \gamma_{ijk}=\alpha_{ijk},\, j\neq 1,$$

called a generalized Cauchy-Riemann system of Type II.

A single partial differential equation satisfied by all the component functions φ_i in a Type I [Type II] system is called a generalized Laplace differential equation of Type I [Type II]. Principal results: Theorem 1: The differential equation $\sum_{s=1}^t A_s(\partial/\partial\xi_1,\partial/\partial\xi_2,\cdots,\partial/\partial\xi_m)\varphi=0$, where A_s is a homogeneous polynomial over K of degree s, is a generalized Laplace differential equation of Type I if and only if $A_s(e_1,\cdots,e_m)=0$, $s=1,\cdots,t$. Theorem 2: The system of generalized Laplace operators of Type II forms a principal ideal of $K[\partial/\partial\xi_1,\partial/\partial\xi_2,\cdots,\partial/\partial\xi_m]$ whose generating element is the "g.c.d." of the elements of the adjoint of the matrix of Δ.

If A is alternative, K the real field, and $m=n$, the regularity conditions lead to elliptic Cauchy-Riemann systems. It is shown that the only non-equivalent such systems arise when A is (1) the complex field, (2) the real quaternions, (3) the Cayley numbers.

R. F. Rinehart (Cleveland, Ohio)

Mizohata, Sigeru 2518

Sur l'analyticité de la fonction spectrale de l'opérateur Δ relatif au problème extérieur.

Proc. Japan Acad. **39** (1963), 352–357.

Let $G(P,Q\,|\,\lambda)$ be Green's function for the equation $(\lambda^2-\Delta)\psi=0$ with $\operatorname{Re}\lambda>0$ in the exterior of a body on which $\partial\psi/\partial n$ is prescribed. Let $E(P-Q;\lambda)$ be the fundamental solution vanishing at ∞. It is shown that $K_c=G-E$ can be continued analytically beyond $\operatorname{Re}\lambda=0$, and a representation for G is found. Following Carleman, G is related to the spectral function θ by

$$\int_S G(P,Q\,|\,\lambda)f(Q)\,dQ=\int_{-\infty}^0\frac{1}{\lambda^2-\mu}\,d_\mu\int_S\theta(P,Q\,|\,\mu)f(Q)\,dQ$$

with $\partial f/\partial n=0$ on S. The function θ is analytic in λ except at 0 and satisfies, for bounded continuous ω,

$$\int_{-\infty}^0\omega(\lambda)\,d_\lambda\int\theta(P,Q\,|\,\lambda)f(Q)\,dQ=\int_{-\infty}^0\omega(\lambda)\left[\int t(P,Q\,|\,\lambda)f(Q)\,dQ\right]d\lambda,$$

where t is analytic in λ near the real negative axis, $\lambda\neq 0$.

It follows that solutions of the wave equation in the exterior of S, on which the normal derivative vanishes, decay in time if their initial data are in C^2 and of compact support (see Lax and Phillips [Bull. Amer. Math. Soc. **68** (1962), 47–49; MR **24** #A913]).

C. S. Morawetz (New York)

Loges, Frido 2519

Partikuläre Integrale der Wellengleichung $\Delta u+k^2u=0$ in Koordinaten des elliptischen Zylinders und ihre Eigenschaften. 1. Mitteilung: Integrale der Form

$$U_j=U_j\left[C_j\int_0^{\rho_i}e^{-i\pi\tau^2/2}\,d\tau+C_j'\right].$$

Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe **11** (1962), 543–552.

In previous papers [see, e.g., Optik **19** (1962), 323–326] the author introduced particular integrals of the wave equation in elliptic coordinates in order to construct approximate solutions of the two-dimensional problem of scattering of a plane wave by a slit. The present paper discusses properties of these particular integrals (which are defined on a two-sheeted Riemann surface). It is demonstrated, for example, that they satisfy radiation and edge conditions associated with the slit diffraction problem. As preliminaries, the author reviews the transformation from cartesian to elliptic cylinder coordinates and the resultant form of the wave equation, the radiation condition, and of the edge condition.

J. E. Burke (Mountain View, Calif.)

Herdan, R. 2520

The asymptotic dispersion relationship for a general linearized system of differential equations of fluid-dynamical type.

Quart. J. Mech. Appl. Math. **17** (1964), 351–367.

Consider the system of n equations

$$\sum_{j=1}^n\left\{a_{ij}\frac{\partial f_j^{(1)}}{\partial t}+\sum_{\lambda=1}^3 b_{ij,\lambda}\frac{\partial f_j^{(1)}}{\partial x_\lambda}+h_{ij}f_j^{(1)}\right\}=0$$

for n functions $f_j^{(1)}$ in the independent variables t,x_1,x_2,x_3; the coefficients depend on position only.

Choose any direction in x_λ-space given by the unit vector (n_1,n_2,n_3). When the characteristic speeds u_m of the system, given by $\det|u_m a_{ij}-\sum_{\lambda=1}^3 n_\lambda b_{ij,\lambda}|=0$, are all finite, conditions are examined for which plane-wave solutions, propagating locally in the given direction, of the form

$$f_j^{(1)}=f_{j0}^{(1)}e^{i(\omega t-\kappa z)},$$

with $f_{j0}^{(1)}=\text{const}$, $z=n_1x_1+n_2x_2+n_3x_3$, can be formally determined. An algebraic algorithm for finding a suitable formal expansion for ω in decreasing powers of κ,

$$\omega=\varepsilon_1\kappa+\varepsilon_0+\varepsilon_{-1}/\kappa+\cdots$$

with $\varepsilon_1 = u_m$, is given for each characteristic speed u_m, when the coefficients of the system are assumed to be constant in regions comparable with the wave lengths $2\pi/\kappa$ for large κ. No proof of convergence of the formal series are given.

In the paper there are illustrative examples, and the importance of this work to the linearized equations of fluid dynamics and other fields is stressed.

A. D. Solomon (New York)

Leray, J. 2521

The functional transformations required by the theory of partial differential equations.

SIAM Rev. **5** (1963), 321–334.

A summary of a theory of a general Laplace transform due to the author [Bull. Soc. Math. France **90** (1962), 39–156; MR **26** #1625]. The theory is based on the unitary solution $U(\xi, y)$ of an analytic differential operator $a \neq a(y, \partial/\partial y)$ of order m defined by $aU=1$, $U=0$, m times on $\xi \cdot x = \xi_0 + \xi_1 x_1 + \cdots + \xi_l x_l = 0$. The function $U(\xi, x)$ is multivalued and homogeneous of order 0 in ξ. It can be uniformized and a suitable Laplace transform of it, which integrates U on a compact cycle in ξ-space depending on x, gives an elementary solution $E^*(x, y)$ of the adjoint of a. A similar transform operating on the product of U by an analytic function $v(y)$ gives the solution of Cauchy's problem $a^*u = v$, $u=0$, m times on an analytic noncharacteristic hypersurface S.

L. Gårding (Lund)

Gasymov, M. G.; Levitan, B. M. 2522

The asymptotic behaviour of the spectral functions of the Schrödinger operator near a planar part of the boundary. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 527–552.

Consider the eigenvalue problem

(1) $$-\Delta u + c(x)u = \lambda u, \qquad (\partial u/\partial n - h(x)u)|_{x\in\Gamma} = 0,$$

where $x = (x_1, x_2, x_3)$ is a point in a finite simply connected region D^+ with boundary Γ. Suppose that $c(x)$ is summable over D^+ and that Γ contains a plane piece Γ_p located in the plane $x_3 = 0$. Without loss of generality, suppose that the problem has positive eigenvalues $\mu_1^2 \leqq \mu_2^2 \leqq \cdots$ with corresponding orthonormal set of eigenfunctions $\varphi_1(x)$, $\varphi_2(x), \cdots$. The spectral function $\omega(x, y, \mu)$ of the problem is given by $\omega(x, y, \mu) = \sum_{\mu_n < \mu} \varphi_n(x)\varphi_n(y)$. Let $\omega_0(x, y, \mu)$ be the spectral function of the boundary problem $\Delta u + \lambda u = 0$, $\partial u/\partial x_3|_{x_3=0} = 0$ in the semispace $x_3 > 0$. Suppose that D^- is the reflection of D^+ in $x_3 = 0$, that $D = D^+ + \Gamma_p + D^-$, that D_ε is the set of points of D whose distances to the boundary of D are not less than ε, where ε is an arbitrary positive number, and that $D_\varepsilon^+ = D_\varepsilon \cap \{D^+ + \Gamma_p\}$. It is shown that in D_ε^+, $\omega(x, y, \mu) = \omega_0(x, y, \mu) + O(\mu^2 \ln \mu)$ as $\mu \to \infty$. The proof involves solving the mixed Cauchy problem

(2) $$\Delta u - c(x)u = \partial^2 u/\partial t^2,$$

$$u(x, 0) = \varphi(x), \qquad u_t(x, 0) = 0, \qquad (\partial u/\partial n - h(x)u)|_{x\in\Gamma} = 0,$$

first by the method of Fourier and then by the method of S. L. Sobolev [*Some applications of functional analysis in mathematical physics* (Russian), Izdat. Leningrad. Gos. Univ., Leningrad, 1950; MR **14**, 565; English transl., see #2624 below].

In order to apply the method of Sobolev, a simple transformation of u is performed, and then the functions involved are continued evenly across $x_3 = 0$ so as to reduce the problem to one of the form $\sum_{i=1}^3 \partial^2 u/\partial x_i^2 + \sum_{i=1}^3 b_i(x)\partial u/\partial x_i - q(x)u = \partial^2 u/\partial t^2$, $u(x, 0) = 0$, $u_t(x, 0) = f(x)$ in D. Since $b_3(x)$ is discontinuous, it is necessary to adapt Sobolev's method to this case, and this is done in the first four sections of the article. In these sections it is also shown that the solution $u(x, t)$ of (2) can be written in the form $u(x, t) = u_0(x, t) + (\partial^2/\partial t^2)u_1(x, t)$, where $|u_1(x, t)| \leqq Ct^3$ for small t and $u_0(x, t)$ is the solution of $\Delta u = \partial^2 u/\partial t^2$, $u(x, 0) = \varphi(x)$, $u_t(x, 0) = 0$, $\partial u/\partial x_3|_{x_3=0} = 0$. By use of the above results and the methods of preceding articles of B. M. Levitan [Trudy Moskov. Mat. Obšč. **4** (1955), 237–290; MR **17**, 372; ibid. **5** (1956), 269–298; MR **18**, 313; Izv. Akad. Nauk SSSR Ser. Mat. **17** (1953), 269–284; MR **15**, 316; ibid. **20** (1956), 337–376; MR **18**, 312; ibid. **20** (1956), 437–468; MR **18**, 312], the following expansion theorem is proved: Under suitable restrictions on φ and for $x \in \Gamma_p + D^+$,

$$\lim_{\mu\to\infty} \int_0^\mu (1 - \gamma^2/\mu^2)^2 \, d_\gamma \rho(x, \gamma) = 0,$$

where

$$\rho(x, \gamma) = \int_{D^+} \varphi(y)[\omega(x, y, \gamma) - \omega_0(x, y, \gamma)] \, dy.$$

A number of remarks are made on how the asymptotics given in the article may be sharpened and on what kinds of results may be obtained if D^+ is a region with a corner or if D^+ is a parallelepiped. In the latter case a sharper version is given of a known asymptotic formula for the number of eigenvalues of problem (1) less than μ.

R. C. Gilbert (Fullerton, Calif.)

Eastham, M. S. P. 2523

The distribution of the eigenvalues in certain eigenvalue problems.

Quart. J. Math. Oxford Ser. (2) **15** (1964), 251–257.

The author has already studied the problem of determining whether there are functions $\psi(r, \theta)$ (not identically zero) which are harmonic in $r < 1$, continuous in $r \leqq 1$, and satisfy on $r = 1$ the boundary condition

$$\partial\psi/\partial n = -\{\lambda p(\theta) - q(\theta)\}\psi(1, \theta),$$

where λ is a parameter ((1) for the case $p(\theta) = 1$ [same J. (2) **13** (1962), 304–320; MR **27** #455]; (2) for the case of general $p(\theta)$ under certain conditions [ibid. (2) **14** (1963), 259–272; MR **27** #3822]). There is an infinite non-decreasing sequence of real eigenvalues, each of which is associated with just one eigenfunction.

For the case $p(\theta) = 1$, he has obtained in (1) the asymptotic law $N(\lambda) \sim 2\lambda$, where $N(\lambda)$ denotes the number of eigenvalues not exceeding λ. In the present paper, he tries to generalize this asymptotic law to the case of a general positive $p(\theta)$ under certain conditions. In order to deduce the final result, first he studies the similar problem of finding non-vanishing harmonic functions in a sector with radius 1 and angle α, satisfying the boundary conditions $\partial\psi/\partial n = -(\lambda p - q)\psi$ on the arc, and either $\psi = 0$ or $\partial\psi/\partial n = 0$ on the radii of this sector. For the case $p(\theta) = k$ (a constant), it is shown that $N(\lambda) \sim \lambda k\alpha/\pi$. Now divide the interval $[0, 2\pi]$ by the points α_i, where $0 = \alpha_1 < \alpha_2 < \cdots < \alpha_{j+1} = 2\pi$; then the unit circular disk is divided into the sectors D_i ($r \leqq 1$, $\alpha_i \leqq \theta < \alpha_{i+1}$). In each D_i, consider the problem of determining non-vanishing harmonic functions with the

boundary conditions $\partial\psi/\partial n = -(\lambda P_i - q)\psi$ on the arc, and $\partial\psi/\partial n = 0$ on the radii of D_i; also the similar problem with the boundary conditions $\partial\psi/\partial n = -(\lambda r_i - q)\psi$ on the arc, and $\psi = 0$ on the radii of D_i, where P_i and r_i denote the upper and lower bounds of $p(\theta)$ in $[\alpha_i, \alpha_{i+1}]$, respectively. By the above result, these problems have eigenvalues with the asymptotic laws $N_i(\lambda) \sim \lambda P_i(\alpha_{i+1} - \alpha_i)/\pi$ and $N_i(\lambda) \sim \lambda r_i(\alpha_{i+1} - \alpha_i)/\pi$, where $N_i(\lambda)$ is the number of eigenvalues of these problems in D_i not exceeding λ. The author compares the set of all the eigenvalues of these problems with the set of eigenvalues of the original problem, and gives the asymptotic law

$$N(\lambda) \sim \frac{\lambda}{\pi} \int_0^{2\pi} p(\theta)\, d\theta,$$

where $N(\lambda)$ denotes the number of eigenvalues of the original problem not exceeding λ. *T. Uno* (Tokyo)

Suyama, Yukio; Ono, Akira 2524

On the Cauchy problem of the linear elliptic partial differential equations.

Mem. Fac. Sci. Kyushu Univ. Ser. A **15** (1961/62), 171–177.

The authors consider the Cauchy problem in a half-plane for an elliptic equation in two independent variables having constant coefficients and no lower-order terms. They make extensive use of the techniques developed by S. Agmon [Comm. Pure Appl. Math. **10** (1957), 179–239; MR **21** #5057], who obtains an explicit solution of the Dirichlet problem in a half-plane by constructing the corresponding Poisson kernels. The authors show that if the Cauchy data have compact support and are sufficiently differentiable, then the solution may be represented as a sum of convolutions with the Agmon kernel functions.

L. M. Sibner (Stanford, Calif.)

Gårding, Lars 2525

Eigenfunction expansions.

Partial Differential Equations (*Proc. Summer Seminar, Boulder, Col.*, 1957), *pp.* 301–325. *Interscience, New York*, 1964.

This is a very illuminating elementary discussion of the spectral theorem for self-adjoint operators in direct integral form. Thus the guiding point of view is taken that any self-adjoint operator A can be diagonalized by a unitary transformation, and one looks for such a unitary map U from the Hilbert space H to some $L^2(\mu, \nu)$ space L of functions $\vec{f}(\xi)$ on a set Q ($\nu(\xi)$ is the dimension function, i.e., $\vec{f} = (f_1, \cdots, f_{\nu(\xi)})$), such that $UAU^*: L \to L$ is multiplication by a function $\lambda(\xi)$. The map U is given by $(\overrightarrow{Uf})(\xi) = (f, \vec{\varphi}(\xi))$ when eigenfunctions $\vec{\varphi}(\xi)$ exist (ξ is a labeling index here and of course is to be thought of as the spectrum of A when $\lambda(\xi) = \xi$). Some nice examples and motivations are given in illustrating this situation. Then the case of generalized eigenfunctions is illustrated by important examples and, in one fairly general situation, for almost all $\xi \in Q$, the existence of a linearly independent set $\{\varphi_k(\xi)\}$ of distributions playing the role of generalized eigenfunctions is proved (modulo a few technical details). (Thus for f in a dense set in H one has $(\overrightarrow{Uf})(\xi) = \langle f, \vec{\varphi}(\xi)\rangle$.) A few results for elliptic and hypo-elliptic differential operators are indicated. Then the proof of an eigenfunction theorem for operators of Carleman type on an L^2 space is sketched. Finally, the Weyl-Stone-Titchmarsh-Kodaira formula is obtained via the point of view mentioned, and some results for ordinary differential operators are derived.

R. Carroll (Urbana, Ill.)

Brown, Robert D. 2526

Reflection laws of fourth order elliptic differential equations in two independent variables.

J. Math. Mech. **13** (1964), 365–383.

Si consideri l'operatore a coefficienti costanti $L = \sum_{0 \le i+j \le 4} \alpha_{ij}\, \partial^{i+j}/\partial x^i \partial y^j$ e si faccia l'ipotesi che la parte principale di L si spezza in quattro fattori lineari distinti. Sia D un dominio convesso adiacente ad un segmento aperto σ dell'asse x e posto nel semipiano $y < 0$. Sia $u \in C^4(D \cup \sigma)$ una funzione analitica reale di due variabili reali definita in D soluzione del problema ai limiti: $L(u) = 0$ in D, $u(x, 0) = f(x)$, $u_y(x, 0) = g(x, 0)$ su σ dove $f(x)$ e $g(x)$ sono analitiche in x per $(x, 0) \in \sigma$. In questo lavoro si determina esplicitamente un dominio convesso $\hat{D}$ adiacente a σ posto nel semipiano $y > 0$ in modo che u possa essere prolungata, sotto opportune condizioni su $f(x)$ e $g(x)$, in una soluzione analitica dell'equazione $L(u) = 0$ in $D \cup \sigma \cup \hat{D}$.

Identificato il piano (x, y) col piano $z = 0$ dello spazio reale (x, y, z), si usa l'analiticità di u per trasformare il problema del prolungamento di u in un opportuno problema misto per un sistema iperbolico di forma speciale che viene risolto grazie ad un teorema generale concernente sistemi di questo tipo.

Si dimostra inoltre che $\hat{D}$ dipende solo dalla parte principale di L e, grazie ad un opportuno esempio, che $\hat{D}$ non coincide sempre con la riflessione di D attraverso σ.

G. Geymonat (Pavia)

Bazley, Norman W.; Fox, David W. 2527

Lower bounds for energy levels of molecular systems.

J. Mathematical Phys. **4** (1963), 1147–1153.

It is shown that the Hamiltonian H for a molecule in which the nuclei are regarded as fixed has the form $H = \sum H_\alpha + H'$, where each H_α has known eigenvalues and eigenvectors, and H' is a positive multiplicative operator.

A procedure, related to those of A. Weinstein [*Étude des spectres des équations aux dérivées partielles de la théorie des plaques élastiques*, Memor. Sci. Math., Fasc. 88, Gauthier-Villars, Paris, 1937], of N. Aronszajn [Proc. Sympos. Spectral Theory Differential Problems, pp. 179–202, Oklahoma Agricultural and Mechanical College, Stillwater, Okla., 1951; MR **13**, 469], and of the reviewer [Inst. Fluid Dynamics Appl. Math. Univ. Maryland Tech. Note BN-183 (1959)] is developed, by which arbitrarily close lower bounds for the eigenvalues of such an operator can be computed in finite terms.

H. F. Weinberger (Minneapolis, Minn.)

Mackevič, I. P. 2528

Solution of a boundary-value problem for the biharmonic equation in a half-plane with m elliptic holes. (Russian)

Dokl. Akad. Nauk BSSR **8** (1964), 209–212.

Let D be a half-plane with m elliptic holes. Let L denote the boundary straight line and E_j $(j = 1, \cdots, m)$ denote

the boundary of the jth hole. Let (x_j, y_j) be the rectangular co-ordinate system, whose origin coincides with the center of the ellipse E_j and whose axis o_jx_j coincides with the major axis of the ellipse E_j. Let (δ_j, η_j) denote an elliptic co-ordinate system corresponding to the ellipse E_j such that $x_j+iy_j=h_j \operatorname{ch}(\delta_j+i\eta_j)$, where h_j is equal to half the focal distance of the ellipse E_j. Let $\partial U/\partial v$ denote the directional derivative which on the line L coincides with the normal derivative and $\partial U/\partial v=\partial U/\partial\delta_j$ on E_j. Let (r, φ) be a polar system of co-ordinates. Denote by C the family of all functions which are of class C^1 on the closed domain $\bar{D}$ and of class C^4 in the domain D; let the following derivatives ΔU, $\partial\Delta U/\partial\delta_j$ $(j=1, \cdots, m)$, $r^{-1}U$, $\partial U/\partial r$, $r^{-2}\Delta U$, $r^{-3}\,\partial\Delta U/\partial r$ be bounded in D. The author investigates the problem of existence and uniqueness of the solution $U\in C$ satisfying the biharmonic equation $\Delta^2 U=0$ in D and the boundary condition $U=g$, $\partial U/\partial v=h$ on the boundary ∂D, where g and h are functions with sufficiently regular Fourier expansions in the variable η_j on the set E_j and vanish on the line L. An outline of the proofs is given. The author constructs a sequence of functions $U_n\in C$ which allows him to expand the solution U into a series of the form $\sum_n t_nU_n$. Thus, he reduces the problem of the existence of the solution to a problem on the existence of the inverse operator of a linear operator in the space of convergent sequences.

W. Bogdanowicz (Washington, D.C.)

Mel'nik, Z. O. 2529

On a special mixed problem. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 577–580.

The author proves the existence and uniqueness of the solution of a mixed problem for a special hyperbolic system in a right cylinder in the space R^{n+1} with coordinates $x=(x_1, x_2, \cdots, x_{n+1})$, namely,

$$(1)\qquad A_i(x, D)u_i+\sum_{s=1}^{N}\sum_{|\alpha|<2p} a_\alpha^{i,s}(x)D_\alpha u_s=f_i(x).$$

Here $D_\alpha=D_{\alpha_1}\cdots D_{\alpha_m}$ $(D_i=\partial/\partial x_i)$, $|\alpha|$ is the sum of the orders of the indices, and the operator $A_i(x, D)$ has the form

$$A_i(x, D)=\prod_{j=1}^{p} A_i^j(x. D),$$

$$A_i^j(x, D)=D_1^2-C_i^j\sum_{k,l=2}^{n+1} a_{kl}^i(x)D_kD_l=D_1^2-C_i^ja^i(x, D),$$

where C_i^j are constants and $1\leqq i\leqq N$, $1\leqq j\leqq p$. If Ω is a domain with boundary S in the subspace R_1^n with coordinates $x'=(x_2, \cdots, x_{n+1})$, and if Z denotes the cylinder $\{0\leqq x_1\leqq T;\ x'\in\bar{\Omega}\}$, then, for all $f_i(x)\in L_2(Z)$ $(1\leqq i\leqq N)$, there is a unique solution $u=(u_1, \cdots, u_N)\in W_2^{(2p-1)}(Z)$ of the boundary-value problem consisting of (1) and the boundary conditions $D_1^mu_i|_{x_1=0}=0$ $(0\leqq m<2p-1;$ $1\leqq i\leqq N)$ and $u_i|_S=a^iu_i|_S=\cdots=(a^i)^{p-1}u_i|_S=0$ $(1\leqq i\leqq N)$.

A. J. Lohwater (Providence, R.I.)

Džavadov, M. G. 2530

The Cauchy problem for a hyperbolic equation with a small parameter multiplying the highest derivatives. (Russian. Azerbaijani summary)

Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Mat. Tehn. Nauk **1963**, *no.* 6, 3–9.

Let $x=(x_1, x_2, \cdots)$, $D_1=\partial/\partial x_1$, and consider $L_\varepsilon U=\varepsilon a(x)U+b(x)U=f$, with the boundary conditions (1) $D_1^kU|_{x_1=0}=0$ $(k=0, 1, \cdots, m+1)$. $a(x)=D_1^{m+1}+\cdots$ and $b(x)=b_1^0(x)D_1^m+\cdots$ are totally hyperbolic operators of orders $m+1$ and m, respectively.

Following Višik and Ljusternik, an expansion $U=\sum_0^n\varepsilon^iW_i+\varepsilon^m\sum_0^n\varepsilon^iv_i+\varepsilon^{n+1}Z_n$ is considered, where $bW_0=f$, $bW_i=-aW_{i-1}$ and $(D_1^{m+1}+b_1^0D_1^m)v_i$ consists of terms linear in $v_0, \cdots, v_{i-1}$. The v_i are needed to satisfy (1) for $k=m+1$, and if $b_1^0>0$, they become negligible outside a boundary layer.

Assuming the sheets of the normal cone of $b(x)$ separate those of $a(x)$, the author shows Z_n is bounded as $\varepsilon\to 0$. He uses the Leray-Gårding inequality. There are misprints, one of which equates n with the number of space dimensions.

P. Ungar (New York)

Haĭrullina, S. P. 2531

Solution of the Cauchy problem for a hyperbolic equation with data on a degenerate curve. (Russian)

Dokl. Akad. Nauk BSSR **8** (1964), 361–364.

The Cauchy problem at $y=0$ for the equation

$$y^2u_{xx}-u_{yy}+au_x+bu_y+cu=f$$

is treated for the case, left open in previous work [see, e.g., M. H. Protter, Canad. J. Math. **6** (1954), 542–553; MR **16**, 255], when $\lim_{y\to 0}a(x, y)=0$ is not assumed.

P. Ungar (New York)

Lax, Peter D. 2532

Development of singularities of solutions of nonlinear hyperbolic partial differential equations.

J. Mathematical Phys. **5** (1964), 611–613.

The author makes use of two theorems in ordinary differential equations to place upper and lower bounds on the time interval in which the solution of an initial-value problem for the non-linear hyperbolic system

$$u_t+au_x+bv_x=0,$$

$$v_t+cu_x+dv_x=0,$$

a, b, c and d being functions of u and v, exists. He then derives a result for a mixed initial-boundary-value problem for the equation $y_{tt}=K^2(y_x)y_{xx}$ and rederives a result of Zabusky [same J. **3** (1962), 1028–1039; MR **26** #4067] for the critical number of vibrations of a non-linear string.

D. H. Tucker (Salt Lake City, Utah)

John, Fritz 2533

Hyperbolic and parabolic equations.

Partial Differential Equations (Proc. Summer Seminar, Boulder, Col., 1957), *pp.* 1–129. *Interscience, New York,* 1964.

The introductory chapter gives some intuitive notions underlying the study of hyperbolic and parabolic equations. In particular, the idea of such equations representing equations of evolution is emphasized. Chapter 2 is on the wave equation; explicit formulas for the solution are given in one and three space dimensions, and such ideas as characteristics, domain of dependence, Huygen's principle, and continuous dependence on initial data are discussed, with interesting examples. The method

of descent is indicated and the non-homogeneous problem treated. Then the Cauchy problem for general initial surfaces is broached and shown to be solvable for space-like, nowhere characteristic surfaces under sufficient regularity of data. Next, the use of energy integrals and associated inequalities to obtain uniqueness theorems is indicated for various regions, and a priori estimates are discussed. To conclude the chapter, the general linear equation (C^∞) with the wave operator as principal part is solved using the estimates just mentioned. Also some mixed problems are discussed.

In Chapter 3 the Cauchy problem for a system of equations $Lu = \sum_{|\alpha| \leq m} A_\alpha(x) D^\alpha u = B(x)$ is treated for the A_α square matrices. The analytic case and the Cauchy-Kowalewski theorem are discussed, and the Holmgren uniqueness theorem is proved. Finally, there is a section on the propagation of discontinuities. Chapter 4 deals in the beginning with a single mth-order hyperbolic equation with constant highest-order terms and with data given on a non-characteristic hyperplane $x_0 = 0$. Reductions to standard form are examined. Then techniques for solving the homogeneous equation involving only the principal part of the operator are given, which yield a priori estimates and enable one to solve the more general equation using these estimates. Thus the Fourier method and the method of decomposition into plane waves are examined. For the case of a strictly hyperbolic equation (i.e., distinct characteristic roots) the Fourier method leads to the desired a priori estimates indicated above and the general equation is then shown to be solvable by an iteration technique in the C^∞ case; uniqueness is also proved. For first-order hyperbolic systems it is next shown that adequate a priori estimates can be obtained by the Fourier method without strict hyperbolicity under suitable conditions (e.g., a symmetric system). Then symmetric hyperbolic systems with variable coefficients are studied by a method of Friedrichs, and a uniqueness theorem is obtained for lens-shaped regions bounded by $x_0 = 0$ and a space-like surface. Other variations are indicated and examples given.

Chapter 5 is on the heat equation. A maximum principle is proved and applied to uniqueness questions. Then the initial-value problem is solved and the backwards problem is discussed; the boundary-initial-value problem for a rectangle is next solved. Chapter 6 is on finite-difference methods for some parabolic and hyperbolic equations.

This brief sketch should indicate that there is contained in these lectures an interesting assortment of material which is designed to give an understanding of some problems and techniques involving hyperbolic and parabolic equations. The emphasis is on illustrating part of the subject without attempting to survey it. The point of view is classical, and this serves well in furnishing insight into the subject; it also makes it possible for the lectures to be read by someone familiar with only the fundamentals of real and complex analysis. *R. Carroll* (Urbana, Ill.)

Kamynin, L. I. 2534

The existence of a solution of boundary-value problems for a parabolic equation with discontinuous coefficients. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 721–744.

This paper contains the completely detailed version of a result announced by the author earlier in abbreviated form [Dokl. Akad. Nauk SSSR **139** (1961), 1048–1051; MR **24** #A1519]. It is proved that classical solutions of the first, second and third boundary-value problems exist for the system

$$a_i(x, t)\frac{\partial^2 u_i}{\partial x^2} = \frac{\partial u_i}{\partial t} + b_i(x, t)\frac{\partial u_i}{\partial x} + c_i(x, t)u_i + f_i(x, t)$$

$$(i = 1, 2)$$

under conditions too protracted to list here (but summarized in the review cited). The method consists in reducing the general problem to two simpler ones, the first being a system of singular Volterra integral equations having a solution in the form of a single-layer potential, and the second admitting the construction of a Green's function. The author leans heavily upon Gevrey's pioneering work on parabolic equations, now more than 50 years old. *R. N. Goss* (San Diego, Calif.)

Gruzewska, H. Milicer [Milicer-Grużewska, H.] 2535

Quelques propriétés des solutions fondamentales du système parabolique d'équations.

Publ. Sém. Géom. Univ. Neuchâtel (1) *Fasc.* 3 (1963), 3–9.

The author uses the "method of substitutions" developed earlier [Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **10** (1962), 239–246; MR **27** #476] to answer a question raised in Remarque (1.2) of the author's paper [Mem. Accad. Sci. Torino Parte I (3) **4** (1959/61), 255–280 (= no. 5, i + 24 pp.); MR **25** #327]. *J. Elliott* (New York)

Gruzewska, H. Milicer [Milicer-Grużewska, H.] 2536

Une application de la méthode "des substitutions" aux processus stochastiques.

Publ. Inst. Statist. Univ. Paris **12** (1963), 63–66.

The author shows that her proofs in a previous paper [same Publ. **9** (1960), 377–390; MR **26** #4058] can be greatly simplified by the "method of substitutions" developed by her [Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **10** (1962), 239–246; MR **27** #476]. *J. Elliott* (New York)

Deckert, K. L.; Maple, C. G. 2537

Solutions for diffusion equations with integral type boundary conditions.

Proc. Iowa Acad. Sci. **70** (1963), 354–361.

The authors prove the existence and uniqueness of the solution to the boundary-value problem:

$$(1) \qquad u_{xx} - u_t = F(x, t) \qquad (0 < x < 1, t > 0),$$

$$(2) \qquad u(x, 0) = g(x) \qquad (0 \leqq x \leqq 1),$$

$$(3) \qquad u(0, t) = f(t) \qquad (t \geqq 0),$$

$$(4) \qquad \int_0^1 u(x, t)\,dx = h(t) \qquad (t \geqq 0),$$

where F, g, f, h are continuous and g is representable by its Fourier cosine series. *J. Elliott* (New Brunswick, N.J.)

Kanel', Ja. I. 2538

Stabilization of solutions of the Cauchy problem for certain linear parabolic equations. (Russian)

Uspehi Mat. Nauk **18** (1963), *no.* 2 (110), 127–134.

The author considers the behavior as $t \to \infty$ of the solution of the initial-value problem

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) - b(x,t)\frac{\partial u}{\partial x}, \quad -\infty < x < \infty, \quad 0 < t,$$

$$u(x,0) = u_0(x), \qquad -\infty < x < \infty.$$

It is assumed that $a(x,t) \geqq a_0 > 0$ for some constant a_0, and that a and b have bounded derivatives up through order 6 and 5, respectively. It is assumed that u_0 is bounded and measurable.

Theorem 1: Let there exist positive constants ε, x_0 such that

$$(1) \qquad \frac{b(x,t)}{a(x,t)} > \frac{1+\varepsilon}{x}, \qquad x > x_0, \quad t > 0,$$

$$(2) \qquad \frac{b(x,t)}{a(x,t)} < \frac{1+\varepsilon}{x}, \qquad x < -x_0, \quad t > 0.$$

Then there exists a finite constant u^0 such that $\lim_{t\to\infty} u(x,t) \equiv u^0$. In case $u_0(x) \geqq 0$ and $u_0(x) > 0$ on some interval, then $u^0 > 0$. Theorem 2: Let (1) hold, and also

$$\frac{b(x,t)}{a(x,t)} \geqq \frac{1}{x}, \qquad x < -x_0,$$

and assume the existence of $\lim_{x\to-\infty} u_0(x) = u_0$. Then $\lim_{t\to\infty} u(x,t) \equiv u_0$.

Some results are obtained also in the case of a quasilinear equation. *B. Frank Jones, Jr.* (Houston, Tex.)

Jiang, Li-shang [Chiang, Li-shang] 2539

The two-phase Stefan problem. I.

Acta Math. Sinica **13** (1963), 631–646 (*Chinese*); *translated as Chinese Math.* **4** (1964), 686–702.

Suppose a semi-infinite metallic rod is heated at one end. If initially both the solid and liquid phases exist, then the two-phase Stefan problem occurs in the study of the temperature distribution $u_i(x,t)$ $(i=1,2)$ and the variation at the interface $h(t)$. It is the following boundary-value problem:

$$(1) \quad \frac{\partial}{\partial x}\left(k_1\frac{\partial u_1}{\partial x}\right) = c_1\rho_1\frac{\partial u_1}{\partial t}, \quad 0 < x < h(t), \ 0 < t \leqq T,$$

$$(2) \quad \frac{\partial}{\partial x}\left(k_2\frac{\partial u_2}{\partial x}\right) = c_2\rho_2\frac{\partial u_2}{\partial t}, \quad h(t) < x < \infty, \ 0 < t \leqq T,$$

$$(3) \qquad -k_1\frac{\partial u_1(0,t)}{\partial x} = g(t), \qquad 0 \leqq t \leqq T,$$

$$(4) \qquad u_1(x,0) = \phi_1(x), \qquad 0 \leqq x \leqq b,$$

$$u_2(x,0) = \phi_2(x), \qquad b \leqq x < \infty,$$

$$(5) \quad u_1(h(t),t) = u_2(h(t),t) = u_0, \qquad 0 \leqq t \leqq T,$$

$$(6) \quad L\rho_2\frac{dh}{dt} = -k_1\frac{\partial u_1(h(t),t)}{\partial x} + k_2\frac{\partial u_2(h(t),t)}{\partial x}, \qquad 0 \leqq t \leqq T,$$

$$(7) \quad h(0) = b; \quad u_2(x,t)\frac{\partial u_2(x,t)}{\partial x} = O(1) \quad \text{as } x \to \infty,\ t \in [0,\tau].$$

Here k_i, c_i, ρ_i are the thermoconductivity, specific heat, and density of the rod; L is the latent heat during melting, u_0 is the melting temperature, and b is the initial interface location.

The author proves that under certain monotonicity assumptions on the boundary and initial values, the interface $h(t)$ is a monotone increasing function of t. The method of successive approximations is used to prove the existence of a unique solution in a finite closed domain.

J. Elliott (New Brunswick, N.J.)

Jiang, Li-shang [Chiang, Li-shang] 2540

The two-phase Stefan problem. II.

Acta Math. Sinica **14** (1964), 33–49 (*Chinese*); *translated as Chinese Math.* **5** (1964), 36–53.

A continuation and generalization of the results in Part I [#2539]. The case in which the interface $h(t)$ is not necessarily a monotone increasing function of t is treated. The main result is proved by using the Leray-Schauder theorem. *J. Elliott* (New Brunswick, N.J.)

Pogorzelski, Witold 2541

Problème aux limites pour un système parabolique d'équations aux dérivées partielles.

Ann. Sci. École Norm. Sup. (3) **80** (1963), 173–191.

In this extensive work, the boundary-value problem for a quasilinear parabolic system of the form

$$(1) \qquad \Psi u = F[X, t, u, (D^1u), \cdots, (D^{M-1}u)]$$

with general boundary conditions is solved. In (1) Ψ is a matrix operator,

$$\Psi_{\alpha,\beta} = \sum_{k_1,\cdots,k_\nu}^{0\le\nu\le M} A_{\alpha\beta}{}^{k_1,\cdots,k_\nu}(X,t)\frac{\partial^\nu}{\partial x_{k_1}\cdots\partial x_{k_\nu}} - \delta_{\alpha\beta}\frac{\partial}{\partial t} \qquad (\alpha,\beta = 1,\cdots,N),$$

u is a vector, F is a vector of functions $F_1, \cdots, F_N$, $(D^\nu u)$ are matrices, elements of which are derivatives of $u_1, \cdots, u_N$ of order ν,

$$D_\beta{}^\nu u_\alpha = \frac{\partial^\nu u_\alpha}{\partial x_{k_1}\cdots\partial x_{k_\nu}},$$

β being determined by the numbers $k_1, \cdots, k_\nu$.

The solution of the problem is assumed in the form of a sum of the space potential

$$-\int_0^t \iiint_\Omega \Gamma(X,t;Y,\tau) \times F[Y,\tau,u(Y,\tau),(D_Y{}^1u),\cdots,(D_Y{}^{M-1}u)]\,dY d\tau,$$

of the potential of the simple layer

$$\int_0^t \iint_S \Gamma(X,t;Q,\tau)\varphi(Q,\tau)\,dQd\tau$$

with unknown density φ and of the Poisson-Weierstrass integral

$$\iiint_\Omega \Gamma(X,t;Y,0)f(Y)\,dY.$$

The problem is reduced to the solution of an integro-differential equation, the existence of the solution of which is proved by means of Schauder's fixed-point theorem. The verification of the assumptions justifying

the use of this theorem is rather complicated, and is carefully worked out in the paper. *R. Výborný* (Prague)

Szarski, J. 2542
Sur un système non linéaire d'inégalités différentielles paraboliques.
Ann. Polon. Math. **15** (1964), 15–22.

This note is concerned with the two non-linear systems of inequalities

$$(1)\quad z_t^i \leqq f^i(t, X, Z, z_x^i, z_{xx}^i)$$
$$(2)\quad z_t^i \geqq f^i(t, X, Z, z_x^i, z_{xx}^i) \qquad (i = 1, \cdots, m)$$

in an open subset Ω of $(0, T)\times R^n$. Here $X=(x_1, \cdots, x_n)$, $Z=(z^1, \cdots, z^m)$, $z_x^i=(z_{x_1}^i, \cdots, z_{x_n}^i)$, $z_{xx}^i=(z^i_{x_1x_1}, z^i_{x_1x_2}, \cdots, z^i_{x_nx_n})$. The author shows that under certain conditions on f^i, it follows that for two sequences of solutions u^i, v^i of (1) and (2), respectively, the initial inequalities $u^i(0, X)\leqq v^i(0, X)$ plus certain boundary inequalities imply $u^i\leqq v^i$ in Ω. This work extends some previous results of M. Mlak [Ann. Polon. Math. **3** (1957), 349–354; MR **19**, 558].
J. Elliott (New Brunswick, N.J.)

Łojczyk-Królikiewicz, I. 2543
L'allure asymptotique des solutions des problèmes de Fourier relatifs aux équations linéaires normales du type parabolique dans l'espace $\mathscr{E}^{m+1}$.
Ann. Polon. Math. **14** (1963), 1–12.

Let

$$(1)\qquad F[u] = \sum_{i,j=1}^{m} a_{ij}u_{x_ix_j} + \sum_{k=1}^{m} b_k u_{x_k} - u_t + cu$$

be a parabolic operator for $X=(x_1, \cdots, x_m)\in S_0$ and $t\in[0, \infty)$. The author investigates conditions under which solutions to $F[u]=f(x, t)$ tend uniformly to 0 in $\bar S_0$ as $t\to\infty$. The results obtained are similar to those of A. Friedman [J. Math. Mech. **8** (1959) 57–76; MR **21** #765], but the hypotheses on the coefficients are weaker.
J. Elliott (New Brunswick, N.J.)

Lipko, B. Ja. 2544
On $2\vec{b}$-parabolic systems with increasing coefficients. (Ukrainian. Russian and English summaries)
Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 35–38.

Author's summary: "A fundamental matrix of the solutions is found for $2\vec{b}$-parabolic systems, and is used to investigate Cauchy's problem. It is proved that a theorem of the Liouville theorem type holds true for the solution of such systems, and so does the uniqueness theorem of the Täcklund type."

Rektorys, Karel 2545
Die Lösung der gemischten Randwertaufgabe und des Problems mit einer Integralbedingung "im Ganzen" für eine nichtlineare parabolische Gleichung mit der Netzmethode. (Russian summary)
Czechoslovak Math. J. **13 (88)** (1963), 189–208.

Finite-difference techniques are used to establish existence theorems for several types of boundary-value problems for parabolic differential equations. Two of the results proved are as follows, where the precise smoothness condition on the coefficients in the differential equation have been omitted for brevity. These conditions may be found in a previous paper of the author [same J. **12 (87)** (1962), 69–103].

1. Let $f(x)$ be piecewise continuous on $[0, 1]$, $g(t)$ piecewise continuous on $[0, \tau]$, and $h(t)$ of class C^1 on $[0, \tau]$. Then there exists a unique solution $u(x, t)$ of the system

$$u_t = A(x,t,u)u_{xx} + B(x,t,u)u_x + C(x,t,u)u + D(x,t,u),$$
$$u(x, 0) = f(x), \qquad 0 < x < 1,$$
$$u(1, t) = g(t), \qquad 0 < t < \tau,$$
$$u_x(0, t) = k[u(0, t) - h(t)], \qquad 0 < t < \tau \quad (k > 0).$$

2. Let $f(x)$ be piecewise continuous on $[0, 1]$, and $g(t)$, $h(t)$ piecewise continuous on $[0, \tau]$. Then there exists a unique solution $u(x, t)$, $z(x, t)$ of the system

$$u_t = A(x, t, u)u_{xx} + B(x, t, u)u_x + C(x, t, u, z),$$
$$u(x, 0) = f(x), \qquad 0 < x < 1,$$
$$u(0, t) = g(t), \qquad 0 < t < \tau,$$
$$u(1, t) = h(t), \qquad 0 < t < \tau,$$
$$z(x, t) = \int_0^t C[x, \zeta, u(x, \zeta), z(x, \zeta)]\, d\zeta, \qquad 0 < t < \tau.$$

B. Frank Jones, Jr. (Houston, Tex.)

Bodrecova, L. B. 2546
Solution of a diffusion problem with variable temperature. (Russian)
Mat. Sb. (N.S.) **64 (106)** (1964), 223–233.

In the paper, the diffusion problem

$$\frac{\partial u}{\partial \tau} = D(\tau)\frac{\partial^2 u}{\partial x^2} \qquad (x > 0, \tau > 0),$$

$$u_{\tau=0} = c_0, \qquad \left.\frac{\partial u}{\partial x}\right|_{x=0} = \beta_1(\tau)u_{x=0},$$

with D and β_1 varying with time, is solved. After the transformation $t=\int_0^\tau D(\tau_1)\,d\tau_1$, which transforms the coefficient of $\partial^2u/\partial x^2$ to the value 1, the problem is solved approximately. Three methods are used for this purpose: expansion in series, successive approximations and the operational method. Convergence of these methods is proved. *K. Rektorys* (Prague)

Napetvaridze, O. I. 2547
On a fundamental contact boundary-value problem in heat conduction. (Russian. Georgian summary)
Soobšč. Akad. Nauk Gruzin. SSR **33** (1964), 271–278.

The author considers heat conduction in all of R^3, in which the medium is of one type in a region B_1 bounded by a smooth closed surface S and is of another type in $B_2=R^3-(B_1\cup S)$. If k_i is the coefficient of thermal conductivity, c_i the specific heat, and ρ_i the density of the medium in B_i, and if $a_i=(k_i/c_i\rho_i)^{1/2}$, then the following boundary-value problem is considered:

$$\Delta u = a_1^{-2}\,\partial u/\partial t \quad \text{in } B_1\times(0, \infty),$$
$$\Delta u = a_2^{-2}\,\partial u/\partial t \quad \text{in } B_2\times(0, \infty),$$
$$(u)_1 - (u)_2 = g(x, t) \quad \text{on } S\times(0, \infty),$$
$$(k_1\,\partial u/\partial n)_1 - (k_2\,\partial u/\partial n)_2 = h(x, t) \quad \text{on } S\times(0, \infty),$$
$$u(x, 0) = f(x) \quad \text{in } R^3.$$

Here the notation $(V)_i$ stands for the limiting value of V as points on S are approached from B_i.

A uniqueness theorem is established by a standard application of Green's identity. To prove existence, the author first reduces the problem to an integro-differential equation for u. Existence is then established under the additional assumption that

$$\left|1-\frac{a_2^2}{a_1^2}\right|+\frac{3}{2}\left|1-\frac{k_2}{k_1}\right| < 1,$$

which the author states is satisfied for the cases in which the media are iron and steel; bronze and iron; bronze and nickel; and silver and copper.

B. Frank Jones, Jr. (Houston, Tex.)

Milgram, A. N. 2548

Parabolic equations.

Partial Differential Equations (Proc. Summer Seminar, Boulder, Col., 1957), *pp.* 327–339. *Interscience, New York*, 1964.

Some of the methods yielding existence-uniqueness results for boundary-value problems for parabolic equations are discussed. First, three methods for solving $\Delta u - u_t = f$ are given, namely, a direct Hilbert-space approach yielding a weak solution, a potential-theoretic method leading to an integral equation of Volterra type, and a Perron-type method based on the maximum principle. Some observations about the methods and various boundary-value problems are made, and the lecture concludes with some remarks on parabolic equations of higher order.

R. Carroll (Urbana, Ill.)

Redheffer, R. M. 2549

Elementary remarks on problems of mixed type.

J. Math. and Phys. **43** (1964), 1–14.

The author extends uniqueness, comparison and stability theorems which he derived previously [Bull. Amer. Math. Soc. **69** (1963), 497–500; MR **26** #6512] for the third boundary-value problem to problems of mixed type. The equations are parabolic or elliptic, or a mixture of the two. Both the equation and the boundary condition are nonlinear, and the boundary is allowed to have corners and edges if they are not infinitely sharp. The region B is bounded in E^n, and is such that some $c \in C^{(2)}$ has a normal derivative $c_\nu > 0$ on that part of ∂B where w_ν is prescribed, w being the unknown function.

A comparison theorem follows. By p is meant $+$ or $-$. On ∂B, let $Ru = k(x) - k(x, u, u_\nu)$ and $\Delta k = k(x, u, u_\nu) - k(x, v, v_\nu)$. With $\Gamma \subset \partial B$, suppose $p(u-v) > 0$ on ∂B, $p(u_\nu - v_\nu) \leqq 0$ on Γ and $p(u_\nu - v_\nu) < 0$ on $\partial B - \Gamma$ implies $p\Delta k < 0$. Let $Tu = \varphi(x) - f(x, u, u_i, u_{ij})$, where f is monotone with respect to u, strictly so in $B_1 \subset B$. With $I(r, s) = p[f(x, s, r_i, r_{ij}) - f(x, s, s_i, r_{ij})]$, suppose $|I(r, s)| \leqq g\|r_i - s_i\|$, where g is continuous, non-decreasing, $g(0) = 0$, and $\int_0^1 [g(t)]^{-1}\, dt = \infty$. Suppose $f_{ij} = (\partial/\partial s_{ij}) f(x, v, v_i, s_{ij}) \in C$. With u, v fixed, define $s_{ij}(x)$ by $p[f(x, v, v_i, u_{ij}) - f(x, v, v_i, v_{ij})] = p \sum f_{ij}(x, v, v_i, s_{ij})(u_{ij} - v_{ij})$, and let $m_{ij}(x) = f_{ij}(x, v, v_{ij}, s_{ij})$. The author proves the following theorem. Suppose $(m_{ij}) \geqq 0$. Suppose $S_c(m_{ij}) \subset \Gamma_1 \cup B_1$, where $S_c = \{x \mid x \in \partial B, c_\nu \leqq 0$, or $x \in B \cup \partial B$ and $\liminf \sum m_{ij} c_{ij} = -\infty$ or $\liminf \sum m_{ij} c_i c_j \leqq 0\}$. Under suitable differentiability assumptions, $p(Tu - Tv) \leqq 0$ and $p(Ru - Rv) \leqq 0$ implies $p(u - v) \leqq 0$.

L. Sarason (Stanford, Calif.)

Sobolev, S. 2550

Sur les problèmes mixtes pour les équations aux dérivées partielles à deux variables indépendantes.

Calcutta Math. Soc. Golden Jubilee Commemoration Vol. (1958/59), *Part II, pp.* 447–484. *Calcutta Math. Soc., Calcutta*, 1963.

The author considers the equation in two independent variables with constant coefficients,

$$(*) \qquad \frac{\partial^n u}{\partial t^n} + \sum_{\substack{l \leq m \\ k < n}} A_{kl} \frac{\partial^{k+l} u}{\partial t^k \partial x^l} = F(x, t), \qquad 0 \leqq t < \infty,$$

with Cauchy data prescribed at $t = 0$. Mixed problems for this equation are considered in either of the two regions (a) $0 \leqq x < \infty$, or (b) $0 \leqq x \leqq 1$. The results will be described for the case $0 \leqq x \leqq \infty$, in which the author treats the mixed problem of finding a solution of the differential equation which satisfies the given Cauchy data, together with the boundary conditions with constant coefficients,

$$\sum_{l=0}^{m-1} B_l^{(s)} \left.\frac{\partial^l u}{\partial x^l}\right|_{x=0} = 0, \qquad s = 1, 2, \cdots, q_-.$$

Let $\Delta(\lambda, \alpha) = \lambda^n + \sum A_{kl} \lambda^k \alpha^l$. The condition of Petrovsky for the well-posedness of the Cauchy problem is that there exists a number σ_0 such that for purely imaginary α the roots λ of $\Delta(\lambda, \alpha) = 0$ satisfy $\operatorname{Re} \lambda \leqq \sigma_0$. The given equation is assumed to satisfy this condition.

Under these conditions, for each λ such that $\operatorname{Re} \lambda > \sigma_0$, the equation $\Delta(\lambda, \alpha) = 0$ has the same number r_- of roots α such that $\operatorname{Re} \alpha < 0$. The author's result for the mixed problem described above is contained in the statement of the rough principle that the mixed problem is well-posed if and only if $q_- = r_-$, but this is only an approximate statement of the result.

The analysis is based on a detailed analysis of Green's functions, with Laplace transform techniques.

It is stated that analogous results are valid for systems of equations of the type (*), where u and F are vectors of dimension N and each A_{kl} is an $N \times N$ constant matrix.

There are numerous misprints.

B. Frank Jones, Jr. (Houston, Tex.)

Karatoprakliev, G. 2551

A generalization of the Tricomi problem. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 271–274.

The generalization considered is one in which the solution and its derivative normal to the line of parabolic degeneracy are permitted to have discontinuities of the form

$$u(x, 0^-) = \alpha(x) u(x, 0^+) + \gamma(x)$$

at that line. The Tricomi extremum principle is found to hold for the new problem, and an existence proof is given.

D. A. Lee (Dayton, Ohio)

Fényes, Tamás 2552

Über restriktive lineare partielle Differential-Differenzengleichungen mit konstanten Koeffizienten. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 13–25.

Consider the linear partial differential-difference equation with constant coefficients

$$(1)\quad \sum_{\mu=0}^{m}\sum_{\nu=0}^{n}\alpha_{\mu\nu}\frac{\partial^{\mu+\nu}x(\lambda,t)}{\partial\lambda^{\mu}\partial t^{\nu}}+\sum_{\mu=0}^{m_1}\sum_{\nu=0}^{n_1}\beta_{\mu\nu}\frac{\partial^{\mu+\nu}x(\lambda,t-\tau)}{\partial\lambda^{\mu}\partial t^{\nu}}=\varphi(\lambda,t),$$

$\lambda_1\leqq\lambda\leqq\lambda_2$, $0\leqq t<\infty$, $\tau>0$ and $x(\lambda,t)=0$, when $t<0$, subject to conditions of the form

$$(2)\quad \begin{aligned}&\sum_{\mu=0}^{m}\sum_{\nu=0}^{\kappa}\alpha_{\mu,n-\kappa+\nu}\frac{\partial^{\mu+\nu}x(\lambda,0)}{\partial\lambda^{\mu}\partial t^{\nu}}=g_{\kappa}^{1}(\lambda),\quad \kappa=0,1,\cdots,n-1,\\ &\sum_{\mu=0}^{m_1}\sum_{\nu=0}^{\kappa}\beta_{\mu,n_1-\kappa+\nu}\frac{\partial^{\mu+\nu}x(\lambda,0)}{\partial\lambda^{\mu}\partial t^{\nu}}=g_{\kappa}^{2}(\lambda),\quad \kappa=0,1,\cdots,n-1.\end{aligned}$$

Having defined what is meant by a solution of (1) subject to (2), as well as the notion of restrictiveness, a criterion for the restrictiveness of (1) is given. After deriving an operational property of (1), the author illustrates the theorems proved by means of an example. The work follows closely that of J. Mikusiński [*Operatorenrechnung*, VEB Deutscher Verlag Wiss., Berlin, 1957; MR **20** #1230] and J. Wloka [J. Reine Angew. Math. **202** (1959), 107–128; MR **22** #148].

E. J. Scott (Urbana, Ill.)

Kapilevič, M. B. 2553

The approximation of singular solutions of the Chaplygin equation. (Russian)

Dokl. Akad. Nauk SSSR **154** (1964), 258–261.

The author considers the equation $z_{\theta\theta}-z_{\sigma\sigma}-b(\sigma)z_\sigma=0$ with $b(\sigma)=\sum b_n\sigma^{2(n-1)/3}$ and $z(\theta,0)=\tau(\theta)$, $z_\eta(\theta,0)=\nu(\theta)$, $\eta=-(\frac{3}{2}\sigma)^{2/3}$. By a change of dependent variable he obtains a partial differential equation that is satisfied by an integral $W(\theta,\sigma)$ of a kernel function in a certain integral representation of z. Two different iterative procedures are derived for computation of successive approximations of $W(\theta,\sigma)$. *Č. Masaitis* (Aberdeen, Md.)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 2233, 2368, 2369, 2396, 2552, 2588, 2594, 2939, 3270.

Coifman, Ronald 2554

Sur l'équivalence des familles d'itérées et quelques définitions d'itération régulière.

C. R. Acad. Sci. Paris **259** (1964), 284–286.

A family $f_\sigma(x)$, $\sigma\in R$, of iterates of $f(x)$ is a family for which $f_1(x)=f(x)$, $f_{\sigma+\mu}(x)=f_\sigma(f_\mu(x))$. It is assumed here that $f_\sigma(x)$ is continuous and strictly monotone in σ and x and that $0<f_\sigma(x)<x$ for $\sigma>0$, $x\in(0,\alpha_0)$. The author restates a definition of P. Lévy [Ann. Mat. Pura Appl. (4) **5** (1928), 269–298, p. 282] to declare families f_σ and g_σ equivalent when there exists $\alpha\neq0$ such that, for all $\sigma>0$, $\varepsilon>0$, there is an $X=X(\varepsilon,\sigma)$ with $g_{\alpha\sigma+\varepsilon}(x)<f_\sigma(x)<g_{\alpha\sigma-\varepsilon}(x)$ for $0<x<X$. Necessary and sufficient conditions for equivalence are given, and it is shown that definitions of regular families of iterates given by G. Szekeres [Acta Math. **100** (1958), 203–258; MR **21** #5744; J. Austral. Math. Soc. **2** (1961/62), 301–320; MR **25** #5302] can be restated in terms of equivalence to a standard family.

I. N. Baker (London)

Kuczma, Marek 2555

Note on Schröder's functional equation.

J. Austral. Math. Soc. **4** (1964), 149–151.

The paper contains an elementary proof of the following theorem: Suppose that the real function $f(x)$ defined in $\langle 0,a)$, $a\leqq\infty$, satisfies (i) f is continuous and strictly increasing in $\langle 0,a)$, (ii) $f(0)=0$ and $0<f(x)<x$ in $(0,a)$, (iii) $\lim_{x\to0+0}f(x)/x=s$, $0<s<1$. Let $f_n(x)$ denote the nth iterate of $f(x)$, $n=1,2,\cdots$. Then Schröder's equation $\phi[f(x)]=s\phi(x)$ possesses at most a one-parameter family of solutions $\phi(x)$ such that $\phi(x)/x$ is monotonic in $(0,a)$. If these solutions exist, they are given by

$$\phi(x)=c\lim_{n\to\infty}f_n(x)/f_n(d),$$

where c is an arbitrary constant and d is arbitrary in $(0,a)$. If, moreover, $f(x)/x$ is monotonic in $(0,a)$, then this formula actually defines a one-parameter family of solutions such that $\phi(x)/x$ is monotonic in $(0,a)$.

I. N. Baker (London)

Lundberg, Anders 2556

On iterated functions with asymptotic conditions at a fixpoint.

Ark. Mat. **5**, 193–206 (1964).

The author continues investigations of G. Szekeres [Acta Math. **100** (1958), 203–258; MR **21** #5744] and H. Michel [Publ. Math. Debrecen **9** (1962), 13–46; MR **25** #4050] concerning iterations of real functions.

Let S be the class of functions f which are continuous and strictly increasing on $[0,\infty)$, $f(0)=0$, $f(\infty)=\infty$, $f(x)\neq x$ in $(0,\infty)$, supplemented by the identity function $f(x)\equiv x$. A one-parameter family of functions $f_s\subset S$ is called an iteration group (i.g.) of $f(x)$ if $f_s(f_t(x))=f_{s+t}(x)$ holds for $x\in[0,\infty)$, $s,t\in(-\infty,+\infty)$ and $f_1(x)=f(x)$. An i.g. is called regular [cf. Szekeres, loc. cit.] whenever $f_s'(0)$ exists for every s and is different from 0 and 1, except when $s=0$. (Such an i.g. is unique [cf. Szekeres, loc. cit., Michel, loc. cit.].) The author's main result is the following. If $f\in S$ and $f'(0)=k$, $0<k<1$, then f has a regular i.g. $f_s(x)$, continuous with respect to s, if and only if $\varphi(x)=\lim f_n(x)/f_n(a)$ (where a is an arbitrarily fixed number from $(0,\infty)$ and $f_n(x)$ are the natural iterates of $f(x)$) exists and is continuous and strictly increasing on $(0,\infty)$. ($\varphi(x)$ is then a solution of the Schröder equation $\varphi[f(x)]=k\varphi(x)$ and $f_s(x)=\varphi^{-1}[k^s\varphi(x)]$.) The author then rediscovers the condition $f'(x)=k+O(x^\delta)$, $\delta>0$, $x\to0$, due to G. Szekeres [loc. cit.], finds the condition that $f(x)$ be convex or concave (independently of the reviewer [Rozprawy Mat. **34** (1963)]; cf. also M. K. Fort, Jr. [Proc. Amer. Math. Soc. **6** (1955), 960–967; MR **18**, 326]; see also the reviewer: #2555 above) and the condition that $f(x)/x$ be increasing, as sufficient conditions for the existence of a regular i.g. of such an $f(x)$. In the last section the author continues the investigations of Michel [loc. cit.] concerning the uniqueness of i.g.'s. However, a number of questions are still left open. *M. Kuczma* (Katowice)

Hosszú, Miklós 2557

On a functional equation treated by S. Kurepa. (Serbo-Croatian summary)

Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II **18** (1963), 59–60.

The author shows that the most general solution of the functional equation (*) $f(x+y,z)+f(x,y)=f(x,y+z)+f(y,z)$ is of the form $f(x,y)=S(x,y)+A(x,y)$, where S is a symmetric solution of (*) and A an antisymmetric one, additive in each place. If f is a continuous, measurable or bounded solution of (*), then $A=0$. Thus all such solutions are symmetric and, by a result of J. Erdős [same Glasnik **14** (1959), 3–5; MR **21** #7375], must be of the form $f(x,y)=F(x+y)-F(x)-F(y)$. The equation (*) has also been considered by S. Kurepa [ibid. **11** (1956), 3–5; MR **18**, 217]. *B. Schweizer* (Tucson, Ariz.)

Gheorghiu, Oct. Em. [Gheorghiu, Octavian Emil] 2558

Über eine Klasse von Funktionalgleichungen.

Enseignement Math. (2) **10** (1964), 245–247.

The author considers the functional equation

$$(*)\ F(x+y)=\frac{(cA+aB)F(x)F(y)+C(aF(x)+aF(y)+b)}{D[-aF(x)F(y)-bF(x)-bF(y)]+dA-bB},$$

where $A=ad-bc$, $B=ab+cd$, $C=b^2+d^2$, $D=a^2+c^2$ (a,b,c,d real), and $A\neq 0$. By means of a sequence of formal substitutions he reduces (*) to a system of functional equations previously solved by M. Ghermănescu [Bul. Inst. Politech. Iaşi **4** (1949), 362–368; MR **20** #187] and finds that the most general measurable solution of (*) is given by $F(x)=(b-d\tan\lambda x)/(c\tan\lambda x-a)$. Special cases of (*) have previously been considered by V. Alaci [Mathematica (Timişoara) **19** (1943), 23–25; MR **5**, 185]. *B. Schweizer* (Tucson, Ariz.)

Apostol, Tom M.; Zuckerman, Herbert S. 2559

On the functional equation

$$F(mn)F((m,n))=F(m)F(n)f((m,n)).$$

Pacific J. Math. **14** (1964), 377–384.

The authors study the characterization of all numerical functions F defined on the positive integers which satisfy the functional equation

$$(1)\qquad F(mn)F((m,n))=F(m)F(n)f((m,n)),$$

where f is a completely multiplicative arithmetical function, $f(mn)=f(m)f(n)$, and (m,n) denotes the greatest common divisor of m and n. It is noteworthy that in the case $f(n)\equiv n$, equation (1) is satisfied by the Euler φ function.

For the given function $f(n)$ the value $f(1)$ may be only 0 or 1. If $f(1)=0$, $f(n)$ vanishes identically and (1) reduces to

$$(2)\qquad F(mn)F((m,n))=0.$$

For this case the authors establish that $F(n)$ satisfies (2) if and only if there exists a set of positive integers A, no one of which is divisible by the square of any other, such that $F(n)=0$ for all n not in A. If $f(1)=1$ and $F(1)\neq 0$, the solutions may be normalized so that $F(1)=1$. In this case the authors find that $F(n)$ satisfies (1) if and only if $F(n)$ is multiplicative and satisfies the equation

$$F(p^{a+b})F(p^b)=F(p^a)F(p^b)f(p^b)$$

for all primes p and all integers $a\geqq b\geqq 1$. A number of subsequent theorems furnish a complete classification of all solutions of (1) for this case. If $f(1)=1$ and $F(1)=0$, every solution of (1) must satisfy

$$(3)\qquad F(m)F(n)=0,\qquad (m,n)=1.$$

A function $F(n)$ is shown to be a solution of (3) if there exists a set S of positive integers, all greater than unity and no pair relatively prime, such that $F(n)$ is not zero for any n in S, but vanishes for all other values. Additional specifications for S are obtained for the case $f(n)$ is never zero. *P. E. Guenther* (Cleveland, Ohio)

Haruki, Hiroshi 2560

On the functional inequality

$$\left|f\left(\frac{x+y}{2}\right)\right|\leqq\frac{|f(x)|+|f(y)|}{2}.$$

J. Math. Soc. Japan **16** (1964), 39–41.

The author shows that the only entire functions $f(z)$ satisfying $|f((x+y)/2)|\leqq\frac{1}{2}(|f(x)|+|f(y)|)$ are $(\alpha z+\beta)^n$ and $\exp(\alpha z+\beta)$, where α, β are arbitrary complex numbers and n is an arbitrary natural number. Applications are made to finding the entire solutions of a number of functional equations. *I. N. Baker* (London)

Mitrinović, Dragoslav S. 2561

Sur une équation fonctionnelle binome.

C. R. Acad. Sci. Paris **258** (1964), 5577–5580.

If S is a non-empty set, L an operator, f a function, C the set of complex numbers, the author remarks that the functional equation $f(z)=af(Lz)$, where $z\in S$, $L:S\to S$, $f:S\to C$, $L^n=1$, $a^n=1$, has the general solution $f(z)=\sum_{\nu=1}^n a^\nu g(L^\nu z)$, where $g:S\to C$ is an arbitrary function. If $a^n\neq 1$, there is only the trivial solution $f\equiv 0$. Several other functional equations are reduced to the one above. *I. N. Baker* (London)

SEQUENCES, SERIES, SUMMABILITY

See also 2392, 2598.

Kazandzidis, G. S. 2562

On a conjecture of Moessner and a general problem.

Bull. Soc. Math. Grèce (N.S.) **2** (1961), no. 1, 23–30.

Let k be a fixed positive integer. Delete all multiples of k from the sequence (1): $1,2,3,\cdots$; of the remaining sequence $1,2,\cdots,k-1,k+1,\cdots,2k-1,2k+1,\cdots$ write down the sequence of partial sums (2): $s_1,s_2,s_3,\cdots$. Now delete the terms of (2) of index divisible by $k-1$ and again write down the sequence of partial sums of the remaining sequence; call this new sequence (3): $t_1,t_2,t_3,\cdots$. Continuing in this way we get the sequences (1), (2), (3), $\cdots$, (k). A. Moessner [S.-B. Math.-Natur. Kl. Bayer Akad. Wiss. **1951**, 29; MR **14**, 353] conjectured that the sequence (k) is $1^k,2^k,3^k,\cdots$. The present paper contains a simple proof of this conjecture. In addition, the author states a more general conjecture, the proof of which will appear in a later paper.

It is noted that a proof of Moessner's conjecture was given by O. Perron [ibid. **1951**, 31–34; MR **14**, 353]. *L. Carlitz* (Durham, N.C.)

Manolov, Spas 2563

Some properties of a class of recurrently defined infinite sequences. (Bulgarian)

Fiz.-Mat. Spis. Bŭlgar. Akad. Nauk. **7 (40)** (1964), 48–51.

Avec une suite périodique $\{a_{10}, a_{20}, \cdots, a_{k0}, a_{10}, a_{20}, \cdots, a_{k0}, \cdots\}$, $k \geqq 3$, on forme une nouvelle suite a_{sn} par la formule de récurrence

$$a_{sn} = \tfrac{1}{3}(a_{s+p,n-1} + a_{s+q,n-1} + a_{s+r,n-1})$$
$$(0 \leqq p < q < r;\ s = 1, 2, \cdots, k;\ n = 1, 2, 3, \cdots).$$

On examine le cas où $\lim_{n\to\infty} a_{sn}$ est égale à $k^{-1}(a_{10} + a_{20} + \cdots + a_{k0})$. *M. Tomić* (Belgrade)

Azatjan, K. G. 2564

On the sum of series which do not converge absolutely. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 3 (40), 3–6.

The main result is the following elementary theorem. Let p, q be natural numbers. Let f be a positive non-increasing function on $[1, \infty)$, and put $a_n = f(n)$. Let the series $\sum_{n=1}^{\infty} (a_n - b_n)$ have a finite sum s, and let

$$t = \lim_{r\to\infty} \int_{pr}^{qr} f(x)\,dx.$$

Then

$$\sum_{r=1}^{\infty} (a_{(r-1)p+1} + \cdots + a_{rp} - (b_{(r-1)q+1} + \cdots + b_{rq})) = s + t.$$

J. Mařík (Prague)

Dzagnidze, O. P. 2565

Representation of measurable functions of two variables by double series. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **34** (1964), 277–282.

Soient $\Phi(u)$ et $\Psi(v)$ des fonctions complémentaires au sens de Young, telles que $\Phi(2u) \leqq K\Phi(u)$ pour $u > 0$ et $\Psi(2v) \leqq K\Psi(v)$ pour $v > 0$, où K est une constante positive. Désignons par $L_\Phi(R)$, $R = [0, 1; 0, 1]$, l'ensemble des fonctions réelles mesurables $f(x, y)$, telles que

$$\iint_R \Phi(|f(x, y)|)\,dxdy < +\infty.$$

La norme de $f(x, y) \in L_\Phi(R)$ est définie par

$$\|f\|_\Phi{}^R = \sup \left| \iint_R f(x, y) g(x, y)\,dxdy \right|,$$

où la borne supérieure est prise par rapport à toutes les $g(x, y)$ pour lesquelles

$$\iint_R \Psi(|g(x, y)|)\,dxdy \leqq 1.$$

On démontre sept théorèmes, dont voici trois. Théorème 3: Pour chaque fonction $f(x, y)$ mesurable et presque partout finie sur R, on peut définir, par rapport à un système de Haar, une série double qui converge vers $f(x, y)$ presque partout sur R. Théorème 6: Le système de Haar est complet par rapport à $L(R)$. Théorème 7: Le système de Haar forme une base de $L_\Phi(R)$.

S. Marcus (Bucharest)

Dzagnidze, O. P. 2566

On universal double series. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **34** (1964), 525–528.

Considérons la série

$$(1) \qquad \sum_{m,n=1}^{\infty} f_{m,n}(x, y)$$

où $f_{m,n}(x, y)$ $(m, n = 1, 2, \cdots)$ sont presque partout finies et mesurables sur l'intervalle fermé $R = [0, 1; 0, 1]$. La suite (1) est universelle au sens habituel si pour toute fonction f mesurable, presque partout finie sur R ou égale à $+\infty$ ou $-\infty$ sur un ensemble de mesure positive, il existe une suite croissante $\{p_n\}$ de nombres naturels, telle que

$$\lim_{n\to\infty} \sum_{i=1}^{p_n} \sum_{k=1}^{p_n} f_{i,k}(x, y) = f(x, y)$$

presque partout sur R. La série (1) est dite universelle par rapport à la permutabilité dans la classe des fonctions presque partout finies et mesurables au sens de la λ-convergence en mesure [V. G. Čelidze, Akad. Nauk Gruzin. SSR Trudy Tbiliss. Mat. Inst. Razmadze **16** (1948), 1–37; MR **14**, 159] si pour toute fonction f presque partout finie et mesurable sur R les termes de la série (1) peuvent être permutés de manière que la série double ainsi obtenue soit λ-convergente en mesure sur R vers f. On donne quatre théorèmes, dont voici le dernier: Si la série (1) est universelle par rapport à la permutabilité dans la classe des fonctions presque partout finies et mesurables au sens de la λ-convergence en mesure pour $\lambda = 1$, alors les termes de (1) peuvent être permutés de manière que la série double ainsi obtenue soit universelle au sens habituel. *S. Marcus* (Bucharest)

Borwein, D.; Shawyer, B. L. R. 2567

On absolute Riesz summability factors.

J. London Math. Soc. **39** (1964), 455–465.

Given a series $\sum_{n=1}^{\infty} a_n$, an unboundedly increasing sequence $\{\lambda_n\} \equiv \lambda$ of positive numbers defines, for $x \geqq \lambda_1$, the Riesz mean $x^{-k} \sum_{\lambda_n \leqq x} (x - \lambda_n)^k a_n$ of type λ and order k $(\geqq 0)$, and the series is said to be absolutely Riesz summable, or summable $|R, \lambda, k|$, if the Riesz mean is of bounded variation in (λ_1, ∞). The authors' object is "to obtain manageable conditions sufficient to ensure, when k is not an integer, the truth of the proposition: $\sum_{n=1}^{\infty} a_n \psi(\lambda_n)$ is summable $|R, \phi(\lambda), k|$ whenever $\sum_{n=1}^{\infty} a_n$ is summable $|R, \lambda, k|$". They succeed to the extent of obtaining a set of conditions each of which is verifiable in particular cases, quite easily if the functions involved are not too complicated. To the reviewer's knowledge, the first paper in this direction was that by Chandrasekharan [J. Indian Math. Soc. (N.S.) **6** (1942), 168–180; MR **5**, 63] in 1942, and further contributions came from Tatchell [J. London Math. Soc. **29** (1954), 49–59; MR **15**, 305], Dikshit [Indian J. Math. **1** (1958), no. 1, 33–40; MR **21** #3695; Proc. Nat. Inst. Sci. India Part A **26** (1960), 86–94; MR **23** #A3396], Prasad, Pati, as well as the reviewer [J. London Math. Soc. **31** (1956), 300–311; MR **19**, 135; ibid. **31** (1956), 311–319; MR **19**, 135], references to most of which can be found in the present paper. The main result obtained by the authors, too long to quote here, includes some of the earlier results and extends the range of applicability of the reviewer's result.

U. C. Guha (Singapore)

Daniel, E. C. 2568
On absolute summability factors of infinite series.
Proc. Japan Acad. **40** (1964), 65–69.

Let $\{\mu_n\}$ be a positive sequence such that $\Lambda_n=\sum_{v=1}^n \mu_v\to\infty$ as $n\to\infty$. An infinite series $\sum a_n$ with partial sums s_n is said to be summable $|R, \mu_n|$ if the sequence

$$t_n = \frac{1}{\Lambda_n}\sum_{v=1}^n \mu_v s_v, \qquad n=1,2,3,\cdots,$$

is of bounded variation (i.e., absolutely convergent). For arbitrary sequences $\{\lambda_n\}$ and $\{\varphi_n\}$ the following result is obtained. If

$$\sum_{n=2}^\infty |s_n-a_1|\,|\lambda_n|\,|\varphi_n-\varphi_{n+1}|<\infty,$$

$$\sum_{n=2}^\infty |s_n-a_1|\,|\lambda_n|\,|\varphi_n|\frac{\mu_n}{\Lambda_n}<\infty,$$

$$\sum_{n=2}^\infty |s_n-a_1|\,|\varphi_{n+1}|\,|\lambda_n-\lambda_{n+1}|<\infty,$$

then $\sum_{n=1}^\infty a_n\lambda_n\varphi_n$ is summable $|R, \mu_n|$.

J. D. Buckholtz (Lexington, Ky.)

Lorentz, G. G.; Zeller, K. 2569
Summation of sequences and summation of series.
Proc. Amer. Math. Soc. **15** (1964), 743–746.

Let $\{s_k\}$ denote a sequence, $\sigma_n=A_ns=\sum_k a_{nk}s_k$, $\tau_n=A_n'u=\sum_k a_{nk}'u_k$, where $u_k=s_k-s_{k-1}$, $s_{-1}=0$. The series matrix A' can be called a dual of the sequence matrix A if σ_n becomes τ_n, or conversely, under the application of formal summation by parts. In many instances dual methods of summation are equivalent in the sense that they have the same summability fields. (A summability field of a sequence or series method B is the set of all sequences which are summable by B.) In this paper the authors show that the equivalence property is not universal by showing that there exist regular sequence summation methods for which the summability field is not contained in the summability field of any series summation method, and vice versa.

B. E. Rhoades (Berkeley, Calif.)

Martić, Branislav 2570
Two inclusion theorems for a class of summability methods. (Serbo-Croatian. English summary)
Bull. Soc. Math. Phys. Macédoine **13** (1962), 13–20.

The summability methods $S^{\alpha,\beta}$ were introduced by the reviewer [Acad. Serbe Sci. Publ. Inst. Math. **12** (1958), 125–136; MR **23** #A1964] and then studied in several papers by the author [Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke **325** (1962), 127–163; MR **27** #3972; Publ. Inst. Math. (Beograd) (N.S.) **2 (16)** (1962), 93–98]. In the first paper the author proved the inclusion $S^{\alpha,\beta}\subset S^{\alpha+\varepsilon,\beta}$ for $\alpha>-1$, $\beta>0$, $\alpha+\beta>0$ and $\varepsilon>0$. In the second paper he established the inclusion $S^{\alpha,\beta}\subset S^{\alpha,\theta\beta}$ for $\alpha\geqq 0$, $\beta>0$ and $0<\theta<1$. The superposition of both theorems gives the inclusion $S^{\alpha,\beta}\subset S^{\alpha+\varepsilon,\theta\beta}$ for $\alpha\geqq 0$, $\beta>0$, $\varepsilon>0$ and $0<\theta<1$. Here the author gives, under the same conditions, a short and unified proof of the last inclusion. He discusses also the series-to-series version of the quoted methods and proves that the $S^{\alpha,\beta}$-summability of the series $\sum_{n=0}^\infty u_n$ implies the $S^{\alpha,\beta}$-summability of the series $\sum_{n=0}^\infty v_n$, where $v_0=0$ and $v_n=u_{n-1}$ for $n=1,2,3,\cdots$ to the same limit (but not conversely).

V. Vučković (Notre Dame, Ind.)

Nikolenko, V. N. 2571
On a method of summation of series. (Russian)
Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 2 (39), 127–135.

Let $A_\alpha{}^k$ be the operator defined by

$$(A_\alpha{}^k\varphi)(x)=\frac{1}{\Gamma(k)}\int_0^\infty \varphi(xe^{-\tau})e^{\alpha\tau}\tau^{k-1}\,d\tau$$

defined on the class of continuous functions φ satisfying the conditions $\varphi(x)=O(x^s)$ as $x\to 0$, where $s\geqq 0$, $s\geqq \operatorname{Re}\alpha$, $k>0$. The inverse operator

$$(A_\alpha{}^{-k}f)(x)=\frac{(-1)^{k_0+1}}{\Gamma(k_0-k+1)}\int_0^\infty \tau^{k_0-k}\frac{d^{k_0+1}}{d\tau^{k_0+1}}[f(xe^{-\tau})e^{\alpha\tau}]\,d\tau,$$

where $k_0=[k]$, acts on functions satisfying the above conditions and is differentiable k_0 times. The formula for composition of operators $A_\alpha{}^k$ and their inverse operators is established. This enables the author to evaluate the sums of the series $\sum_{n=s}^\infty c_nP(n)x^n$, where

$$P(n)=\frac{M_0+M_1(n-\beta_1)^{r_1}+\cdots+M_q(n-\beta_q)^{r_q}}{(n-\alpha_1)^{k_1}\cdots(n-\alpha_p)^{k_p}}$$

in terms of integrals over infinite intervals in the case when the sum of the entire function $\sum_{n=s}^\infty c_nx^n$ is known.

A. Alexiewicz (Poznań)

Geisberg, S. P. 2572
On the absolute summability of gap series by the Riemann method. (Russian)
Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 4 (41), 39–46.

Definition: A sequence $\{\varphi_n\}$ of integers is $|R,\alpha|$-lacunary if every $\sum a_k$ converges which is $|R,\alpha|$-summable and for which $a_k=0$ if $k\neq\varphi_n$. Theorem: If $\sum a_k$ is $|R,\alpha|$-summable and $\sum_{k=0}^m a_k=O(1)$, then $\sum a_k$ is absolutely Abel summable. Theorem: A sequence $\{\varphi_n\}$ of natural numbers is $|R,\alpha|$-lacunary if and only if $\liminf(\varphi_{n+1}/\varphi_n)>1$.

J. Mayer-Kalkschmidt (Albuquerque, N.M.)

Elin, M. V. 2573
Integral summability methods which are absolutely translative for bounded functions. (Russian)
Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 3 (40), 51–58.

An integral transformation (T): $\varphi(s)=\int_0^\infty K(s,t)f(t)\,dt$ is called uniformly regular if it is regular and transforms every bounded and uniformly convergent set of measurable functions onto a uniformly convergent one. A regular transformation (T) is uniformly regular if and only if $\int_0^B |K(s,t)|\,dt\to 0$ $(s\to\infty)$ for every $B>0$. A transformation (T) is called absolutely translative for bounded functions (a.t.b.f.) if $\int_0^\infty K(s,t)(f(t+u)-f(t))\,dt\to 0$ $(s\to\infty)$ for every bounded measurable function f and every $u>0$. A regular transformation (T) is a.t.b.f. if (and, for a uniformly regular transformation, only if)

$$\int_0^\infty |K(s,t)-K(s,t-u)|\,dt\to 0 \qquad (s\to\infty)$$

for every $u>0$ (we put $K(s,t)=0$ for $t\leq 0$). The author constructs a regular a.t.b.f. transformation not fulfilling this condition. Further, some related problems are investigated, e.g., a necessary and sufficient condition is given for a transformation (T) with the kernel $K(s,t)=p(s-t)/P(s)$ $(t\leq s)$, $K(s,t)=0$ $(t>s)$, where $0<P(s)=\int_0^s p(t)\,dt$, $p(t)\geqq 0$, to be a.t.b.f. *J. Mařík* (Prague)

Berg, I. David 2574

A Banach algebra criterion for Tauberian theorems.

Proc. Amer. Math. Soc. **15** (1964), 648–652.

For an infinite complex matrix $A=(a_{ij})_{i,j=1}^{\infty}$ and a sequence $x=(x(k))_{k=1}^{\infty}$ the transformed sequence Ax is defined by $Ax(j)=\sum_{k=1}^{\infty} a_{jk}x_k$. A matrix A is conservative if Ax is convergent for all convergent x. The set Γ of conservative matrices is known to be a Banach algebra under the usual algebra operations and with $\|A\|=\sup\{\sum_{j=1}^{\infty}|a_{ij}|: 1\leq i<\infty\}$. Let Δ be the Banach subalgebra of Γ consisting of all $(a_{ij})\in\Gamma$ such that $a_{ij}=0$ for $j>i$. J. Copping [Studia Math. **21** (1961/62), 177–194; MR **25** #3296] showed that if $A\in\Delta$, A is on the boundary of the group of invertible elements of Δ, A is non-invertible, and A is not a left divisor of zero, then A sums a bounded divergent sequence. The author proves Copping's theorem with Δ replaced by Γ, and more, as follows. Theorem 1: If A sums a bounded divergent sequence, then A is a left topological divisor of zero. Theorem 2: If A is a left topological divisor of zero and not a left divisor of zero, then A sums a bounded divergent sequence. *Edwin Hewitt* (Seattle, Wash.)

Slepenčuk, K. M. 2575

Theorems of Tauberian type for $(C_\theta^{(\alpha)},\lambda)$-methods of summing series. (Russian)

Izv. Vysš. Učebn. Zaved Matematika **1964**, *no.* 3 (40), 131–135.

The summability method that is considered in the paper is of generalised Hölder type, defined as follows. Let $\lambda_0=0$, $\lambda_n\uparrow+\infty$, $\theta\geqq 0$; write $C_n^{(0)}=\sum_0^n a_k$, and, for $\alpha=1,2,3,\cdots$,

$$C_n^{(\alpha)}=(\lambda_n+\theta)^{-1}\sum_0^{n-1}(\lambda_{k+1}-\lambda_k)C_k^{(\alpha-1)}+\theta(\lambda_n+\theta)^{-1}C_n^{(\alpha-1)}.$$

Then $\sum a_k$ is $(C_\theta^{(\alpha)},\lambda)$-summable to S if $C_n^{(\alpha)}\to S$ as $n\to\infty$, and $|(C_\theta^{(\alpha)},\lambda)|$-summable if also $\sum|C_{n+1}^{(\alpha)}-C_n^{(\alpha)}|<\infty$. When $\alpha=1$, $\theta>0$, $\lambda_n=n$, one has the C_θ-method treated by B. A. Kaufman [Učen. Zap. Čkalovsk. Gos. Ped. Inst. **11** (1957), 243–248], and when $\theta=0$ the $(C_\theta^{(\alpha)},\lambda)$-method coincides with the $(H^{(\alpha)},\lambda)$-method previously considered by the author [Dopovidi Akad. Nauk Ukraïn. RSR **1962**, 729–732; MR **25** #2350].

Four elementary Tauberian theorems are proved. Theorem 1 states that if $\sum a_k=S(C_\theta^{(\alpha)},\lambda)$, then the condition $(\lambda_n+\theta)^{-1}\sum_1^n\lambda_k a_k\to 0$ as $n\to\infty$ is necessary and sufficient for the convergence of $\sum a_k$ to S. Theorem 2 is a trivial modification of Theorem 1. Theorem 3 gives a necessary and sufficient condition for absolute summability to imply absolute convergence. Theorem 4 gives a condition for summability $(H^{(k+m)},\lambda)$, with $m\geqq 0$, to imply summability $(H^{(k)},\lambda)$. This last theorem generalises a result (Theorem 1) of T. A. Newton [Pacific J. Math. **3** (1953), 807–822; MR **15**, 416].

J. H. Williamson (Cambridge, England)

APPROXIMATIONS AND EXPANSIONS

See also 2400, 2589, 2602, 2658, 2940, 2967.

Jager, H. 2576

A multidimensional generalization of the Padé table. IV, V, VI.

Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 227–239, 240–244, 245–249.

Each of these three papers develops in some detail a special case of the more general function system considered in the first three papers of the series [same Proc. **67** (1964), 193–198, 199–211, 212–225; MR **29** #402]. These special cases are (i) the binomial function system, (ii) the exponential function system, and (iii) the logarithmic function system. *C. W. Clenshaw* (Teddington)

Topoljanskiĭ, D. B. 2577

An application of interpolation polynomials for approximating the solution of boundary-value problems. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 299–302.

Author's summary: "An approximate solution of boundary value problems is sought by various direct methods in the class of interpolational polynomials. In the case of one-dimensional boundary value problems the solution is sought in the class of Hermite's interpolational polynomials, which makes it possible to obtain directly approximate values of the function and its derivatives at points given beforehand."

Aramă, Lia 2578

An estimate of the remainder in the approximation of functions by Bernstein polynomials. (Romanian. Russian and French summaries)

An. Univ. București Ser. Ști. Natur. Mat.-Fiz. **11** (1962), *no.* 34, 165–178 (1964).

In 1962 the reviewer presented a paper [Math. Comp. **17** (1963), 270–278] concerning the evaluation of the remainder terms in some approximation formulas by Bernstein polynomials in one and two variables. Among other results there has been given an integral expression for the remainder of the approximation formula of a function $f(x)$, defined on the interval $[0,1]$ and with its second derivative R-integrable on this interval, by the corresponding Bernstein polynomial of degree n. The author of the present paper finds the same result using another method; as is indicated in the bibliographical reference of the heading, the date of publication of the author's paper is 1964. *D. D. Stancu* (Cluj)

Sendov, Blagovest 2579

On best approximation with algebraic polynomials relative to Hausdorff distance. (Bulgarian. German summary)

Annuaire Univ. Sofia Fac. Sci. Phys. Math. Livre 1 *Math.* **56** (1961/62), 195–207 (1963).

Let F be a closed and bounded subset of the plane whose projection on the real axis is an interval Δ. For every polynomial P let $r(F,P)$ be the Hausdorff distance between F and that portion of the graph of P which lies

above Δ. Let $E_n^*(F) = \inf r(F, P)$ taken over all polynomials P of degree at most n. If every vertical line through Δ intersects F in a point or an interval, then $E_n^*(F) = O(n^{-1}\log n)$ as $n\to\infty$. If $E_n^*(F) = o(1)$, then every vertical line through Δ intersects F in a point or an interval. If $E_n^*(F) = o(n^{-1})$, then every vertical line through the interior of Δ intersects F in a point. If $E_n^*(F) = o(n^{-2})$, then every vertical line through Δ intersects F in a point. *L. de Branges* (Lafayette, Ind.)

Talbot, A. 2580

On a class of Tchebysheffian approximation problems solvable algebraically.

Proc. Cambridge Philos. Soc. **58** (1962), 244–267.

Soit une fonction $g(x)$ continue dans $a\le x\le b$, et une fonction de forme donnée $f(x, p_j)$, à valeurs réelles, continue en x et en p_j. Le problème de Tchebyscheff consiste à déterminer les p_j pour que $\max_{a\le x\le b}|f(x,p_j)-g(x)|$ soit minimum.

L'objet de ce travail est d'employer une nouvelle méthode pour déterminer ce minimum dans le cas où f et g sont des polynômes ou des fractions rationnelles. Ce cas avait été traité par Tchebyscheff lui même, mais cette nouvelle méthode permet de résoudre les problèmes d'approximation chaque fois que cette résolution est possible algébriquement. Les calculs sont assez compliqués, mais ne présentent pas de difficultés théoriques. L'auteur donne des applications très intéressantes de ce procédé: meilleure approximation de zéro par un polynôme, par une fraction rationnelle de dénominateur donné; approximation, par un polynôme, de la fonction $1/(x-\lambda)$, d'un polynôme par une fraction rationnelle (à remarquer ici une très élégante solution par matrices). Ici la méthode donne une approximation de meilleure qualité que celle de Tchebyscheff. *R. Campbell* (Caen)

Hadžimullaev, F. S. 2581

On a theorem of D. Stancu. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 1061–1062.

In a detailed paper published in 1960 [Acad. R. P. Romîne Fil. Iaşi Stud. Cerc. Şti. Mat. **11** (1960), 221–233; MR **24** #A374a] the reviewer has stated and proved the following theorem: If the function $f(x, y)$ is bounded on the triangle $\Delta: x+y\le 1,\ x\ge 0,\ y\ge 0$ and if at an interior point (x, y) of Δ the second total differential $d^2f(x, y)$ exists, then denoting by $B_n(f; x, y)$ the Bernstein polynomial of degree n, we have the following asymptotic formula

$$(*)\quad B_n(f;x,y) = f(x,y)+\frac{x(1-x)}{2n}f''_{x^2}(x,y) -\frac{xy}{n}f''_{xy}(x,y)+\frac{y(1-y)}{2n}f''_{y^2}(x,y)+\frac{\varepsilon_n}{n},$$

where $\varepsilon_n\to 0$ as $n\to\infty$.

In a previous note [the reviewer, Dokl. Akad. Nauk SSSR **134** (1960), 48–51; MR **23** #A3401] the validity of this formula required only the boundedness of the second partial derivatives (although in the context it was mentioned that these partial derivatives are continuous) of the function $f(x, y)$ on the triangle Δ.

The author of the present paper gives an example of a function $f(x, y)$ for which (*) does not hold at the point $(\frac12,\frac12)$. At this point the function considered does not possess the differential $d^2f(x, y)$, the partial derivatives f_x' and f_y' not being differentiable. He does not refer to the first paper mentioned above.

{There are several misprints: p. 1061, formula (2); p. 1062, formula (3); p. 1062, line 7 from above.} *D. D. Stancu* (Cluj)

Motornyĭ, V. P. 2582

On the best approximation of functions of two variables of the form $\varphi(x)+\psi(y)$. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **27** (1963), 1211–1214.

Given a set Φ in the Euclidean plane and a bounded function f defined on Φ, with modulus of continuity ω, the author gives sufficient conditions on the set Φ for existence of two functions φ, ψ with the same modulus of continuity for which

$$\inf_{\varphi,\psi}\sup_{(x,y)\in\Phi}|f(x,y)-[\varphi(x)+\psi(y)]|$$

is attained. *A. Alexiewicz* (Poznań)

Birkhoff, Garrett; de Boor, Carl 2583

Error bounds for spline interpolation.

J. Math. Mech. **13** (1964), 827–835.

Let $f(x)\in C^1[0,1]$ and let $\pi: 0=x_0<x_1<\cdots<x_n=1$ be a partition. $f(x)$ is now interpolated by a cubic spline function $s(x)$, which means that $s(x)$ is a cubic polynomial in each interval (x_i, x_{i+1}), $s(x)\in C^2[0,1]$, while it is known that $s(x)$ is uniquely defined by the conditions

$$f(x_i) = s(x_i)\qquad (i=0,\cdots,n),$$
$$f'(0)=s'(0),\qquad f'(1)=s'(1).$$

The object of the study is the error function $e(x) = f(x)-s(x)$ and its derivatives as we refine the partition π. A detailed investigation of the fundamental functions of the interpolation process yields the following results. (1) Let $f(x)\in C^4[0,1]$. There are constants $K_r(M)$ $(r=0,1,2,3)$, depending only on $M>0$, such that

$$\|e^{(r)}\|\le\|f^{(4)}\|K_r(M)|\pi|^{4-r}\qquad (r=0,1,2,3),$$

provided that $M_\pi\le M$. Here we have used the following notations: $\Delta x_i = x_{i+1}-x_i$, $|\pi| = \max\Delta x_i$, $M_\pi = |\pi|/(\min\Delta x_i)$, and $\|g\| = \max g(x)$.

The authors go a step further and prove (2): Let $f(x)\in C^3[0,1]$ while $f'''(x)$ is absolutely continuous in [0, 1]. If we write $\pi=\pi_n$ and denote by $\{\pi_n\}$ a sequence of partitions such that $|\pi_n|\to 0$ as $n\to\infty$, while $M_{\pi_n}\le M$, then $\|e_n'''\|\to 0$.

A recent paper by J. H. Ahlberg and E. N. Nilson [J. Soc. Indust. Appl. Math. **11** (1963), 95–104; MR **27** #2763], devoted to the same general subject, obtained convergence results for the second derivative by means of matrix inversion rather than the direct study of the fundamental functions. *I. J. Schoenberg* (Madison, Wis.)

Császár, Á. [Császár, Ákos]; Czipszer, J. 2584

Sur des critères généraux d'approximation uniforme.

Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **6** (1963), 17–26.

The authors consider generalizations of the Stone-Weierstrass theorem following G. Nöbeling and H. Bauer [Jber. Deutsch. Math. Verein. **58** (1955), Abt. 1, 54–72;

MR 17, 605]. The main result is: "(A) Soient E un ensemble quelconque et S une classe de fonctions réelles définies sur E, non vide et satisfaisant: (1) Les fonctions $f \in S$ sont bornées; (2) Les fonctions constantes appartiennent à S; (3) $f \in S$ implique $f+c \in S$ pour tout nombre réel c; (4) Si $f, g \in S$, on a max $(f, g) \in S$ et min $(f, g) \in S$. Soit F une fonction réelle, bornée, et définie sur E. Alors pour que la fonction F puisse être approchée uniformément aussi près que l'on veut par des fonctions appartenant à S, il faut et il suffit qu'à chaque $\varepsilon > 0$ il corresponde un $\delta > 0$ et des fonctions $f_1, \cdots, f_n \in S$ de façon que (5) $x, y \in E$, $F(y) - F(x) \geqq \varepsilon$ entraînent max $(f_i(y) - f_i(x)) \geqq \delta$".

Variations of this result are considered, certain ways of weakening the hypothesis are shown to be impossible, and the relationship of classes such as S with function algebras is discussed. *J. R. Rice* (W. Lafayette, Ind.)

Császár, Ákos 2585

Sur un critère d'approximation uniforme. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 413–416 (1964).

The author continues the study in the paper reviewed above [#2584]. It is shown that the hypothesis that F be bounded on E is superfluous, but all other hypotheses are essential for the validity of the result. Note that in view of (3), one can superficially weaken (2) to read "La fonction $f(x) \equiv 0$ appartient à S".

J. R. Rice (W. Lafayette, Ind.)

Wagner, Eberhard 2586

Asymptotische Untersuchungen von Parameterintegralen mit positivem Integranden. II.

Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe **11** (1962), 553–565.

Part I appeared in Math. Nachr. **24** (1962), 371–383 [MR **27** #1755]. A "Hauptsatz" on the expansion of integrals of the form $\int_{x(s)-w(s)}^{x(s)+w(s)} \exp[-g(s,t)]\,dt$ was obtained using the Taylor expansion in a neighborhood of a maximum $t = x(s)$ of the integrand [ibid. **17** (1958), 101–135; MR **20** #5390]. In this paper the author gives a sufficient condition on $g(s, t)$ for the applicability of the Hauptsatz. He also treats the case where $g(s, t)$ has two maxima in the interval of integration. *R. R. Goldberg* (Evanston, Ill.)

Bugrov, Ja. S. 2587

Approximation of a class of functions with dominant mixed derivative. (Russian)

Mat. Sb. (N.S.) **64 (106)** (1964), 410–418.

S. M. Nikol'skiĭ [Dokl. Akad. Nauk SSSR **150** (1963), 484–487; MR **26** #6826] has introduced the Banach spaces of functions of n real variables $S_p^{(r)}H(R_n)$ $(1 \leqq p \leqq \infty)$ and has proved that $f \in S_p^{(r)}H(R_n)$ if and only if $f(x) = \sum_{e \subseteq e_n} \sum_{k^e \geqq 0} Q_{k^e}(x)$, where $e = (1, 2, \cdots, n)$, $k = (k_1, \cdots, k_n)$, $k^e = (k_1^e, \cdots, k_n^e)$ with $k_j^e = k_j$ for $j \in e$ and $k_j^e = 0$ for $j \in e_n \setminus e$, $Q_{k^e}(x)$ $(x = (x_1, \cdots, x_n) \in R_n)$ is an integral function of degree $2^{k_j^e}$ in the variable x_j $(j = 1, 2, \cdots)$ and $\|Q_{k^e}\|_p \leqq c2^{(-k,r^e)}$, where $(k, r) = \sum_{i=1}^n k_i r_i$, $c = \text{const}$. The author proves that for $f \in S_{(p)}{}^r H(R_n)$ $(1 \leqq p \leqq \infty)$ we have $\|f - S_N(f; \rho)\| \leqq cN^{-1}[\ln N]^{\nu-1}$, where

$$S_N(f;\rho) = \sum_{e \subseteq e_n} \sum_{2^{(k^e,\rho)} \leqq N} Q_{k^e}(x)$$

and where c is a constant which depends on n and r. The author also proves some approximation theorems for certain other classes of functions.

Ivan Singer (Bucharest)

Sjuĭ, L. S. [Hsu, L. C.]; Van, Ž. H. [Wang, J. H.] 2588

General "increasing multiplier" methods and approximation of unbounded continuous functions by certain concrete polynomial operators. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 264–267.

From the authors' introduction: "In the present paper we develop the method of multiplier enlargement [cf. L. C. Hsu, Studia Math. **21** (1961/62), 37–43; MR **25** #2357] and apply it to problems concerning polynomial approximations of continuous functions. For this purpose we introduce a class of the most effective polynomial operators and give some generalizations of the Fejér-Hermite interpolating polynomials. Some apparently new asymptotic formulae for the rapidity of the approximation are also established."

The authors consider approximations of unbounded continuous functionals $f(x)$ on Banach spaces by sequences of polynomial operators $L_n[f(t); x]$ in the form

$$\lim_{n\to\infty} L_n[f(a_n t); a_n^{-1}x] = f(x),$$

where a_n is a suitably chosen sequence of positive numbers (multipliers) tending to infinity. The general theorem is specified to obtain a number of results on the approximation of continuous functions of exponential growth on Euclidean spaces E_k. (Restating the results would amount to repeating the paper.) In the one-dimensional case $(E_1 = (-\infty, +\infty))$ the results apply to functions $f(x) = O(\exp^m |x|)$, where

$$\exp^1 |x| = e^{|x|}, \qquad \exp^{m+1}|x| = \exp^1(\exp^m |x|).$$

M. Kuczma (Katowice)

FOURIER ANALYSIS

See also 2229, 2343, 2446, 2565, 2661, 2975.

Schoenberg, I. J. 2589

On trigonometric spline interpolation.

J. Math. Mech. **13** (1964), 795–825.

This paper defines trigonometric splines and gives a complete development of the theory of trigonometric splines in parallel with the classical and recent theory of polynomial splines. Theorems and results are too numerous to mention individually, but the chapter headings are: (1) The spline functions of the operator $\Delta_m = D(D^2+1^2)\cdots(D^2+m^2)$; (2) The periodic analogue of Peano's theorem; (3) Trigonometric spline functions and interpolation; (4) On best approximation of linear operators; and (5) The construction of explicit formulas.

The classical (polynomial) splines are associated with the differential operator D^3, and recent papers have considered, more generally, D^m. The author comments that the present development could be carried out for a general linear differential operator with continuous coefficients. Presumably then, we can look forward to seeing the

classical splines with their intriguing variety of properties take their place as a special case of an abstract and elegant theory concerning differential operators in certain Hilbert spaces. The present paper is certainly a step in that direction. *J. R. Rice* (W. Lafayette, Ind.)

Kokilašvili, V. M. 2590

An estimate for best approximations and moduli of smoothness in various Lebesgue spaces of periodic functions with transformed Fourier series. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **35** (1964), 3–8.

For $f \in L_p[0, 2\pi]$ let $\omega_k(f, h)$ denote the modulus of smoothness of order k for f and let $E_n(f)$ denote the best approximation of f by trigonometric polynomials of order $\leq n$, where both ω_k and E_n are taken in the L_p sense. Let $f \sim \sum (a_m \cos mx + b_m \sin mx)$ and let

$$F \sim \sum \lambda_m(a_m \cos mx + b_m \sin mx)$$

for a suitable sequence $\{\lambda_m\}$. The author establishes inequalities relating $\omega_k(F, h)$ and $E_n(F)$ to $\omega_k(f, h)$ and $E_n(f)$. *R. R. Goldberg* (Evanston, Ill.)

Anselone, P. M. 2591

On the asymptotic behavior of the sum of a "non-harmonic Fourier series".

Enseignement Math. (2) **10** (1964), 236–244.

One of the typical theorems is as follows. Let

$$(1) \qquad \varphi(t) = \sum_{n=-\infty}^{\infty} c_n e^{z_n t},$$

$$t \geqq 0, \quad c_n = a_n + ib_n, \quad z_n = x_n + iy_n,$$

where $x_n < 0$, and $M = \sup_n |z_n - 2\pi ni| < \log 2$. Then the series (1) converges in $L_2(0, s)$ for each $s > 0$ if and only if $\sum |c_n|^2 < \infty$, in which case $\int_s^{s+\tau} |\varphi(t)|^2\, dt \to 0$ as $s \to \infty$, for each fixed $\tau > 0$, and $s^{-1} \int_0^s |\varphi(t)|^2\, dt \to 0$ as $s \to \infty$. It has already been known that $\varphi(t)$ is not necessarily bounded. Application to the asymptotic behavior of the solution of a difference-integral equation is also observed. *G. Sunouchi* (Sendai)

Katznelson, Y. 2592

Sets of uniqueness for some classes of trigonometrical series.

Bull. Amer. Math. Soc. **70** (1964), 722–723.

Let T be the circle group. The Fourier coefficients of a distribution μ on T are denoted by $\hat{\mu}(n)$. If B is a space of sequences, $\mathscr{F}B$ is the space of distributions μ on T for which $\hat{\mu}(n) \in B$; if B is normed, let $\|\mu\|_{\mathscr{F}B} = \|\{\hat{\mu}(n)\}\|_B$. A closed set E is a set of uniqueness for B if E carries no element of B. In this note the author constructs sets of positive Lebesgue measure which are sets of uniqueness for l_p, $1 \leqq p < 2$. He does this by constructing a closed set $E_{\varepsilon,p} \subseteq T$, $\varepsilon > 0$, $1 \leqq p < 2$, with the following properties. (1) meas $(E_{\varepsilon,p}) > 2\pi - \varepsilon$. (2) If μ is carried by $E_{\varepsilon,p}$ then $\|\mu\|_{\mathscr{F}l^\infty} \leqq \varepsilon \|\mu\|_{\mathscr{F}l^p}$. Once these sets are constructed, he takes a sequence ε_n, $p_n = 2 - \varepsilon_n$ and lets $E = \bigcap_1^\infty E_{\varepsilon_n, p_n}$, E being the desired set of uniqueness. *J. H. Neuwirth* (Greenbelt, Md.)

Talaljan, A. A. 2593

Complete systems of unconditional convergence in the weak sense. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 713–720.

Theorem: There exists a complete orthonormal system $\{\varphi_n\}_{n=1}^\infty$ for $L^2[0, 1]$ which has the following property: for any function $f \in L^2[0, 1]$ and $\varepsilon > 0$, there exists $\psi \in L^2[0, 1]$ such that ψ differs from f on a set of measure less than ε, and the Fourier series of ψ relative to the system $\{\varphi_n\}$ converges absolutely almost everywhere.

It is not known whether the usual trigonometric functions have this property. *T. W. Gamelin* (Cambridge, Mass.)

Tranter, C. J. 2594

An improved method for dual trigonometrical series.

Proc. Glasgow Math. Assoc. **6**, 136–140 (1964).

The author obtains the solution of

$$\sum_{n=0}^{\infty} (2n+1)A_n \cos (n+\tfrac{1}{2})x = f(x), \qquad 0 < x < c;$$

$$\sum_{n=0}^{\infty} A_n \cos (n+\tfrac{1}{2})x = 0, \qquad c < x < \pi$$

(f is a given function) in the form

$$A_n = \frac{1}{2} \int_0^c F(\theta) P_n(\cos \theta) \sin \theta \, d\theta,$$

where

$$F(\theta) = 2^{1/2}\pi^{-1} \int_0^\theta (\cos x - \cos \theta)^{-1/2} f(x)\, dx.$$

He also solves three other similar dual series relations. *A. Erdélyi* (Edinburgh)

Golinskiĭ, B. L. 2595

Local limit relations and asymptotic formulae for polynomials which are orthogonal on the unit circle. (Russian)

Mat. Sb. (*N.S.*) **64** (**106**) (1964), 321–356.

Let $q \in L[-\pi, \pi]$ and let $\{P_n\}_{n=1}^\infty$ be the sequence of polynomials orthonormal on the unit circle with respect to q. Let

$$\pi(z) = \exp\left\{-(4\pi)^{-1} \int_{-\pi}^{\pi} \ln q(t)[(e^{it}+z)/(e^{it}-z)]\, dt\right\}$$

and let $\rho_n(z) = |\pi(z) - P_n^*(z)|$, where $P_n^*(z) = z^n \overline{P_n(z^{-1})}$. There are two types of known results on the order of $\rho_n(e^{it})$ as $n \to \infty$: (a) global conditions on q sufficient to ensure that $\rho_n(e^{it}) = o(1)$ as $n \to \infty$ uniformly on $-\pi \leqq t \leqq \pi$, and (b) weaker global conditions plus local conditions on q in a neighborhood of a single point t_0 sufficient to ensure that $\rho_n(e^{it_0}) = o(1)$ as $n \to \infty$ [e.g., G. Freud, Acad. Serbe Sci. Publ. Inst. Math. **11** (1957), 19–32; MR **20** #1147]. In this paper the author generalizes and strengthens the known results of type (b). *R. R. Goldberg* (Evanston, Ill.)

Krall, H. L.; Sheffer, I. M. 2596

A characterization of orthogonal polynomials.

J. Math. Anal. Appl. **8** (1964), 232–244.

The authors characterize, in terms of their generating

functions, those polynomial sets $\{P_n(x): n=0, 1, 2, \cdots;$ $P_n(x)$ a polynomial of degree exactly $n\}$ that are orthogonal sets in the sense that there exists a function $\alpha(x)$ of bounded variation on a finite or infinite interval (a, b) such that $\int_a^b P_n(x)x^k\,d\alpha(x)$ is zero for $k<n$ and different from zero for $k=n$. This characterization theorem is then used to determine those polynomial sets which are simultaneously orthogonal sets and one of the following: (i) polynomial sets satisfying a linear differential equation of a certain form; (ii) polynomial sets of type zero. These specific problems were solved at a much earlier date by the individual authors using different methods [see H. L. Krall, Duke Math. J. **4** (1938), 705–718; I. M. Sheffer, ibid. **5** (1939), 590–622; MR **1**, 15].

W. T. Reid (Norman, Okla.)

Petrovskaja, M. B. 2597

Null series with respect to a Haar system and sets of uniqueness. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 773–798.

Let $\{\varphi_n\}$ $(n=0, 1, \cdots)$ be a certain system of functions defined on $[0, 1]$. A series $\sum_{n=0}^{\infty} a_n\varphi_n(t)$ is called a null series with respect to the system $\{\varphi_n\}$ if it converges almost everywhere on $[0, 1]$ to zero and if not all coefficients a_n are zero. Let $\{\chi_m\}$ be the Haar system. The main theorems proved are the following. (i) For every positive monotone sequence $\{a_n\}$ fulfilling the conditions $\sum_{n=0}^{\infty} a_n{}^2=\infty$ and $\sqrt{(2^n)}a_{2^n}\downarrow$, there exists a sequence $\{b_n\}$ such that $|b_n|\leqq a_n$ and such that $\sum_{n=1}^{\infty} b_n\chi_n(t)$ is a null series with respect to the system $\{\chi_m\}$. (ii) For every positive function ω, defined on the set of integers and with $\omega(n)\uparrow\infty$ $(n\to\infty)$, there can be constructed a null series $\sum b_i\chi_i(t)$ with respect to the system $\{\chi_m\}$ such that $\sum_{i=1}^{n} b_i{}^2\leqq\omega(n)$ $(n=1, 2, \cdots)$. (iii) Let $\lim_{m\to\infty} a_m\chi_m(t)=0$ for all $t\in[0, 1]$, and let $\sum_{m=1}^{\infty} a_m\chi_m(t)$ be everywhere convergent on $[0, 1]$, except possibly on a countable set, to a bounded measurable function f. Then this series is the Fourier series of f. The author remarks in a footnote that she found, in the meantime, that this theorem holds even if only $f\in L[0, 1]$.

G. Goes (Chicago, Ill.)

Ishiguro, Kazuo 2598

On the quasi-Hausdorff means whose weight function has jumps.

Proc. Japan Acad. **40** (1964), 59–64.

It is shown that certain quasi-Hausdorff means of the series $\sum_{n=1}^{\infty} (\sin nt/n)$ exhibit the Gibbs phenomenon. These include regular quasi-Hausdorff means whose weight functions are step functions continuous at the origin, and a more general class in which the weight function is such that its continuous part satisfies an integrability condition.

J. D. Buckholtz (Lexington, Ky.)

Sunouchi, Gen-ichirô 2599

On the class of saturation in the theory of approximation. II.

Tôhoku Math. J. (2) **13** (1961), 112–118.

Pour la première partie voir même J. (2) **12** (1960), 339–344 [MR **23** #A1987]. L'auteur a déjà donné antérieurement la classe de saturation pour plusieurs procédés de sommation en théorie des séries de Fourier et de l'intégrale de Fourier. Il donne ici la classe de saturation pour une approximation locale (on prend comme modèle de procédé de sommation celui de Cesàro). Il établit que si $\sigma_n(x)-f(x)=O(1/n)$ uniformément dans $[a, b]$, alors $f(x)=$ const. Il en déduit d'autres résultats pour le cas où $f(x)$ est intégrable et aussi relativement à l'intégrale singulière de Landau qui s'écrit

$$\int_n (x, f) = \frac{1}{k_n}\int_0^1 f(t)[1-(x-t)^2]^n\,dt.$$

R. Campbell (Caen)

Cheng, Nai-dong [Cheng, Nai-tung] 2600

On the constructive property of a class of uniformly almost-periodic functions.

Acta Math. Sinica **13** (1963), 441–446 (*Chinese*); *translated as Chinese Math.* **4** (1964), 478–484.

The class S consists of the uniformly almost periodic (u.a.p.) functions f with Fourier series $\sum_{-\infty}^{\infty} A_k e^{i\Lambda_k x}$ such that $\Lambda_0=0$, $\Lambda_{-k}=-\Lambda_k$, $(\Lambda_k)_{k\geq 1}$ decreases to 0, $A_0=0$, $|A_k|+|A_{-k}|>0$ $(k\neq 0)$. For any $\varepsilon>0$,

$$E_\varepsilon(f) = \inf_{F\in Q_\varepsilon}\{\sup_x |f(x)-F(x)|\},$$

where Q_ε is the class of u.a.p. functions with Fourier indices λ_k such that $|\lambda_k|>\varepsilon$; and, for $N>0$,

$$\Omega_f{}^*(N) = \sup_{|T_2|\geq|T_1|\geq N}\left\{\sup_x\left|\frac{1}{T_1T_2}\int_0^{T_2}\int_0^{T_1} f(x+s+t)\,dtds\right|\right\}.$$

The principal theorem of the paper is the following: When $f\in S$, a necessary and sufficient condition for the existence of a constant B such that $E_\varepsilon(f)\leqq B\varepsilon$ for all $\varepsilon>0$ is that $\Omega_f{}^*(N)=O(1/N)$. Less complete results of a similar type had previously been given by E. A. Bredihina [Mat. Sb. (N.S.) **56** (**98**) (1962), 59–76; MR **25** #2383].

H. Burkill (Sheffield)

Kovan'ko, O. S. [Kovan'ko, A. S.]; Lisevič, L. M. [Lisevič, L. N.] 2601

Some properties of the indefinite integral and derivative of an S^p-almost-periodic function. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 705–706.

Authors' summary: "The authors consider the conditions under which the indefinite integral of an S^p-almost-periodic function is also an S^p-almost-periodic function. In particular, the necessary and sufficient conditions are given. A definition is also given of the 'D_{S^p} derivative in the mean' of an S^p-almost-periodic function, and some of its properties are established."

Grišin, V. B. 2602

On the approximation of periodic functions of two variables by de La Vallée Poussin sums. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1433–1437.

The author considers the class $K_1K_2W^{(r,l)}H^{(\alpha,\beta)}$ of periodic functions $f(x, y)$ such that the rth derivative with respect to x and the lth derivative with respect to y (in the Weyl sense) satisfy

$$|\varphi(x_1, y_1)-\varphi(x_2, y_2)| \leqq K_1|x_2-x_1|^\alpha+K_2|y_2-y_1|^\beta.$$

He obtains upper bounds and also asymptotic expressions

for the best approximation to such functions by de La Vallée Poussin sums

$$\frac{1}{(p+1)(q+1)}\sum_{k=m-p}^{m}\sum_{v=n-q}^{n} S_{k,v},$$

where $S_{k,v}$ are the rectangular partial sums of the Fourier series of f, when $p=o(m)$, $q=o(n)$, under various assumptions on the relative growth of m and n.

R. P. Boas, Jr. (Evanston, Ill.)

INTEGRAL TRANSFORMS, OPERATIONAL CALCULUS

See also 2420, 2552, 2573, 2832.

Saxena, R. K. 2603

A theorem on generalised Laplace transform.

Bull. Calcutta Math. Soc. **53** (1961), 155–160.

Wenn die Laplace-Transformation durch

$$\psi(p) = p\int_0^\infty e^{-pt}f(t)\,dt \equiv L[f]$$

und die von R. S. Varma angegebene Verallgemeinerung durch

$$\varphi(p) = p\int_0^\infty e^{-(1/2)pt}(pt)^{m-1/2}W_{k,m}(pt)f(t)\,dt \equiv V[f]$$

gegeben ist, so gilt der Satz: Aus $L[f]=\psi(1/p)$ und $V[t^{k-m-1/2}e^{-b/t}\psi(t)]=\varphi(p)$ folgt

$$\varphi(p) = 2p^{m+1}\int_0^\infty (t+b)^{k-1/2}K_{2m}(2\sqrt{(pt+pb)})f(t)\,dt.$$

Mit Hilfe dieses Satzes werden zwei spezielle Integrale über K-Funktionen ausgerechnet. *G. Doetsch* (Freiburg)

Singh, S. P. 2604

Some properties of a generalised Laplace transform.

J. Sci. Res. Banaras Hindu Univ. **13** (1962/63), 344–360.

Für die von V. P. Mainra angegebene verallgemeinerte Laplace-Transformation

$$\Phi(s) = s\int_0^\infty (st)^{-\lambda-1/2}e^{-st/2}W_{k+1/2,m}(st)f(t)\,dt$$
$$\equiv W[f(t);\,k+\tfrac12,\,\lambda,\,m]$$

werden zunächst zu 8 Originalfunktionen $f(t)$ wie $t^\nu e^{-at}$, $t^\mu J_{\nu+\mu}(a\sqrt{t})J_{\nu-\mu}(a\sqrt{t})$, usw. die Bildfunktionen ausgerechnet. Dann werden drei Sätze von folgendem Typus bewiesen: Wenn eine Funktion mit zwei anderen in gewisser Weise durch die W-Transformation verknüpft ist, so besteht ein Zusammenhang zwischen diesen beiden Funktionen, z.B.: Aus $W[f(t);\,k+\frac12,\,\lambda,\,m]=\Phi(s)$ und $W[(1/\sqrt{t})\Phi(1/t);\,k+\frac12,\,\lambda,\,m]=g(s)$ folgt

$$W[f(t^2);\,2k+\tfrac12,\,2\lambda,\,2m] = \frac{2^{2k-2\lambda+1}}{(\sqrt{\pi})s}\,g\left(\frac{s^2}{4}\right).$$

G. Doetsch (Freiburg)

v. Wolfersdorf, L. 2605

Berechnung einiger Laplace- und Fourier-Integrale.

Z. Angew. Math. Mech. **43** (1963), 565–567.

The author considers the integral

$$(1)\qquad K_\alpha(p) = \int_0^\infty e^{-pt}t^\alpha\cos\sqrt{(a^2t^2-2abt)}\,dt$$
$$(\operatorname{Re} p>0,\ \operatorname{Re}\alpha>-1,\ a>0).$$

He shows that

$$(2)\qquad K_\alpha(p) = 2^{\alpha+1}\Gamma(\alpha+1)(p+\sqrt{(p^2+a^2)})^{-(\alpha+1)}$$
$$\times\Phi_1\left(\alpha+1;\,\alpha+\frac32;\,\frac12;\,\frac{p-\sqrt{(p^2+a^2)}}{p+\sqrt{(p^2+a^2)}},\,\frac{ab}{p+\sqrt{(p^2+a^2)}}\right),$$

where Φ_1 is a confluent form of Appell's double hypergeometric function F_1.

He also obtains a similar expression for

$$L_\alpha(p) = \int_0^\infty e^{-pt}t^\alpha\frac{\sin\sqrt{(a^2t^2-2abt)}}{\sqrt{(a^2t^2-2abt)}}\,dt.$$

The integrals

$$M_\alpha(y) = \int_0^\infty x(\sqrt{(x^2+\lambda^2)}-\lambda)^\alpha e^{-p\sqrt{(x^2+\lambda^2)}}\cos(xy)\,dx,$$

$$N_\alpha(y) = \int_0^\infty (\sqrt{(x^2+\lambda^2)}-\lambda)^\alpha e^{-p\sqrt{(x^2+\lambda^2)}}\sin(xy)\,dx$$
$$(y>0,\ \operatorname{Re} p>0,\ \operatorname{Re}\alpha>-1,\ \operatorname{Re}\lambda>0)$$

are also considered.

The author finally uses (1) and (2) to evaluate the generating functions for the Bessel polynomials

$$\sum_{n=0}^\infty\frac{(-1)^n2^n}{(2n)!}\Gamma(n+\alpha+1)Y_n^{(\alpha)}(x)t^n$$

and

$$\sum_{n=0}^\infty\frac{(-1)^n2^n}{(2n+1)!}\Gamma(n+\alpha+1)Y_n^{(\alpha)}(x)t^n$$

also in terms of the Φ_1 functions.

W. A. Al-Salam (Durham, N.C.)

Al-Bassam, M. A. [Bassam, M. A.] 2606

On certain types of H-R transform equations and their equivalent differential equations.

J. Reine Angew. Math. **216** (1964), 91–100.

The author defines a differential equation or an integro-differential equation or integral equation to be equivalent to a transform equation of successive operators if the former can be expanded in the form of the latter. He then shows how the differential equation $y'+[f'(x)+g'(x)]y=0$ can be expressed in the form of a transform equation of successive operators. He then establishes a necessary and sufficient condition that the second-order linear differential equation

$$(a+bx)(c+dx)Y''+(a_1+b_1x)Y'+d_1Y = 0$$

will be equivalent to the transform equation

$$\overset{x}{\underset{x_0}{I}}{}^{-w}(a+bx)^{1-p}(c+dx)^{1-q}\overset{x}{\underset{x_0}{I}}{}^{-1}(a+bx)^p(c+dx)^q\overset{x}{\underset{x_0}{I}}{}^{w-1}Y = 0.$$

He then applies the method to verify that the solution of the differential equation satisfies Legendre's equation. He then finds the conditions that the differential equation

$$(a+bx)(c+dx)y''+(A_1x^2+B_1x+C_1)y'$$
$$+(A_2x^2+B_2x+C_2)y = 0$$

can be represented by an equivalent H.R. transform. He applies the method to show the differential equation

$$(1-x^2)y''-2\lambda(x^2+(x/\lambda)-1)y' -[\lambda^2(x^2-1)+2\lambda x-\alpha(\alpha+1)]y = 0,$$

where $\lambda \neq 0$ and α are constants, is equivalent to the transform equation

$$\overset{x}{\underset{x_0}{I}}{}^{\alpha}(x^2-1)^{-\alpha}\overset{x}{\underset{x_0}{I}}{}^{-1}(x^2-1)^{\alpha+1}\overset{x}{\underset{x_0}{I}}{}^{-\alpha-1}e^{\lambda x}y = 0.$$

D. P. Banerjee (Tirupati)

Somayajulu, A. L. N. 2607

Humbert and hyperbesselien functions.

J. Sci. Res. Banaras Hindu Univ. **13** (1962/63), 361–381.

It was shown by B. M. Mehrotra [Proc. London Math. Soc. (2) **34** (1932), 231–240] that if

$$P(x) = (2\pi i)^{-1}\int_{k-i\infty}^{k+i\infty} 2^s\Gamma(\tfrac14+\tfrac12\mu+\tfrac12 s)\Gamma(\tfrac14+\tfrac12\nu+\tfrac12 s)\lambda(s)x^{-s}\,ds,$$

where $0<k<1$ and $\lambda(s)=\lambda(1-s)$, then the kernel $P(x)$ transforms R_μ into R_ν; that is, if $f(x)$ is its own Hankel transform of order μ ($f(x)\in R_\mu$), then the function $\int_0^\infty f(y)P(xy)\,dy$ belongs to R_ν.

The author shows that if $J_{\lambda_1,\lambda_2}(x)$ denotes the Humbert function

$$\frac{(\frac13 x)^{\lambda_1+\lambda_2}{}_0F_2(\lambda_1+1,\ \lambda_2+1;\ -x^3/27)}{\Gamma(\lambda_1+1)\Gamma(\lambda_2+1)},$$

then, for any $P(x)$ of the above form, the function $x^{-1}\int_0^\infty y^{(3-2\nu)/6}J_{(1/2)\mu+(1/2)\nu,\mu}(\frac32 y^{2/3})P(y/x)\,dy$ belongs to R_μ. He also shows that, if $f(x)\in R_\mu$ and if

$$Q(x) = x^\alpha\int_0^\infty y^\alpha J_{(1/2)\mu+(1/2)\nu,\nu}(-(xy)^{2/3})f(y)\,dy,$$

then (i) when $\alpha=\frac12-\frac43\mu+\nu$, the kernel $Q(x)$ transforms $R_{2\mu-\nu}$ into $R_{2\nu-\mu}$; (ii) when $\alpha=\frac12-\frac13\mu$, $Q(x)$ transforms R_ν into itself. He mentions various particular cases and applications of these results. *P. Heywood* (Edinburgh)

Kauczor, Richard 2608

Anwendung der *Z*-Transformationen auf Vierpolketten und Leitungen.

Wiss. Z. Hochsch. Verkehrswesen "Friedrich List" Dresden **11** (1964), 7–11.

Boehme, T. K. 2609

On approximate solutions to the convolution equation on the half-line.

Bull. Amer. Math. Soc. **69** (1963), 847–849.

The author considers the algebra $\mathscr{L}$ of locally integrable functions on $(0,\infty)$, with convolution as multiplication and with its topology defined by the seminorms

$$\|x\|_T = \int_0^T |x(t)|\,dt.$$

He gives a new proof of the following result of C. Foiaş [Studia Math. **21** (1961/62), 73–74; MR **25** #3334]. If k and r belong to $\mathscr{L}$, and k vanishes on no neighbourhood of the origin, then the equation $k*x=r$ has an approximate solution, in the sense that, given $T>0$ and $\varepsilon>0$, there exists $x\in L(0,T)$ such that $\|r-k*x\|_T<\varepsilon$. He obtains a construction for such a solution by considering the operator K defined in $L^2(0,T)$ by $Kx=k*x$, and noting that K is compact. He shows also that every closed proper ideal in $\mathscr{L}$ is of the form $\mathscr{L}_\alpha=\{x:x\in\mathscr{L},\ \|x\|_\alpha=0\}$ for some α. *F. Smithies* (Cambridge, England)

INTEGRAL EQUATIONS

See also 2403, 2493, 2494, 2495, 2609, 2679, 2680, 3005, 3212.

Stonic'kiĭ, A. A. 2610

Approximate solution, by the method of Ju. D. Sokolov, of an infinite system of integral equations of Volterra type depending on a parameter. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1555–1559.

An approximate solution is obtained for an infinite system of linear integral equations by Ju. D. Sokolov's method. A sufficient condition for the convergence of the process is given, and the error is estimated.

H. P. Thielman (Alexandria, Va.)

Vinokurov, V. R. 2611

Approximation on an infinite interval to a system of Volterra integral equations by a system of algebraic equations. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1963**, *no.* 5 (36), 24–29.

The author compares the integral equation

$$(1)\qquad y_1(x) = f(x)+\int_0^x K(x,s)y_1(s)\,ds$$

on the interval $[0,\infty)$ with an approximating system of linear equations

$$(2)\qquad y_2(x_m) = f(x_m)+\sum_{p=0}^{m-1} B_{mp}K(x_m,x_p)y_2(x_p),$$

constructed as in an earlier paper [same Izv. **1963**, no. 4 (35), 33–43; MR **27** #4030]. Here (x_m) is an increasing sequence tending to $+\infty$, and $|x_{m+1}-x_m|\leq h$, where h is independent of m.

He shows by examples that the stability, as defined in his earlier paper, of the solution of (1) does not imply that of (2), even for small h, and that even if both (1) and (2) have stable solutions, it does not follow that $y_1(x_m)-y_2(x_m)\to 0$ uniformly in m. He then shows that all is in fact well under appropriate and rather complicated conditions on $K(x,s)$ and $f(x)$.

All results extend to systems of equations, in terms of which the paper is actually written.

F. Smithies (Cambridge, England)

Kisilevs'kiĭ, G. E. 2612

Some criteria for unicellularity of dissipative Volterra operators. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 710–713.

Author's summary: "The following theorems are established. If the functions

$$\varphi_1(t), \varphi_2(t), \cdots, \varphi_N(t) \qquad \left(\sum_{i=1}^{N} \varphi_i(t)\overline{\varphi_i(t)} \equiv 1\right)$$

satisfy Lipschitz's condition on $[0, l]$, then

$$Af = i\int_x^l \sum_{i=1}^{N} \varphi_i(t)\overline{\varphi_i(x)}f(t)\,dt \qquad (f(x) \in L^2(0, l))$$

is a unicellular integral operator. If the functions $\varphi_1(t)$, $\varphi_2(t), \cdots, \varphi_N(t)$ satisfy Lipschitz's condition on each segment $[\varepsilon, l-\varepsilon]$ $(\varepsilon > 0)$, then A is a unicellular operator."

Ehrmann, Hans 2613

Zum Einschliessungssatz von Collatz für die Eigenwerte von Integralgleichungen.

Math. Z. **83** (1964), 67–71.

In 1941 L. Collatz [Math. Z. **47** (1941), 395–398; MR **4**, 83] proved the following enclosure theorem for eigenvalues of certain integral equations. "Suppose there is given a linear integral equation of the form

$$(0) \qquad \varphi(s) = \lambda \int_a^b K(s, t)\varphi(t)\,dt,$$

where $K(s, t)$ is a given symmetrical, square-integrable kernel which is continuous in the mean.

"Let $F_0(s)$ be a continuous, not identically vanishing function defined on the closed interval $[a, b]$, and let

$$F_1(s) = \int_a^b K(s, t)F_0(t)\,dt.$$

"If the function $G(s) = F_0(s)/F_1(s)$ lies between finite limits $G_{\min}$ and $G_{\max}$, and if G has a constant sign on the fundamental interval, then $G_{\min}$ and $G_{\max}$ enclose at least one eigenvalue of (0): $G_{\min} \leqq \lambda_k \leqq G_{\max}$."

The max-min-definition of the eigenvalues employed above fails, of course, if the function $G(s)$ changes sign in the fundamental interval. Even the refinement of Collatz's theory due to R. Iglisch [Math. Ann. **118** (1941), 263–275; MR **5**, 267] is inapplicable when $G(s)$ changes sign and the kernel is undefined at one or more points of the interval. The present author remedies these deficiences in the previous treatments by considering solutions of the equation

$$(1) \qquad u - \lambda A u = 0,$$

where u is in a real Hilbert space H of square-integrable, real functions $u(s)$ over $[a, b]$, and A is a linear, self-adjoint, fully continuous, non-null operator on H. He shows that the Collatz theorem is, in general, valid without the restriction upon $G(s)$ to uniformity of sign. His essential result, embodied in the first of two theorems of his own, is that an arbitrary, not identically vanishing function $F_0(s)$ which is contained in the domain of definition of the above-mentioned integral operators gives rise to a Collatz function $G(s)$ whose range encloses an eigenvalue of (1).

The proof of the foregoing suggests his second theorem, which asserts that the equation $u - GAu = 0$, $u \in H$, where $G = G(s)$ is real-valued and defined over $[a, b]$, has only the trivial solution $u = 0$ when G does not enclose an eigenvalue of (1). The theorem gives the conditions under which this circumstance obtains.

W. P. De Witt (Washington, D.C.)

Hopf, Eberhard 2614

An inequality for positive linear integral operators.

J. Math. Mech. **12** (1963), 683–692.

This paper is concerned with some extensions of Jentzsch's theorem on the existence of a positive eigenfunction for a positive integral operator K. The two main results are essentially as follows, various qualifying conditions being omitted.

If K satisfies the "cross-ratio" condition

$$\frac{K\varphi(x')}{K\varphi(x)}\Big/\frac{K\psi(x')}{K\psi(x)} \leqq \kappa^2$$

for all x and x' and all positive functions φ and ψ, then

$$\operatorname{osc}\frac{Kf}{Kp} \leqq \frac{\kappa-1}{\kappa+1}\operatorname{osc}\frac{f}{p},$$

where f is arbitrary and p is positive. The constant $(\kappa-1)/(\kappa+1)$ is best possible.

If $K\varphi(x')/K\varphi(x) \leqq \kappa$ for all x and x' and all positive φ, and λ_0 is the eigenvalue giving a positive eigenfunction, then any other eigenvalue λ satisfies the inequality

$$|\lambda| \leqq \frac{\kappa-1}{\kappa+1}\lambda_0;$$

again the constant is best possible.

F. Smithies (Cambridge, England)

Parasjuk, E. M. [Parasjuk, E. N.] 2615

On the index of a class of systems of singular integral equations. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 711–714.

Considered are singular integral equations which arise in the study of boundary-value problems for elliptic systems of partial differential equations in the plane. The boundary of the region in which the problem is treated is permitted to contain an infinite number of singular points. A formula is given for the index of the integral equations considered.

H. P. Thielman (Alexandria, Va.)

Rakovščik, L. S. 2616

On the theory of integral equations of convolution type. (Russian)

Uspehi Mat. Nauk **18** (1963), *no.* 4 (112), 171–178.

The author applies the theory of paired integral equations developed by Gohberg and Kreĭn [Ž. Teoret. i Prikl. Mat. No. 1 (1959), 58–81] to the integral equation

$$(1) \qquad a(t)\varphi(t) + \sum_{j=1}^{n} b_j(t)\int_{-\infty}^{+\infty} k_j(t-\tau)\varphi(\tau)\,d\tau = f(t)$$

under quite general conditions on $a(t)$ and $b_j(t)$. Two theorems are proved which state that, under certain hypotheses, the theorems of F. Noether apply to the operator given by the left side of equation (1), and that equation (1) is equivalent, under certain solvability conditions, to the Riesz-Schauder equation.

H. P. Thielman (Alexandria, Va.)

Shamir, Eliahu 2617

Reduced Hilbert transforms and singular integral equations.

J. Analyse Math. **12** (1964), 277–305.

On the positive real line R_+ the equation

$$(1) \qquad T_\varphi \equiv (CH^+ + DH^-)\varphi = \psi$$

is considered, where φ and ψ are m vector-valued functions and C and D constant $m \times m$ matrices, while H^+ and H^- are Hilbert transforms:

$$(H^\pm\varphi)(x) = \lim_{\varepsilon \downarrow 0} \int_{R_+} \frac{\varphi(y)}{x \pm i\varepsilon - y}\, dy.$$

Introduce the following norms:

$$\|\varphi\|_1 = \left(\int_{R_+} |x|^{-\sigma p}|\varphi(x)|^p\, dx\right)^{1/p},$$

$$\|\varphi\|_2 = \left(\iint_{R_+ \times R_+} |x-y|^{-\sigma p-1}|\varphi(x)-\varphi(y)|^p\, dxdy\right)^{1/p},$$

$$\|\varphi\|_3 = \left(\int_{R_+} |(D+i0)^\sigma \varphi(x)|^p\, dx\right)^{1/p},$$

where, in the last case, the operator $(D+i0)^\sigma$ is defined with the aid of the Fourier transform (on the whole real line R), p and σ being real numbers, $1<p<\infty$, $0<\sigma<1$, $\sigma \neq 1/p$. Assume that the eigenvalues of the matrix $-C^{-1}D$ lie outside the ray $\arg \lambda = 2\pi(1/p-\sigma)$ in the complex plane. Then the inequality

$$\|\varphi\|_\nu \leqq C\|T\varphi\|_\nu, \qquad \nu = 1, 2 \text{ or } 3,$$

holds, where C is a constant, for any $\varphi \in C_0^\infty(R_+)$, with, if $\nu=1$ and $\sigma > 1/p$, the additional restriction

$$\int_{R_+} x^{-1}\varphi(x)\, dx = 0.$$

If $\nu=1$, the proof depends on the classical results of M. Riesz and an explicit diagonalization formula for the operator $x^{-\sigma}Tx^\sigma$, generalizing a result due to J. Schwartz [Comm. Pure Appl. Math. **15** (1962), 75–90; MR **29** #480] corresponding to the special case $p=2$, $\sigma=0$, which follows easily with the aid of the Fourier transform (on R_+ considered as a multiplicative group). Since

$$\|\varphi\|_2 = \left(\int_{R_+} |1-y|^{-\sigma p-1}\|\varphi(x)-\varphi(xy)\|_1{}^p\, dy\right)^{1/p},$$

the result in the case $\nu=1$ at once carries over to the case $\nu=2$. The case $\nu=3$ is likewise a straightforward consequence of the case $\nu=1$. Furthermore, the index of T in the space corresponding to the "non-homogeneous" norm

$$\|\varphi\|_2 + \left(\int_{R_+} |\varphi(x)|^p\, dx\right)^{1/p}$$

is determined. Finally, there is a brief discussion of (1) on a finite interval.

The results of the paper, which were previously announced in two notes [C. R. Acad. Sci. Paris **254** (1962), 3621–3623; MR **25** #3268; ibid. **255** (1962), 448–449; MR **26** #582], have important applications to the mixed problem for elliptic partial differential equations in the plane [cf. the author, Ann. Scuola Norm. Sup. Pisa (3) **17** (1963), 117–139; MR **28** #329]. In the (rather simple) special case $p=2$, many of them are contained at least implicitly in the work of the reviewer [Comm. Pure Appl. Math. **14** (1961), 711–731; Ann. Scuola Norm. Sup. Pisa (3) **15** (1961), 337–353; MR **25** #4240]. *J. Peetre* (Lund)

Wiener, Klaus 2618a

Belastete Integralgleichungen mit Hadamard-Integralen.

Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe **12** (1963), 31–37.

Wiener, Klaus 2618b

Über die Lösung einer Abelschen Integralgleichung mit Hadamard-Integralen.

Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe **12** (1963), 39–42.

Wiener, Klaus 2618c

Nichtlineare Integralgleichungen mit Hadamard-Integralen.

Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe **12** (1963), 43–49.

These papers present theorems concerning integral equations of several common varieties but in which the integrals are interpreted as finite parts in the sense of Hadamard. *H. Widom* (Ithaca, N.Y.)

v. Wolfersdorf, Lothar 2619

Zur Lösung der einfachsten zweidimensionalen singulären Integralgleichung.

Univ. e Politec. Torino Rend. Sem. Mat. **22** (1962/63), 215–226 (1964).

The author considers the convolution equation

$$Au(x) = \mu u(x) + \int_{E_2} \frac{f(\theta)}{r^2}\, u(y)\, dy,$$

where μ is a constant, $x, y \in E_2$, $r=|x-y|$, $y_1-x_1=r\cos\theta$, $y_2-x_2=r\sin\theta$, $\int_0^{2\pi} f(\theta)\, d\theta=0$, and the $\int_{E_2}$ is a principal-value integral. Instead of applying Fourier transforms, he carries out the approach of Tricomi [Math. Z. **27** (1927), 87–133] and constructs the kernel of A^{-1}, which has the same form as A. He shows that the main condition of validity for his calculations is equivalent to the non-vanishing of the Fourier transform of the kernel of A. *R. T. Seeley* (Waltham, Mass.)

Burlak, J. 2620

On Gordon's method of solving dual integral equations.

Proc. Glasgow Math. Assoc. **6**, 117–122 (1964).

The dual integral equations

$$\text{(A)} \quad \begin{aligned} &\int_0^\infty u^\alpha \psi(u) J_\nu(xu)\, du = f(x), && 0 \leqq x < 1, \\ &\int_0^\infty \psi(u) J_\nu(xu)\, du = g(x), && x > 1, \end{aligned}$$

where J_ν is the Bessel function of order ν, $f(x)$ and $g(x)$ are given, and $\psi(x)$ is the unknown function, have recently been a subject of wide interest. Gordon [J. London Math. Soc. **29** (1954), 360–363; MR **16**, 261] and Copson [Proc. Glasgow Math. Assoc. **5** (1961), 21–24] used different methods to solve (A) with $g(x)=0$. The present author solves (A) with $f(x)=0$ using Gordon's method and then obtains the solution of (A) by the superposition of his solution and that of Copson's. The method used is formal and the frequent interchanges of orders of integration are not justified, so that the range attributed to the parameters involved in the solution may not be valid. *A. E. Danese* (Buffalo, N.Y.)

Smagina, V. I. 2621
Exceptional cases of integral equations of convolution type and the corresponding equations of first kind in the class of functions of exponential growth. The equation of class (B). (Russian)
Dokl. Akad. Nauk BSSR **7** (1963), 76–79.
The dual integral equations

$$\lambda_1\varphi(x)+(2\pi)^{-1/2}\int_{-\infty}^{\infty} k_1(x-t)\varphi(t)\,dt = f(x) \qquad (x>0),$$

$$\lambda_2\varphi(x)+(2\pi)^{-1/2}\int_{-\infty}^{\infty} k_2(x-t)\varphi(t)\,dt = f(x) \qquad (x<0),$$

where the functions k_1, k_2, and f are of exponential growth at $\pm\infty$, were considered by F. D. Gahov and Ju. I. Čerskiĭ [Izv. Akad. Nauk SSSR Ser. Mat. **20** (1956), 33–52; MR **18**, 134], who reduced them to a Riemann boundary problem on a contour consisting of two parallel straight lines and obtained solutions under certain supplementary conditions on the coefficient functions in the Riemann problem. These conditions were removed by F. D. Gahov and the present author [Dokl. Akad. Nauk SSSR **136** (1961), 1277–1280; MR **24** #A989] for the case when the given functions vanish at $\pm\infty$. In this paper the author removes these conditions in the general case, obtaining information about the number of linearly independent solutions and about conditions of solubility.
F. Smithies (Cambridge, England)

Manžeron, D. [Mangeron, D.]; Krivošein, L. E. 2622
The solution of integro-differential equations by a polynomial method. (Russian)
Mathematica (Cluj) **4 (27)** (1962), 271–287.
This paper has already been published in another journal [Bul. Inst. Politehn. Iaşi (N.S.) **7 (11)** (1961), no. 3–4, 27–42; MR **26** #2846]. *V. Lakshmikantham* (Aurangabad)

Szarski, J. 2623
On an integro-differential equation.
Ann. Polon. Math. **14** (1963/64), 321–333.
The author treats the integro-differential equation

$$(1)\qquad \left[\int_a^b u(t,y)\,dy\right]u_t'(t,x) = -\lambda(t,x)u(t,x)$$

subject to the initial condition (2) $u(0,x)=v(x)$ for almost every x on (a,b), where $v(x)$ and $\lambda(t,x)$ are given functions with $\lambda(t,x)\geqq 0$. A function $u(t,x)$ defined for t on some interval $0\leqq t<t_0$ and almost every x in (a,b) is termed a solution of (1) if it satisfies this equation almost everywhere, and $u(t,x)$ is absolutely continuous in t for almost every x, $u(t,x)$ is Lebesgue integrable with respect to x, and $[\int_a^b u(t,y)dy]^{-1}$ is integrable with respect to t in every compact subinterval of $0\leqq t<t_0$. Under the assumption that $v(x)$ is a non-negative Lebesgue integrable function in (a,b) with $\int_a^b v(x)\,dx>0$, while $\lambda(t,x)$ is non-negative almost everywhere on $0<t$, $a<x<b$, and bounded and Lebesgue integrable on arbitrary finite rectangles $0<t<t_1$, $a<x<b$, the author proves, by the method of successive approximations, that if t_0 is sufficiently small, there is a unique solution of (1), (2) on $0\leqq t<t_0$, $a<x<b$. There are established various basic properties of solutions of (1), (2), including their continuous dependence on v and λ. An equation of the form (1) has been derived by J. Bodziony for the description of the process of screening of granular bodies, and for the case in which $\lambda(t,x)$ is independent of t, and has at most one point of discontinuity as a function of x. J. Bodziony and S. Gołąb [Arch. Mech. Stos. **13** (1961), 529–554; MR **24** #B2299] have established the existence of a unique solution and derived some of its properties.
W. T. Reid (Norman, Okla.)

FUNCTIONAL ANALYSIS
See also 2143, 2271, 2280, 2294, 2325, 2327, 2352, 2364, 2393, 2412, 2413, 2421, 2431, 2455, 2456, 2518, 2522, 2528, 2574, 2609, 2617, 2714, 2931, 3002, 3023, 3137, 3138, 3140, 3142, 3144, 3160, 3262.

Sobolev, S. L. 2624
★Applications of functional analysis in mathematical physics.
Translated from the Russian by F. E. Browder. Translations of Mathematical Monographs, Vol. 7. *American Mathematical Society, Providence, R.I.*, 1963. vii + 239 *pp.* $6.70.
The table of contents is given in the review of the original Russian edition [Izdat. Leningrad. Gos. Univ., Leningrad, 1950; MR **14**, 565]. The translation is exceedingly smooth.
B. Frank Jones, Jr. (Houston, Tex.)

Andalafte, E. Z.; Blumenthal, L. M. 2625
Metric characterizations of Banach and Euclidean spaces.
Fund. Math. **55** (1964), 23–55.
The authors consider what conditions should be placed on a metric space in order that it be (a) a Banach space, (b) a euclidean space. In each case, the norm of the vector space is to yield a metric which coincides with the given one. Complete characterizations are given. Consider the following sets of axioms on an abstract set M. (I) Metricity. M is a metric space. (II) Finite compactness. Every bounded infinite subset of M has an accumulation element. (III) Convexity. If $p, r\in M$ with $p\neq r$, there exists $q\in M$ such that $p\neq q\neq r$ and $pq+qr=pr$ (here xy denotes the distance from x to y). (IV) External convexity. If $p, q\in M$, $p\neq q$, there exists an $r\neq p$, $\neq q$ such that q is between p and r (meaning $rq+qp=rp$). (V) Two-triple property. Given four distinct elements of M, the linearity of any two triples implies the linearity of the other two (linearity of a triple means that one element is between the other two). (VI) Young postulate. If $p, q, r\in M$ and q', r' are mid-elements of p and q and p and r, respectively, then $q'r'=\frac{1}{2}qr$. (VII) Ficken postulate. If f is a foot of an element p on a line L (p not on L), and if q and r are elements of L with $fq=fr$, then $sq=sr$ for any s of a line joining p and f (the foot of a point on a line is the point having least distance to the given point). The second characterization theorem states that the above postulates characterize a euclidean space of finite dimension. The first main theorem states that a space satisfying postulates (I) through (VI) is a Banach space. *E. R. Lorch* (New York)

Deutsch, Nimet 2626a
Interpolation dans les espaces vectoriels topologiques localement convexes.
C. R. Acad. Sci. Paris **257** (1963), 3796–3799.

Deutsch, Nimet 2626b
Construction duale de la construction d'une note précédente.
C. R. Acad. Sci. Paris **257** (1963), 4114–4116.

Deutsch, Nimet 2626c
Espaces intermédiaires, conditions d'existence d'espaces intermédiaires non triviaux.
C. R. Acad. Sci. Paris **258** (1964), 41–43.

Deutsch, Nimet 2626d
Espaces intermédiaires entre l'espace des suites à croissance lente et l'espace de suites à décroissance rapide.
C. R. Acad. Sci. Paris **258** (1964), 2468–2470.

An interpolation pair is a pair of locally convex topological vector spaces (E_1, E_2) together with continuous injections $j_i: E_i \to \mathscr{E}$, where $\mathscr{E}$ is a separable topological vector space. These form a category by taking appropriate pairs of morphisms. An interpolation functor is a functor whose value on (E_1, E_2) is a subspace of $\mathscr{E}$ between $E_1 \cap E_2$ and $E_1 + E_2$ [see also J.-L. Lions and J. Peetre, #2627 below]. In the first paper several interpolation functors are defined, and a number of interesting properties are stated. The second paper identifies the duals of the spaces constructed in the first paper. The third paper considers intermediate spaces, i.e., spaces E with $E_1 \cap E_2 \subset E \subset E_1 + E_2$. With suitable hypotheses, it is asserted that every such space can be expressed in terms of the spaces of the first paper. (There seems to be an adjoint functor somewhere in this construction.) Also, conditions are given for the existence of non-trivial intermediate spaces. Finally, the fourth paper gives examples. In particular, it is shown that intermediate spaces exist for pairs of spaces of distributions. *J. W. Gray* (Urbana, Ill.)

Lions, J.-L.; Peetre, J. 2627
Sur une classe d'espaces d'interpolation.
Inst. Hautes Études Sci. Publ. Math. No. 19 (1964), 5–68.

The authors define "mean" interpolation spaces between Banach spaces. Several definitions are given which are shown to be equivalent to each other, as well as to that of the "trace" spaces. One definition is as follows. Let A_0 and A_1 be two Banach spaces, each continuously embedded in a topological vector space $\mathscr{A}$. For $0 \leqq p_i < \infty$ and $\xi_0\xi_1 < 0$, the space $A = S(p_0, \xi_0, A_0; p_1, \xi_1, A_1)$ consists of those $a \in \mathscr{A}$ such that

$$(1) \qquad a = \int_{-\infty}^{\infty} u(x)\, dx$$

for some vector-valued function $u(x)$ satisfying

$$(2) \qquad \max_{i=0,1} \int_{-\infty}^{\infty} \|e^{\xi_i x} u(x)\|_{A_i}\, dx < \infty,$$

and the norm of a is the infimum of (2) taken over all $u(x)$ satisfying (1).

Some properties of mean spaces are proved. The dual space of A is shown to be equivalent to

$$S(p_0', -\xi_0, A_0'; p_1', -\xi_1, A_1'),$$

where $1/p_i + 1/p_i' = 1$. Moreover, it is shown that

$$\|a\|_A \leqq \text{const}\cdot \|a\|_{A_0}{}^{1-\theta}\|a\|_{A_1}{}^{\theta}, \qquad \theta = \frac{\xi_0}{\xi_0 - \xi_1},$$

for all $a \in A_0 \cap A_1$. In addition, for each $t > 0$ and each $a \in A$ there are $a_i \in A_i$, $i = 0, 1$, such that $a = a_0 + a_1$ and

$$\|a_i\|_{A_i} < \text{const}\cdot t^{i-\theta}\|a\|_A, \qquad i = 0, 1.$$

A space having these properties is said to be of class $\mathscr{K}_\theta(A_0, A_1)$. Next, a reiteration theorem is proved. It states that if X_i is a Banach space of class $\mathscr{K}_{\theta_i}(A_0, A_1)$, $i = 0, 1$, then $S(q_0, \eta_0, X_0; q_1, \eta_1, X_1) = A$, where $\eta_i = (1-\theta_i)\xi_0 + \theta_i\xi_1$, $1/q_i = (1-\theta_i)/p_0 + \theta_i/p_1$, $i = 0, 1$. Moreover, if π is a linear mapping which takes X_i continuously into itself, $i = 0, 1$, then π takes A boundedly into itself for $\theta_0 < \xi_0/\xi_0 - \xi_1 < \theta_1$.

Inclusion and compactness theorems are considered and generalizations of the convexity theorems of M. Riesz and J. Marcinkiewicz are proved. Other applications and generalizations are given. *M. Schechter* (New York)

James, Robert C. 2628
Weakly compact sets.
Trans. Amer. Math. Soc. **113** (1964), 129–140.

In an earlier paper [Studia Math. **23** (1963/64), 205–216], the author proved an old conjecture that a Banach space B is reflexive if and only if each member of its dual B^* attains a maximum on the unit sphere of B. This result can be restated as follows: The unit sphere S of B is weakly compact if and only if each member of B^* attains a maximum on S. In the present paper the author goes even further. He proves that a weakly closed subset C of a Banach space B is weakly compact if and only if each member of B^* attains a maximum on C. The proof is rather long and delicate. *I. Namioka* (Seattle, Wash.)

Landsberg, Max 2629
Über die Fixpunkte kompakter Abbildungen.
Math. Ann. **154** (1964), 427–431.

The various fixed-point theorems of this paper are all known or are close to known results, but in most cases the author's reasoning is simpler or his hypotheses are weaker than those employed previously. His discussion is very efficiently organized. The following three theorems may be cited as typical. (3) Suppose that E is a topological linear space, W is a closed neighborhood of 0 such that $]0, 1[\cap W \subset \operatorname{int} W$, and ϕ is a compact mapping of W into E which can be uniformly approximated by mappings with finite-dimensional range. If ϕ has no fixed point, there exist $t_0 \in]0, 1[$ and $x_0 \in \partial W$ such that $x_0 = t_0\phi(x_0)$. (5) Suppose that E is a locally convex space and A is a closed subset of E which is a retract of a convex subset of E. Then every compact mapping of A into A has a fixed point. (6) Suppose that F is a metric linear space which is an extension space for the class of all metric spaces and A is a retract of F. Then every compact mapping of A into A has a fixed point. *Victor Klee* (Seattle, Wash.)

Ruston, A. F. 2630
Auerbach's theorem and tensor products of Banach spaces.
Proc. Cambridge Philos. Soc. **58** (1962), 476–480.

The author presents a proof of Auerbach's theorem that there exist complete normed bi-orthogonal systems in any finite-dimensional Banach space. Two applications to the theory of tensor products of Banach spaces are given.

(i) If $\mathfrak{M}$ is an n-dimensional subspace of a Banach space $\mathfrak{B}$, then there exists a projection P of $\mathfrak{B}$ onto $\mathfrak{M}$ with trace-norm n. (ii) Let $\mathfrak{B}_1 \otimes \mathfrak{B}_2$ be the algebraic tensor product of Banach spaces $\mathfrak{B}_1$ and $\mathfrak{B}_2$, let γ be the greatest cross-norm on $\mathfrak{B}_1 \otimes \mathfrak{B}_2$, and let λ be the least cross-norm on $\mathfrak{B}_1 \otimes \mathfrak{B}_2$ whose associate is also a cross-norm. Then $\gamma(u) \leqq r\lambda(u)$ for any element $u \in \mathfrak{B}_1 \otimes \mathfrak{B}_2$ of rank r.

{Another proof of Auerbach's theorem has been given by A. E. Taylor [Bull. Amer. Math. Soc. **53** (1947), 614–616; MR **8**, 588; errata, MR **8**, 709]. See also M. M. Day [Trans. Amer. Math. Soc. **62** (1947), 315–319; MR **9**, 246].}

A. Wiweger (Warsaw)

Simons, S. 2631

A new definition of linear topological space.

Proc. London Math. Soc. (3) **13** (1963), 1–19.

The new definition proposed by the author involves replacing the continuity of λx at $(0, 0)$ by another concept, indecomposability, which, it is claimed, fits more naturally in the context of spaces over a general field. If X, Y, Z are topological spaces and f maps $X \times Y$ into Z, f is called indecomposable at (x_0, y_0) if and only if, whenever $f(x_0, y_0)$ is adherent to the net $f(x_\alpha, y_\alpha)$, either x_0 is adherent to (x_α) or y_0 to (y_α). Given a field F, topologised so as to make addition continuous in one variable, a linear space X over F is called a *LTS if and only if it is a T_1 additive topological group, the mapping $\lambda \to \lambda x$ (with $x \neq 0$) is continuous at $\lambda = 0$ and the mapping $(\lambda, x) \to \lambda x$ is indecomposable at $(0, 0)$. Many of the standard elementary properties of real or complex spaces, for example, those concerning products, quotients and finite-dimensional subspaces, hold for *LTS's over a general field.

The author examines the relation between *LTS's and the linear topological spaces over (suitably restricted) non-discrete topological spaces described in Bourbaki [*Les structures fondamentales de l'analyse*, Livre V, *Espaces vectoriels topologiques*, Chapitre I, Actualités Sci. Indust., No. 1189, Hermann, Paris, 1953; MR **14**, 880].

A. P. Robertson (Glasgow)

Singer, Ivan 2632

Completeness and aw^*-topologies.

Arch. Math. **15** (1964), 42–45.

The author discusses four properties of a locally convex space E which have previously been characterized in terms of a mixture of the w^* and aw^* topologies of the conjugate space E^*. By routine arguments he shows that they can be described in terms of the aw^* topology alone. The (successively stronger) properties are (A) the B_r-completeness, (B) the B-completeness of Ptak [Czechoslovak Math. J. **3** (78) (1953), 301–364; MR **16**, 262], (C) the hypercompleteness of Kelley [Michigan Math. J. **5** (1958), 235–246; MR **21** #2173], and (D) the strict hypercompleteness introduced in the present note. Let E be a locally convex space and let F denote the space E^* in its aw^* topology. Then E has one of the properties (A)–(D) if and only if E is complete and the space F satisfies (respectively) the following conditions: (A) every closed linear proper subspace of E is contained in a closed hyperplane, (B) every closed linear subspace of F can be strictly separated from any outside point by means of a closed hyperplane, (C) every circled closed convex subset of F can be strictly separated from any outside point by means of a closed hyperplane, (D) every closed convex subset of F can be strictly separated from any outside point by means of a closed hyperplane. It is known that (A) does not imply (B), and Kelley has asked whether (B) implies (C) or (C) implies (D). The author does not answer these questions, but he raises the same problems for general topological linear spaces (rather than those of the form (E^*, aw^*)). {Reviewer's note: Let τ_1 be the usual topology for the space S of all measurable functions on $[0, 1]$, let τ_2 be the finest locally convex topology for S, and let $\sigma = \sup(\tau_1, \tau_2)$. Then (S, σ) has property (C) but not property (D). By use of the continuum hypothesis, it is possible to construct a space which has property (B) but not property (C).}

Victor Klee (Seattle, Wash.)

Singer, Ivan 2633

Weak compactness, pseudo-reflexivity and quasi-reflexivity.

Math. Ann. **154** (1964), 77–87.

Let E be a Banach space, V a linear subspace of the conjugate space E^*, and ϕ and π the canonical mappings of E into V^* and E^{**}, respectively. The space E is said to be V-pseudo-reflexive provided ϕ is a homeomorphism of E onto V^*, and to be quasi-reflexive of order n provided $\operatorname{codim}_{E^{**}} \pi(E) = n$. The author notes that E is quasi-reflexive of order n if and only if there exists a total closed linear subspace V of E^* with $\operatorname{codim}_{E^*} V = n$ such that E is V-pseudo-reflexive. He then proceeds to characterize pseudo-reflexivity in ways which extend some of the most important characterizations of reflexivity. His main results are in terms of sequential $\sigma(E, V)$-compactness, quasi-reflexivity properties of separable subspaces, or intersection properties of sequences of bounded $\sigma(E, V)$-closed convex sets.

Victor Klee (Seattle, Wash.)

Amir, D. 2634

Projections onto continuous function spaces.

Proc. Amer. Math. Soc. **15** (1964), 396–402.

A Banach space X is injective if for every Banach space Y containing X as a subspace, there exists a projection from Y onto X; and it is said to have the λ-projection property, $X \in P_\lambda$, if (it is injective and) such a projection can always be found with norm $\leqq \lambda$ [D. A. Goodner, Trans. Amer. Math. Soc. **69** (1950), 89–108; MR **12**, 266; J. R. Isbell and Z. Semadeni, ibid. **107** (1963), 38–48; MR **22** #4169].

Let S be a compact Hausdorff space, $M(S)$ the Banach space of bounded real-valued functions on S (supremum norm), and $C(S)$ the subspace of all continuous functions on S. Let Σ be a field of subsets of S which contains an open basis and is closed under complementation, finite unions, and the closure operation; and let $B(S, \Sigma)$ be the subspace of $M(S)$ spanned by the characteristic functions of sets in Σ. $C(S) \subset B(S, \Sigma) \subset M(S)$. For $s \in S$, define $\rho_1(s, \Sigma) = \sup(1 - 1/n)$, where the supremum is taken over all n such that there exist n disjoint open sets $G_1, \cdots, G_n$ in Σ with $s \in \bigcap_{i=1}^n \overline{G_i}$; and for $k > 1$, let $\rho_k(s, \Sigma) = \sup\{(1 - 1/n) + \inf_u \min_{1 \leqq i \leqq n} \sup_{t \in G_i \cap u} \rho_{k-1}(t, \Sigma)\}$, where the first supremum is taken over all finite families $\{G_1, \cdots, G_n\}$ of disjoint open sets in Σ such that $s \in \bigcap_{i=1}^n \overline{G_i}$ and the infimum is taken over all neighborhoods u of s. Finally, let $\rho(s, \Sigma) = \sup_{k \geqq 1} \rho_k(s, \Sigma)$ and $\rho(S, \Sigma) = \sup_{s \in S} \rho(s, \Sigma)$. When Σ is the set of all subsets of S, $\rho(S, \Sigma)$ is shortened to $\rho(S)$.

The author asserts (Theorem 1) that if P is a projection from $B(S,\Sigma)$ onto $C(S)$, then $\|P\| \geqq 1+2\rho(S,\Sigma)$; and thus (Theorem 1′) if $C(S)$ is a P_λ space, then $\lambda \geqq 1+2\rho(S)$. These results are used to obtain others concerning P_λ spaces $C(S)$; e.g., (Theorem 2) if $C(S) \in P_\lambda$, then S contains a dense, open, extremally disconnected subset; and (Corollary 2) if X is an infinite discrete space, then $C(\beta X - X)$ is not a P_λ space (answering Question 5 of Isbell and Semadeni [loc. cit.]).

The final expression given for the function F constructed in the proof of Theorem 1 does not appear to be correct, but a verification that $|F| \leqq 1$, for which purpose the expression was derived, is possible. A more troublesome defect is to be found: it has been tacitly assumed in the proof of Theorem 1 that $\rho(S,\Sigma)$ is finite (given $\varepsilon > 0$, $s \in S$ is chosen so that $\rho(S,\Sigma) \leqq \rho(s,\Sigma)+\varepsilon/8$, and, then k is chosen so that $\rho(s,\Sigma) \leqq \rho_k(s,\Sigma)+\varepsilon/8$; but $\rho_k(s,\Sigma) \leqq k$ at each point $s \in S$). Later on in the paper (Remark b, p. 401) one finds a statement to the effect that if $C(S) \in P_\lambda$, then $\rho(S)$ is finite, but no proof or reference is given (nor is the reviewer able to find or prove this result). This matter is crucial in that if finiteness of $\rho(S)$ is not a necessary condition in order that $C(S) \in P_\lambda$, then arguments used in the proofs of each of the results of the paper are invalid. Theorem 2, however, is reported to have been proved by other methods [the author, Bull. Res. Council Israel Sect. F **10F** (1962), 133–138; MR **26** #592]. *T. R. Jenkins* (Los Altos, Calif.)

Garkavi, A. L. 2635

On Chebyshev and almost-Chebyshev subspaces. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 799–818.

This paper consists of the proofs of the results announced earlier in the author's paper of the same title [Dokl. Akad. Nauk SSSR **149** (1963), 1250–1252; MR **26** #6737]. In addition, a table is given which summarizes the known results (by the author and the reviewer) concerning the existence of Čebyšev subspaces of finite dimension and of finite defect in the classical Banach spaces.

R. R. Phelps (Seattle, Wash.)

Grisvard, Pierre 2636

Semi-groupes faiblement continus et interpolation.

C. R. Acad. Sci. Paris **259** (1964), 27–29.

This is a sequel to a previous note [same C. R. **258** (1964), 4900–4902; MR **29** #456]; we refer to the review of that note as (I). In particular, several interesting concrete applications in the same direction are given. It is shown, by taking $n=1$, that

(1) $$[C^{s_0}, B^{s_1}_{P,p}]_\eta = B^s_{P/\eta,p/\eta},$$

(2) $$(C^{s_0}, B_p{}^{s_1})_{\eta,p/\eta} = B^s_{P/\eta,p/\eta},$$

where $B^s_{P,p}$ is as in (I) and C^s is the space of functions whose derivatives up to order $[s]$ are bounded and satisfy a Hölder condition of exponent $s-[s]$; it is assumed that $1<P, p<\infty$, $0<\eta<1$, s_0 non-integer >0, s_1 arbitrary, $s=s_0(1-\eta)+s_1\eta$. Note that (1) and (2) are essentially the limiting cases as $P_0\to\infty$, $p_0\to\infty$ of (1′) and (2′) of (I). Using an imbedding theorem due to Il'in [Trudy Mat. Inst. Steklov. **53** (1959), 128–144; MR **22** #A11084] the author next gives the following interpolation theorem. Let π be a continuous linear mapping from (i) L^{p_0} into C^α and (ii) L^{p_1} into L^q $(q>p_1)$. Then π is continuous from L^p $(1/p=(1-\theta)/p_0+\theta/p_1)$ into (1°) C^μ if $\mu>0$, (2°) the Orlicz space L^φ with $\varphi=\sum_{k\geq q/\theta} t^k/k!$ if $\mu=0$, (3°) $L^{-1/\mu}$ if $\mu<0$, where $\mu=(1-\theta)\alpha-\theta/q$. A similar result with L^q replaced by the Lorentz space M_σ $(\sigma=1-1/q)$ and periodicity is given. It is stated that these results also hold for $n>1$. Then one gets back the main results of Stampacchia [Comm. Pure Appl. Math. **17** (1964), 293–306]. *J. Peetre* (Lund)

Frivaldszky, Sándor 2637

Theory of generalized functions of Sobolev in the case of general (not starshaped) regions. (Russian. German summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 309–316 (1964).

L'auteur démontre pour l'espace de Sobolev $W_p{}^{(k)}(\Omega)=E(u\in L_p(\Omega),\ D^\alpha u\in L_p(\Omega),\ |\alpha|=k)$ et pour $\Omega_0\subset\Omega$ les divers théorèmes de l'immersion du type $|u|_{W_p{}^{(k-1)}(\Omega_0)}\leq c\cdot|u|_{W_p{}^{(k)}(\Omega)}$. Ces résultats sont bien connus et utilisés par plusieurs auteurs. On ajoute encore deux contre-exemples montrant le fait connu que les théorèmes de l'immersion, valables pour Ω avec la propriété du cône, ne sont pas valables en général. *J. Nečas* (Prague)

Kaliničenko, D. F. 2638

Some properties of functions of the spaces $W_p{}^m$ and $W^m_{p,\alpha_1,\cdots,\alpha_n}$. (Russian)

Mat. Sb. (N.S.) **64 (106)** (1964), 436–457.

In this paper the author obtains bounds for $(\int_\Omega |u|\,d\Omega)^p$ or for $\int_\Omega |u|^p\,d\Omega$, where $u(x_1,\cdots,x_n)$ is in the Sobolev spaces mentioned in the title, for various convex domains Ω in E_n, including the unit cube, the Lipschitz "cube" and the "cube" of tangent planes $E_{n-1}(M)$ constructed for each point M of a smooth $(n-1)$-dimensional manifold bounding Ω. The principal tool used to derive all of these estimates is the relation

$$u(\bar{\bar{x}})-u(\bar{x}) = \sum_{i=1}^n \int_{\bar{x}_i}^{\bar{\bar{x}}_i} \frac{\partial u(\hat{x})}{\partial x_i}\,dx_i$$

for two points $\bar{\bar{x}}$ and $\bar{x}$ of Ω, where the mark ^ on the point $\hat{x}$ is shorthand for a certain systematic pairing of coordinates having double bars with those having a single bar. *R. N. Goss* (San Diego, Calif.)

Lizorkin, P. I. 2639

Functions of Hirschman type and relations between the spaces $B_p{}^r(E_n)$ and $L_p{}^r(E_n)$. (Russian)

Mat. Sb. (N.S.) **63 (105)** (1964), 505–535.

The author characterizes the spaces $L_p{}^\alpha(E_n)$ and $B_p{}^\alpha(E_n)$ in terms of Poisson integrals, and from this derives the known inclusions $B_p{}^\alpha(E_n)\subset L_p{}^\alpha(E_n)$ for $1<p\leqq 2$, with the reverse for $2\leqq p<\infty$. Here $L_p{}^\alpha(E_n)$ is the space of functions $f=G_\alpha * \varphi$ for some φ in L^p, where the Fourier transform of G_α is $(1+|x|^2)^{-\alpha/2}$; and $B_p{}^\alpha(E_n)$ consists, for $0<\alpha<2$, of those f in L^p for which

$$\int_{E_n}\int_{E_n} |f(x+y)+f(x-y)-2f(x)|^p|y|^{-n-\alpha p}\,dx\,dy<\infty.$$

For f in $L^p(E_n)$, U is defined on $E_n\times\{t\geqq 0\}$ as the solution of $U=\sum_1^n \partial^2U/\partial x_i{}^2+\partial^2U/\partial t^2$ with boundary value f, and

vanishing as $t \to +\infty$. Then L_p^α is characterized as those f in L^p for which

$$\int_{E_n}\left[\int_0^\infty t^{1-2\alpha}(|\text{grad } U|^2+|U|^2)\,dt\right]^{p/2}dx$$

is finite; for B_p^α the corresponding requirement is given as

$$\int_{E_n}\int_0^\infty t^{p-1-\alpha p}(|\text{grad } U|^2+|U|^2)^{p/2}\,dtdx < \infty.$$

The proofs employ results of Benedek, Calderón, and Panzone [Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 356–365; MR **24** #A3479], together with numerous classical inequalities.

This characterization of B_p^α is nearly the same as that given by Taibleson [Bull. Amer. Math. Soc. **69** (1963), 487–493; MR **27** #577]; the same paper gives the inclusions between B_p^α and L_p^α, as well as a thorough investigation of several related questions, for slightly more general spaces. *R. T. Seeley* (Waltham, Mass.)

Makovoz, Ju. I. 2640

Chebyshev subspaces of the space C. (Russian)

Uspehi Mat. Nauk **19** (1964), *no.* 4 (118), 185–188.

A linear subspace Γ of $C(T)$ (T compact Hausdorff) is said to be a Čebyšev subspace if to each element x of $C(T)$ there exists a unique nearest element x_0 in Γ. The author proves (Theorem 1) that if T consists of a finite number m of connected components and if Γ is a Čebyšev subspace of $C(T)$ such that $C(T)/\Gamma$ is finite-dimensional, then $\dim C(T)/\Gamma \leq m$. This generalizes a result ($m=1$) of the reviewer [Trans. Amer. Math. Soc. **95** (1960), 238–255, Theorem 3.4; MR **22** #3964]. Using the same methods and hypotheses, Theorem 2 relates the number of isolated points in T to the Čebyšev rank r_Γ of a subspace Γ of T. [If $x \in C(T)$, let $r_\Gamma(x)$ denote the dimension of the (convex) set of nearest points in Γ to x and let r_Γ be the sup over all x of the $r_\Gamma(x)$.] {This paper was received by the editors in May, 1961 and appeared in August, 1964. In the interim (and independently), A. L. Garkavi [Mat. Sb. (N.S.) **62** (**104**) (1963), 104–120; MR **27** #6076; Dokl. Akad. Nauk SSSR **155** (1964), 513–516; MR **28** #3320] and the reviewer [Pacific J. Math. **13** (1963), 647–655; MR **27** #6069] have obtained numerous results concerning Čebyšev subspaces of finite deficiency in $C(T)$. In particular, they have shown (compare with Theorem 1) that if $C(T)$ contains a Čebyšev subspace Γ with $1<\dim C(T)/\Gamma<\infty$, then T contains no nondegenerate connected open set.}

R. R. Phelps (Seattle, Wash.)

Pym, John S. 2641

Positive functionals, additivity, and supports.

J. London Math. Soc. **39** (1964), 391–399.

Let $\mathfrak{F}$ be a sublattice and linear subspace of the space of all bounded real functions on a set X with $1 \in \mathfrak{F}$. For a nonnegative functional T on $\mathfrak{F}$, the author calls a non-void subset E of X a "partial support" of T if $Tf=0$ for every f in $\mathfrak{F}$ vanishing on E, while $f \in \mathfrak{F}$, $f \geqq 0$, and $f(x)>0$ for some x in E imply $Tf>0$. If such a partial support exists, there is a largest one, called the support of T; if not, T is said to be "entirely without support".

The author's concern is in showing when such a support exists. He shows the support exists if T is "net additive" (i.e., $Tf_\delta \to 0$ for every decreasing net in $\mathfrak{F}$ converging pointwise to 0), but the converse fails and, in fact, T is net additive if and only if T is appropriately "disjoint" from every functional entirely without support. Various related results and examples are given.

I. Glicksberg (Seattle, Wash.)

Swong, Khyson 2642

A representation theory of continuous linear maps.

Math. Ann. **155** (1964), 270–291; *errata, ibid.* **157** (1964), 178.

The author extends the results of the reviewer [Bull. Sci. Math. (2) **82** (1958), 29–40; MR **20** #3447; Rev. Math. Pures Appl. **4** (1959), 391–401; MR **22** #5883] concerning the representation of continuous linear functionals on the Banach spaces $B(T,E)$ of all totally measurable functions (in the sense of M. Gowurin) and $C(T,E)$ of all continuous functions defined on a compact space T with values in a Banach space E (both $B(T,E)$ and $C(T,E)$ being endowed with the uniform norm), and the results of C. Foiaş and the reviewer [ibid. **5** (1960), 729–752; MR **24** #A1618] concerning the representation of continuous linear mappings of $C(T,E)$ into a Banach space F by means of vector measures of bounded semi-variation defined on the Borel sets of T and with values in $\mathscr{L}(E,F)$ (the space of all continuous linear mappings of E into F) to the following more general situation: E, F are locally convex Hausdorff spaces, $B(T,E)$ and $C(T,E)$ are endowed with the topology of uniform convergence, $[B(T,E)]^*$ is replaced by $\mathscr{L}(B(T,E),F)$ and the norm of $U \in \mathscr{L}(E,F)$ is replaced by its "generalized norm" in the sense of G. Marinescu [C. R. Acad. Sci. Paris **241** (1955), 1693–1695; MR **17**, 986]. In the last paragraph of the paper the author also considers the case when E and F are endowed with order structures. For similar extensions of some other representation theorems, see A. and C. Ionescu-Tulcea [Ann. Mat. Pura Appl. (4) **53** (1961), 63–87; MR **23** #A2039].

Ivan Singer (Bucharest)

Gelfand, I. M. [Gel'fand, I. M.]; 2643
Schilow, G. E. [Šilov, G. E.]

★Verallgemeinerte Funktionen (Distributionen). III. Einige Fragen zur Theorie der Differentialgleichungen.

Hochschulbücher für Mathematik, Band 49.

VEB Deutscher Verlag der Wissenschaften, Berlin, 1964. 195 *pp.* *DM* 29.50.

Part II appeared in 1962 [MR **26** #6765]. This is a translation of the authors' *Generalized functions*, No. 3, *Some questions in the theory of differential equations* (Russian) [Fizmatgiz, Moscow, 1958; MR **21** #5142b].

H. Federer (Providence, R.I.)

Bognár, J. 2644

Some relations among the negativity properties of operators in spaces with indefinite metric. (Russian. English summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 201–212.

Let H_k be a Hilbert space with an indefinite inner product in the sense of L. S. Pontrjagin [Izv. Akad. Nauk SSSR Ser. Mat. **8** (1944), 243–280; MR **6**, 273] and I. S. Iohvidov and M. G. Kreĭn [Trudy Moskov. Mat. Obšč. **5** (1956), 367–432; MR **18**, 320]. The author is interested in various nonnegativity conditions for a continuous self-adjoint operator

on H_κ. Several implication relations between these conditions have been announced without proof in a previous paper [Ann. Acad. Sci. Fenn. Ser. A I No. 336/10 (1963); MR **28** #4368]. The present paper contains the proofs of some of these results. *Ky Fan* (Evanston, Ill.)

Brodskiĭ, M. S.; Šmul'jan, Ju. L. 2645
Invariant subspaces of a linear operator and divisors of its characteristic function. (Russian)
Uspehi Mat. Nauk **19** (1964), *no.* 1 (115), 143–149.

A colligation $\alpha = \begin{pmatrix} A & K & J \\ \mathfrak{H} & & \mathfrak{G} \end{pmatrix}$ consists of a pair of Hilbert spaces $\mathfrak{H}$ and $\mathfrak{G}$ and a triad of bounded linear operators A, K, and J acting, respectively, on $\mathfrak{H}$, from $\mathfrak{G}$ to $\mathfrak{H}$, and on $\mathfrak{G}$, and such that (1) K is completely continuous, (2) $J = J^*$, $J^2 = E$ (identity operator), (3) $KJK^* = (A - A^*)/2i$. The projection of a colligation on a subspace of $\mathfrak{H}$ is defined in the obvious way. Further, a colligation is called simple if the closed linear hull of all vectors A^nKg ($g \in \mathfrak{G}$; $n = 0, 1, 2, \cdots$) is $\mathfrak{H}$ itself. {In the paper, A^nKg is misprinted as $A''Kg$.} The characteristic operator-function corresponding to α is then $W_\alpha(\lambda) = E - 2iK^*(A - \lambda E)^{-1}KJ$. A class Ω_J of operator-functions related to J is then introduced, containing all characteristic functions (the exact conditions are rather too lengthy to be reproduced here). Divisibility in Ω_J is defined; W_2 is a divisor of W_1 ($W_2(\lambda) \leqq W_1(\lambda)$) if there exists $W_3 \in \Omega_J$ with $W_1(\lambda) = W_2(\lambda)W_3(\lambda)$, and W_2 is an admissible divisor of W_1 ($W_2(\lambda) \underline{\leqq} W_1(\lambda)$) if both W_2 and W_3 are characteristic functions.

Results already proved (in a slightly different form) about characteristic operator-functions can now be applied to give information about the invariant subspaces of A. Suppose A is a bounded linear operator on $\mathfrak{H}$ such that $(A - A^*)/2i$ is completely continuous. It is possible to choose $\mathfrak{G}$, K, and J so that $\alpha = \begin{pmatrix} A & K & J \\ \mathfrak{H} & & \mathfrak{G} \end{pmatrix}$ is a simple colligation. Let $W(\lambda)$ be the characteristic function of α and denote by $\mathfrak{M}$ the set of all admissible divisors of W. Define a map Φ from the subspaces of $\mathfrak{H}$ that are invariant under A to $\mathfrak{M}$ by requiring $\Phi(\mathfrak{H}_0)$ to be the characteristic operator-function of the projection of α on the subspace $\mathfrak{H}_0$. Then the following results hold. Theorem 1: The map Φ is one-to-one. Theorem 2: Let $\Phi(\mathfrak{H}_i) = W_i(\lambda)$ $(i = 1, 2)$; in order that $\mathfrak{H}_1$ should be a subspace of $\mathfrak{H}_2$ it is necessary and sufficient that $W_1(\lambda) \leqq W_2(\lambda)$. Theorem 3: The operator A is unicellular if and only if the set $\mathfrak{M}$ is totally ordered.
J. H. Williamson (Cambridge, England)

Foguel, S. R. 2646
A counterexample to a problem of Sz.-Nagy.
Proc. Amer. Math. Soc. **15** (1964), 788–790.

The author constructs a Hilbert space operator with uniformly bounded powers which is not similar to a contraction. (For details see the paper reviewed below [#2647].) In the proof of the properties of the example a previous lemma of the author [Pacific J. Math. **13** (1963), 551–562; MR **29** #473] is used. *A. Korányi* (Princeton, N.J.)

Halmos, P. R. 2647
On Foguel's answer to Nagy's question.
Proc. Amer. Math. Soc. **15** (1964), 791–793.

This is a simplified formulation of the example given in the paper reviewed above [#2646]. H_0 is a Hilbert space with orthonormal basis $\{e_0, e_1, e_2, \cdots\}$; S is the unilateral shift operator on H_0; J is an infinite set of natural numbers such that $i, j \in J$ and $i < j$ imply $2i < j$; Q is the projection from H_0 onto the span of all the e_j's with $j \in J$. The example is $\begin{pmatrix} S^* & Q \\ 0 & S \end{pmatrix}$ acting on $H = H_0 \oplus H_0$. The paper is self-contained, as it also includes a new proof of Foguel's lemma used in the paper reviewed above.
A. Korányi (Princeton, N.J.)

Hadeler, Karl-Peter 2648
Estimates for the spectrum of normal operators. (Russian)
Dokl. Akad. Nauk SSSR **157** (1964), 284–287.

The spectrum of a linear operator A on various B-spaces of functions is related to the range of the functions Af/f, f an element of the B-space. Example: Let A be a closed, densely defined, normal operator on $L(d\mu)$, and let $0 \not\equiv f \in D(A)$. Set $g(s) = Af(s)/f(s)$ for $f(s) \neq 0$, $g(s) = \infty$ for $f(s) = 0$ and $Af(s) \neq 0$, and let $g(s)$ remain undefined if both $f(s) = 0$ and $Af(s) = 0$. If the values of g are located within a closed disc K in the extended complex plane, then K meets the spectrum of A.
T. W. Gamelin (Cambridge, Mass.)

Iohvidov, I. S. 2649
Singular linear manifolds in the spaces Π_κ. (Russian. English summary)
Ukrain. Mat. Ž. **16** (1964), 300–308.

The space Π_κ is a Hilbert space $\mathfrak{H}$ endowed with a form $[x, y] = (Jx, y)$, where $J = P - Q$, P and Q are orthogonal projections in $\mathfrak{H}$, $P + Q = \mathrm{I}$, and min (dim $P\mathfrak{H}$, dim $Q\mathfrak{H}$) $= \kappa < \infty$. A linear manifold $\mathfrak{L} \subset \mathfrak{H}$ is called definite if the form $[x, x]$ ($x \in \mathfrak{L}$) has a constant sign. A definite linear manifold $\mathfrak{L} \subset \mathfrak{H}$ is called singular if the norms $(x, x)^{1/2}$ and $|[x, x]|^{1/2}$ are non-equivalent on $\mathfrak{L}$. The author studies the properties of singular linear manifolds in Π_κ spaces and proves the following theorem. If $\mathfrak{L}$ is an infinite-dimensional linear space with a positive Hermitian form $[x, y]$, complete with respect to the norm $[x, x]^{1/2}$, then for every positive integer κ there is a Π_κ space $\mathfrak{H}$ such that $\mathfrak{L}$ can be embedded (with the preservation of the form $[x, y]$) into $\mathfrak{H}$, and $\mathfrak{L}$ proves to be a singular linear manifold in $\mathfrak{H}$ with a given measure of singularity $m \leqq \kappa$. *A. Wiweger* (Warsaw)

Kreĭn, M. G.; Langer, G. K. [Langer, Heinz] 2650
On the spectral function of a self-adjoint operator in a space with indefinite metric. (Russian)
Dokl. Akad. Nauk SSSR **152** (1963), 39–42.

Let Π_κ be a Pontrjagin space, as axiomatized by I. S. Iohvidov and M. G. Kreĭn [Trudy Moskov. Mat. Obšč. **5** (1956), 367–432; MR **18**, 320], with an indefinite inner product (x, y) having the property that in a suitable system of coordinates $(x, x) = \sum \varepsilon_j |\xi_j|^2$ ($\varepsilon_j = \pm 1$), where precisely κ terms are negative. The concepts of self-adjoint operator and orthogonal projection are defined with respect to (x, y). An operator function E_λ, the values of which are orthogonal projections in Π_κ, is called a "spectral function with critical points $\alpha_1, \alpha_2, \cdots, \alpha_n$ ($-\infty < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \infty$) if and only if E_λ is defined for all real λ different from $\alpha_1, \alpha_2, \cdots, \alpha_n$, and the following

conditions are valid: (1) $E_\lambda E_\mu = E_\mu E_\lambda = E_{\min(\lambda,\mu)}$; (2) for any $x \in \Pi_\kappa$, the function $(E_\lambda x, x)$ is non-decreasing on each interval not containing a critical point, and in the sense of strong convergence, $E_\lambda = E_{\lambda-0}$, $E_{-\infty}=0$, $E_\infty = I$; (3) for each α_j and any $\varepsilon>0$ there exists an element $x_j = x_j(\varepsilon)$ such that $(E_{\alpha_j-\varepsilon}x_j, x_j) > (E_{\alpha_j+\varepsilon}x_j, x_j)$. It is shown that to every self-adjoint operator A with real spectrum there corresponds a unique spectral function with critical points E_λ (called "the proper spectral function of A") such that if particular care is taken at the critical points, then E_λ is related to A in a manner analgous to the relation between a self-adjoint operator in Hilbert space and its spectral function. Using the proper spectral function of the self-adjoint operator A, a number of concepts are defined for A, and a number of statements are made concerning the structural properties of A. Also considered is the question of when a given spectral function with critical points is the proper spectral function of some self-adjoint operator. It is claimed that the results of the paper may be used to construct an operational calculus for A.

R. C. Gilbert (Fullerton, Calif.)

Gohberg, I. C.; Kreĭn, M. G. 2651

Completeness criteria for the system of contraction root vectors. (Russian)

Ukrain. Mat. Ž. **16** (1964), 78–82.

The object of the authors is to establish a theorem about the completeness of root vectors of a contraction operator. They consider a ring of bounded linear operators in a separable Hilbert space and its two-sided ideal, and introduce the concept of a nuclear operator.

P. K. Ghosh (Calcutta)

Kurpel', M. S. [Kurpel', N. S.] 2652

An approximate method of solving linear operator equations in Hilbert space. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1275–1279.

Author's summary: "This paper deals with an iteration method of solving linear equations in a Hilbert space, the essence being that the initial equation $u=f+Au$ is replaced by a new equivalent equation

$$u = f + PA(I-A)^{-1}f + (A-PA)u,$$

to which the ordinary method of successive approximations is applied. The operation of projecting P is carried out in a definite way. The sufficient conditions for convergence of the process of successive approximations are presented, and an evaluation is given of the error of the nth approximation. One more generalization of the constructed algorithm is considered."

Mejlbo, Lars C. 2653

On the solution of the commutation relation $PQ-QP=-iI$.

Math. Scand. **13** (1963), 129–139.

The commutation relation $[P,Q]=-iI$ is studied in topological vector spaces with inner product. An example is L. Schwartz's space $(\mathcal{S})$, with $P\varphi(t) = -i\,d\varphi/dt$ and $Q\varphi(t) = t\varphi(t)$, with the usual $\mathcal{L}_2$-inner product. Theorem: If $\mathcal{W} \neq \{0\}$ is any complex inner product space with everywhere-defined operators P, Q satisfying the commutation relation, and such that (a) $(P\pm iI)\mathcal{W} = (Q\pm iI)\mathcal{W} = \mathcal{W}$; (b) $\mathcal{W}$ is complete in the topology determined by the seminorms $\varphi \to \|K\varphi\|$, K ranging over all polynomials in P and Q; (c) There is no nontrivial subspace of $\mathcal{W}$ closed in the above topology and invariant under $P, Q, (P\pm iI)^{-1}$, $(Q\pm iI)^{-1}$, then there is an isomorphism from $\mathcal{W}$ to $(\mathcal{S})$ carrying over P, Q and the inner product to the right things. The topology of (b) becomes the usual topology of $(\mathcal{S})$. This is an improvement on another result of P. Kristensen, L. Mejlbo and E. T. Poulsen (to appear). The uniqueness goes over to finitely many degrees of freedom: $[P_l, Q_m] = -i\delta_{l,m}$, $[P_l, P_m] = [Q_l, Q_m] = 0$, $l, m = 1, \cdots, n$. If one assumes the extra axiom that $(P_m - iQ_m)\varphi = 0$ has a nonzero solution ("existence of vacuum state"), then the uniqueness holds in infinitely many dimensions as well, although the model in this case is unwieldy.

J. Feldman (Berkeley, Calif.)

Medvedev, V. A. 2654

On convergence of the projection method in eigenvalue problems. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 258–260.

Let H_1, H_2 be Hilbert spaces, and let $L(\lambda)$ be a bounded linear operator from H_1 to H_2 depending analytically on the parameter λ varying in a region of the complex plane. Given two projectively complete sequences $\{R_n\}$ and $\{M_n\}$ of finite-dimensional subspaces of H_1 and H_2, respectively, such that $\dim R_n = \dim M_n$, and let P_n be the projection operator on R_n. The author proves some theorems relating the eigenvalues of the equation (*) $L(\lambda)x=0$ with the eigenvectors of the approximating equation (**) $P_nL(\lambda)x=0$. Let a sequence $\{\lambda_n\}$ of eigenvalues of (**) converge to an eigenvalue λ_0 of (*), and let x_n be the corresponding eigenvectors of (**) of norm one. Let S be the operator projecting on the eigenspace of λ_0, and let $Q=I-S$. Theorem 1: Qx_n converges weakly to zero. Theorem 2: If $L(\lambda_0) = L_0 + T$ with T a compact operator and L_0 satisfying $\liminf_{n\to\infty} \min_{z\in R_n} (\|P_nL_0z_n\|/\|z_n\|) > 0$, then $Qx_n \to 0$. Let now $H_1 = H_2$. Theorem 3: Suppose each sequence $\{z_n\}$ of elements of norm one converging weakly to zero satisfies $\liminf_{n\to\infty} |(L(\lambda_0)z_n, z_n)| > 0$, then $Qx_n \to 0$.

A. Alexiewicz (Poznań)

Nakamura, Masahiro; Yoshida, Midori 2655

On Bückner's inclusion theorems for Hermitean operators.

Proc. Japan Acad. **40** (1964), 180–182.

The authors obtain some inclusion theorems relating the spectrum of a Hermitian operator A to the Schwartz constants $a_n = (A^nu, u)$ of some unit vector u. With the additional hypothesis that A is completely continuous, these theorems appear in H. Bückner [*Survey of numerical analysis*, pp. 439–467, McGraw-Hill, New York, 1962; MR **24** #B1776]. *R. G. Douglas* (Ann Arbor, Mich.)

Schwartz, Jack 2656

Some non-selfadjoint operators.

Comm. Pure Appl. Math. **13** (1960), 609–639.

This paper contains a very interesting study of spectral resolutions of perturbations of self-adjoint operators. The principal feature is the ideal of spectral singularity. It

consists in the idea of so-called exceptional points (or intervals) outside of which the spectral resolution behaves "as usual", i.e., it is countably additive and uniformly bounded. More specifically, the setting is Hilbert space, and the spectral resolution of an operator is, as usual, a family $E(e)$ of projection-valued functions on the Borel sets e of the real line. This function $E(e)$ is given by a Stieltjes inversion formula from the resolvent. General references are N. Dunford and J. T. Schwartz [*Linear operators*, Part I, Interscience, New York, 1958; MR **22** #8302; Part II, 1963; as well as N. Dunford, Bull. Amer. Math. Soc. **64** (1958), 217–274; MR **21** #3616].

The introductory Section 1 deals with the background of the problem and illustrates the occurrence of an isolated single spectral singularity or exceptional point λ_0 by calculating in detail what happens when a bounded self-adjoint operator is perturbed by a one-dimensional operator P: there may exist a point λ_0 such that if $E_1(e)$ is the spectral resolution of $T_1 = T + P$, then (1) $E_1(e)$ is countably additive on the set of all e whose closure does not contain λ_0; (2) $\|E_1(e)\| \leq M < \infty$ for all e whose distance from λ_0 is at least equal to a given positive number a and $M = M(a)$; (3) $\|E_1(e)\| \to \infty$ as a tends to zero. In this case the functional calculus is also worked out.

The second section explores in detail the situation where the self-adjoint operator T acting on $L_2(a, b)$ is just multiplication by the independent variable and is perturbed by an operator P which is the product of two integral operators. Under suitable conditions imposed on the kernels of these integral operators there exists a finite set of exceptional points and a spectral resolution of $T_1 = T + P$ such that the above three statements are true relative to the finite set of exceptional points. The exceptional points are determined explicitly as roots of certain Fredholm determinants. The spectral singularities which arise behave like poles of finite order: they can be "removed" by constructing another set function which is countable additive and uniformly bounded for all e and which is obtained by multiplying $E_1(e)$ by a suitable polynomial of finite degree and then integrating. The completeness of this spectral resolution is explored, and it turns out that if $E_1(e)g = 0$ for all e whose closure fails to meet the set of exceptional points, then g is a linear combination of functions h satisfying an equation $(T_1 - \mu_1)^n = 0$ for some n and μ_1. The foregoing is then generalized and applied to the case where T is the Laplace operator in more than two dimensions and T is a suitable multiplication operator; cases where exceptional points may arise are discussed. Finally, finite-dimensional perturbations are discussed.

The third section deals with an exposition and application of some of Friedrichs' work on perturbation [K. O. Friedrichs, Comm. Pure Appl. Math. **1** (1948), 361–406; MR **10**, 547; Math. Ann. **115** (1938), 249–272]. First a theorem on small perturbations is proved concerning the similarity of the operator T of multiplication by the independent variable on $L_2(a, b)$ and $T + P$, where P is a sufficiently small integral operator; the operator S implementing the similarity is expressed as usual. This is then combined with the work of the preceding section to cover "large" perturbations to the effect that the perturbed operator remains "spectral at infinity", i.e., that it has a spectral resolution which is countably additive outside a sufficiently large bounded interval. This is then again applied to perturbations of the Laplace operator in E_n. The section closes with two theorems combining the foregoing with the earlier work on perturbations of the Laplace operator. The two theorems deal with the spectrum and the spectral resolution. The latter turns out to be countably additive on the spectrum under suitable hypotheses or to present a well-defined finite set of exceptional points.

The next to the last section contains an example, the one-dimensional Laplacian, showing explicitly the occurrence of an exceptional point and otherwise illustrating the preceding work. The closing section discusses the sensitivity of the growth conditions imposed on the perturbation operators involved in the second section and in Friedrichs' method. It is pointed out that a result by M. A. Naĭmark [Uspehi Math. Nauk **8** (1953), no. 4 (56), 174–175; MR **15**, 530] can be obtained by the methods of this paper. *G. K. Kalisch* (Minneapolis, Minn.)

Gel'man, A. E. 2657

Analytic solutions of a class of operator equations. (Russian)

Dokl. Akad. Nauk SSSR **153** (1963), 1234–1237.

The author studies perturbations of linear equations in Banach spaces which have originally non-unique solutions. He states three general theorems concerning the analytic dependence of the solutions of the perturbed equations on a parameter determining the perturbation. He also investigates estimates involving convergence and remainder terms of series solutions. The paper closes with two applications to differential and integral equations.

G. K. Kalisch (Minneapolis, Minn.)

Loginov, B. V. 2658

An estimate of the precision of the method of perturbations. (Russian. Uzbek summary)

Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk **1963**, *no.* 6, 14–20.

The author considers a linear operator A in Hilbert space depending analytically on a complex parameter ε in some neighbourhood of $\varepsilon = 0$:

$$A = A(\varepsilon) = A_0 + B(\varepsilon) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots + \varepsilon^n A_n + \cdots.$$

He considers the cases where the unperturbed operator A_0 is bounded and where it is unbounded. He relaxes certain conditions given in an earlier paper [same Izv. **1963**, no. 5, 21–25]. But for this generalization some price has to be paid: the estimate is less accurate, as is noted by the author. *P. K. Ghosh* (Calcutta)

Sigalov, A. G. 2659

A new algorithm in the theory of perturbations of the continuous spectrum. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 49–52.

These explicit calculations in the perturbation theory of selfadjoint transformations are closely related to the recent announcements of M. Š. Birman and S. B. Èntina [same Dokl. **155** (1964), 506–508; MR **29** #1887] and of S. T. Kuroda [Bull. Amer. Math. Soc. **70** (1964), 556–560], who make a time-dependent calculation of the scattering

operator. See also this reviewer [Amer. J. Math. **84** (1962), 543–560; MR **27** #4083] who does not, however, identify the unitary operator with the scattering operator.

L. de Branges (Lafayette, Ind.)

Stampfli, Joseph G. 2660

Sums of projections.

Duke Math. J. **31** (1964), 455–461.

The author considers the type of linear transformation which may be expressed as a sum of projections. An essential hypothesis is that the underlying space is a Hilbert space. A projection is a bounded transformation E such that $E^2 = E$. Specifically, E need not be self-adjoint. If the space is finite-dimensional, then it is easy to show that if $E_1 + \cdots + E_n = I$ (the identity), it follows that $E_iE_j = 0$, $i \neq j$, that is, the projections are disjoint. In the infinite-dimensional case, this is no longer so. It is first proved that the equation $E_1 + E_2 + E_3 = \alpha I$ is possible precisely for the cases $\alpha = 0, 1, 2, 3, 3/2$. The case $\alpha = 3/2$ requires a special construction. Next, the author shows that $E_1 + \cdots + E_4 = I$ is possible for non-pairwise disjoint projections. Indeed, if T is any tridiagonal operator, then T is the sum of four projections. By a tridiagonal operator is meant one whose matrix representation with respect to an orthonormal set has zero entries except in the main diagonal and the two immediately adjacent "diagonals". The proof, which is based on the piecing together of two special types of projections, is ingenious but requires the careful fitting together of many details. If A is self-adjoint, then A is tridiagonal (if A has a cyclic vector, this is easy to see and the general case follows in a straightforward way). Thus each self-adjoint operator is the sum of four projections. If T is arbitrary, then $T = A_1 + iA_2$, where A_1 and A_2 are self-adjoint, and since iA_2 is tridiagonal (as well as A_2), T is the sum of eight projections.

E. R. Lorch (New York)

Verblunsky, S. 2661

A uniqueness theorem for exponential series.

J. London Math. Soc. **39** (1964), 65–71.

Let $a \leqq v \leqq b$ be an assigned finite interval with $a \leqq 0 \leqq b$ and S a complex Banach space consisting of functions $\varphi(v) \in L(a, b)$ such that (i) the set of all polynomials is everywhere dense in S; (ii) if $\varphi_n \in S$, the uniform convergence of $\{\varphi_n\}$ in (a, b) implies convergence in norm; (iii) if $\{\varphi_n\} \to \varphi$ in norm, then $\int_a^b |\varphi_n - \varphi|\, dv \to 0$. If $A(\varphi)$ denotes a non-null linear functional on S and z a complex parameter, let the entire function $A(e^{zv})$ of z have zeros λ_n (arranged in the order of non-decreasing modulus) of multiplicity m_n and let $O_\nu(\varphi; t)$ denote the linear functional defined by

$$O_\nu(\varphi; t) = \operatorname{res}_{\lambda_\nu} e^{(z-\lambda_\nu)t} A\left(\int_0^v \varphi(u) e^{z(v-u)}\, du\right) \Big/ A(e^{zv}).$$

For each ν let $c_\nu(t)$ denote a polynomial of degree $< m_\nu$. Using properties of the bi-orthogonal system of functionals $O_\nu(r_n(v)e^{\lambda_n v}; t)$ previously established by the author [Quart. J. Math. Oxford Ser. (2) **14** (1963), 1–8; MR **26** #6759] the following uniqueness theorem is proved: If $m_\nu = O(1)$, $\operatorname{Re} \lambda_\nu = O(1)$ and $\sum c_\nu(t) e^{\lambda_\nu t}$ converges for $a < v < b$ to $f(t) \in L(a, b)$, then $c_\nu(t) = O_\nu(f; t)$.

M. V. Subbarao (Edmonton, Alta.)

Kurepa, Svetozar 2662

On operator-roots of an analytic function. (Serbo-Croatian summary)

Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II **18** (1963), 49–51.

Let T be a bounded operator on a Hilbert space, and let $\mathfrak{F}(T)$ denote the set of all functions f which are analytic in some neighborhood of $\sigma(T)$, the spectrum of T. It is shown that if there exists a function f in $\mathfrak{F}(T)$, $f \neq 0$, having simple zeros in the disk $|\lambda| \leqq \|T\|$ and satisfying $f(T) = 0$, then $\sigma(T)$ is a finite set and T is similar to a normal operator.

C. R. Putnam (Lafayette, Ind.)

Kurepa, Svetozar 2663

On ergodic elements in Banach algebras. (Serbo-Croatian summary)

Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II **18** (1963), 43–47.

Let B be a Banach algebra with unit element. An element $a \in B$ is said to be ergodic if the sequence

$$s_n(a) = (1/n) \sum_{k=0}^{n-1} a^k$$

converges in B to an element $s(a)$ as $n \to \infty$. Consider in B the discrete parameter semigroup $\{a^n : n = 0, 1, 2, \cdots\}$ and the continuous parameter semigroup $\{a(t), t \geqq 0\}$. A continuous parameter semigroup is said to be ergodic if $s = \lim_{T\to\infty} (1/T) \int_0^T a(t)\, dt$ exists. The main result of this note can be stated as follows: A semigroup of operators in B is ergodic if it "passes through" an ergodic element. In the case of a continuous parameter semigroup, "passes through" means $a(t_0) \in B$ is ergodic for at least one $t_0 > 0$, and in the case of a discrete parameter semigroup $a \in B$ is an ergodic element if a^p is ergodic for some natural number p.

A. T. Bharucha-Reid (Detroit, Mich.)

Greenleaf, Frederick P. 2664

Norm decreasing homomorphisms of group algebras.

Bull. Amer. Math. Soc. **70** (1964), 536–539.

P. J. Cohen [Amer. J. Math. **82** (1960), 213–226; MR **24** #A3232] has completely described the homomorphisms from $L_1(G)$, where G is a locally compact abelian group, into the algebra of complex measures on another such group. Now the author announces the form, when the groups are not (necessarily) abelian, of those special homomorphisms of norm $\leqq 1$. The result follows from a determination of the multiplicative groups in the unit ball of measures on a locally compact group. Proofs will be given in a subsequent paper.

I. Glicksberg (Seattle, Wash.)

Lindberg, John A., Jr. 2665

Factorization of polynomials over Banach algebras.

Trans. Amer. Math. Soc. **112** (1964), 356–368.

Let A be a commutative semi-simple Banach algebra, and let $a(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial over A ($a_0, \cdots, a_n$ belong to A). The modified zero-set $Z(a)$ is the class of pairs (ξ, λ) where $\xi \in \operatorname{Hom}(A, \mathbf{C})$, $\lambda \in \mathbf{C}$, and $\xi(a_0 + a_1\lambda + \cdots + a_n\lambda^n) = 0$ but $\xi(a_0), \cdots, \xi(a_n)$ are not all 0. The main theorem is as follows. Suppose $Z(a)$ has an open subset K which is compact and yet projects onto

Hom$(A, \mathbf{C})$. Then $a(x)$ has a monic polynomial factor $b(x)$ such that $Z(b) = K$ and $Z(a/b) = Z(a) - Z(b)$.

R. Arens (Los Angeles, Calif.)

Mallios, Anastasios 2666

On the spectrum of a topological tensor product of locally convex algebras.

Math. Ann. **154** (1964), 171–180.

The author extends results of J. Tomiyama [see, for example, Tôhoku Math. J. (2) **12** (1960), 147–154; MR **22** #5910] and the reviewer [see, for example, Trans. Amer. Math. Soc. **103** (1962), 525–548; MR **25** #2406] on the spectrum of the tensor product of Banach algebras. Considering the more general class of Q-algebras, the author is able to conclude that the formula relating the spectrum of the tensor product of Banach algebras to the direct product of the individual spectra is extensible to a fairly large class of commutative locally m-convex algebras that are not necessarily normed algebras. Furthermore, the results are shown to be valid for any topology that is imposed on the tensor product of two algebras so long as this topology satisfies compatibility conditions relating tensorial and topological notions. Extensions of this work to the case where the algebras in question are special kinds of Q-algebras are promised.

B. R. Gelbaum (Irvine, Calif.)

Nussbaum, A. E. 2667

On the reduction of C^*-algebras.

Proc. Amer. Math. Soc. **15** (1964), 567–573.

In this paper, the author studies the decomposition theory of *-representations of a separable C^*-algebra over its center into irreducible ones. The main theorem is stated as follows. Theorem 3: Let R be a C^*-algebra with identity on a Hilbert space and cyclic vector x, $\|x\| = 1$, and E a diagonal algebra of R. Then there exists an isometric isomorphism of $\mathfrak{H}$ onto a topological direct integral ${}^s\!\int_T \oplus \mathfrak{H}(t)\, d\nu(t)$ such that if we identify $\mathfrak{H}$ with $\int_T \oplus \mathfrak{H}(t)\, d\nu(t)$, then (a) E is the diagonal algebra of $\int_T \oplus \mathfrak{H}(t)\, d\nu(t)$; (b) $R = \int_T \oplus R(t)\, d\nu(t)$, where $R(t)$ is irreducible for all $t \in T$; (c) S contains the cyclic vector x, and for every $t \in T$, $x(t)$ is a cyclic vector for $R(t)$ and $\|x(t)\| = 1$; (d) for every $t_1 \neq t_2$ in T there exists at least one $A \in R$ such that $(A(t_1)x(t_1)|x(t_1)) \neq (A(t_2)x(t_2)|x(t_2))$.

The author gives complete proofs to the statements (a) and (b). Their essential ideas are as follows: D, as the spectrum of the W^*-algebra E, is a hyper-Stone space, and ρ is a normal measure on D whose support is D. Then a subset M of D is nowhere dense if and only if M is a ρ-null set. Now let $R = \int_D \oplus R(t)\, d\rho(t)$ be the decomposition of R. $R(t)$ is irreducible for all $t \in D - N$; here $\rho(N) = 0$ and $\bar{N}$ is a ρ-null set, where $\bar{N}$ is the closure of N. Put $T = D - \bar{N}$; then a decomposition $R = \int_T \oplus R(t)$ satisfies the statements (a) and (b).

On the other hand, the author asserts that the other statements are obvious. However, the statement (d) is not obvious; in fact, it is false in the locally compact space T as above. For example, let E be a maximal abelian W^*-algebra on a separable Hilbert space and R a uniformly separable C^*-subalgebra with identity of E such that the weak closure of $R = E$. In this case, $\bar{N} = (\varnothing)$, so that $T = D$. If the statement (d) is true, the spectrum of R is also D, a contradiction. The statement corresponding to (d) is the essential part of Tomita's theorem [Math. J. Okayama Univ. **3** (1954), 147–173; MR **15**, 968]. In fact, other statements were already proved by Mautner, Segal and Godement [Mautner, Ann. of Math. (2) **51** (1950), 1–25; MR **11**, 324; Segal, Mem. Amer. Math. Soc. No. 9 (1951); MR **13**, 472; Godement, Ann. of Math. (2) **53** (1951), 68–124; MR **12**, 421]. Moreover, it has the following important meaning (compare with the theorems of Choquet and Bishop and de Leeuw [Bishop and de Leeuw, Ann. Inst. Fourier (Grenoble) **9** (1959), 305–331; MR **22** #4945]): Let R be a separable C^*-algebra with unit, S the state space, V the set of all extreme points of S, and φ a positive linear functional on R. Then there is a positive Radon measure ν on the compact space S such that $\varphi = \int_S \psi\, d\nu(\psi)$ and $\nu(S) = \nu(V)$. Therefore it is an interesting question whether or not (d) in the author's theorem is positive in such situations that it can give a further refinement to Tomita's theorem.

Various applications of Theorem 3 are given.

S. Sakai (Philadelphia, Pa.)

Takesaki, Masamichi 2668

On the cross-norm of the direct product of C^*-algebras.

Tôhoku Math. J. (2) **16** (1964), 111–122.

Let $\mathfrak{A}_1 \odot \mathfrak{A}_2$ denote the algebraic direct product of two C^*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$. A norm β on $\mathfrak{A}_1 \odot \mathfrak{A}_2$ is said to be compatible if the completion of $\mathfrak{A}_1 \odot \mathfrak{A}_2$ by β is a C^*-algebra and $\|x_1 \otimes x_2\|_\beta \leqq \|x_1\| \cdot \|x_2\|$, for $x_1 \in \mathfrak{A}_1$, $x_2 \in \mathfrak{A}_2$. Turumaru [cf. same J. (2) **4** (1952), 242–251; MR **14**, 991] has introduced a compatible norm in $\mathfrak{A}_1 \odot \mathfrak{A}_2$ which we henceforth call the T-cross norm. In this paper the author proves that the T-cross norm is smallest among the compatible norms. He then proceeds to examine the conditions under which the T-cross norm is the unique compatible norm. For this purpose we say a C^*-algebra $\mathfrak{A}_1$ has property (T) if, for every C^*-algebra $\mathfrak{A}_2$, the T-cross norm of $\mathfrak{A}_1 \odot \mathfrak{A}_2$ is the unique compatible norm. The author proves that (1) every type I C^*-algebra has property (T), (2) there exist C^*-algebras having property (T) which are not of type I, and (3) there exist C^*-algebras which do not have property (T).

J. Ernest (Rochester, N.Y.)

Browder, A.; Wermer, J. 2669

A method for constructing Dirichlet algebras.

Proc. Amer. Math. Soc. **15** (1964), 546–552.

The authors present methods for constructing Dirichlet algebras with unexpected properties. For example, they find a Dirichlet algebra properly a subalgebra of A_0, the functions continuous on the disc which are holomorphic on the interior. A by-product of these investigations is this generalization of Weierstrass's approximation theorem: Let Γ be the unit circle $\{|z| = 1\}$, and let $w: \Gamma \to \Gamma$ be an orientation-reversing homeomorphism. Then any continuous function on Γ can be uniformly approximated by linear combinations of $\{\cdots, w^3, w^2, w, 1, z, z^2, z^3, \cdots\}$.

R. Arens (Los Angeles, Calif.)

Wermer, J. 2670

Approximation on a disk.

Math. Ann. **155** (1964), 331–333.

Let f be a continuous complex-valued function on the

unit disk D of the complex plane. The author gives a simple but ingenious argument which shows that if f has a representation of the form $f(z)=\bar{z}+R(z)$, where $|R(z_1)-R(z_2)|<|z_1-z_2|$ for all z_1, z_2 in D, then every continuous function on D can be uniformly approximated by polynomials in the functions z and $f(z)$. This elegant result is specialized to give the following corollary. If g is a function on D with continuous partial derivatives of the second order for which $\partial g(0)/\partial\bar{z}\neq 0$, then there exists a closed neighborhood of 0 in D on which every continuous function is approximable by polynomials in the functions z and $g(z)$. *E. A. Bishop* (Berkeley, Calif.)

Forster, Otto 2671

Primärzerlegung in Steinschen Algebren.

Math. Ann. **154** (1964), 307–329.

The $\mathscr{C}$-algebra of all holomorphic functions on a holomorphically complete space can be given in a canonical way the structure of a topological $\mathscr{C}$-algebra. A topological $\mathscr{C}$-algebra A that is isomorphic to such an algebra of holomorphic functions is called a Stein algebra. The category of Stein algebras is easily seen to be closed under countable direct products. Also, $A/\mathfrak{a}$, where $\mathfrak{a}$ is a closed ideal in A, is a Stein algebra. Every closed primary ideal $\mathfrak{y}$ in A satisfies $\sqrt{\mathfrak{y}}=\operatorname{rad}\mathfrak{y}$, and $(\sqrt{\mathfrak{y}})^k=\mathfrak{y}$ for some k; in addition, for every closed prime ideal $\mathfrak{z}$ in A, $\mathfrak{z}=\operatorname{rad}\mathfrak{z}$ holds. In A, the ascending [descending] chain condition is valid for closed primary [closed prime] ideals. The main theorem reads: Every closed ideal $\mathfrak{a}$ in a Stein algebra can be written as a locally finite, irredundant intersection of closed primary ideals $\mathfrak{y}_j$ whose associated prime ideals $\mathfrak{z}_j=\sqrt{\mathfrak{y}_j}$ are mutually distinct. The $\mathfrak{z}_j$'s, as well as the isolated ideals of $\mathfrak{a}$, are uniquely determined by $\mathfrak{a}$. As a consequence, each Stein algebra is isomorphic to some closed subalgebra of a countable direct product of Stein algebras in each of which the zero ideal is primary. Moreover, given the locally finite set M of closed primary ideals in A, Hilbert's Nullstellensatz is valid for $\bigcap\{\mathfrak{y}:\mathfrak{y}\in M\}$ if and only if there exists an integer k such that for every $a\in\bigcap\{\mathfrak{y}:\mathfrak{y}\in M\}$, $a^k\in\operatorname{rad}\bigcap\{\mathfrak{y}:\mathfrak{y}\in M\}$ holds. In particular, if Hilbert's Nullstellensatz is valid for the closed ideal $\mathfrak{a}$, then it is valid with an exponent that depends only on $\mathfrak{a}$ and not on the element in question. A few remarks about Stein modules conclude the paper.

H. Röhrl (Minneapolis, Minn.)

Friedman, Avner 2672

★Generalized functions and partial differential equations.

Prentice-Hall, Inc., Englewood Cliffs, N.J., 1963. xii + 340 *pp.* $10.00.

The author presents for the first time in book form in English results (up to 1961) of the theory of generalized functions in the area of partial differential equations. The basic sources were the two volumes of L. Schwartz [*Théorie des distributions*, Tome I, Actualités Sci. Indust., No. 1091, Hermann, Paris, 1950; MR **12**, 31; Tome II, Actualités Sci. Indust., No. 1122, Hermann, Paris, 1951; MR **12**, 833] and the first three volumes of I. M. Gel'fand and the reviewer [*Generalized functions*, Part 1, Fizmatgiz, Moscow, 1958; MR **20** #4182; Part 2, 1958; MR **21** #5142a; Part 3, 1958; MR **21** #5142b]; in what follows the last three volumes will be referred to as GS-I, GS-II, and GS-III, respectively. The author also makes use of some journal articles, in particular, his own.

A short summary by chapters, of which there are eleven, is as follows. Chapter I. Linear topological spaces. Countably normed and complete spaces. Countable unions (with the inductive limit topology and only with convergence without topology).

Chapter II. Spaces of fundamental and generalized functions, in particular, the spaces $K(M_p)$ and $Z(M_p)$ introduced in GS-II, 2. Sufficient conditions for the completion of these spaces (i.e., compactness of bounded sets). The author corrects an error in GS-II, p. 141, by imposing on the functions $M_p(x)$ the additional condition that for every p there exists a $p'>p$ such that whenever $|x'-x|\leqq 1$, the inequality $M_p(x)/M_{p'}(x')\leqq C_p$ holds.

Chapter III. Theory of distributions (i.e., generalized functions over the special spaces $\mathscr{D}$ and S of L. Schwartz); the general form of distributions, tensor products, convolutions, the Fourier transform. The author introduces the spaces $K_r(M_p)$ with norm

$$\|\varphi\|_{pr}=\sup_{|\alpha|\leq p}\left[\int_{R_n}M_p(x)|\mathscr{D}^\alpha\varphi(x)|^r\,dx\right]^{1/r};$$

for $r=\infty$ we obtain the space $K(M_p)$, for $M_p(x)\equiv 1$ the space $\mathscr{D}_{L^r}$ of L. Schwartz (the question of the conditions for the coincidence of all $K_r(M_p)$ for fixed $M_p(x)$ and r varying from 1 to ∞ apparently remains open).

Chapter IV. The Fourier transform and the convolution of generalized functions over arbitrary base spaces.

Chapter V. Theorems on analytic functions (GS-II, 4, § 7). If $|f(z)|\leqq C_1e^{b_1|z|^p}$ for $z\in C^n$ and $|f(x)|\leqq C_2e^{b_2|x|^h}$ for $x\in R^n$, then there exists a domain $H_\mu=\{z:|y_j|\leqq C_3(1+|x_j|)^\mu\}$, $\mu=1-(p-h)$, $j=1,\cdots,n$, in which $|f(z)|\leqq C_3e^{b_3|z|^h}$. {Reviewer's remark: It was shown by E. A. Gorin [Uspehi Mat. Nauk **16** (1961), no. 1 (97), 91–118; MR **24** #A1269] that such an inequality holds in the domain $\{z:|y|\leqq C_4(1+|x|)^\mu\}$.} If $f(z)$ is a function of one variable of order p such that $|f(z)|\leqq C_3e^{b_3|z|^h}$ in H_μ, then

$$|f(z)|\leqq C_4e^{b_4|x|^h+b_5|y|^{p/\mu}}$$

for all $z\in C^1$. The author indicates that this theorem cannot be extended to the case of functions of several variables for the domains $H_\mu=\{x:|y_j|\leqq C_3(1+|x_j|)^\mu$, $j=1,\cdots,n\}$. {Reviewer's remark: However, this theorem is always valid if H_μ is the domain $\{z:|y|\leqq C_4(1+|x|)^\mu\}$.} The author also discusses the spaces W introduced by V. L. Gurevič (cf. GS-III, 1).

Chapter VI. Theorems on Fourier transforms of finite order of growth with various assumptions about the behaviour in a real domain (slow or rapid increase or decrease as $|x|\to\infty$). Certain results (for example, for the Fourier transform of a function of order $p>1$ with slow growth in a real domain) belong to the author. {Reviewer's remark: A systematic account of these and more general questions has been given from a unified point of view by V. P. Palamodov [Trudy Moskov. Mat. Obšč. **11** (1962), 309–350; MR **28** #4401]. The result concerning the resolution of the identity in the space S^β on p. 154 follows from an old result of S. Mandelbrojt [J. École Polytechn. (2) **32** (1934), 227–277].}

Chapter VII. The Cauchy problem for systems of linear partial differential equations. A comparison with GS-III, 2–3, shows that some results have been sharpened. In particular, it is shown that ordinary functions $u(t,x)$ which are solutions of the system $\partial u/\partial t=P(i\partial/\partial x)u$ in the sense of generalized functions are solutions in the classical

sense. At the same time the initial conditions are fulfilled, as before, only in the sense of generalized functions. {Reviewer's remark: There are interesting papers of M. V. Fedorjuk which clarify the asymptotic behaviour of the Green's matrix as $t \to 0$, $|x| \to \infty$ [Dokl. Akad. Nauk SSSR **132** (1960), 63–66; MR **23** #A1942; ibid. **134** (1960), 1027–1029; MR **23** #A1139].} Certain existence theorems for the case of several space variables are given correctly in a more circumspect formulation than in GS-III, considering the remark mentioned about Chapter V.

Chapter VIII. Exposition of results of the author [J. Math. Mech. **11** (1962), 859–889; MR **26** #4045] concerning the Cauchy problem for an equation with several time variables of the form

$$\frac{\partial u}{\partial t_j} = \sum_{k=1}^{\nu} P_{jk}\left(t, i\frac{\partial}{\partial x}\right)u_k(x, t),$$

$$\frac{\partial^\nu u(x, t)}{\partial t_1 \cdots \partial t_\nu} = P\left(i\frac{\partial}{\partial x}\right)u(x, t).$$

Chapter IX. Spaces of type S [GS-II, 4]. {Reviewer's remark: The equalities (3.11)–(3.13) are incorrect; one should replace them by an inclusion of the type $\subset$. Indeed, for $\alpha = 0$ and $\beta = 0$ equality prevails but without the additional factor $\exp(1/AB)$ [cf. the reviewer, Izv. Vysš. Učebn. Zaved. Matematika **1961**, no. 3 (22), 174–176; MR **29** #5055]. These equalities are used by the author in the following chapter.}

Chapter X. Further applications to differential equations; a Phragmén-Lindelöf type theorem [GS-III, 2], a theorem of Liouville type [the reviewer, Izv. Vysš. Učebn. Zaved. Matematika **1961**, no. 4 (23), 137–147; MR **25** #2443], the theorem of Hörmander, Malgrange and Ehrenpreis on the existence of fundamental solutions, and the method of Radon. {Reviewer's remark: The calculation (1.11) perpetuates the error of GS-II, 2; the derivation is valid under the additional hypothesis that the matrix $P(\sigma)$ commutes with its derivatives.}

Chapter XI. The theory of hypo-ellipticity of Hörmander and the solution of equations with a free term.

The brevity and systematic character are the real merits of the book in comparison with GS-II and GS-III.

G. E. Shilov (Moscow)

Kac, I. S. 2673

Correction to I. S. Kac's paper "Spectral multiplicity of a second-order differential operator and expansion in eigenfunctions". (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 951–952.

A number of misprints in the author's earlier article [same Izv. **27** (1963), 1081–1112; MR **28** #3196] are corrected. *R. C. Gilbert* (Fullerton, Calif.)

Khoan, Vo-Khac 2674

Q-solutions faibles d'équations différentielles opérationnelles non linéaires.

C. R. Acad. Sci. Paris **258** (1964), 5569–5572.

Les notations non définies sont celles des notes précédentes [mêmes C. R. **257** (1963), 3800–3803; MR **28** #1480; ibid. **258** (1964), 47–50; MR **28** #4257; ibid. **258** (1964), 3161–3164; MR **29** #499; ibid. **258** (1964), 3430–3433; MR **29** #500]. H est un espace de Hilbert séparable réel, avec le produit scalaire $\langle 1 \rangle$. A est un opérateur linéaire de H, autoadjoint, positif. B est un opérateur non linéaire de H tel que $Bu = \beta\|u\|^\sigma u$, $\beta, \sigma > 0$, $f(t) \in \tilde{Q}(0, \infty; H)$ et a une $\mathscr{B}^2$-dérivée $\in Q(0, \infty; H)$. $\mathscr{A}(f; \mathscr{P}, H)$ est l'espace engendré par les translatées de f. $\mathfrak{M}$ est l'opérateur de moyenne temporelle $\lim_{T\to\infty} (1/T)\int_0^T \varphi(t)\,dt$.

On cherche une fonction $u(t)$ possédant une $\mathscr{B}^2$-dérivée $u'(t)$ telle que $A^{1/2}u$ et $A^{1/2}u' \in \mathscr{A}(f; \mathscr{P}, H)$ et qui soit solution faible de l'équation différentielle opérationnelle

$$u' + Au + Bu = f,$$

c'est-à-dire, telle que

$$\mathfrak{M}\{\langle u'|\Phi\rangle + \langle A^{1/2}u|A^{1/2}\Phi\rangle + \langle Bu|\Phi\rangle\} = \mathfrak{M}\langle f|\Phi\rangle,$$

pour toute fonction Φ telle que $A^{1/2}\Phi \in \mathscr{A}(f; \mathscr{P}, H)$.

On démontre l'unicité des solutions éventuelles, et, à l'aide de la méthode de Galerkine, leur existence.

J. Bass (Paris)

Khoan, Vo-Khac 2675

Q-solutions fortes d'équations différentielles opérationnelles non linéaires.

C. R. Acad. Sci. Paris **258** (1964), 6323–6326.

Les notations non définies sont celles des notes précédentes (voir les citations dans #2674 ci-dessus). H est un espace de Hilbert, muni du produit scalaire $\langle 1 \rangle$ et de la norme $\|\ \|$. A est un opérateur linéaire dans H, auto-adjoint, de domaine $\mathscr{D}(A)$, tel que $A \geqq \alpha I$, $\alpha > 0$. B est un opérateur non linéaire dérivant d'un opérateur p-linéaire $\mathscr{L}$ (soumis à quelques restrictions) par la formule $Bh = \mathscr{L}(h, \cdots, h)$. $C(0, \infty; H)$ est l'espace des fonctions définies sur $\overline{R^+}$, à valeurs dans H, bornées et continues, normé par $|||f||| = \sup_t \|f(t)\|$. $f(t)$ est une $\tilde{Q}$-fonction appartenant à $C(0, \infty; H)$ ainsi que sa dérivée. Ses translatées engendrent l'espace $\mathscr{A}(f; \mathscr{P}, H)$.

Après quelques lemmes, on considère d'abord l'équation différentielle opérationnelle linéaire

$$u' = Au + f,$$

où $u(0) \in \mathscr{D}(A)$ est donné.

Ce problème admet une solution et une seule, ayant les propriétés suivantes: (a) $u(t)$ est une application continue de $\overline{R^+}$ dans $\mathscr{D}(A)$, une fois continûment différentiable de $\overline{R^+}$ dans H; (b) $u'(t)$ et $A^\theta u(t) \in \mathscr{A}(f; \mathscr{P}, H)$, $0 \leqq \theta \leqq 1$.

Cette solution admet une représentation explicite

$$u = G(t)u(0) + \int_0^t G(t-s)f(s)\,ds,$$

où G est un noyau de Green. Cela permet ensuite de transformer l'équation non linéaire $u' = -Au + Bu + f$ (où $u(0) \in \mathscr{D}(A)$ est donné) en une équation intégrale opérationnelle. On en cherche une solution telle que $A^\theta u \in \mathscr{A}(f; \mathscr{P}, H) \cap C(0, \infty; H)$ pour tout θ $(0 \leqq \theta < 1)$. On démontre un théorème d'unicité, puis par approximations successives, un théorème d'existence valable si $u(0)$ et $f(t)$ sont "assez petits". *J. Bass* (Paris)

Seeley, R. T. 2676

Extension of C^∞ functions defined in a half space.

Proc. Amer. Math. Soc. **15** (1964), 625–626.

Si Ω est un ouvert de R^m, et $W_p{}^k(\Omega)$ un espace de Sobolev d'ordre k sur Ω (relativement à $L_p(\Omega)$) on peut, lorsque la frontière de Ω est assez régulière (le problème de la

régularité minimum n'est pas ici l'objet de l'auteur), construire un prolongement de $W_p^k(\Omega)$ à $W_p^k(R^m)$, i.e., un opérateur linéaire continu, soit π, du premier espace dans le second, tel que $\pi u = u$ presque partout sur Ω, pour tout $u \in W_p^k(\Omega)$. L'auteur montre que si $\Omega = R_x^n \times \{t > 0\}$, on peut trouver un π valable pour tout k (et tout p); la construction élégante est la suivante: si f est définie dans Ω. on définit πf pour $t < 0$ par

$$\pi f(x, t) = \sum_{k=0}^{\infty} a_k \phi(b_k t) f(x, b_k t),$$

où ϕ est C^∞ dans $t \geqq 0$, $\phi(t) = 1$ si $t \in [0, 1]$, $\phi(t) = 0$ si $t \geqq 2$, et où les a_k et b_k sont choisis de façon que (l'auteur montre que c'est possible) $b_k < 0$, $\sum |a_k| \, |b_k|^n < \infty$ pour tout entier $n \geqq 0$, $\sum a_k b_k^n = 1$ pour tout entier $n \geqq 0$, et $b_k \to -\infty$.

J. L. Lions (Paris)

Skaček, B. Ja. 2677

Asymptotic behaviour of the negative part of the spectrum of higher-order singular differential operators. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 14–17.

Author's summary: "Let Ω be an n-dimensional Euclidean space and L an operator defined in $L_2(\Omega)$ by the operation

$$l(u) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(p\frac{\partial u}{\partial x_i}\right) - qu.$$

By means of variational methods an asymptotic formula is obtained for the number $m(\varepsilon)$ of eigenvalues of the operator L, lying to the left of ε $(\varepsilon > 0)$.

Markovskiĭ, A. I. 2678

On the domain of values of a maximal differential operator in $L^2[0, \infty)$. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1130–1131.

Solvability conditions in $L^2(0, \infty)$ are determined for an ordinary non-homogeneous linear differential equation with constant coefficients.

H. P. Thielman (Alexandria, Va.)

Schreiber, M. 2679

Remark on a paper of Kalisch.

J. Math. Anal. Appl. **7** (1963), 62–63.

The reviewer proved [same J. **5** (1962), 176–183; MR **25** #4307] (we use the same notations and definitions in this review) that Titchmarsh's theorem on convolution and the theorem of the unicellularity of the basic Volterra operator S on $L_2(0, 1)$ are equivalent. The author simplifies and unifies the reviewer's proofs.

G. K. Kalisch (Minneapolis, Minn.)

Sahnovič, L. A. 2680

Spectral analysis of Volterra operators prescribed in the vector-function space $L_m^2[0, l]$. (Russian)

Ukrain. Mat. Ž. **16** (1964), 259–268.

In the present paper the author extends the results of his previous paper [Izv. Akad. Nauk SSSR Ser. Mat. **21** (1957), 235–262; MR **19**, 970] by significant relaxation in the conditions on the kernel treated there. He establishes a new theorem which facilitates investigation of a system of differential equations associated with such kernels.

P. K. Ghosh (Calcutta)

Ckirija, K. E. 2681

The conditional extremum of a functional in Hilbert space for a continual number of relations. (Georgian. Russian summary)

Soobšč. Akad. Nauk Gruzin. SSR **31** (1963), 529–535.

The author considers the problem of finding functions $x(s)$ and $\lambda(s)$ in L^2 satisfying the Euler-Lagrange equation

$$L(x(s)) = \int_0^1 L(t, x(s))\lambda(t)\, dt,$$

where $L(x) = \operatorname{grad} f(x) \in L^2$, $L(t, x) = \operatorname{grad} \varphi(t, x)$, and $\int_0^1 \varphi^2(t, x)\, dt < \infty$.

Hellwig, Günter 2682

★Differentialoperatoren der mathematischen Physik. Eine Einführung.

Springer-Verlag, Berlin, 1964. xi + 253 *pp.* *DM* 36.00.

This book is a comprehensive introduction to the general theory of self-adjoint operators in Hilbert space, and to the application of this theory to the spectral analysis of second-order ordinary and partial differential operators. Topics: (1) Geometry of Hilbert space, orthonormal expansions; (2) Generalities about linear operators, semi-boundedness of linear operators, applications to the ordinary second-order operator, the Laplacian in an arbitrary domain, and the Schrödinger operator in n-dimensional space (including singular potentials); (3) Eigenfunctions and eigenvalues of completely continuous operators, application to the ordinary second-order operator and the Laplacian in bounded n-dimensional domains; (4) General spectral theory of self-adjoint operators, essential self-adjointness, proofs of essential self-adjointness for various Schrödinger operators, minimax principle; (5) The Weyl boundary-value theory of the singular ordinary second-order operator; criteria for the limit-point and limit-circle case.

The exposition is pleasant and readable throughout. The author has been at pains to treat those points of the theory of special interest to physicists, and this book should be of particular use to physicists wishing to inform themselves about the methods and results of the abstract Hilbert space theory.

J. T. Schwartz (New York)

CALCULUS OF VARIATIONS

See also 2364, 2681, 3298, 3304, 3307, 3316a–c.

Tokarev, P. I. 2683

An invariant form of the Jacobi equations in the case of a centroplanar indicatrix for the simplest Lagrange variational problem. (Russian. Kazak summary)

Izv. Akad. Nauk Kazah. SSR Ser. Fiz.-Mat. Nauk **1963**, *no.* 3, 17–20.

In his previous work [Trudy Sem. Vektor. Tenzor. Anal. **9** (1952), 431–455; MR **14**, 1124] the author derived the expression for the second variation of the simplest

Lagrange variational problem (call it I). In the present note the author derives the Jacobi equation for the problem described in the title. To achieve this the author begins with the variational principle corresponding to I, with a complementary condition expressible in the form of a differential equation for the function sought. The problem reduces to the Bolza problem and practically to the evaluation of the coefficients appearing in the resulting equations. The Jacobi equation is expressed in terms of a system of linear ordinary differential equations. Some remarks on the subject of the relation between the Euler and Jacobi equations close the note.

M. Z. v. Krzywoblocki (E. Lansing, Mich.)

Nauryzbaev, K. Ž. 2684

A variational method for an equation with constant coefficients. (Russian)

Izv. Akad. Nauk Kazah. SSR Ser. Fiz.-Mat. Nauk **1963**, *no.* 3, 21–25.

The author begins with a variational principle applied to a functional in n-dimensional space involving the expression $I=\sum \alpha_{kl}(x)f^{(k)}(x)f^{(l)}(x)$ with $x=(x_1,\cdots,x_n)$. The expression I is bounded. In the next step the author considers Fourier transforms, and using Parseval's formula, constructs an integral involving an expression analogous to (I) with Fourier transforms in place of functions $f^{(k)}$ (call it II). After discussing some properties of (II), there is proposed a correlation between (I) and (II) by means of a straightforward formula $I=II$ plus the remaining functional. To this the author applies classical considerations of variational analysis, demonstrating such properties as boundedness, existence of a limit, uniqueness, etc. By a proper substitution it is demonstrated that the variational principle involving (I) is equivalent to an operator $Lu\equiv\sum\alpha_{kl}(-1)^{|l|}u^{k+l}=O(III)$, $\alpha_{kl}=$ constants, thus proving that solutions of (I) give solutions of (III). The equation (III) may be of an elliptic, hyperelliptic or partially hyperelliptic type. No particular application of the proposed technique is provided.

M. Z. v. Krzywoblocki (E. Lansing, Mich.)

Al'ber, S. I. 2685

Higher-dimensional problems in global variational calculus. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 727–730.

Es sei Ω ein beschränkter Bereich in R^n mit der Berandung S.

$$(1)\quad J(u)=\int_\Omega F(x,u,u_x)\,dx,\qquad u|_S=\varphi,\ u\in W_m{}^1(\Omega);$$

F genügt gewissen Regularitätsbedingungen. $u\in W_m{}^1(\Omega)$, für welches die erste Variation $\delta J(u,\eta)=0$ für alle $\overset{0}{W}_m{}^1(\Omega)$, heisst stationäre Lösung (s.L.) von (1). Es sei

$$(2)\quad \frac{\partial u}{\partial t}=\frac{\partial}{\partial x_i}F_{p_i}-F_u,\qquad u|_S=\varphi,\ u|_{t=0}=\psi;\ \varphi,\psi\in C^{2,\alpha}.$$

Es werden interessante Sätze angekündigt, die besagen, dass falls die Existenz und Stabilisation (für $t\to\infty$) der Trajectorien von (2) gesichert ist, dann existieren s.L. entsprechender Variationsaufgabe (1). Anwendungen: (1) Es sei M^n ein Riemannscher Raum negativer Krümmung und $x:K_2\to M^n$ sei geschlossene Fläche der Klasse $C^{2,\alpha}$. Falls u eine beliebige Fläche der Klasse $C^{2,\alpha}$ mit derselben Randbedingung: $x|_\Gamma=y|_\Gamma$ ist, wobei

$$\max_{t\in K}\|y(t)-x(t)\|<\rho_0,$$

dann ist $D(y)-D(x)>0$ (D, Dirichletsches Integral). D.h., es gibt bloss eine einzige harmonische Fläche in einer ρ_0-Umgebung von x. (2) Räume geschlossener Flächen eines fixierten topologischen Typus im kompakten Riemannschen Raum negativer Krümmung bestehen aus homotop-trivialen Komponenten. In jeder Komponente gibt es genau eine einzige harmonische und einzige minimale geschlossene Fläche.

K. Maurin (Warsaw)

GEOMETRY

See also 2160, 2225, 2244, 2625, 2718, 2724, 2726, 2733.

Kline, Morris 2686

Geometry.

Sci. Amer. **211** (1964), 60–69.

An expository article on the nature of geometry.

Delessert, André 2687

Une construction de la géométrie élémentaire fondée sur la notion de réflexion.

Enseignement Math. (2) **10** (1964), 1–124.

The author presents an axiomatic development of Euclidean geometry over any real field which contains the square root of each of its positive elements. It is a group-theoretic approach, similar to that of F. Bachmann [*Aufbau der Geometrie aus dem Spiegelungsbegriff*, Springer, Berlin, 1959; MR **21** #6557]. Its scope is smaller than the work of Bachmann in the sense that it does not include the elliptic and hyperbolic geometries. On the other hand, the author does not confine himself to the plane (as Bachmann does), but, after a treatment of the plane case, extends the development to n dimensions.

Intended as a text in the foundations of Euclidean geometry, the work proceeds smoothly and lucidly to construct the framework of the system. Seven axioms are used to construct the geometry in the plane and another two to generalize to n dimensions. The starting point is the introduction of a group G generated by a set Σ of involutory elements. The object, of course, is to place restrictions on G and Σ in such a way that G becomes the group of isometries of the desired geometry and the elements of Σ become reflections in lines of this geometry. Axioms are introduced as needed, with the relevant consequences of each one given before another appears. Any algebraic or topological concept which is needed is explained in an appendix, making the work self-contained.

A noteworthy feature is a section in which the author examines his first seven axioms. To illustrate the independence of the axioms, he gives, for each k from 1 to 6, an example of a geometrical system which satisfies the first k axioms but not the next.

This work is carefully done in a thoroughly enjoyable style. It could be used with great profit as the basis for a rigorous course in the foundations of geometry.

F. A. Sherk (Toronto, Ont.)

Pejas, W. 2688
Eine algebraische Beschreibung der angeordneten Ebenen mit nichteuklidischer Metrik.
Math. Z. **83** (1964), 434–457.
Das im Titel erwähnte Problem wurde schon vom Verfasser behandelt in einer früheren Arbeit [Math. Ann. **143** (1961), 212–235; MR **25** #4393] unter der zusätzlichen Bedingung der freihen Beweglichkeit. Jetzt wird diese beseitigt. Jede metrisch-nichteuklidische Ebene T im Sinne des Axiomensystems von Bachmann [*Aufbau der Geometrie aus dem Spiegelungsbegriff*, Springer, Berlin, 1959; MR **21** #6557] lässt sich bekanntlich in eine ordinäre projektiv-metrische Ebene E über einem kommutativen Körper K einbetten. Ist T angeordnet, so besitzt K eine Anordnung, bezüglich welcher T konvex ist, und umgekehrt.

Die Frage ist nun, in den ordinären projektiv-metrischen Ebenen über angeordneten Körpern die konvexen metrischen Teilebenen algebraisch zu beschreiben. Es sei E gegeben mittels eines Vektorraums über K mit nichtsingulärer symmetrischer Bilinearform F. c_{AB} bezeichne das Quadrat des Kosinus der von den projektiven Punkten A, B gebildeten Winkel im Sinne von F, R_0 den Unterring der ganzzählig-einschliessbaren Elemente von K, und P_0 das maximale Ideal von R_0. Die vollständige Beschreibung aller T wird gewonnen mittels Unterringe R von K, die zugleich R_0-Moduln sind und abgeschlossen gegen Multiplikation mit $(1+\{\alpha\})^{-1}$, wo $\{\alpha\}$ die Formwert eines Punktes A von T ist. Es gibt drei Fälle (X bezeichne die Punkte von T): (a) $R\subset P_0$, $c_{AX}\in 1+R$; (b) $R\supset R_0$, $\alpha>0$, $c_{AX}\in 1+R_+$ (positive Elemente von R); (c) $R\supset R_0$, $\alpha<0$, $\{\alpha\}$ enthält keine Einheit von R, und $c_{AX}\in 1-P_0$ oder $c_{AX}\in 1+R_+$. *G. J. Schellekens* (Utrecht)

Guggenheimer, H. 2689
Topology and elementary geometry. II. Symmetries.
Proc. Amer. Math. Soc. **15** (1964), 164–173.
Im Anschluss an eine frühere Arbeit [Math. Japon. **5** (1958/59), 51–57; MR **23** #A561] formuliert der Verfasser ein Axiomensystem ebener absoluter Geometrie, das sich von dem Hilbertschen (Hilberts Axiomen I 1–3, II, III) dadurch unterscheidet, dass die in den Anordnungsaxiomen (Axiomgruppe II) enthaltenen topologischen Forderungen verschärft und die Kongruenzaxiome (Axiomgruppe III) durch die folgenden Forderungen ersetzt sind: Zu jeder Geraden gibt es eine Spiegelung (einen involutorischen Automorphismus, der die Gerade punktweise festlässt), und es gilt das Hjelmslevsche Axiom "Ein Spiegelungsprodukt, welches eine Gerade punktweise festlässt, ist die Identität oder eine Spiegelung". Das entscheidende über Hilbert hinausgehende Axiom II 5 "Die Halbebenen-Topologie ist vollständig" bedarf einer Präzisierung, da offenbar nicht eine beliebige Uniformisierung der Halbebenen-Topologie zugelassen werden soll. Sowohl unter Voraussetzung eines euklidischen als eines hyperbolischen Parallelenaxioms (IV E bzw. IV H) gewinnt der Verfasser, indem er durch ein Produkt von Spiegelungen ineinander überführbare Mengen kongruent nennt, die in den Hilbertschen Kongruenzaxiomen enthaltenen Aussagen. Ein Axiom V soll schliesslich zur Kennzeichnung der (reellen) euklidischen und hyperbolischen Ebene führen. Ohne weitere Begründung wird in dem abschliessenden Theorem behauptet, dass das Axiom "Die Menge der Punkte hat die Mächtigkeit des Kontinuums" dies leistet. *F. Bachmann* (Kiel)

Iwata, Shikō 2690
Polygons in the complex plane. (Japanese summary)
Sci. Rep. Fac. Lib. Arts Ed. Gifu Univ. Natur. Sci. **3** (1962/63), 161–167.

Mitrinović, D. S. 2691
A simple procedure for the determination of the axes of symmetry and metrical elements of the conics.
Boll. Un. Mat. Ital. (3) **19** (1964), 208–215.

Schröder, Eberhard 2692
Überdeckungskurven.
Wiss. Z. Techn. Univ. Dresden **12** (1963), 1613–1626.
Die vorliegende Arbeit erläutert zuerst einen von H. Steinhaus [*Kaleidoskop der Mathematik*, S. 315, VEB Deutscher Verlag Wiss., Berlin, 1959; MR **22** #7918] bemerkten optischen Effekt, der bei der Betrachtung zweier (mit gleichen oder verschiedenen) Winkelgeschwindigkeiten um eine Achse rotierender windschiefer Stäbe bei Beobachtung gegen einen dunklen Hintergrund entsteht. Die Netzhaut des Auges registriert eine dunkle Kurve k, die sich aus den unterschiedlichen Überdeckungszeiten der Punkte des dunklen Hintergrundes durch die helleren rotierenden Stäbe ergibt. Idealisiert man die Stäbe durch zwei rotierende Geraden e_1, e_2 und legt man an diese Geraden aus dem festen Auge 0 die Treffgeraden $t=[e_1, e_2]$, so ist die Kurve k der Ort der Treffpunkte (Überdeckungspunkte) $T_1=[te_1]$, $T_2=[te_2]$, und besteht daher aus zwei "Überdeckungskurven" $k_1=\{T_1\}$ und $k_2=\{T_2\}$, die auf den von den Geraden g_1 und g_2 erzeugten einschaligen Drehhyperboloiden liegen. Von diesen Überdeckungskurven werden (für allgemeine Drehbewegungen) mit Hilfe der Formeln von Cayley mittels des Chaslesschen Korrespondenzprinzips die charakteristischen Zahlen ermittelt und für spezielle Lagen des Auges 0 mehrere Sonderfälle studiert.

Sodann wird eine zur Drehachse normale Bildebene π eingeführt und die (gemeinsame) Projektion k' der beiden Überdeckungskurven k_1 und k_2 auf π aus dem Auge 0 studiert. Die charakteristischen Zahlen von k' werden mittels der Plückerschen Formeln hergeleitet. Es zeigt sich, daß man durch diesen Mechanismus viele bekannte rationale zirkulare ebene Kurven vierter Ordnung erhalten kann.

Hält man die Bildebene π fest und ändert man das Auge 0, so erhält man eine Abbildung der Raumpunkte 0 auf ein System von Kurven k' der Ebene π. Die Treffgeraden t bilden dabei einen Strahlkomplex, dessen charakteristische Größen untersucht werden. Stehen die Drehgeschwindigkeiten der Geraden e_1, e_2 in rationalem Verhältnis, so erhält man einen algebraischen Strahlkomplex. Als Sonderfall kann man auch einen der acht zirkularen quadratischen Komplexe vom Referenten [Akad. Wiss. Wien Math.-Natur. Kl. S.-B. IIa **145** (1936), 657–680; ibid. **147** (1938), 37–47] erhalten, nämlich einen tetraedralen Komplex der Segreschen Elementarteilercharakteristik [(11)(11)(11)]. *K. Strubecker* (Karlsruhe)

Brennan, J. G. 2693
Another generalization of Pascal's theorem.
J. London Math. Soc. **39** (1964), 449–450.
Consider in S_n a quadric primal V_{n-1}^2 and a simplex with

vertices A_i none of which lie on the quadric. Let a_i be the prime face of the simplex opposite A_i. Through A_i there pass n edges of the simplex each of which meets the quadric in a pair of points. On each such line choose one such point and let b_i be defined by the n points chosen, each point serving to define only one prime. There thus results a second simplex with prime faces b_i and opposite vertices B_i. Each simplex is derivable from the other in the same fashion. The generalization in question asserts that the $n+1$ intersections $a_i \cap b_i$ are associated, i.e., any line which meets any n of them also meets the $(n+1)$st.

If $n=2$, Pascal's theorem results, and for $n=3$ a theorem of Chasles. *L. M. Kelly* (Cambridge, England)

Derry, Douglas 2694a
Sugli spazi osculatori dei poligoni d'ordine n di L_n.
Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **33** (1962), 253–259.

Derry, Douglas 2694b
Sui poligoni aperti d'ordine n in uno spazio proiettivo reale di dimensione n. II.
Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **33** (1962), 405–411.

Derry, Douglas 2694c
Approssimazione di una curva di L_n d'ordine n con un poligono dello stesso ordine.
Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **34** (1963), 511–516.

Let L_m denote an m-flat in projective n-space L_n. The flat spanned by the union of the points or sets $E_1, E_2, \cdots$ is denoted by $[E_1, E_2, \cdots]$. The symbol AB denotes one of the two closed segments bounded by A and B. The polygon $A_1A_2\cdots A_r$ has the sides $A_1A_2, \cdots, A_{r-1}A_r$. The point $B\in L_{n-1}\cap A_iA_{i+1}$ is called a point of intersection of L_{n-1} with $A_1\cdots A_r$ if $B=L_{n-1}\cap A_iA_{i+1}$ or $B=A_i$ or $B=A_{i+1}$. A polygon π_n of order n in L_n spans L_n and intersects no L_{n-1} more than n times. If $\pi_n=A_1\cdots A_r$ is open, i.e., $A_r\neq A_1$, the segment A_rA_1 can be chosen in one and only one way such that the closed polygon $A_1\cdots A_rA_1$ is of order n.

Let (1) $\pi_n=A_1\cdots A_rA_1$ be closed. Define $A_{i+k}A_{i-1}$ such that $A_{i-1}A_i\cdots A_{i+k}A_{i-1}$ becomes a π_{k+1}. Then $L_k=L_{k,i}$ is called an osculating space if $L_k\cap \operatorname{int}(A_{i+k}A_{i-1})=\varnothing$ and

$$[A_i, A_{i+1}, \cdots, A_{i+k-1}] \subset L_k \subset [A_{i-1}, A_i, \cdots, A_{i+k-1}, A_{i+k}];\quad 0<k<n$$

(indices are computed (mod r)). The set of all the intersections $\bigcap_{i=r-n+1}^{r+1} L_{n-1,i}$ is an open n simplex $S(A_r, A_1)$ with the vertices

$$P_i = [A_1, A_2, \cdots, A_i]\cap[A_r, A_{r-1}, \cdots, A_{r-n-1+i}],\quad i=1,2,\cdots,n+1.$$

Let $\pi_{n-1}=B_1B_2\cdots B_{r-1}B_1\subset L_{n-1}$ be closed. Choose any segment B_nA_{n+1} with $A_{n+1}\notin L_{n-1}$. Define $A_{n+1}B_{n+1}$ such that B_nB_{n+1}, B_nA_{n+1}, and $A_{n+1}B_{n+1}$ form an odd triangle. If the segment A_iB_i has been defined, any point $A_{i+1}\in\operatorname{int}(A_iB_i)$ defines a subsegment $A_{i+1}B_i$. Define $A_{i+1}B_{i+1}$ such that the triangle $A_{i+1}B_i$, B_iB_{i+1}, $A_{i+1}B_{i+1}$ is even; $i=n+1, \cdots, r-1$. Then $B_1B_2\cdots B_nA_{n+1}\cdots A_rB_1$ is a closed π_n, and every closed π_n can be obtained this way. The vertices of $S(A_rB_1)$ are those of $S(B_{r-1}B_1)$ together with A_r.

Given (1), then $A_1\cdots A_rA_{r+1}\cdots A_{r+k}A_1$ has the order n if and only if $P_1P_2\cdots P_{n+1}A_{r+1}\cdots A_{r+k}P_1$ has that order. Here each segment P_iP_{i+1} is chosen outside $S(A_rA_1)$. This result makes it possible to construct the most general closed π_n containing a given open π_n.

Let C_n denote a differentiable closed curve of real order n in L_n. Thus C_n is the homeomorphic image of a circle and meets no L_{n-1} in more than n points; if any $k+1$ mutually distinct points $s_1, \cdots, s_{k+1}$ of C_n converge to a point s, then $[s_1, \cdots, s_{k+1}]$ converges to the "osculating k-space" (k, s) of C_n at s, $0\leqq k<n$. If an arc of C_n bounded by s_1 and s_2 and the segment s_1s_2 form an even curve, this segment is said to be inscribed in C_n. This way, a closed polygon can be inscribed in C_n. Studying the extensions of such a polygon, the author proves: Suppose the points $s_1, s_2, \cdots, s_p$ lie on C_n. Then there exists a polygon (1) with p mutually disjoint subpolygons $A_{m_i}A_{m_i+1}\cdots A_{m_i+n-1}$ such that

$$(k, s_i) = [A_{m_i}, A_{m_i+1}, \cdots, A_{m_i+k}],\quad 0\leqq k<n,\ 1\leqq i\leqq p.$$

By means of this result and an earlier one on the π_n's [Ann. Mat. Pura Appl. (4) **54** (1961), 285–293; MR **24** #A2280], the following 30-year old conjecture is proved: No L_{n-2} meets more than $2(n-1)$ tangents $(1, s)$ of a C_n. Actually, the intersections $(1, s)\cap L_{n-2}$ can be provided with multiplicities: Let L_{n-2} be fixed, $s\in C_n$. Let $e(s)$ denote the largest k such that $[L_{n-2}, (k, s)]\neq L_n$. Let $j(s)$ be the smallest k such that $(k, s)\not\subset L_{n-2}$ ($j(s)=0$ if $s\notin L_{n-2}$). Then $\sum_s (e(s)+j(s))\leqq 2(n-1)$.
P. Scherk (Toronto, Ont.)

Heyting, A. 2695
★Projectieve meetkunde [Projective geometry].
P. Noordhoff N.V., Groningen, 1964. viii+126 *pp.* *Dfl.* 13.90; $3.90.

This introductory textbook is primarily intended for use in a first course in geometry for students preparing for the Dutch state examinations for high school teachers. The theory is based on linear algebra, but synthetic and algebraic methods are used in elegant interplay. Subjects treated are projective spaces, coordinates, duality in the plane, cross ratios, harmonic relationship, Desargues's theorem, the real plane, projectivities, fixpoints, involutions, correlations, conics, Pascal's theorem, polarity, affine and metric geometry. The final chapter treats bundles of conics. The pace is slow and utmost rigor is observed. The text contains some 50 figures and well over 200 exercises. *G. J. Schellekens* (Utrecht)

Perron, Oskar 2696
Seiten und Diagonalen eines Kreisvierecks in der hyperbolischen Geometrie.
Math. Z. **84** (1964), 88–92.

Es seien a, b, c, d die Längen der Seiten, und e, f die Längen der Diagonalen eines Kreisvierecks. In der Euklidischen Geometrie gelten die Formeln $e^2(ad+bc)=(ab+cd)(ac+bd)$, $f^2(ab+cd)=(ad+bc)(ac+bd)$ und der

Satz von Ptolemaeus $ac+bd=ef$. Der Verfasser beweist dieselben Formeln in der hyperbolischen Geometrie für die Grössen sinh $(a/2), \cdots,$ sinh $(f/2)$ statt $a, \cdots, f$.

G. J. Schellekens (Utrecht)

Segre, Beniamino 2697

Arithmetische Eigenschaften von Galois-Räumen. I.

Math. Ann. **154** (1964), 195–256.

Con questa memoria dei Math. Annalen e con le altre due pubblicate negli [Ann. Mat. Pura Appl. (4) **64** (1964), 1–76] e nelle [Math. Notae **19** (1964), 1–10], l'autore fornisce una brillante ed interessante esposizione degli ultimi risultati delle sue ricerche sui legami tra le teoria algebriche e geometriche [per i risultati precedenti vedasi il volume: *Lectures on modern geometry*, Edizioni Cremonese, Rome, 1961; MR **24** #A1045].

Come l'autore dichiara nella introduzione, in questa memoria (per la cui traduzione in lingua tedesca Egli ringrazia W. Groebner) si occupa soprattutto di problemi di carattere numerico ed aritmetico relativi alle varietà algebriche di uno spazio di Galois $S_{r,q}$, ossia di uno spazio proiettivo di dimensione r su di un corpo finito di dimensione q.

Il lavoro è diviso in tre parti. Nella parte I, dedicata alle equazioni algebriche in una variabile, l'autore si occupa dapprima (§ 1) del carattere quadratico del discriminante D di un polinomio $f(x)$ di grado n sopra un corpo γ, tanto nel caso che la caratteristica p di γ sia 2, oppure $\neq 2$. Nel caso $p\neq 0$, si ritrova per altra via un teorema di L. Carlitz [Simon Stevin **31** (1956), 27–30; MR **18**, 285]. Nel § 2 l'autore considera i casi singolari per il determinante Hessiano di una forma binaria e fa vedere come in un corpo di caratteristica $p\neq 0$ ogni equazione algebrica di grado p, il cui gruppo Hessiano svanisca identicamente $(a=0)$ o si riduca ad un unico punto (contato $2p-4$ volte), si possa con una sostituzione lineare (che manda quel punto all' ∞) trasformare in una equazione trinomia del tipo (10): $x^p=ax+b$. Nel § 3 si esamina in modo particolare il caso in cui il corpo γ è finito e di ordine $q=p^h$, determinando una condizione necessaria e sufficiente perchè la (10) non abbia soluzioni in γ. Se invece ne ha, puo averne una $(p\neq 2)$ oppure p, che possono esprimersi razionalmente mediante b, una radice $(p-1)$-esima di a ed una arbitraria delle p radici che ha in γ l'equazione fissa (nella incognita k):

$$k^{p^{h-1}}+k^{p^{h-2}}+\cdots+k^p+1 = 0.$$

Nel § 4 si esamina il caso in cui il corpo finito γ sia di caratteristica $p=2$ e perciò di ordine $q=2^h$. Allora gli elementi del corpo γ ed i polinomi $f(x)$ su γ si suddividono in due categorie distinte e non vuote, che per i polinomi irriducibili in γ sono quelle dei polinomi di grado dispari o di grado pari. Se invece $f(x)$ è riducibile (e a discriminante non nullo) appartiene alla 1ª od alla 2ª categoria se tra i suoi fattori irriducibili ve n'è un numero pari o dispari di grado pari. Fra l'altro se ne deduce che, anche se $p=2$, un polinomio su γ a discriminante non nullo si spezza, in una qualsiasi estensione quadratica di γ, in un prodotto di fattori irriducibili, tutti di grado dispari. Comunque le cosiderazioni precedenti si applicano nel § 5 alla deduzione di alcune proprietà delle equazioni algebriche su un corpo finito γ, in forma indipendente dalla caratteristica. In particolare (se $p\neq 3$) per un polinomio cubico $f(x)$ a discriminante non nullo, se x_1, x_2, x_3 sono i suoi zeri e z_1, z_2 quelli del determinante hessiano $h(x)$ di $f(x)$, questi 5 numeri distinti si ripartiscono in γ ed in alcune sue estensioni quadratiche o cubiche in quattro modi possibili, secondo che $q\equiv 1 \pmod 3$, oppure $q\equiv 2 \pmod 3$, oppure z_1, z_2 appartengano o no a γ. Da questo teorema si deduce un procedimento per il calcolo effettivo di x_1, x_2, x_3 che ha molte analogie con le classiche formule di Cardano. Nel § 6 l'autore si occupa diffusamente delle equazioni biquadratiche $f(x)=0$ a discriminante non nulle e della distribuzione in γ delle radici di una equazione siffatta e di quelle della sua risolvente cubica $\varphi(\lambda)$, con particolare riferimento al caso equianarmonico.

Nella parte II, dedicata alle equazioni omogenee di grado n, del tipo $f(x_1, \cdots, x_r)=g(y_1, \cdots, y_s)$, l'autore si occupa dapprima in generale (§ 7) del numero N delle soluzioni di un'equazione siffatta (determinate a meno di un fattore non nullo) che cadono nel corpo finito γ di ordine $q=nt+1$ dei coefficienti. Le formule ottenute vengono applicate nel § 8 al caso particolare in cui la forma f si riduce ad una costante $k\in\gamma$ e, in una estensione di ordine n di γ, la forma g si spezzi nel prodotto di n forme lineari coniugate e linearmente indipendenti. Si passa poi al caso in cui f è una forma arbitraria su γ, ottenendo per N una espressione in cui figura come parametro il numero F dei punti di $S_{r,q}$ giacenti sulla forma $f=0$. Nel § 9 si considera la superficie cubica $f(x_1, x_2)=f(y_1, y_2)$ dell' $S_{3,q}=\{(x_1, x_2, y_1, y_2)\}$. Quando $(q-1, 3)=1$ e la superficie non è singolare, il numero N dei punti (con le coordinate in γ) che giacciono su di essa è dato da $N=q^2+q[(F-1)^2+1]+1$, dove $F=0, 1, 2, 3$ è il numero degli zeri, contenuti in γ, della forma f, ciascuno contato una sola volta. Se il discriminante D della f è $\neq 0$, e $q\equiv 1 \pmod 3$, la formula precedente va sostituita con l'altra: $N=q^2+q[(F-1)^2+3]+1$.

Infine l'autore fa vedere nel § 10 come dalle precedenti considerazioni si possa dedurre (limitatamente ai valori 0, 1 del genere g) una semplice dimostrazione di un teorema di H. Hasse [Convegno Sci. Fis. Mat. Natur. [1939], pp. 85–140, Reale Accad. Italia, Rome, 1943; MR **12**, 47] ed A. Weil [*Sur les courbes algébriques et les variétés qui s'en déduisent*, Actualités Sci. Indust., No. 1041, Hermann, Paris, 1948; MR **10**, 262] su una limitazione superiore del numero dei punti appartenenti ad una curva algebrica assolutamente irriducibile su di un corpo finito.

Nella parte III si studiano gli stessi problemi già affrontati nella parte II, nel caso particolare che le forme f, g siano combinazioni lineari di potenze n-esime. Sfruttando un algoritmo opportuno, introdotto nel § 11, si studiano successivamente nei §§ 12, 13, 14, 15 i casi in cui $n=2, 3, 4$ ed $n\geqq 5$. È praticamente impossibile riassumere in breve i numerosi risultati conseguiti dall'autore, che ne illustra anche i collegamenti con altre teorie, cosicchè sembra preferibile lasciarli degustare al Lettore interessato.

E. Morgantini (Padova)

Keller, Ott-Heinrich 2698

Ein neuer Beweis eines Satzes von Salmon über Kurven 3. Ordnung mit Hilfe elliptischer Funktionen.

Math. Z. **84** (1964), 277–278.

Eine ebene Kubik k vom Geschlecht 1 besitzt bekanntlich eine Parameterdarstellung mit elliptischen Funktionen. Aus dem Parameter u des Punktes P auf k ergeben sich dann die Parameter der Berührungspunkte der 4 von P aus an k gelegten Tangenten in der Form $-\frac{1}{2}u$,

$\frac{1}{2}(-u+\omega_1)$, $\frac{1}{2}(-u+\omega_2)$, $\frac{1}{2}(-u+\omega_1+\omega_2)$. Diese 4 Tangenten sind stets voneinander verschieden, d.h. ihr Doppelverhältnis nimmt niemals die Werte 0, 1, ∞ an. Da dies Doppelverhältnis anderseits eine elliptische Funktion ist, muß es nach Liouville konstant sein, womit der Salmonsche Satz bewiesen ist. *W. Burau* (Hamburg)

Baur, Arnold 2699
Rationale Punkte auf Kurven dritter Ordnung vom Geschlecht Eins.
Math.-Phys. Semesterber. **10** (1963/64), 241–251.

K sei eine ebene Kubik vom Geschlecht 1 über dem rationalen Zahlkörper mit dem rationalen Punkt R. Dann ist bekanntlich auch der Tangentialpunkt T von R, rational, und man erhält so eine Kette von rationalen Punkten, die endlich oder unendlich sein kann und nur bei einem Wendepunkt aufhört. Der Verfasser nennt K Zwillingskurve, wenn es auf ihr einen rationalen Punkt T gibt, der Tangentialpunkt von 2 rationalen Punkten ist; gibt es sogar 4 rationale Tangenten von T aus, so heißt die Zwillingskurve Quadrupelkurve. Ebene Kubiken, die keine Zwillingskurven sind, nennt er Einfachkurven. Als Grenzfall gehören auch die Kubiken mit einem rationalen Wendepunkt zu den Zwillingskurven. Diese lassen sich auf die Normalgestalt

$$ax^2y+by^2+2cxy+dx+ey = 0$$

mit rationalen Koeffizienten transformieren. Der Verfasser beweist, daß auf ihnen alle rationalen Punkte paarweise auftreten. Im allgemeinen besitzt die Zwillingskurve bei $ae-bd\neq 0$ 6 rationale Punkte. Der Verfasser gibt Beispiele für Zwillingskurven mit 1, 3, 4, 8, 9 und unendlich vielen rationalen Punkten. Es folgen dann Beispiele von Quadrupelkurven, die geschlossene Ketten von je 12, 16 und 26 rationalen Punkten besitzen. Bei Einfachkurven ist die Frage offen, ob es solche mit einer geschlossenen Kette von endlich vielen rationalen Punkten gibt. Doch gibt es Einfachkurven mit unendlich vielen rationalen Punkten. Hinreichend ist dafür die Existenz eines rationalen Anfangspunktes R_0 auf K mit 4 irrationalen Tangenten durch R_0 an K.
W. Burau (Hamburg)

Reich, Ludwig 2700
Beiträge zur algebraischen Geometrie. I.
Österreich. Akad. Wiss. Math.-Natur. Kl. S.-B. II **172** (1963), 1–42.

The following theorem is proved. Let D be any finite set of lines, conics, and twisted cubics in three-dimensional space, over an algebraically closed field of characteristic zero. Let P be a finite set of points such that (i) every curve in the set D contains at least one of the points P, and (ii) P includes every point common to two (or more) curves of D. Then there exists an algebraic surface which meets every curve of D in the points P lying on it, and in no other points.

The proof is essentially elementary, though complicated. The idea is to construct, for each component of D, a surface containing the remaining components and meeting the component in question in the points P lying on it, and nowhere else, and then to form a suitable linear combination of powers of the forms determining these surfaces. The complications come from the numerous relationships that can exist between two curves of the set D. *J. A. Todd* (Cambridge, England)

Ostrom, T. G. 2701
Planar loops as direct sums.
Arch. Math. **14** (1963), 369–372.

Let π be an arbitrary affine plane, O a fixed point on π; call O-lines the lines through O. In a previous paper [same Arch. **12** (1961), 151–158; MR **24** #A2871] the author introduced an addition, $P+Q=Q+P=R$, between points P and Q not lying on the same O-line by use of the "parallelogram law" for the "vectors" OP and OQ; he called an algebraic structure $\mathfrak{L}(+)$ of this kind a "planar half-loop", and he characterized such structures (associated with an affine plane) by a complete set of postulates. If the addition is defined for each pair of points, the author calls $\mathfrak{L}(+)$ a "planar loop" (a planar loop need not be abelian, but if $P+Q\neq Q+P$, then P and Q must lie on the same O-line). In the present paper the author investigates a finite planar loop $\mathfrak{L}$ which is the direct sum of two loops $\mathfrak{R}$ and $\mathfrak{S}$; that is, (1) each element of $\mathfrak{L}$ can be written uniquely in the form $R+S$, $R\in\mathfrak{R}$, $S\in\mathfrak{S}$; (2) $(\bar{R}+\bar{S})+(R+S)=(\bar{R}+R)+(\bar{S}+S)=(\bar{R}+S)+(R+\bar{S})$; $R,\bar{R}\in\mathfrak{R}$; $S,\bar{S}\in\mathfrak{S}$. The principal results of the investigation are: Theorem 1: If $\mathfrak{L}$ is the direct sum of $\mathfrak{R}$ and $\mathfrak{S}$, where the orders of $\mathfrak{L}$, $\mathfrak{R}$, $\mathfrak{S}$ are, respectively, n^2, r, s, and if $s\leqq n$, then $\mathfrak{S}$ is associative; Theorem 2: If $\mathfrak{S}$ contains points on more than one O-line of $\mathfrak{L}$, then $\mathfrak{L}$ is power-associative and every element of $\mathfrak{L}$ has the same order. *L. Lombardo-Radice* (Rome)

Crowe, D. W. 2702
The trigonometry of $GF(2^{2n})$ and finite hyperbolic planes.
Mathematika **11** (1964), 83–88.

Let $\mathrm{GF}(q^2)$ denote the field having $q^2=2^{2n}$ elements. The automorphism $z\to\bar{z}=z^q$ defines a conjugate in this field. The real elements r for which $r=\bar{r}$ constitute a subfield, called the real subfield, which can be denoted by $\mathrm{GF}(q)$. If c is an arbitrary element of $\mathrm{GF}(q^2)$ and r is an arbitrary real element of $\mathrm{GF}(q^2)$, then the set of elements $z\in\mathrm{GF}(q^2)$ for which $(z-c)(\overline{z-c})=r^2$ is called a circle with centre c and radius r. The points of the unit circle, $z\bar{z}=1$, are readily verified to be the $q+1$ distinct powers of $\delta=\lambda^{q-1}$, where λ is a generator of the multiplicative group $\mathrm{GF}^*(q^2)$ of $\mathrm{GF}(q^2)$.

In the present note it is shown that the elements of $\mathrm{GF}(q^2)$ can be represented in "polar form" in such a way that $\mathrm{GF}(q^2)$ acts like an "Argand diagram" over its "real subfield" $\mathrm{GF}(q)$. From this polar representation it is easy to develop a trigonometry of the plane $\mathrm{GF}(q^2)$, including definitions of circles and orthogonality. As an application of these ideas it is shown that the circles and lines orthogonal to a circle yield a new model satisfying Graves's axioms for finite homogeneous hyperbolic planes.
R. G. Stanton (Waterloo, Ont.)

Lüneburg, Heinz 2703
Endliche projektive Ebenen von Lenz-Barlotti Typ I-6.
Abh. Math. Sem. Univ. Hamburg **27** (1964), 75–79.

A projective plane is (P, g)-transitive for given point P and line g if its collineation group contains all possible central collineations with center P and axis g. A plane is

said to be of Lenz-Barlotti Type I-6 if there is a line r, a point $R \in r$, and a 1-1 mapping μ of the points of $r-\{R\}$ on the lines through R and different from r, such that the plane is (P, g)-transitive precisely when $P \in r-\{R\}$ and $g = P^{\mu}$. The author proves that there is no such finite plane of odd order. *F. A. Sherk* (Toronto, Ont.)

Manolov, S.; Kolarov, D. 2704
A method of dividing a space into regions. (Bulgarian)
Fiz.- Mat. Spis. Bŭlgar. Akad. Nauk. **6 (39)** (1963), 31–37.

The authors solve the problem, which is analogous to a generalisation in their former paper [Annuaire Univ. Sofia Fac. Sci. Phys. Math. Livre 1 **52** (1957/58), 165–181] of a theorem of D. Pompeiu [see S. V. Pavlovič, Mathesis **65** (1956), 211–214; MR **18**, 500]: Let ABC be an arbitrary triangle in a three-dimensional Euclidean space and d_a, d_b, d_c the distances of straight lines BC, CA, AB from any point $M \in E_3$. The task is to determine the set of points M such that with line-segments of lengths d_a, d_b, d_c one can construct a triangle. The authors find (1) conditions for $M \in S$, and (2) the equation of the surface which is a boundary of S. *Z. Nádeník* (Prague)

Fejes Tóth, L. 2705
★Regular figures.
A Pergamon Press Book.
The Macmillan Co., New York, 1964. xi+339 *pp.* (12 *anaglyphs with viewer*) $12.00.

The appeal of symmetry, displayed in a regular polyhedron or a wall frieze pattern, is universal. The mathematics behind the symmetry, while having its roots in antiquity, is by no means exhausted. These are the two main impressions that one is left with after reading this work.

The book is divided into two parts. Part One, entitled "Systematology of the Regular Figures", outlines the classical theory. This is, of course, a group-theoretic approach. The author begins by considering the discrete symmetry groups of the Euclidean plane. All seventeen plane crystallographic groups are illustrated. The three regular and eight semiregular (archimedean) tessellations are introduced. This leads naturally to circle-packings and circle-coverings, which occupy much of the author's attention later on. There follow chapters dealing with regular figures on the sphere and in the hyperbolic plane. The author then turns to Euclidean space of three dimensions, with a survey of regular and semiregular polyhedra, and the parallelohedra. Part One concludes with an introduction to regular polytopes of higher dimensions, especially those of dimension four.

Part Two, "Genetics of the Regular Figures", aims at exhibiting the regular figures as solutions of various extremum problems. To cite the best-known example, the arrangement of circles in the densest close-packing of equal circles in the Euclidean plane is such that the centres are the vertices of the regular tessellation {3, 6}. This part of the book fairly bristles with tantalizing theorems, the statements of which are disarmingly simple, but whose proofs are often long and intricate. The author considers problems of this nature in the Euclidean plane, on the sphere, in the hyperbolic plane, and in Euclidean spaces of three and higher dimensions, thus paralleling the material in Part One. There are, of course, many gaps in this approach; it is not nearly as complete as the classical theory given in Part One. But enough is given to make the approach convincing and attractive. It will surely wet the appetites of geometers for many years to come.

Apart from the selection of the material itself, other noteworthy features of the book are the historical remarks and the many illustrations, which include twelve anaglyphs, complete with a viewer. There is a large bibliography and an adequate index.
F. A. Sherk (Toronto, Ont.)

CONVEX SETS AND GEOMETRIC INEQUALITIES

See also 2705, 2796.

Dubins, Lester; Hirsch, Morris W.; Karush, Jack 2706
Scissor congruence.
Israel J. Math. **1** (1963), 239–247.

Two convex bodies E_+ and E_- in the Euclidean plane P are said to be scissor-congruent provided there exist (for some n) nonoverlapping topological disks $D_1, \cdots, D_n$ and nonoverlapping topological disks $D_{-1}, \cdots, D_{-n}$ such that $E_\pm = \bigcup_{i=1}^n D_{\pm i}$ and D_i is equivalent to D_{-i} under a rigid motion of the plane. Scissor-congruence of the boundaries of convex bodies is similarly defined, involving partitioning of the boundaries into arcs. The authors' principal result asserts that if both E_+ and E_- are strictly convex, then they are scissor-congruent if and only if they have the same area and their respective boundaries are scissor-congruent. (This is in marked contrast to Bolyai's theorem asserting that every convex polygon of unit area can be cut by a finite number of line segments into pieces which can be rearranged to form the unit square.) The proof is based on sixteen lemmas. There is an extension to finite unions of disjoint strictly convex bodies, and a sharpened form for the ellipse, as follows: If an ellipse E_+ is scissor-congruent to a convex body E_-, then there is a rigid motion carrying E_+ onto E_-. It is unknown whether any other plane convex bodies have this property.
Victor Klee (Seattle, Wash.)

Firey, W. J.; Groemer, H. 2707
Convex polyhedra which cover a convex set.
J. London Math. Soc. **39** (1964), 261–266.

The authors prove a number of results concerning the covering of convex sets by polyhedra. One of the most interesting is the following: Let T' be a (not necessarily regular) simplex in Euclidean n-space and C a convex set of mean width β; there exists a simplex T similar to T' covering C such that if r is the radius of the largest sphere in T, $r \leqq \frac{1}{2}\beta$. Another is a sharpening of Borsuk's problem in the plane: Any convex set in the plane of mean width β can be subdivided into three sets of diameter not exceeding $\frac{1}{2}\beta\sqrt{3}$. *John W. Green* (Los Angeles, Calif.)

Bilinski, Stanko 2708
Die primitivste Form des Vierscheitelsatzes. (Serbo-Croatian summary)
Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II **18** (1963), 85–93.

The four-vertex theorem for sufficiently smooth plane convex curves states that the curvature, if it is not constant, has at least four extrema. The author proves the following analogue of this theorem, adapted to polygons, from which the four-vertex theorem for curves follows by a limiting process. Let $A_1, A_2, \cdots, A_n$ be a convex n-gon in the plane, all of whose vertex angles are obtuse. Let ρ_i be the radius of the circle through A_{i-1}, A_i, A_{i+1}, enumerating the A's in cyclic fashion. Then in the cyclic sequence of differences $\rho_2-\rho_1, \rho_3-\rho_2, \cdots, \rho_1-\rho_n$ there are at least four changes of sign unless all differences are zero.

John W. Green (Los Angeles, Calif.)

Post, K. A. 2709

Star extension of plane convex sets.

Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 330–338.

H. Brunn [Math. Ann. **73** (1913), 436–440] (see also H. Hadwiger and H. Debrunner [*Kombinatorische Geometrie in der Ebene*, pp. 30, 78, 79, Inst. Math. Univ. Genève, Geneva, 1960; MR **22** #11310]) proved that the kernel K_S of a plane set S (that is, the subset of all points $p \in S$ such that for all points $q \in S$ the segment $pq \subset S$) is either empty or convex. In this paper the author discusses the following problem related to the indicated theorem. Given a convex set K in the euclidean plane, when does there exist a set $S \supset K$ such that $K_S = K$? If $S \neq K$, $K = K_S$, and also the closures $\bar{S}$, $\bar{K}$ are different, then S is called a proper K-star and the main theorems in the paper are the following. A strictly convex set K with a regular boundary admits a proper K-star if and only if its boundary contains an arc whose intersection with the complement of K is countable. A convex closed set admits a proper K-star if and only if it is neither a half-plane nor a two-way infinite strip.

Z. Nádeník (Prague)

Hammer, Joseph 2710

On a general area-perimeter relation for two-dimensional lattices.

Amer. Math. Monthly **71** (1964), 534–535.

For each convex domain D in E^d, let $V(D)$, $F(D)$, and $n(D)$ denote, respectively, the d-measure of D, the $(d-1)$-measure of the boundary of D, and the number of lattice points in D. For each $\mu > 0$, let $N(d, \mu)$ denote the minimum of $n(D)$ for D such that $V(D)/F(D) > \mu$. Although the author treats a more special problem, his reasoning shows that the function $N(d, \cdot)$ is superadditive, that is, $N(d, \mu_1+\mu_2) \geqq N(d, \mu_1)+N(d, \mu_2)$. In conjunction with a theorem of E. A. Bender [same Monthly **69** (1962), 742–744], this implies that $N(2, r/2) \geqq r$ for each positive integer r.

Victor Klee (Seattle, Wash.)

Rockafellar, R. T. 2711

Duality theorems for convex functions.

Bull. Amer. Math. Soc. **70** (1964), 189–192.

This note announces the following theorem: Let F and G be finite-dimensional partially-ordered real vector spaces in which non-negative cones $P(F)$ and $P(G)$ are polyhedral, let A be a linear transformation from F to G, and let f and g be proper convex and proper concave functions on F and G, respectively; then

$$\inf\{f(x)-g(y) \mid x \geq 0, Ax \geq y\} = \max\{g^*(y^*)-f^*(x^*) \mid y^* \geq 0, A^*y^* \leq x^*\},$$

provided there exists at least one $x \in \mathrm{ri}(\mathrm{dom}\, f)$ such that $x \geqq 0$ and $Ax \geqq y$ for some $y \in \mathrm{ri}(\mathrm{dom}\, g)$. Here $\mathrm{ri}(\mathrm{dom}\, f)$ stands for relative interior of $\{x \mid f(x) < \infty\}$, * for conjugate and adjoint, and proper convex on F means a function f everywhere defined such that $-\infty < f(x)$ for all x, $f(x) < \infty$ for at least one x, and $f(\lambda x_1+(1-\lambda)x_2) \leqq \lambda f(x_1)+(1-\lambda)f(x_2)$ for all x_1 and x_2 when $0<\lambda<1$. The theorem includes several results of convex programming, and an outline of the proof is given.

U. D'Ambrosio (Providence, R.I.)

Klee, Victor 2712

Extreme points of convex sets without completeness of the scalar field.

Mathematika **11** (1964), 59–63.

Let E be a finite-dimensional vector space over an ordered field $\mathfrak{F}$. The author solves the problem of describing a class $\mathscr{K}_E$ of convex subsets of E satisfying the following conditions: (a) The description of $\mathscr{K}_E$ is independent of the scalar field; (b) if $K \in \mathscr{K}_E$ and K is linearly bounded (the intersection with each line is contained in a segment), then $K = \mathrm{con\ ex}\ K$ (the convex hull of the extreme points); (c) when $\mathfrak{F}$ is the real field, $\mathscr{K}_E$ is the class of closed convex subsets of E; (d) for arbitrary F, $\mathscr{K}_E$ includes all convex polytopes in E. A subset K of E is called segmentally closed provided the intersection of K with an arbitrary closed segment is itself a closed segment. The author proves that if K is a segmentally closed, linearly bounded subset of E, then $K = \mathrm{con\ ex}\ K$. That (a) and (c) hold is readily proved for the collection of segmentally closed, convex sets. A ray R is called an extreme ray of the convex set K if and only if $R \subset K$ and R contains every (closed) segment $[u, v]$ such that $[u, v] \subset K$ and the (open) interval $]u, v[$ intersects R. The principal result is the theorem that if K is a segmentally closed subset of E that contains no line, then K is the convex hull of the union of its extreme points and extreme rays. This theorem shows that segmental closedness in place of linear closedness suffices for all the reasoning of the author [J. Math. Anal. Appl. 7 (1963), 425–427; MR **27** #6188], and in particular, that if K is n-dimensional, segmentally closed and contains no line, then each point of K is a convex combination of some $n+1$ extreme points of K or is a convex combination of n points of K, each an extreme point or in an extreme ray of K.

E. Hemmingsen (Syracuse, N.Y.)

Skorobogat'ko, V. Ja.; Bobik, O. I. 2713

On the sphere of maximum radius inscribed in a given region. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1567–1570.

Authors' summary: "A method is outlined for determining a maximum sphere inscribed in a given region."

Fréchet, Maurice 2714

Les définitions de la somme et du produit scalaires en termes de distance dans un espace abstrait.

Calcutta Math. Soc. Golden Jubilee Commemoration Vol. (1958/59), *Part I, pp.* 151–160. *Calcutta Math. Soc., Calcutta*, 1963.

A preliminary exposition of results discussed in more detail in another paper [the author, Ann. Sci. École Norm. Sup. (3) **75** (1958), 223–255; MR **21** #4394].

F. Smithies (Cambridge, England)

DIFFERENTIAL GEOMETRY

See also 2325, 2383, 2406, 2505, 2685, 2694a–c, 2708, 2815, 2869, 2870, 3225.

Strubecker, Karl 2715

★Differentialgeometrie. Band I: Kurventheorie der Ebene und des Raumes.

Zweite, erweiterte Auflage. Sammlung Göschen, Bänder 1113/1113a.

Walter de Gruyter & Co., Berlin, 1964. 253 *pp.* *DM* 5.80.

The first edition (1955) was reviewed in MR **16**, 954. From the first to the second edition the text has increased by more than 100 pages. The general tenor of the book has not been changed: A very careful and detailed treatment of curves in two and three dimensions, with many examples. There is a heavy emphasis on complex geometry and isotropic curves, both in the plane and in space. This is probably the most complete book on the subject published in recent years and should appeal in particular to engineering students. The drawings are beautiful.

H. W. Guggenheimer (Minneapolis, Minn.)

Bompiani, E. 2716

Some extensions of Meusnier and Euler theorems.

Tensor (*N.S.*) **14** (1963), 268–276.

Let O be a fixed point in euclidean 3-space. A cap σ^k [a curvilinear differential element E^k] at O is an equivalence class of surfaces [of curves] which have contact of order k with each other at O. Both σ^k and E^k can be described by the canonical expansion of representatives.

$$\sigma^k : z = \sum_2^k \frac{1}{n!} \varphi_n(x, y) + \cdots,$$

where $\varphi_n(x, y)$ is a homogeneous form of degree n, $\varphi_2(x, y) = R_1^{-1}x^2 + R_2^{-1}y^2$.

$$E^k: Y = \sum_2^k \frac{A_n}{n!} X^n + \cdots, \qquad Z = -\sum_3^k \frac{B_n}{n!} X^n + \cdots,$$

where $A_2 = \rho$, $A_3 = \dot{\rho}$, $B_3 = \rho\tau$, $\cdots$.

If $E^k \in \sigma^{k'}$, the orthogonal transformation from the (x, y, z)- to the (X, Y, Z)-coordinate system can be expressed in terms of the angles between the x- and X-axes and the z- and Z-axes. This yields certain relations between these angles and the coefficients of the canonical expansions. These relations are computed in the case $k = k' = 4$.

Given σ^3 and $E^2 \in \sigma^3$. The $E^3 \in \sigma^3$ with $E^2 \subset E^3$ form a one-parametric family which is studied in detail. The paper also touches upon the two-parametric family of the $E^3 \in \sigma^3$ containing a given $E^1 \in \sigma^3$, i.e., with a given tangent.

P. Scherk (Toronto, Ont.)

do Carmo, Manfredo Perdigão 2717

★Geometria diferencial local [Local differential geometry].

Textos de Matemática, No. 12.

Instituto de Física e Matemática, Universidade do Recife, Recife, 1963. 128 *pp.*

A very classical treatment of local curves and surfaces in three-space, about to the extent the same topic is treated in Willmore's book [*An introduction to differential geometry*, Clarendon, Oxford, 1959; MR **28** #2482].

H. W. Guggenheimer (Minneapolis, Minn.)

Penna, Anna Maria 2718

Quadriche di rotazione osculatrici ad una curva sghemba.

Univ. e Politec. Torino Rend. Sem. Mat. **22** (1962/63), 111–113 (1964).

A quadric of revolution is constructed which has contact of fourth order with a curve in space in a generic point and which has a given direction for its axis of revolution. The two special cases of a paraboloid and of a parabolical cylinder are discussed (axis at infinity).

A. Aeppli (Minneapolis, Minn.)

Benneton, Gaston 2719

Sur les formes différentielles extérieures liées aux surfaces.

C. R. Acad. Sci. Paris **259** (1964), 42–44.

Here it is shown that the fundamental formulas for a surface in 3-dimensional euclidean space, such as the equations of Gauss and Codazzi, Meusnier's theorem, Enneper's formula, and so on, can be represented briefly by the use of vectorial differential forms and determinants in which multiplications are exterior in a certain order.

M. Kurita (Nagoya)

Rembs, Eduard 2720

Über Paare von Parallelflächen.

Math. Nachr. **27** (1963/64), 299–301.

The author proves the following theorem. Let H and K be, respectively, the mean curvature and the Gaussian curvature of a surface S in 3-dimensional euclidean space. When $c = (H^2 - K)/HK$ is a constant, the surface parallel to S in the distance c has Gaussian curvature H^2. This surface S is a Weingarten surface, and so the equations of Codazzi can be integrated, which is actually done. Finally, the principal curvatures of the surface S, which is, moreover, a surface of rotation, are given.

M. Kurita (Nagoya)

Ripianu, Dumitru 2721

Une question de géométrie plane.

Mathematica (*Cluj*) **4** (**27**) (1962), 309–382.

The author solves the following question. Two plane curves C_1 and C_2 are given. Find a curve C in the same plane, every point M of which has the property that M is the middle point of a segment M_1M_2, where M_1 and M_2 are points of intersection, nearest to M, of the normal of C at M with the curves C_1 and C_2, respectively. The author establishes differential equations and solves the problem for two particular cases: (1) C_1 and C_2 are both straight lines, (2) C_1 is a straight line and C_2 is a half-ellipse with its major axis on C_1. Precise considerations are given to the solution curves, and the curves in each particular case are drawn.

M. Kurita (Nagoya)

Bottema, O. 2722

Ein Problem der affinen Kinematik.

Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 290–300.

The author considers some properties of a certain area-preserving, one-parametric affine motion on a plane. If four points A_i ($i = 1, 2, 3, 4$) fixed on a moving plane V' describe straight lines g_i on the plane V by the motion, then there is a conic K on V' every point of which describes a straight line g, and the correspondence between the points of K and the directions of g is projective. The straight lines g form a curve of class 3 on V. On the basis of the above considerations, detailed studies are given for the cases that K is an ellipse, a hyperbola and a parabola.
M. Kurita (Nagoya)

Lane, N. D. 2723
Arcs of parabolic order four.
Canad. J. Math. **16** (1964), 321–338.

An arc in the real affine plane (continuous image of a real interval) is said to be of parabolic order four if every parabola intersects it at most four times. In general, four points of an arc determine a pair of parabolas that may converge if some of the points approach a given point along the arc and the other points remain fixed. This parabolic differentiation was studied by the author in a previous paper [same J. **15** (1963), 546–562; MR **27** #656]. From the parabolic order four, information is obtained on the various types of parabolic differentiability of interior and end points, and related properties, of these arcs.
G. J. Schellekens (Utrecht)

Villa, Mario 2724
Un'osservazione sulla geometria affine delle trasformazioni puntuali.
Boll. Un. Mat. Ital. (3) **19** (1964), 53–59.

An analytic transformation T between two affine p-dimensional spaces $S_p(x_1, x_2, \cdots, x_p)$, $S_p'(x_1', x_2', \cdots, x_p')$, in which the origins O, O' correspond, can be represented by equations of the form

$$T: x_i' = f_i(x_1, x_2, \cdots, x_p) + \cdots \qquad (i = 1, 2, \cdots, p),$$

where the f's are polynomials of degree n in the x's free from constant terms and the dots are for terms of degree greater than n. Here it is remarked that the rational integral transformation of degree n between S_p, S_p':

$$T^*: x_i' = f_i(x_1, x_2, \cdots, x_p) \qquad (i = 1, 2, \cdots, p)$$

is intrinsically associated to T, O, O'; the result is obvious, since T^* is the unique transformation of its type which approximates T of order n in the neighbourhood of O, O' [cf. the reviewer, Ann. Mat. Pura Appl. (4) **28** (1949), 107–139; MR **12**, 200].

In the special case when $p = n = 2$, conditions are then obtained for T^* to be Cremonian or integral Cremonian.
B. Segre (Rome)

Cruceanu, V. 2725
Sur la théorie des courbes dans l'espace centro-affine parabolique. (Romanian. Russian and French summaries)
Acad. R. P. Romîne Fil. Iași Stud. Cerc. Ști. Mat. **13** (1962), 313–322.

This is a study of the theory of 3-dimensional curves with respect to the subgroup of the projective group which leaves a plane and a point on it fixed (i.e., the parabolic centro-affine group). The method used is that of the moving frame by E. Cartan. *R. Blum* (Saskatoon, Sask.)

Porcu, Livio 2726
Sulle calotte a centri allineati di superficie algebriche.
Ist. Lombardo Accad. Sci. Lett. Rend. A **97** (1963), 499–514.

Necessary and sufficient conditions in order that n second-order (hypersurface) differential elements of S_r, with centres on a line l and not touching l, should lie on an algebraic hypersurface of order n not passing through l have been given [cf. the reviewer, *Some properties of differentiable varieties and transformations*, §§ 39, 42, Springer, Berlin, 1957; MR **19**, 679]. In the case $r = 3$, those conditions are three in number and were first obtained analytically by E. Bompiani [Atti Accad. Italia Rend. Cl. Sci. Fis. Mat. Natur. (7) **1** (1940), 93–101; MR **1**, 267].

The present paper goes deeper in the last special case. The three conditions are stated in a simple projective form and several additional questions are dealt with, with particular regard to the case ($n = 3$) of triplets of elements lying on a cubic surface of S_3. *B. Segre* (Rome)

Tevzadze, G. N. 2727
On the tensor theory of a congruence of lines in a projective space. (Russian. Georgian summary)
Soobšč. Akad. Nauk Gruzin. SSR **35** (1964), 9–14.

The author applies the method of normalized surfaces of Norden [Trudy Sem. Vektor. Tenzor. Anal. **6** (1948), 125–224; MR **15**, 61]. Relations for asymptotic nets on focal surfaces of the given congruence are obtained. Conditions for W-congruences are found. The relation between interior conjugate connections on focal surfaces of a W-congruence are established, and a necessary and sufficient condition that the focal surfaces are in a correlative correspondence is given. Also formulae for conjugate connections on focal surfaces of congruences of general form are given. Finally, relations among Čebyšev vectors of the asymptotic nets of focally normalized focal surfaces are given.
A. Goetz (Wrocław)

Marcus, F. 2728
Sur les surfaces qui admettent un groupe continu de déformations projectives en elles-mêmes. (Romanian. Russian and French summaries)
Acad. R. P. Romîne Fil. Iași Stud. Cerc. Ști. Mat. **13** (1962), 307–311.

The main result of this paper is the theorem: The trajectories of a continuous group of projective deformations of a surface into itself cannot belong to a congruence D of straight lines as defined by Finikoff [Bull. Sci. Math. (2) **56** (1932), 117–136]. *R. Blum* (Saskatoon, Sask.)

Degen, Wendelin 2729
Zur projektiven Differentialgeometrie der Flächen, die von einer einparametrigen Schar von Kegelschnitten erzeugt werden. I. Grundzüge der Theorie im elliptischen und hyperbolischen Fall.
Math. Ann. **155** (1964), 1–34.

Wie bekannt, für die Theorie der Flächen, die von einer einparametrigen Schar von Kegelschnitten erzeugt werden (die Kegelschnittsflächen), ist wesentlich, ob die Erzeugenden der von den Kegelschnittsebenen eingehüllten Torse die erzeugenden Kegelschnitte in verschiedenen Punkten schneiden (hyperbolischer bzw. elliptischer Fall—je nachdem die Schnittpunkte reell oder konjugiert komplex sind) oder berühren (parabolischer Fall). In diesem ersten Teil werden die elliptischen und hyperbolischen Kegelschnittsflächen gleichzeitig behandelt.

Nach der passenden Wahl des Kalküls und analytischen Aufbau der Theorie werden zunächst die gefundene Grundformeln für allgemeine Kegelschnittsfläche geometrisch interpretiert. Es zeigt sich, dass dabei die bekannte Sterntransformation von G. Bol sehr nützvoll ist. Anschliessend wird ein vollständiges System von Halbinvarianten aufgestellt und die geometrische Deutung für das Verschwinden einiger von ihnen gegeben.

Weiter werden auf Kegelschnittsflächen Systeme von Kurven, die die erzeugenden Kegelschnitte projektiv aufeinander abbilden (die Doppelverhältnisscharen) untergesucht. Eine Doppelverhältnisschar wird dadurch charakterisiert, dass die Tangenten ihrer Kurven längs jedes erzeugenden Kegelschnitts (der gewissen allgemeinen Klasse von Kegelschnittsflächen) eine Regelfläche 4. Ordnung bilden.

Zulätzt wird auf der Kegelschnittsflächen die zur Schar der erzeugenden Kegelschnitte konjugierte Kurvenschar betrachtet. *A. Urban* (Prague)

Vangeldere, J. 2730

Recherches sur la géométrie projective différentielle des V_3 de S_5.

Mém. Soc. Roy. Sci. Liège Coll. in-8° (5) **10** (1964), *no.* 1, 117 *pp.*

Dans ce mémoire l'auteur fait une étude approfondie des variétés (V_3) à trois dimensions de l'espace projectif (S_5) à cinq dimensions. Le cas particulier où la V_3 est un système point au sens de C. Guichard [*Sur les systèmes triplement indéterminés et sur les systèmes triple-orthogonaux*, Gauthier-Villars, Paris, 1905] a été examiné par O. Rozet [Bull. Soc. Roy. Sci. Liège **10** (1941), 612–622; MR **7**, 32], et d'autre part E. Togliatti [Ist. Veneto Sci. Lett. Arti Atti **87** (1927/28), 1373–1421] a envisagé les différents types de V_3 possédant trois systèmes distincts d'asymptotiques au plus. C'est du cas général, pour lequel le nombre de systèmes d'asymptotiques est quatre, que traite le présent travail. La méthode employée est celle qui consiste à caractériser la variété étudiée par un système d'équations aux dérivées partielles, mais où, pour conserver aux calculs une interprétation qui dans le cas bidimensionnel résulte de la possibilité de choisir des courbes de référence ayant une signification géométrique déterminée, la notion de différentielle est substituée avec profit à celle d'équation aux dérivées partielles. La façon dont est faite cette substitution est exposée dans une 1ère partie où, à côté de résultats classiques sur les systèmes de ∞^2 courbes ou de ∞^1 surfaces de V_3, se trouvent introduites les notions sur lesquelles l'auteur a fondé son étude. Parmi celles-ci les correspondances biunivoques établies par l'auteur entre les systèmes (C) de ∞^2 courbes de V_3 et les points d'un plan projectif π, ou entre les systèmes de ∞^1 surfaces de V_3 et les droites de π représentant des différentielles totales exactes, ou encore entre les faisceaux d'hyperplans tangents (ξ) à V_3 au point courant (u, v, w) et les faisceaux (Γ_ξ) de coniques images dans π des cônes quadratiques des tangentes en ce point à l'intersection de ξ et de V_3, comme aussi une certaine correspondance Ω établie entre les points de π par l'intermédiaire des faisceaux (Γ_ξ), ($\Gamma_{\xi'}$) relatifs à deux systèmes de courbes (C), (C') de V_3, jouent un rôle important. Elles permettent notamment une classification des V_3 de S_5 suivant le nombre et la nature des points de base du faisceau (Γ_ξ) en sept types, dont le dernier, dit type (1, 1, 1, 1), pour lequel (Γ_ξ) admet quatre points-bases propres comptant chacun pour un, correspond au cas d'existence de quatre systèmes distincts de lignes asymptotiques (et trois systèmes de lignes principales), est ici étudié pour la première fois en toute généralité. L'étude générale en question se fonde essentiellement sur la considération de bases de référence, établies dans V_3 au moyen de trois systèmes quelconques de courbes (R_1), (R_2), (R_3) dont les images (dR_1, dR_2, dR_3) dans π sont linéairement indépendantes, bases permettant l'introduction de coordonnées absolues et relatives, et la définition de différentielles successives pour une fonction $f(u, v, w)$ ou un point P de V_3, jouissant d'une propriété d'interversion analogue à celle dont jouissent les dérivations partielles ordinaires. Dans la 2e partie du mémoire une V_3 de S_5 est caractérisée projectivement par un système (S) d'équations aux dérivées partielles, établi en faisant jouer aux trois systèmes de lignes principales le rôle des systèmes de base (R_i), $i = 1, 2, 3$. A partir de certaines des conditions d'intégrabilité l'auteur peut ainsi aborder la géométrie des variétés générales V_3. Il recherche notamment les conditions pour que la V_3 considérée, soit réglée, soit un système de Guichard, ait des lignes principales planes, admette un hyperplan bitangent ne dépendant que de deux paramètres essentiels ou certains systèmes de courbes définis géométriquement et appartenant deux à deux à un même système de surfaces. La troisième et dernière partie traite essentiellement des doubles systèmes conjugués, dits de 1ère espèce, d'une V_3, extension des systèmes conjugués ordinaires auxquels ils se réduisent dans le cas involutif, et dont l'introduction conduit à la notion d'une transformation de la variété, à rapprocher de la notion classique de transformation de Laplace qu'elle admet pour cas particulier, et présentant sur cette dernière l'avantage d'exister toujours alors que la transformation classique n'existe que dans certains cas particuliers. La notion de double système conjugué de 1ère espèce dont il vient d'être question existe pour toute V_3 de S_5 [non nécessairement du type (1, 1, 1, 1)]. L'auteur établit d'abord un certain nombre de propriétés appartenant au type général, pour étudier ensuite de façon détaillée le type (1, 1, 1, 1). Le cas où le premier système de courbes est formé de lignes principales conduit à distinguer, pour chaque système de lignes principales, trois cas fondamentaux, dont l'étude est rattachée à certaines considérations générales sur les hyperplans bitangents de la V_3 envisagée, et notamment aux relations que présente la variété engendrée par le dual d'un hyperplan bitangent avec les doubles systèmes conjugués de 1ère espèce dont le 1er système de courbes est le système de lignes principales relatif à cet hyperplan. L'étude systématique des trois cas fondamentaux indiqués est faite parallèlement à celles des doubles systèmes fondamentaux et de la duale de la variété engendrée par les hyperplans bitangents correspondants, et conduit à tout un ensemble de résultats d'un réel intérêt géométrique.

P. Vincensini (Marseille)

Kovancov, N. I. 2731
An osculating quadratic complex. (Russian)
Ukrain. Mat. Ž. **16** (1964), 237–245.

Bekanntlich gehört zu allen Elementen eines beliebigen Komplexes ein Büschel linearer Komplexe als Berührungskomplexe erster Ordnung. Die allgemeine Gleichung eines linearen Komplexes enthält 5 wesentliche Koeffizienten. Die Differentialbedingungen für die Komplexumgebungen erster Ordnung reduzieren diese Koeffizienten auf einen einzigen unabhängigen. Wenn der Komplex K in jedem Element einen linearen Komplex von zweiter Berührungsordnung enthält, gestattet er lediglich die Zuordnung eines und desselben linearen Komplexes. Verfasser untersucht Existenz und Eigenschaften der Komplexe K mit quadratischen Berührkomplexen. In der allgemeinen Gleichung treten 20 Koeffizienten auf und genau auf die gleiche Anzahl von Bedingungen für diese Koeffizienten führt das Bedingungssystem einer Umgebung dritter Ordnung der Komplexelemente, nämlich 1 Bedingung für die Umgebung nullter Ordnung, $1+3$ für diejenigen erster Ordnung, $1+3+6$ für diejenigen zweiter Ordnung und $1+3+6+10$ für die der dritten Ordnung.

Wenn diese Bedingungen insgesamt unabhängig von einander vorausgesetzt werden, ergibt sich die Zuordnung der quadratischen Komplexe in eindeutiger Weise. Wie man einen derartigen Komplex kanonisch darstellen kann, hatte Verfasser bereits in einer früheren Arbeit gezeigt [siehe dasselbe Ž. **8** (1956), 140–158; MR **18**, 505]. Die für die 20 Koeffizienten bestehenden notwendigen Bedingungen lassen deren vier unabhängig und führen gleichzeitig auf gewisse Kompatibilitätsbedingungen bestimmter spezieller Klassen von Komplexen K. Die Freiheit in der Wahl der restlichen 4 Koeffizienten kann benutzt werden, Komplexe K_0 mit entsprechend speziellen invarianten Eigenschaften zu gewinnen. (Einer der unabhängigen Parameter verschwindet auf Grund der Plückerschen Identität für die verwendeten Plückerschen Koordinaten.) Die Menge der quadratischen Komplexe, welche Schmiegkomplexe von K_0 darstellen, ist also dreiparametrig. Demnach existiert im allgemeinen Falle eines Komplexes K kein quadratischer Komplex, der mit ihm eine Berührung dritter Ordnung hätte, dagegen eine dreiparametrige Schar solcher Komplexe im Falle K_0. Sie hängt von 6 willkürlichen Funktionen eines Argumentes ab. Mit der analytischen Durchführung des angedeuteten Programmes gewinnt Verfasser 5 Klassen der gesuchten Komplexe (je nach der Vielfachheit ihrer Wendezentralpunkte). *M. Pinl* (Cologne)

Valette, Guy 2732
Quelques propriétés globales des rubans de courbure. (German summary)
Bull. Soc. Math. Belg. **15** (1963), 413–427.

On the basis of his preceding paper [Acad. Roy. Belg. Bull. Cl. Sci. (5) **47** (1961), 880–894; MR **26** #693a] the author proves some global properties in the conformal theory of "rubans", i.e., a strip of plane elements tangent to a curve (support) in 3-dimensional space. The sphere which has contact of order two with the support of a strip R is called a meridian sphere, and the spherical torsion of R is defined in relation to meridian spheres. A principal strip is defined as one whose element of spherical torsion vanishes. A normal vertex of a strip is defined as the point at which the normal curvature is stationary, and the vertex is called effective if it corresponds to an extremum of the curvature. The main results of the paper are: (1) If R is a principal strip for which only normal vertices are isolated and non-effective, then meridian spheres on the points of R are disjoint; (2) Every closed principal strip has at least two normal effective vertices; (3) If R is a closed orientable principal strip whose support is spherically convex, then it has at least four normal effective vertices. *M. Kurita* (Nagoya)

Pikus, D. L. 2733
A group-theoretic construction of a two-dimensional geometry of constant curvature. (Russian)
Trudy Sem. Vektor. Tenzor. Anal. **12** (1963), 365–396.

The author discusses the two-dimensional Helmholtz-Lie-space problem on the basis of Hilbert's axioms. *H. Freudenthal* (Utrecht)

Golab, S. [Gołąb, Stanisław]; Jakubowicz, A.; Kucharczyk, P. 2734
Sur la notion du "champ trîané".
Mathematica (Cluj) **4 (27)** (1962), 247–252.

Les auteurs donnent une démonstration du théorème suivant: La notion du champ traîné des objets $\Omega(\xi)$ par l'intermédiaire de la transformation ponctuelle $q=f(p)$ est une notion invariante qui ne dépend pas du système des coordonnées. *W. Wrona* (Warsaw)

Kucharzewski, M.; Kuczma, M. 2735
Determination of linear differential geometric objects of the first class, with two components, in a two-dimensional space.
Ann. Polon. Math. **15** (1964), 77–84.

The main result of the present paper states that every linear differential geometric object of the first class with two components in a two-dimensional space (with a measurable law of transformation) is equivalent to one and only one of nine geometric objects (more precisely, five single objects and four one-parameter families of objects), the transformation rules of which are written in detail. *A. Urban* (Prague)

Udalov, A. S. 2736
On the theory of curves and surfaces in affine and projective spaces. (Russian)
Izv. Vysš. Učebn. Zaved. Matematika **1963**, *no.* 4 (35), 158–164.

Objects of connection of degree v in the tangent bundle $T_{vm}(X_m)$ in the sense of Vagner ["The theory of differential objects and the foundations of differential geometry" (Russian), Appendix to the Russian translation of Veblen and Whitehead, *The foundations of differential geometry*, IL, Moscow, 1949] are defined, as well as another kind of tangent bundle $T_{l_{vm}}(X_m)$ introduced in the paper. Two operations leading from differential objects of arbitrary class to objects of the first class are associated with the objects of connection. They are permutable with the operation of forming commutants of differential objects. As an application to the theory of curves in affinely connected spaces explicit formulae for connection objects, a natural frame and a fundamental system of invariants of a curve,

whose existence was indicated by Hlavatý [Nederl. Akad. Wetensch. Proc. **52** (1949), 505–517; MR **11**, 54; ibid. **52** (1949), 714–724; MR **12**, 282; ibid. **52** (1949), 977–986; MR **12**, 282], are given. Further, the results are applied to certain classes of surfaces in projective n-space, and a fundamental system of tensors of least order is obtained. {The paper does not contain complete proofs.}

A. Goetz (Wrocław)

Bolder, H. 2737

Alternating multi-affine forms.

Nieuw Arch. Wisk. (3) **11** (1963), 118–141.

The author thoroughly discusses the algebraic content involved in a proof of Stokes's theorem which was presented by F. and R. Nevanlinna [*Absolute Analysis*, Springer, Berlin, 1959; MR **22** #12176]. However, for more clarity, he describes things in terms of points in an affine space rather than in terms of vectors in a linear space.

W. P. Ziemer (Bloomington, Ind.)

Ślebodziński, Władysław 2738

★Formes extérieures et leurs applications. Vol. II.

Państwowe Wydawnictwo Naukowe, Warsaw, 1963. 271 *pp.* $8.00.

This is the second volume of a book on exterior differential forms and their applications. Volume I, which appeared in 1954 [MR **16**, 1082], dealt with purely algebraic questions concerning the Grassmann algebra of forms on a vector space, including extensive discussion of equations given by such forms. In Volume II the author develops the theory and applications of differential forms on a manifold. The book comprises seven chapters as follows. In Chapter I, forms on an open subset of euclidean space are defined and their theory developed from the beginning. The chapter includes proofs of the Poincaré theorem and Frobenius theorem on completely integrable systems; it also contains a discussion of the concept of class of a differential form. In the second chapter, still restricting himself to the case of an open subset of R^n, the author gives an exposition of exterior differential systems, including the Cartan existence theorem (as generalized by Kähler) and the concept of systems in involution. He applies these to the equation of Pfaff and to a quadratic equation to obtain canonical forms of an exterior form. Chapter III defines differentiable manifold, forms on a manifold, Riemannian manifold, parallelizable manifold, and various other geometric notions suitable for applications of the theory.

In the next chapter, after introducing Lie groups and Lie algebras, the author defines, for an arbitrary subgroup Γ of $\mathrm{Gl}(n, R)$, affine connection with structure group Γ. He also defines a certain method of attaching to a transformation group Γ a sequence $\Gamma_1, \Gamma_2, \cdots$ of transformation groups (on successively higher-dimensional spaces) and the notion of prolongation of a connection. These roughly correspond to taking bundles over fibre bundles and the purpose is to establish a geometric framework for Chapter VI (on infinite continuous pseudo-groups). Chapter V deals with the classical connections, i.e., where Γ is a classical group, and includes the Cartan solution of the equivalence problem for Riemannian metrics. Chapter VI, as mentioned, goes extensively into the theory of transitive pseudo-groups of transformations, with some brief indications on the intransitive case and a discussion of abstract infinite pseudo-groups. The author uses the notion of connection to simplify the exposition of the subject. Chapter VII, the last, is on invariant integrals and their applications to mechanics. In this chapter Stokes's theorem and the theory of Lie derivatives are developed; but the main consideration is the application of invariant integrals to mechanics as followed by Poincaré and Cartan.

As will be seen from the above, this book essentially has as its object an exposition of Cartan's work on exterior differential systems and on infinite pseudo-groups, together with applications to various examples of the author's choosing. The reviewer found that this book clarified many portions of that theory for him. Some readers may find a surplus of classical tensor notation (with many indices), but the concepts are very carefully defined and there is not the ambiguity and reliance on geometry here which make Cartan's work so hard for many to read. The book is not long, and because of its concise and careful style it should serve as a convenient source for those interested in a good account of these subjects.

W. M. Boothby (St. Louis, Mo.)

Willmore, T. J. 2739

The definition of Lie derivative.

Proc. Edinburgh Math. Soc. (2) **12** (1960/61), 27–29.

The author states first a general theorem concerned with the extension of differential operators. Let D be a differential operator which maps contravariant vector fields and scalar fields into tensor fields such that (i) D is linear over the real numbers, i.e., $D(aX+bY)=aDX+bDY$ for any vector fields X, Y and any real numbers a, b; (ii) $D(fY)=fDY+(Df)\otimes Y$ for any scalar f and any vector field Y. Then the operator D can be extended in a unique manner to operate on tensor fields of arbitrary type (p, q) in such a way that (iii) it is linear over the real numbers, i.e., $D(aS+bT)=aDS+bDT$ for any tensors S, T of the same type and for any real numbers a, b; (iv) the Leibniz law of derivation of a product is satisfied, i.e., $D(S\otimes T)=(DS)\otimes T+S\otimes(DT)$ for all tensors S, T; (v) the operator commutes with the process of contraction; alternatively, $DI=0$, where I is the Kronecker tensor.

He then applies this to the coordinate-free definition of Lie derivation L_X with respect to a vector field X, starting from $L_Xf=Xf$ and $L_XY=[X, Y]$, where f is a scalar field and Y a contravariant vector field. *K. Yano* (Tokyo)

Lichnerowicz, André 2740

★Théorie globale des connexions et des groupes d'holonomie.

Consiglio Nazionale delle Ricerche Monografie Matematiche, Vol. 2.

Edizioni Cremonese, Rome; Dunod, Paris, 1962. xv + 282 *pp.* *L.* 4.500.

This is a photo-reproduction (without correction of misprints) of an earlier edition by the first publisher [1957; MR **19**, 453].

Hermann, Robert 2741

Cartan connections and the equivalence problem for geometric structures.

Contributions to Differential Equations **3** (1964), 199–248.

Let B be an $(n-m)$-dimensional manifold, G a connected Lie subgroup of $\mathrm{Gl}(n-m, R)$ and assume that a G-structure is given. It is the purpose of this paper to formulate an equivalence problem for this structure and to solve it in one finite type case.

For the solution of the problem it is first of all necessary to have an n-dimensional manifold M, a maximum rank and onto map $\pi: M \to B$, and a parallelization of M by 1-forms (ω_i) satisfying: (1) The annihilators of $\omega_{m+1}, \cdots, \omega_n$ form an involutive distribution whose integral manifolds are the fibres of π; (2) If $d\omega_i = \sum_{j,k=1}^n c_{ijk}\omega_j \wedge \omega_k$, then for each fixed a, $1 \leqq a \leqq m$, $\{c_{auv}\}$, $u, v = m+1, \cdots, n$, is a matrix in $\mathbf{G}$, the Lie algebra of G. The final conditions relate the G-structure to the (ω_i) more directly. Locally, there are independent forms $\{\theta_u\}$, $u = m+1, \cdots, n$, such that $\pi^*\theta_u = \sum_{v=m+1}^n M_{uv}\omega_v$, where the matrix (M_{uv}) takes its values in G. The totality of these sets of forms determines a G-structure on B, and the third condition is that this G-structure coincides with the given one. Finally, it is required that the mapping $X \to \pi_*(X)$ of the Lie algebra of the infinitesimal automorphisms of the parallelization into the Lie algebra of infinitesimal G-structure automorphisms is an isomorphism.

The statement of sufficient conditions for such a solution will require some definitions. First, assume there is a connected Lie group L and a closed subgroup K, with corresponding Lie algebras $\mathbf{L}$, $\mathbf{K}$, such that the Lie algebra of the linear isotropy group of L/K, with respect to a suitable base of $\mathbf{L}$, is $\mathbf{G}$. Let $\mathbf{L}^p = \{X \in \mathbf{L} \mid$ the vector field determined by X on L/K vanishes to the $(p-1)$st order at the identity coset$\}$. Then $\mathbf{L} = \mathbf{L}^0 \supset \mathbf{L}^1 \supset \cdots$; $[\mathbf{L}^p, \mathbf{L}^q] \subset \mathbf{L}^{p+q-1}$, and $\bigcap_{p=0}^{\infty} \mathbf{L}^p = (0)$. Such a configuration is called a filtered Lie algebra. Then $\{\mathbf{L}_1{}^p\} = \{X \in \mathbf{L}^p \mid [X, \mathbf{L}] \subset \mathbf{L}^p\}$ is again a filtered Lie algebra, the first prolongation of $\{\mathbf{L}^p\}$. The Nth prolongation $\{\mathbf{L}_N{}^p\}$ is defined by iteration. The next of the sufficient conditions is that there exists an $N_0 \geqq 0$ such that $\mathbf{L}_{N_0+1}^1 = (0)$ and $\mathbf{L}_{N_0}{}^1 \neq (0)$. The final condition is a fairly technical one concerning the action of the isotropy group of $\mathbf{L}/\mathbf{L}_N{}^1$ for $N \leqq N_0$. The main theorem of the paper is that these three conditions imply that the equivalence problem for the given G-structure has local solutions. A global solution is then pieced together from these.

In addition to the above, there is a considerable amount of more or less expository material on vector field systems, Ehresmann connections, Cartan connections and associated affine connections, and filtered Lie algebras, representing clarification and extension of ideas of E. Cartan. This paper should be compared with the work of I. M. Singer and S. Sternberg [*The infinite groups of Lie and Cartan*, Part I, Mimeographed notes, Harvard and M.I.T., Cambridge, Mass., 1964].

{The reader should be warned that this paper contains several instances of inconsistent notation, as well as many minor typographical errors.}

R. J. Crittenden (Providence, R.I.)

Sen, R. N. 2742

Associate tensors and affine connections.

J. Indian Math. Soc. (*N.S.*) **27** (1963), 45–56.

Let g_{ij}, $\bar{g}_{ij}$ and a_{ij} be non-singular symmetric tensors. The pair $(g_{ij}, \bar{g}_{ij})$ is associate if $\bar{g}_{ij} = a_{is}a_{jt}g^{st}$. Starting from g_{ij} and a_{ij}, the author gives successively associate pairs and their metric (but not symmetric) connections.

Y. Tashiro (Okayama)

Vranceanu, Giorgio [Vranceanu, Georges] 2743

Proprietà globali di certi spazi a connessione affine.

Univ. e Politec. Torino Rend. Sem. Mat. **22** (1962/63), 5–15 (1964).

The author calls a manifold with an affine connection locally euclidean if its curvature and torsion tensors vanish. In response to the problem of determining under what conditions a locally euclidean space is globally equivalent to affine n-space, the author proves the following theorems. In all of them the space is assumed to be locally euclidean. (1) If x^i are coordinates for which the coefficients of the connection are analytic and u^i are affine normal coordinates, then the series for the coordinate transformation $u^i = u^i(x^1, \cdots, x^n)$ converges throughout the domain of convergence of the coefficients of the connection. (2) If the Jacobian determinant of the transformation $u^i = u^i(x^1, \cdots, x^n)$ is non-zero at one point, then it is non-zero everywhere. (3) If all the geodesics through a single point can be extended indefinitely in their affine parametrization, then the inverse transformation $x^i = x^i(u^1, \cdots, u^n)$ exists globally and the space is thus globally affine space. (4) If, further, the u^i are polynomials in x^i, then the inverse equations also are polynomials, and the contracted connection $\Gamma_k = \Gamma_{ik}{}^i$ vanishes.

Some additional results concerning complete locally euclidean manifolds are mentioned without proof.

R. L. Bishop (Urbana, Ill.)

Bliznikas, V. 2744

Curves on surfaces in a space with generalized Euclidean connection. (Russian. Lithuanian and German summaries)

Litovsk. Mat. Sb. **3** (1963), *no.* 1, 21–27.

A generalized Euclidean space $\mathfrak{H}_n$ is a Riemann space equipped with an affine connection with respect to which the metric tensor is a covariant constant. Such a space was investigated by the author [same Sb. **2** (1962), no. 2, 15–32; MR **28** #550]. Now he develops the theory of curves lying on a hypersurface of $\mathfrak{H}_n$. In this space he considers the usual notions of the theory of curves (different kinds of curvatures, geodesics, and asymptotic lines, etc.) and deduces a Meusnier formula, which generalizes the well-known one of R_n. The method used is that of the exterior forms of Cartan.

L. Tamássy (Debrecen)

Houh, Chorng-Shi 2745

Submanifolds in a Riemannian manifold with general connections.

Math. J. Okayama Univ. **12** (1963/64), 1–37.

The paper deals with l-dimensional submanifolds $\mathfrak{X}_l$ of a Riemannian n-manifold $\mathfrak{X}$ with a general connection $(P_j{}^i, \Gamma_{jk}{}^i)$ in the sense of Ōtsuki [same J. **9** (1959/60), 99–164; MR **22** #2954]. For a given injection $\mathfrak{X}_l \to \mathfrak{X}$ expressed in coordinates as $u^i = u^i(x^a)$ $(i, j = 1, 2, \cdots, n;$ $a, b = 1, 2, \cdots, l)$ denote $\theta_a{}^i = \partial u^i/\partial x^a$, $g_{ab} = \theta_a{}^i\theta_b{}^j g_{ij}$, $\theta_i{}^a = g^{ab}g_{ij}\theta_b{}^j$. The induced general connection in $\mathfrak{X}_l$ is defined by the formulae $P_a{}^b = \theta_i{}^b P_j{}^i \theta_a{}^j$, $\Gamma_{bc}{}^a = \theta_i{}^a(P_j{}^i\partial_c\theta_b{}^j + \Gamma_{jk}{}^i\theta_b{}^j\theta_c{}^k)$.

The induced connection preserves parallelism, i.e., if a vector field tangent to $\mathfrak{X}_l$ is parallel along a curve with respect to the given general connection, then it is also parallel along this curve with respect to the induced connection. The induced connections derived from Γ and

$\bar{\Gamma}=A\Gamma A$ [cf. Ōtsuki, Kōdai Math. Sem. Rep. 14 (1962), 40–52; MR 26 #1825] with $A_j{}^i=\theta_a{}^i\theta_j{}^a$ coincide. All further results are obtained under the assumption that the submanifold is adapted to the given general connection, i.e., each tangent space $T_x(\mathfrak{X}_1)$ and its orthogonal complementary space are invariant under the homomorphism given by $P_j{}^i$. In this case, e.g., the induced general connection derived from normal [respectively, regular or regular metric] general connections is normal [regular, regular metric].

It is not possible to summarize all the results of this rather long paper in a short review. They concern relations between the connections Γ and $\bar{\Gamma}$, development of curves, geodesics and relations between the curvature tensors of Γ, $\bar{\Gamma}$ and the induced connection. *A. Goetz* (Wroclaw)

Ba, Boubakar 2746

Sur les transformations conformes de variétés à bord.

C. R. Acad. Sci. Paris **258** (1964), 54–56.

Soit V une variété de Riemann orientable de dimension m et soit M une sous-variété ouverte de V, relativement compacte, dont le bord ∂M est une sous-variété différentiable compacte de dimension $m-1$. L'auteur démontre d'abord les deux propositions suivantes. (1) Si sur M il existe une transformation infinitésimale conforme tangentielle non isométrique, toute forme harmonique tangentielle ou normale de longueur constante est nulle. (2) Les formes harmoniques tangentielles ou normales sont invariantes par les isométries infinitésimales tangentielles.

Il démontre ensuite le théorème suivant. Soit M une variété de Riemann compacte, orientable, dont le bord ∂M est une variété compacte de codimension 1, homogène pour sa structure de variété de Riemann à bord (autrement dit, l'algèbre de Lie des isométries infinitésimales tangentielles est transitive sur M). Alors: (a) ou bien $H^p(M, \partial M)=0$ pour $0<p<m$; (b) ou bien l'algèbre de Lie des transformations infinitésimales conformes tangentielles coïncide avec celle des isométries. *K. Yano* (Tokyo)

Ba, Boubakar 2747

Transformations conformes et presque analytiques des variétés presque kählériennes à bord.

C. R. Acad. Sci. Paris **257** (1963), 3554–3556.

L'auteur considère une variété presque kählérienne à bord et donne d'abord une condition nécessaire et suffisante pour qu'une transformation conforme soit un automorphisme. Il en déduit trois corollaires: (1) Toute transformation conforme tangentielle dont l'image par J (la structure presque complexe) est normale est un automorphisme; (2) Plus particulièrement encore toute transformation infinitésimale conservant le bord point par point et conforme est un automorphisme; (3) Pour qu'une transformation conforme soit un automorphisme, il suffit qu'elle conserve ω (2-forme échangeable avec la métrique) le long du bord.

L'auteur démontre ensuite deux théorèmes: Toute transformation infinitésimale conforme, tangentielle et presque analytique est un automorphisme. Toute transformation infinitésimale fermée tangentielle est un automorphisme.

Il donne aussi une condition nécessaire et suffisante pour qu'une transformation presque analytique conservant l'élément de volume soit un automorphisme et en déduit trois corollaires: (1) Toute transformation infinitésimale presque analytique tangentielle dont l'image par J est normale et conservant l'élément de volume est un automorphisme; (2) Toute transformation presque analytique conservant le volume et conservant point par point le bord est un automorphisme; (3) Pour qu'une transformation infinitésimale presque analytique soit un automorphisme, il suffit qu'elle conserve ω sur le bord. *K. Yano* (Tokyo)

Klingenberg, Wilhelm 2748

Neue Methoden und Ergebnisse in der Riemannschen Geometrie.

Jber. Deutsch. Math.-Verein. **66** (1963/64), *Abt.* 1, 85–94.

This is an expository paper concerned principally with the topological consequences of assumptions on the curvature of Riemannian manifolds. The following theorems are stated, with some indication given as to the nature of their proofs: the Topogonoff triangle theorem and its ancestors, infinitesimal and global, in the theory of surfaces; the theorems of Hadamard and Cartan; the sphere theorem and the recent generalizations to Kählerian, quaternionic, and Cayley manifolds. In the latter theorems the place of Morse theory in the proofs is stressed.

R. L. Bishop (Urbana, Ill.)

Klingenberg, Wilhelm 2749

Manifolds with restricted conjugate locus. II.

Ann. of Math. (2) **80** (1964), 330–339.

In his previous paper under the same title [same Ann. (2) **78** (1963), 527–547; MR **28** #2506], the author derived topological consequences from the structure of the conjugate locus of a point of a complete Riemannian manifold. In this paper, he improves the results of his first paper by weakening the assumption on the conjugate locus. Let M be an n-dimensional, complete, simply connected Riemannian manifold and $G=(p(s))$, $0\leqq s<\infty$, a geodesic ray in M, parametrized by the arc-length. Denote by (π, k) the following statement about the distribution of the first $k+\lambda$ conjugate points of $p(0)$ along G. "There are no conjugate points in $[0,\pi)$; there are k conjugate points in $[\pi, 4\pi/3]$, k odd and $\geqq 1$; there are no conjugate points in $[4\pi/3, 2\pi)$ and there are λ conjugate points in $[2\pi, 8\pi/3)$, $\lambda\geqq k+1$." Then the main theorem states: Assume that there is a point o in M such that (π, k) holds for all geodesic rays starting from o. For $k=1$ assume in addition that the distance between o and its cut locus $C(o)$ is $\geqq\pi$. Then M is compact, and the loop space of M has the homotopy type of the k-sphere to which there are attached cells of dimension $k+\lambda\geqq 2k+1$. An important consequence of this theorem is that if M is a complete Kähler manifold with Kählerian pinching $>9/16=0.5625$, then M is of the same homotopy type as the complex projective space, thus improving the constant 0.64 in his first paper and the constant $4/7=0.5714\cdots$ in the reviewer's paper [Tôhoku Math. J. (2) **15** (1963), 121–139; MR **27** #4185].

S. Kobayashi (Berkeley, Calif.)

Fomenko, V. T. 2750

Infinitesimal deformations of surfaces with sleeve junctions. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 810–813.

Im Koordinatensystem $Oxyz$ sei durch $z=f(x, y)$ eine einfach zusammenhängende Fläche S mit durchwegs positiver Gaußscher Krümmung und mit dem Rande L gegeben; dabei $S \in C_\alpha^3$, $0<\alpha<1$ (für die Definition dieser Klasse siehe Kap. 1 des Buches (*) I. N. Vekua [*Verallgemeinerte analytische Funktionen* (russisch), Fizmatgiz, Moskau, 1959; MR **21** #7288]). Es sei $\mathfrak{D}$ solche Deformation der Fläche S, bei der ihre Randkurve L auf einer Fläche Σ liegt und in dem Prozesse der Deformation von S der Rand L auf der Fläche Σ gleitet ["deformation with sleeve junction"; siehe (*), Kap. 5, § 8]. Die infinitesimalen Deformationen $\mathfrak{D}$ der Fläche S mit gegebener Veränderung des Linienelementes heißen "Gleitdeformationen" längs Σ. Ist S eine Mütze und L ein Kreis, ändert sich der Winkel φ von S und Σ entlang L zwischen gewissen Schranken und ist ein Punkt von S fix, so gestattet S eine und nur eine Gleitdeformation längs Σ. Wenn noch speziell die betrachteten Deformationen die Infinitesimalverbiegungen sind, so gestattet S eine und nur eine triviale Gleitverbiegung längs Σ. Die Schranken von φ hängen nur von dem Winkel ψ $(0<\psi<\frac{1}{2}\pi)$ der Tangentialebenen von S längs L mit der Grundebene, und zwar $-\frac{1}{2}\pi+\psi \leqq \varphi \leqq \frac{1}{2}\pi$. Bei gewissen Flächen kann man die linke Seite dieser Ungleichung noch verkleinern.

Es folgt die Betrachtung der infinitesimalen Verbiegungen $\mathfrak{D}$ solcher Fläche S mit elliptischem Rande, welche sich aus einem Punkte eindeutig projizieren läßt. Für ähnliche Probleme siehe außer (*) noch A. V. Pogorelov [*Infinitesimale Verbiegungen der allgemeinen konvexen Flächen* (russisch), Izdat. Har'kov. Gos. Univ., Kharkov, 1959; MR **22** #2960] und I. H. Sabitov [Dokl. Akad. Nauk SSSR **147** (1962), 793–796; MR **27** #1912]. *Z. Nádeník* (Prague)

Vitner, Čestmír [Vitner, Čestmír] 2751

Curves in spaces with non-singular pseudo-Euclidean metric of arbitrary index. (Russian. German summary)

Czechoslovak Math. J. **14** (89) (1964), 243–253.

Author's summary: "Die Arbeit ist der Differentialgeometrie der Kurven in pseudoeuklidischen und zentropseudoeuklidischen Räumen mit einem beliebigen Index gewidmet. Unter anderem sind hier explizite Formeln für die begleitende n-Kante und für die Krümmungen in gewöhnlichen sowie in aussergewöhnlichen Punkten einer Kurve abgeleitet. Den Ausgangspunkt bilden hier explizite Formeln für den Schmidtschen Orthogonalisationsprozess, welche von dem Autor in seiner Arbeit [Časopis Pěst. Mat. **89** (1964), 31–35] abgeleitet wurden. Sie ermöglichen die bekannten Ergebnisse der euklidischen und der vom Autor aufgebauten zentroeuklidischen Theorie der Kurven auf den pseudoeuklidischen Fall überzutragen [Czechoslovak Math. J. **12** (87) (1962), 119–143; MR **25** #5458]. Ebenso war es möglich die Ergebnisse des Autors, welche die aussergewöhnlichen Punkte in Riemannschen Räumen betreffen, auf den pseudoeuklidischen Fall überzutragen." *A. Goetz* (Wrocław)

Sawaki, Sumio 2752

On infinitesimal transformations of almost-Kählerian space and K-space.

Kōdai Math. Sem. Rep. **16** (1964), 105–115.

The author's aim is to get sufficient conditions on almost Kähler spaces or K-spaces, under which infinitesimal conformal or projective transformations are reduced to isometric transformations without assuming compactness of the spaces. Let the dimension of a space be $2n$, denote the scalar curvature by R and put $R^*=-\frac{1}{2}\varphi^{lk}\varphi^{ji}R_{lkji}$ (φ^{ji} is the almost complex structure and R_{lkji} the curvature tensor). An infinitesimal conformal transformation is reduced to an isometry in (1) an almost Kähler-Einstein space with $R>0$, and in (2) a K-space with constant R satisfying $\frac{1}{2}(n-1)R-R^* \neq 0$. A projective transformation is reduced to an isometry in (1), and in (3) an Einstein K-space with $R<0$. In a K-space with $R \neq R^*$, an analytic conformal transformation is an isometry. *Y. Tashiro* (Okayama)

Dhawan, Mohini; Prakash, Nirmala 2753

Generalisations of Gauss-Codazzi equations in a subspace imbedded in a Finsler manifold.

Tensor (*N.S.*) **15** (1964), 159–167.

Generalized Gauss-Codazzi equations for a hypersurface F_{n-1} imbedded in an n-dimensional Finsler space F_n, which do not involve the so-called "relative" and "mixed" curvature tensors of F_{n-1}, were recently obtained by the reviewer [Tensor (N.S.) **14** (1963), 226–244; MR **28** #1579]. The present paper is devoted to a derivation of the corresponding Gauss-Codazzi equations for m-dimensional subspaces $(0<m<n)$ of F_n. *H. Rund* (Pretoria)

Tonooka, Keinosuke 2754

Subspace theory of an n-dimensional space with an algebraic metric.

J. Fac. Sci. Hokkaido Univ. Ser. I **18** (1964), 34–40.

An n-dimensional Finsler space $F_n^{(p)}$ is considered, the metric function being the pth root of a homogeneous polynomial of order p, where p is an arbitrary integer $(p \geqq 2)$. Independently of the usual theory of Finsler spaces, a connection is defined. This is applied to the theory of subspaces of $F_n^{(p)}$, for which certain relations are obtained which can be regarded as generalizations of the Gauss-Codazzi equations of classical differential geometry. *H. Rund* (Pretoria)

Yano, K.; Davies, E. T. 2755

On the tangent bundles of Finsler and Riemannian manifolds.

Rend. Circ. Mat. Palermo (2) **12** (1963), 211–228.

Denote by (ξ^h, y^h) the local coordinates of the tangent bundle $T(M)$ of an n-dimensional differentiable manifold M, referred to the local coordinate system (ξ^h). A Finsler metric is imposed on M by means of a metric function $L(\xi^h, y^h)$, which determines a globally defined 1-form $\omega=g_{ji}y^j d\xi^i$, where $g_{ji}=\partial^2(\frac{1}{2}L^2)/\partial y^j \partial y^i$. This gives rise to the 2-form $d\omega=\frac{1}{2}F_{CB}d\xi^C \wedge d\xi^B$ $(A, B, \cdots=1, \cdots, 2n)$, in which $\xi^{n+h}=y^h$. Thus, a positive definite Riemannian metric G_{CB} can be introduced in $T(M)$ such that $F_C{}^A=F_{CB}G^{BA}$ defines an almost complex structure since $F_C{}^E F_E{}^A=-\delta_C^A$. It is proved that the Hermitian metric G can be determined entirely from L. The canonical connection associated with G in $T(M)$ yields, by projection onto M, the connection parameters usually denoted by $\mathring{\Gamma}_j{}^i{}_k$ in the Cartan theory of Finsler spaces, while those denoted by $C_j{}^i{}_k$ in the latter theory are obtained only if M is a Finsler space which has an absolute parallelism

of line elements. The latter condition is also necessary and sufficient that M be Kählerian. For the remainder of the paper it is assumed that L defines a Riemannian metric on M. For a given vector field v^i attached to M, the horizontal lift $('v^A)$, the vertical lift $(''v^A)$, and the complete lift $(\tilde{v}^A)$ are introduced. Conditions that $'v$, $''v$ and $\tilde{v}$ be parallel vector fields in $T(M)$ are derived. Similarly, conditions are given which ensure that these fields are harmonic, or that they are Killing vectors in $T(M)$, or that they are almost analytic (the latter being vectors with respect to which the tensor $F_B{}^A$ has a vanishing Lie derivative). Finally the authors discuss the conditions under which the lift in $T(M)$ of a curve in M is a geodesic of M. *H. Rund* (Pretoria)

Gaeta, Federico 2756

On a process of linearization applicable in problems of integral geometry. (Spanish)

Univ. Nac. La Plata Publ. Fac. Ci. Fisicomat. Serie Segunda Rev. **6** *no.* 5, 17–32 (1960).

Se fondant sur le formalisme de la théorie des correspondances en géométrie algébrique, l'auteur étudie l'action d'un groupe topologique G sur un espace E au moyen des graphes, graf$(\alpha) \subset E \times E$, des transformations de E induites par les éléments $\alpha \in G$. Il étudie en particulier les propriétés d'invariance de ces graphes par rapport aux opérations du groupe $G \times G$. Il montre alors comment ces propriétés permettent une interprétation systématique de nombre de formules de géométrie intégrale. Cette dernière partie du travail aurait certainement gagné à être un peu plus explicite.

P. Dedecker (Rhode-St.-Genèse)

Kirk, W. A. 2757

Isometries in G-spaces.

Duke Math. J. **31** (1964), 539–541.

The G-spaces here are those of Busemann. It is shown that a local isometry φ is an isometry if for some point p the sequence $\{\varphi^n(p)\}$ is bounded. Also it is shown that every isometry of a closed, bounded, convex subset of a straight space into itself has a fixed point. If the straight space has convex spheres, then the intersection of all closed convex sets which contain $\{\varphi^n(p)\}$ is bounded if $\{\varphi^n(p)\}$ is bounded. This implies a fixed-point theorem for motions in such a space. *H. W. Guggenheimer* (Minneapolis, Minn.)

GENERAL TOPOLOGY

See also 2189, 2353, 2384, 2404, 2467, 2625, 2629, 2756, 2805.

Whyburn, Gordon Thomas 2758

★Topological analysis.

Second, revised edition. Princeton Mathematical Series, No. 23.

Princeton University Press, Princeton, N.J., 1964. xii+125 *pp.* $5.00.

The book under review is the second revised edition of the author's earlier book (1958) [MR **20** #6081]. With a minor exception the first 74 pages are the same (this includes Chapters 1 to 5 and two pages of Chapter 6) and consist of the now basic background material for topological analysis. There is a considerable change in the study of differentiable functions. In the present book the author exploits the differential quotient function and the maximum principle and easily obtains the existence of higher derivatives and the power series expansion for differentiable functions; the topological properties of analytic functions (e.g., lightness and openess) are proved as by-products of earlier considerations and no longer play a central role in studying the mapping generated by a differentiable function. This represents a considerable simplification over the previous method. All of this is done in Chapter 6 (pp. 72–80) and is based on the development of the topological (circulation) index in Chapter 5. In this connection, an appendix is included where the author considers just the case of differentiable functions, and presents an elegant and simplified development of the circulation index and readily obtains the Maximum Principle.

Chapter 7 includes Chapters 8 and 9 of the previous edition plus the Ascoli and Vitali Theorems. Chapters 8 and 9 of the present edition are devoted to local and global analysis of open maps. Chapter 9 includes the Stoïlow Theorem and the invariance of orientation of 2-manifolds under light-open-normal mappings. These lead to a characterization of connected separable 2-manifolds without boundary that are Riemann surfaces.

The book is well written and the reviewer anticipates that it, like its predecessor, will be the focal point and guiding light for further research in topological analysis. Indeed, work has already been done in the direction of extending many of these techniques to Fréchet differentiable functions on Banach spaces. Finally, it may be worth noting that, from the point of view of classical analysis, the proof of the existence of higher derivatives and the power series expansion for differentiable functions is equivalent to the proof that a bounded analytic function has a continued fraction expansion. The connecting link between these two chains of thought is the classical Schwarz lemma. *P. Porcelli* (Bloomington, Ind.)

Pervin, William J. 2759

★Foundations of general topology.

Academic Press Textbooks in Mathematics.

Academic Press, New York-London, 1964. xi+209 *pp.* $7.95.

This is a basic text on general topology, designed for advanced undergraduates or beginning graduate students. It covers all the usual basic topics. In addition, there are several more advanced topics, including complete metric spaces and the Baire theorem, the various metrization theorems, and a chapter on uniform spaces. In a sense, the emphasis is on those aspects of general topology which are most useful in analysis. The treatment of a few topics, for example, arc-wise connectedness, is perhaps too brief. The text is well-written, and the author has included many good exercises. *D. W. Kahn* (Minneapolis, Minn.)

Fuks, D. B. 2760

Natural mappings of functors in a category of topological spaces. (Russian)

Mat. Sb. (*N.S.*) **62** (**104**) (1963), 160–179.

This is a study of the operator D by which the author expresses the Eckmann-Hilton duality. He here investigates the dual $D(S \circ T)$ of a composite functor and finds conditions under which $D(S \circ T) = DS \circ DT$.

The spaces considered are functional Hausdorff spaces with base points and have the weak topology with respect to compact sets. In the category of such spaces, a functor S is required to take a space consisting only of base points to another such, and the map $Y^X \to (SY)^{SX}$, where $f: X \to Y$ goes to Sf, is to be continuous. For any space A, let Σ_A be the functor for which $\Sigma_A X = X \circledast A$. Each functor S has a "dual" functor DS for which $DSX = \{S \to \Sigma_X\}$, the space of natural transformations of S to Σ_X [cf. the author, Dokl. Akad. Nauk SSSR **141** (1961), 818–821; MR **25** #571].

For any space X, let

$$\alpha \in (DS \circ DT)X = \{SY \to Y \circledast DTX\}_Y.$$

For each Z, α_{TZ} maps $(S \circ T)Z$ to $TZ \circledast DTX$. If $\gamma \in TZ$ and $\beta \in DTX = \{TY \to Y \circledast X\}_Y$, let $\pi(\gamma, \beta) = \beta_Z(\gamma)$. Then π maps $TZ \circledast DTX$ to $Z \circledast X$ and $\pi \circ \alpha_{TZ}$ gives a map of $(S \circ T)Z$ to $\Sigma_X Z$, natural in Z, and hence an element of $D(S \circ T)X$. Thus there is defined a natural transformation $DS \circ DT \to D(S \circ T)$.

A functor T is called locally contractible if TI, where I is the unit segment, is locally contractible and completely regular. Theorem 1: Let S be a functor such that, for each space B and point $b \in SB$, there is a completely regular space A, a point $a \in SA$, and a map $f: A \to B$ such that $Sf(a) = b$. If T is any locally contractible functor, then $DS \circ DT \to D(S \circ T)$ is a natural equivalence.

There is a natural transformation $\kappa_S: S \to DDS$. If κ_S is an equivalence, S is called reflexive. It is shown that the functor J, where JX is the join $X * X$ of two copies of X with base points identified, is reflexive.

If $f: S \to T$ is a natural transformation, the cylinder Z_f of f is the functor for which $Z_f(X)$ is the cylinder of $f_X: SX \to TX$, and R_f is the functor for which $R_f(X)$ is the space of pairs (p, s), where $p \in SX$ and s is a path in TX starting at $f_X(p)$. Then $DZ_f = R_{Df}$ and, if S, T, and R_f are locally contractible, $DR_f = Z_{Df}$.

C. H. Dowker (London)

Hammer, Preston C. 2761

Extended topology: The continuity concept.

Math. Mag. **36** (1963), 101–105.

The author continues his investigation of expansive set functions u [see, e.g., Niew Arch. Wisk. (3) **9** (1961), 74–86; MR **26** #6925; J. Reine Angew. Math. **213** (1963/64), 174–186; MR **29** #576]. These are generalizations of the Kuratowski closure function requiring only that $uA \supseteq A$ and $u(A \cup B) \supseteq uA \cup uB$. In a previous paper continuity of functions for sets with expansive functions was characterized by a separation property. The present paper considers a generalization of this separation property motivated as follows. "Let us consider pairs, (X_1, X_2), of sets from M. What sort of consideration might lead us to say X_2 is separated from X_1? Certainly we will not require that X_2 separated from X_1 will imply X_1 is separated from X_2, since many applications may be expected to be asymmetric. Shall we require $X_1 \cap X_2 = N$? No! For, X_1 and X_2 might have a common part considered trivial from some standpoint. By this reasoning and examples we were led to conclude that the only property we would use was that if (X_1, X_2) are separated and $X_1 \supseteq X_3$, $X_2 \supseteq X_4$, then (X_3, X_4) are separated. Thus we have a tentative picture of a separation as a hereditary binary relation in m, i.e., a separation is a set of ordered pairs of sets satisfying the above hereditary property. But, wait! Could not there be something desirable added if we did not consider simply pairs of sets but also triples, quadruples, or sequences of sets as separated? It became clear that this also was reasonable and that for certain applications pairs of sets would not do. Hence we now may think of a separation as a hereditary subset of a sort of vector space in which each vector has sets as components. To be explicit let us assume S is a set of ordered ennuples $\{(X_1, X_2, \cdots, X_n)\}$ of subsets of M which is hereditary. Then the set of ennuples which are not separated form an ancestral subset of the space of all ordered ennuples and this we call an association or the association dual to S. Let A be an ennary association for M and B an ennary association for M_1. Then t is (A, B)-continuous provided $tA \subseteq B$; i.e., $(X_1, \cdots, X_n) \in A$ implies $(tX_1, \cdots, tX_n) \in B$. A continuous mapping is association preserving!"

M. L. Curtis (Houston, Tex.)

Remage, Russell, Jr. 2762

Cuttings of finite sets of points.

Duke Math. J. **31** (1964), 359–370.

Motivated by recent developments in fine-cyclic element theory, the author "is concerned at first with unifying some of the extensions of cyclic element theory, and then with an examination of fine-cyclic element theory in a more general framework".

F. Burton Jones (Riverside, Calif.)

Doyle, P. H.; Hocking, J. G.; Osborne, R. P. 2763

Local invertibility.

Fund. Math. **54** (1964), 15–25.

Continuing their study of invertibility properties of spaces, the authors here introduce several related concepts: a topological space S is said to be invertible about a point $p \in S$ if, for each open neighborhood U of p, there exists a homeomorphism h of S onto S such that $h(S-U)$ lies in U. The authors show that the only n-manifold with non-empty boundary and non-empty set of invertible points is the n-cell.

S is said to be invertible about a subset A of S if, for each open neighborhood U of A, there is a homeomorphism h of S onto S such that $h(S-U)$ lies in U. The inversion subset A of S is minimal if S is not invertible about a proper subset of A. Theorem: Let T be a metric space which has a bi-collared simple closed curve as a minimal inversion subset; then T is a torus or the Klein bottle.

The authors show that the Hilbert cube, Hilbert space, and the countably infinite product of open intervals are invertible spaces and ask if the product of two finite-dimensional continua can be invertible.

Relationships between local invertibility and local near-homogeneity are pointed out, and these concepts are further discussed when modified by the requirement that the homeomorphisms involved be isotopic to the identity.

E. Nunnally (Baton Rouge, La.)

Flaksmaier, Ju. [Flachsmeyer, J.] 2764
On the system of canonical open (closed) sets. (Russian)
Dokl. Akad. Nauk SSSR **156** (1964), 32–34.
The author extends his previous work [Math. Nachr. **26** (1963), 57–66; MR **28** #4506] on topological projective spaces and canonical open sets, and obtains a characterization of Hausdorff spaces with isomorphic complete Boolean lattices of canonical open sets in terms of the Stone representation space $M(B)$ of the lattice B.
M. A. Geraghty (Iowa City, Iowa)

Al'fiš, N. Š. 2765
On ultraproducts of topological algebras. (Russian)
Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 3 (40), 7–16.
The author defines the strong product of topological spaces S_t ($t \in T$) as the direct product $\mathsf{P}_{t\in T} S_t$ with a topology whose family of open sets is generated by the family of all sets of the form $\mathsf{P}_{t\in T} G_t$, where $G_t \subseteq S_t$ are open. Given an ultrafilter F of subsets of T, he considers the prime reduced product (ultraproduct) $\mathsf{P}_{t\in T} S_t/F$ and induces in it the topology whose open sets are generated by the images of open sets of the strong product $\mathsf{P}_{t\in T} S_t$; this is called the topological ultraproduct (t.u.). Of course, the natural map $\mathsf{P}_{t\in T} S_t \to \mathsf{P}_{t\in T} S_t/F$ is open. The main results are the following. Each of the classes of topological spaces T_0, T_1, and T_2 is closed under t.u. If not all S_t are connected [compact], then $\mathsf{P}_{t\in T} S_t/F$ is not connected [not compact]. If $\mathfrak{R}_t = \langle X_t, <_t\rangle$ are relational systems, where each $<_t$ is an ordering of X_t, and R_t denotes X_t with the interval-topology given by $<_t$, then the interval-topology of the prime reduced product $\mathsf{P}_{t\in T}\mathfrak{R}_t/F$ is equal to the topology of the t.u. $\mathsf{P}_{t\in T} R_t/F$. Neither connectedness nor compactness is preserved by the t.u. *J. Mycielski* (Wrocław)

Isiwata, Takesi 2766
Some classes of countably compact spaces. (Russian summary)
Czechoslovak Math. J. **14 (89)** (1964), 22–26.
Author's summary: "The present paper studies relations among some classes of countably compact spaces introduced by Z. Frolík [same J. **10 (85)** (1960), 329–338; MR **22** #8480; ibid. **10 (85)** (1960), 339–349; MR **22** #7099]." *J. R. Isbell* (New Orleans, La.)

Pervin, William J. 2767
On separation and proximity spaces.
Amer. Math. Monthly **71** (1964), 158–161.
The axioms for proximity spaces (Efremovich), simple symmetric syntopogenic structures (Császár), separation (Wallace), and subordination (Freudenthal and Aleksandrov) are mentioned, with comments.
A. H. Stone (Rochester, N.Y.)

Banaschewski, Bernhard; Maranda, Jean-Marie 2768
Proximity functions.
Math. Nachr. **23** (1961), 1–37.
This is a largely expository paper on proximity spaces, uniform spaces, compactifications and related subjects, based on the unifying idea of proximity functions. A proximity function α on a set E assigns to each subset A of E a filter $\alpha(A)$ in E and satisfies the following axioms: (1) If $B \in \alpha(A)$, then $B \supseteq A$; (2) If $A \subseteq B$, then $\alpha(A) \supseteq \alpha(B)$; (3) If $B \in \alpha(A)$, then $E - A \in \alpha(E - B)$; (4) For any $B \in \alpha(A)$ there exists C such that $B \in \alpha(C)$ and $C \in \alpha(B)$. It is understood that a filter must contain E and may contain $\varnothing$; thus $E \in \alpha(E)$ and, by (3), $\varnothing \in \alpha(\varnothing)$. Among the theorems proved are the following. If αE is the compact extension of E corresponding to a proximity α, and if $\alpha E - E$ has fewer than 2^c points, then there is a unique uniformity compatible with α. If a compact extension αE arises from a normal basis of the space E, the boundary in E of any open set G is dense in the boundary of the greatest open set H in αE for which $H \cap E = G$.
C. H. Dowker (London)

Kim, Chi Young 2769
Uniformizability of a topological space.
Kyungpook Math. J. **5** (1962), 1–7.
The author gives necessary and sufficient conditions for a topological space to be uniformizable. The notation is unnecessarily complicated and the introduction of convergence classes is superficial.
R. W. Bagley (Coral Gables, Fla.)

Dektjarev, I. M. 2770
The closed-graph theorem for ultracomplete spaces. (Russian)
Dokl. Akad. Nauk SSSR **157** (1964), 771–773.
A uniform space X is called ultracomplete if the space $\mathfrak{F}(X)$ of all closed subsets of X with the Hausdorff uniformity is complete. The following theorem is proved: If the uniform space X is ultracomplete, then every uniformly almost open multivalued mapping with a closed graph from X into a uniform space Y is uniformly open.
A. Wiweger (Warsaw)

Atsuji, Masahiko 2771
Uniform continuity of continuous functions on uniform spaces.
Canad. J. Math. **13** (1961), 657–663.
A uniform space S is called a uc-space if every continuous real function on S is uniformly continuous. A sequence of subsets $\{A_n\}$ is said to be discretely normally separated by $\{B_n\}$ if $\{B_n\}$ is discrete, i.e., each point has a neighbourhood meeting at most one B_n, and there is a continuous function f_n with values 1 on A_n and 0 on $S - B_n$.

A uniform space is uc if and only if for any sequence $\{A_n\}$ discretely normally separated by a sequence $\{B_n\}$, there is an entourage V with $V(A_n) \subset B_n$ for all n. A normal uniform space is uc if and only if each discrete sequence of subsets is uniformly discrete. Also a normal uniform space is uc if and only if for any decreasing sequence $F_1 \supset G_1 \supset \cdots \supset F_n \supset G_n \cdots$ of closed F_n and open G_n with vacuous intersection, there is an entourage V with $G_n \supset V(F_{n+1})$ for all n.

A product $\prod S_\alpha$ of metric spaces is uc if and only if (i) each S_α is compact or (ii) all but finitely many S_α are one-point spaces and either all are uniformly isolated or all are finite except for one which is a uc-space.
C. H. Dowker (London)

Hanai, Sitiro 2772

Open mappings and metrization theorems.

Proc. Japan Acad. **39** (1963), 450–454.

The author first extends a theorem of Nagami [same Proc. **37** (1961), 356–357; MR **26** #1853] by proving that a T_1 space X has a σ-point-finite base if and only if X is the image under an open compact map of some 0-dimensional metric space Y (which can be taken to be separable if and only if X has a countable base). It follows from a theorem of Arhangel'skiĭ [Dokl. Akad. Nauk SSSR **145** (1962), 245–247; MR **25** #2579] that X has a σ-point-finite base if and only if it has a uniform base. The other principal results are as follows. If a collectionwise normal space X is the union of a system $\{G_\alpha|\alpha\in A\}$ of open metrizable subspaces, and if the system is either countable (Theorem 5), or σ-locally-finite and such that the boundary of each G is Lindelöf (Theorem 7), then X is metrizable. If a space X is the union of a σ-locally-finite system of open subspaces whose closures are metrizable, then X is metrizable (Theorem 12, rephrased). {The author has apparently overlooked that his Theorem 13, even in stronger form, is an easy consequence of the observation that X is locally metrizable.}

A. H. Stone (Rochester, N.Y.)

Morita, Kiiti 2773

Products of normal spaces with metric spaces.

Math. Ann. **154** (1964), 365–382.

This paper contains a careful exposition, with complete proofs, of results announced earlier by the author [Proc. Japan Acad. **39** (1963), 148–150; MR **27** #721].

E. Michael (Seattle, Wash.)

Kuz'minov, V. 2774

An example of a dimensionally deficient compactum. (Russian)

Dokl. Akad. Nauk SSSR **141** (1961), 312–315.

For any prime number q, one bores out tunnels in a 3-simplex and identifies mod $2\pi/q$ on the boundaries of the tunnels. In a fine subdivision of the resulting complex one bores tunnels crossing all 2-simplexes and again makes a q-fold identification of the boundaries. One repeats, obtaining an inverse sequence of 3-complexes. The inverse limit is a compact space $P(q)$. The cohomology dimension $\mathrm{cd}_G\, P(q)$ is computed for different coefficient groups G. For example, it is 3 if G is Z_q, the integers mod q, and it is 1 if G is Z_p with $p\neq q$. From Bokštein's formulae, $\dim P(q)\times P(q)=6$, but $\dim P(q)\times P(p)=4$ if $p\neq q$.

C. H. Dowker (London)

Ponomarev, V. 2775

Some applications of projection spectra to the theory of topological spaces. (Russian)

Dokl. Akad. Nauk SSSR **144** (1962), 993–996.

A T_1-space X is normal if and only if there is a continuous closed bicompact map f of some X_0 onto X with $\mathrm{ind}\, X_0=0$ and, if $A=f^{-1}fA$, $\mathrm{Ind}_A\, X_0=0$.

A canonical cover α is a locally finite cover by closures of disjoint open sets, and $\alpha<\beta$ if β refines α and each element of α is the union of the elements of β contained in it. If X is normal, $\Delta X\leqq n$ means that there is a directed set of closed canonical covers of order $\leqq n+1$, at least one of which refines an arbitrary open cover. Then $\dim X\leqq \Delta X$, $\dim X=\Delta X$ if X is metric, and $\Delta X=\infty$ if X is not paracompact. If $\dim X=\Delta X$, then $\mathrm{ind}\, X\leqq \mathrm{Ind}\, X\leqq \dim X=\Delta X$ and, if also X is strongly paracompact, then all these dimensions are equal.

C. H. Dowker (London)

Ponomarev, V. 2776

On paracompact and finally compact spaces. (Russian)

Dokl. Akad. Nauk SSSR **141** (1961), 561–563.

If for every cover ω of a space X there is an ω-map of X onto some paracompact space Y, then X is itself paracompact. This also holds with paracompact replaced by finally compact (Lindelöf) or weakly paracompact (metacompact) or strongly paracompact (an S-space).

A new proof is given, not using Urysohn's lemma, that for each cover ω of a paracompact Hausdorff space X there is an ω-map of X onto some metric space.

C. H. Dowker (London)

Kimura, Nobuo 2777

On the covering dimension of product spaces.

Proc. Japan Acad. **40** (1964), 267–271.

Suppose X and Y are normal spaces, with Y separable metrizable. What further requirements will ensure that $\dim(X+Y)\leqq \dim X+\dim Y$ (where dim denotes the covering dimension)? Weakening some conditions obtained by K. Morita [Amer. J. Math. **75** (1953), 205–223; MR **14**, 893], the author shows that this is true whenever $X\times Y$ is normal and countably paracompact. An example of the reviewer [Bull. Amer. Math. Soc. **69** (1963), 375–376; MR **27** #2956] is used to show that it does not suffice to assume that X is hereditarily paracompact.

E. Michael (Seattle, Wash.)

Trnkova, Vera [Šedivá-Trnková, Věra] 2778

On the closure of classes of spaces under ω-mappings. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 272–274.

Let Y and X be Hausdorff spaces and let ω be an open covering of Y. A mapping $f: Y\rightarrow X$ is an ω-mapping provided there exists an open covering ν of X such that $f^{-1}(\nu)$ refines ω. Every class $\mathfrak{X}$ of spaces generates a new class $\mathfrak{C}(\mathfrak{X})$ consisting of all the spaces Y such that, for every covering ω of Y, there exists an ω-mapping $f: Y\rightarrow X$ into some member X of $\mathfrak{X}$. Let I denote the line segment, E the real line, $I^n, E^n, I^\infty, E^\infty$ Cartesian products with n and $\aleph_0$ factors, respectively, D a discrete space, and B a product of $\aleph_0$ discrete spaces. The author determines the classes $\mathfrak{C}(I^\infty)$, $\mathfrak{C}(E^\infty)$, $\mathfrak{C}(\{E^n, n=1,2,\cdots\})$, $\mathfrak{C}(D)$, $\mathfrak{C}(D\times E^\infty)$, $\mathfrak{C}(\{D\times E^n, n=1,2,\cdots\})$, $\mathfrak{C}(D\times I^\infty)$, $\mathfrak{C}(B\times E^\infty)$. Most of the results are restatements of known facts.

{Reviewer's remark. The notion of the class $\mathfrak{C}(\mathfrak{X})$ generated by $\mathfrak{X}$ has been introduced and studied before [e.g., cf. C. N. Maxwell, Trans. Amer. Math. Soc. **99** (1961), 201–204; MR **22** #11374].}

S. Mardešić (Zagreb)

Brown, R. 2779

Function spaces and product topologies.

Quart. J. Math. Oxford Ser. (2) **15** (1964), 238–250.

The product topologies referred to in the title are those introduced by the author in a previous paper [same J. (2) **14** (1963), 303–319; MR **28** #2516]. The product $X \times_S Y$ of that paper is denoted by $X \boldsymbol{\times} Y$ in the paper under review. The exponential law for function spaces with the compact-open topology is studied first. The main result is that the spaces $X^{Z \boldsymbol{\times} Y}$ and $(X^Y)^Z$ are homeomorphic for all X, Y, and Z. Hence the exponential law holds for $Z \times Y$ if $Z \times Y = Z \boldsymbol{\times} Y$, and this contains and also explains many known results. Some new results are also deduced; a typical one is that the spaces $(X^Y)^Z$ and $(X^Z)^Y$ are not naturally homeomorphic. Secondly, the law $(X \times Y)^Z = X^Z \times Y^Z$ is discussed and is shown to fail in general for products other than the cartesian. Thirdly, the category $\mathscr{K}$ of Hausdorff spaces and functions continuous on compact subspaces (these functions are called k-continuous functions) is considered. It is proved that in this category (i) the product of k-identification maps is a k-identification map, (ii) the exponential map $\mu: X^{Z \times Y} \to (X^Y)^Z$ is defined and is a homeomorphism, and (iii) the natural map $\rho: (X \times Y)^Z \to X^Z \times Y^Z$ is a homeomorphism. Finally, the exponential law is generalised to the category of M-ads. An M-ad $\mathbf{X} = (X, M)$ consists of a space X and a family $\{X_m\}_{m \in M}$ of subspaces of X. In the category of M-ads a map $f: \mathbf{X} \to \mathbf{Y}$ of M-ads $\mathbf{X}$, $\mathbf{Y}$ is a k-continuous function $f: X \to Y$ such that $f(X_m) \subseteq Y_m$ $(m \in M)$.

S. Swaminathan (Kanpur)

Edelstein, Michael 2780

On nonexpansive mappings.

Proc. Amer. Math. Soc. **15** (1964), 689–695.

Let f map E^n into itself, increasing no distance. Either the images $f^m(E^n)$ form a locally finite collection or there is a fixed point. In the latter case there is exactly one fixed point in each flat spanned by an orbit.

J. R. Isbell (New Orleans, La.)

Doičinov, D. 2781

Necessary and sufficient conditions for uniform imbedding of a space into Euclidean space. (Russian)

Dokl. Akad. Nauk SSSR **138** (1961), 1276–1279.

A metric separable space R is called non-condensing if there is a positive distance d such that, for every covering β of positive Lebesgue number, there is a refinement α of positive Lebesgue number and a number p_α such that every set of diameter less than d meets at most p_α elements of α. If R has uniform dimension zero, it can be uniformly imbedded in a Euclidean space (indeed, in a line) if and only if it is non-condensing. The dimensional restriction is necessary. For example, if R is a tree, each branch of which proceeds unit distance and then splits in two, and if distance in R means distance along the tree, then R is non-condensing but cannot be uniformly imbedded in a Euclidean space. Necessary and sufficient conditions are found for imbedding a space of positive uniform dimension.

C. H. Dowker (London)

Haupt, Otto 2782

Ein Kriterium für Bogensummen in der Ebene.

Abh. Math. Sem. Univ. Hamburg **27** (1964), 124–141.

Let G be a closed circular disc in the plane. The author considers a family $\mathfrak{k}$ of arcs and simple closed curves in G, the curves meeting bdry G in one point, the arcs meeting bdry G in their endpoints. The set $\mathfrak{k}$ is a metric space as a subspace of the space of compacta in G. It is assumed of $\mathfrak{k}$ that there is an integer $k \geqq 2$ such that if $K \in \mathfrak{k}$, $x_i \in K$, $i = 1, \cdots, k$, $x_i \neq x_j$ unless $i = j$, then there are open sets $U_1, \cdots, U_k$, $x_i \in U_i$, $\bar{U}_i \cap \bar{U}_j = 0$ unless $i = j$, such that if $x_i' \in U_i$, $i = 1, \cdots, k$, then there is a unique element $K' \in \mathfrak{k}$ containing $x_1', \cdots, x_k'$. The set K' is to depend continuously on the x_i', and other continuity conditions are also imposed. Further, there are open sets W and U^* in int G such that $\bar{U}^* \subset \text{int } G - \bar{W}$, and subfamilies $\mathfrak{b}$ of $\mathfrak{k}$ satisfying most conditions required of $\mathfrak{k}$ such that each point of $\bar{W}$ lies on exactly one element of $\mathfrak{b}$ and if $y \in U^*$ there is at least one such subfamily $\mathfrak{b}$ each element of which contains y.

Suppose that C is a continuum in W and that there are integers m', m such that m is the least integer m' such that the set of $K \in \mathfrak{k}$ for which $C \cap K$ has more than m' components is nowhere dense in $\mathfrak{k}$. This is the weak component order of C. The number obtained if the word "component" is replaced by the word "point" is the weak point order. The author proves that if the weak component order of C is m, then the following are equivalent: (1) the weak point order of C is m; (2) C is hereditarily the union of countably many non-overlapping arcs (an arc sum) and is a regular curve; (3) C is an arc sum; (4) the set of K in $\mathfrak{k}$ such that $C \cap K$ is totally disconnected is dense in $\mathfrak{k}$.

M.-E. Hamstrom (Urbana, Ill.)

Mardešić, Sibe 2783

ε-mappings and inverse limits. (Serbo-Croatian summary)

Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II **18** (1963), 195–205.

Let Π be a class of connected polyhedra and X a compact Hausdorff space admitting an α-map onto a member of Π for each open covering α. Then X is the limit space of an inverse system of metric continua, each admitting α-maps onto members of Π, with bonding mappings onto. In short, every Π-like continuum is an onto limit of onto limits of members of Π. *J. R. Isbell* (New Orleans, La.)

Kruming, P. D. 2784

l-r semigroups. (Russian)

Leningrad. Gos. Ped. Inst. Učen. Zap. **238** (1962), 89–94.

For a subset A of a semigroup S, let

$$l(A) = \{x \in S : xS \cap A \neq \varnothing\}$$

and $r(A) = \{x \in S : Sx \cap A \neq \varnothing\}$. A topological semigroup is called an l-r semigroup if for every compact set $A \subset S$, there is a set $B \supset A$ such that $l(B)$ and $r(B)$ are compact. Theorem 1: Let S be a Hausdorff l-r semigroup. Let $S^* = S \cup \{0\}$, where $0x = x0 = 0$ for all $x \in S^*$, and topologize S^* so that S^* is the one-point compactification of S. Then S^* is a topological semigroup. If S is locally compact, S^* is Hausdorff. Using these observations, the author generalizes a number of theorems concerning maximal ideals, idempotents, etc., in compact semigroups to locally compact Hausdorff l-r semigroups. Theorem 2: The finite product of l-r semigroups is an l-r semigroup.

K. A. Ross (New Haven, Conn.)

Kruming, P. D. 2785
Factor semigroups of topological semigroups. (Russian)
Leningrad. Gos. Ped. Inst. Učen. Zap. **238** (1962), 95–101.

Let S be a topological semigroup, ρ a stable (compatible) equivalence relation on S, and ϕ the natural mapping of S onto S/ρ. Topologize S/ρ with the weakest topology making ϕ continuous. Theorem: If ϕ is an open mapping or if ϕ is closed and each equivalence class is compact, then S/ρ is a topological semigroup. Some applications and examples are given. Included are some compact connected Hausdorff semigroups which are regular but not completely regular. {Terms are defined in Ljapin's book [*Semigroups* (Russian), Fizmatgiz, Moscow, 1960; MR **22** #11054; English transl., Amer. Math. Soc., Providence, R.I., 1963; MR **29** #4817].}

K. A. Ross (New Haven, Conn.)

Brown, Dennison R. 2786
On clans of non-negative matrices.
Proc. Amer. Math. Soc. **15** (1964), 671–674.

Let N_n denote the semigroup of $n \times n$ matrices having non-negative entries (endowed with the usual Euclidean topology). The main result of this paper is that a compact connected subsemigroup of N_n with an identity contains a standard thread from the identity to the minimal ideal and is contractible. The proof depends upon the following interesting lemma: A compact subgroup of N_n is finite.

Lee W. Anderson (State College, Pa.)

Neuwirth, Lee 2787
In-groups and imbeddings of n-complexes in $(n+1)$-manifolds.
Bull. Amer. Math. Soc. **70** (1964), 737–738.

The author establishes a non-imbedding criterion for n-dimensional manifolds which can give results when homological arguments fail. For any finite n-complex K^n, a finite set of groups, called out-groups, are defined and can be computed. The main result is that K^n cannot be semi-linearly imbedded in the closed orientable manifold M^{n+1} in such a way that $M^{n+1}-K^n$ is connected unless one of the out-groups is a homomorphic image of $\pi_1(M^{n+1})$. This follows from the result that any semi-linear imbedding of K^n in a closed orientable manifold M^{n+1} (with $M^{n+1}-K^n$ connected) determines a specific out-group, called the in-group. It is proved that the in-group is a homomorphic image of $\pi_1(M^{n+1})$.

As an example, let K^2 be the 2-complex obtained by identifying opposite faces of an octahedron as described by Seifert and Threlfall [*Lehrbuch der Topologie*, p. 213, Teubner, Leipzig, 1934]. One can calculate that every out-group of K^2 has the cyclic group Z_3 as homomorphic image. It follows that K^2 cannot be semi-linearly imbedded in a closed orientable 3-manifold M^3 (with M^3-K^2 connected) unless Z_3 is a homomorphic image of $\pi_1(M^3)$.

M. L. Curtis (Houston, Tex.)

Greathouse, Charles 2788
The equivalence of the Annulus Conjecture and the Slab Conjecture.
Bull. Amer. Math. Soc. **70** (1964), 716–717.

From the author's introduction: "In same Bull. **70** (1964), 415–418 [MR **28** #4526], the author showed that the Slab Conjecture implies the Annulus Conjecture. The purpose of this paper is to show that the Annulus Conjecture implies the Slab Conjecture for $n>3$ and hence the two conjectures are equivalent for $n>3$. R^n, S^n will denote n-space and the n-sphere, respectively. A k-manifold N is embedded in a locally flat manner in an n-manifold M provided each point of N has a neighborhood U in M such that $(U, U\cap N)\approx(R^n, R^k)$. The Annulus Conjecture: Let S_1^{n-1}, S_2^{n-1} be disjoint locally flat $(n-1)$-spheres embedded in S^n and let M be the submanifold of S^n bounded by $S_1^{n-1}\cup S_2^{n-1}$; then $M\approx S^{n-1}\times[0, 1]$. The Slab Conjecture: Let R_1^{n-1}, R_2^{n-1} be disjoint locally flat $(n-1)$-spaces embedded as closed subsets of R^n and let M be the submanifold of R^n bounded by $R_1^{n-1}\cup R_2^{n-1}$; then $M\approx R^{n-1}\times[0, 1]$."

The technique is to use the one-point compactification and apply some results due to Cantrell on almost locally flat embeddings. *M. L. Curtis* (Houston, Tex.)

Bing, R. H. 2789
Spheres in E^3.
Amer. Math. Monthly **71** (1964), 353–364.

In this expository paper the author considers questions relating to the embedding of 2-spheres in Euclidean 3-space E^3. The author first gives a clear description of a wild 2-sphere S in E^3 such that S is pierced at each point by a line segment; this is a modification of an example due to M. K. Fort. Next, he gives a neat proof, using simple integration, of the Tietze extension theorem, which is then used to construct a neighbourhood N of a given 2-sphere S in E^3 and a retraction $N\to S$. He remarks that this explicit construction is lacking in his proof [same Monthly **71** (1964), 481–484; MR **28** #5435] that for any $p\in \operatorname{Int} S$, there is a retraction of $E^3-\{p\}$ onto S. The paper concludes with some remarks about Antoine's necklace, which is used to construct a horned-sphere.

H. B. Griffiths (Birmingham)

Shields, Allen L. 2790
On fixed points of commuting analytic functions.
Proc. Amer. Math. Soc. **15** (1964), 703–706.

Let f and g be two continuous functions which map the closed interval $[0, 1]$ into itself and commute $(f(g(x))=g(f(x))$ for all x). In 1954, Eldon Dyer asked if f and g must have a common fixed point. The same question was raised independently by the author in 1955 and by Lester Dubins in 1956. A more general form was posed by J. R. Isbell in 1957. (For a recent partial solution to this question see H. Cohen [Proc. Amer. Math. Soc. **15** (1964), 293–296].)

In this paper the author considers the above question for commuting functions on the unit disc in the complex plane, and proves the following theorem. Let F be a commuting family of continuous functions mapping the closed unit disc into itself and analytic in the open disc. Then there is a common fixed point for all the functions of the family.

The theorem follows from five lemmas, two of which the author proves by interesting application of results from the theory of topological semigroups.

Anne L. Hudson (Syracuse, N.Y.)

ALGEBRAIC TOPOLOGY

See also 2787, 2813.

Artin, Emil; Braun, Hel 2791

★Vorlesungen über algebraische Topologie.
Ausgearbeitet von Armin Thedy.
Erhältlich durch A. Thedy, Mathematisches Seminar der Universität Hamburg, Hamburg, 1964. v+173 *pp.* *DM* 8.00.

These are the notes to a semester's course given at Hamburg University. First the singular theory is developed, directly without the intervention of the simplicial theory. Then the axioms are given as a generalization of the singular theory, and, except for § 14 (the Jordan-Brouwer separation theorem), the axiomatic theory is used from that point on. The various modern devices (categories, functors, diagrams, tensor product, etc.) that have become imbedded in homology theory are used relentlessly to give an efficient and algebraic development of the theory. However, a real effort has been made to ameliorate this with geometric explanations of what is going on. Furthermore, actual examples are brought in as soon as possible: the homology groups of points, spheres, graphs, and plane domains are calculated in § 7, and their cohomology groups in § 10; the (homology and cohomology) groups of the closed surfaces, real and complex projective spaces, and lens spaces are calculated in § 13; in § 15 is given the way to calculate the groups of what the author calls a spherical complex (essentially a CW-complex). Other applications are the degree of a mapping and Brouwer's fixed-point theorem (§ 7), the winding number and Cauchy's integral theorem (§ 11), the Jordan-Brouwer theorem together with a brief discussion of the Jordan curve theorem and the pathology of 3-dimensional space (§ 14), invariance of domain and of dimension (§ 14). The book ends with the classification of the closed surfaces (§ 19). The reviewer has the feeling that some minor innovations have been made, making for a somewhat more streamlined development than usual.

In general, the famous Artin clarity is evident throughout. However, there is some evidence of carelessness in the writing: on p. 92 there are several occurrences of $ra-sb$ that should have been $sa-rb$, on p. 111 the symbol C is suddenly used (for the complex plane) with no explanation, and confusion is compounded by the simultaneous occurrence of some kind of a misprint; **C** is used for "complement" (pp. 131ff.) without explanation, superscript o is used for "interior" (pp. 134ff.) also without explanation, the construction of the "sum" of two manifolds is made twice (p. 149 and p. 166), and the statement (p. 168) of Rado's theorem about the triangulability of 2-manifolds leaves out the hypothesis of separability.

As far as the reviewer is aware, this is the only book to combine a modern approach with an early introduction to specific calculations and applications, and as such is highly recommended for classroom use. It should be added that the typography is most unattractive.

R. H. Fox (Princeton, N.J.)

Ádám, A.; Čulík, K.; Pollák, G. 2792

Ein Satz über teilweise gerichtete Graphen. (Russian summary)
Czechoslovak Math. J. **13** (88) (1963), 619–621.

Consider a directed graph determined by a binary irreflexive relation $\rho(x,y)$ defined on a set of points. Let $\sigma(x,y)$ denote the relation $\rho(x,y)$ & $\rho(y,x)$. A many-to-one mapping $x\to x'$ of the points of the directed graph G into the points of the directed graph G' is called a strong homomorphism if

$$\sigma(x,y)\Leftrightarrow x'=y'\quad (x\neq y),$$
$$\rho(x,y)\ \&\ \bar\rho(y,x)\Rightarrow \rho(x',y')\ \&\ \bar\rho(y',x'),$$
$$\bar\rho(x,y)\ \&\ \bar\rho(y,x)\Rightarrow \bar\rho(x',y')\ \&\ \bar\rho(y',x').$$

It is shown that G has a strong homomorphism if and only if $\sigma(x,y)$ & $\rho(x,z)\Rightarrow\rho(y,z)$ and $\sigma(x,y)$ & $\rho(z,x)\Rightarrow\rho(z,y)$ for each triplet x,y,z of distinct points of G.

J. W. Moon (Edmonton, Alta.)

Leĭfman, L. Ja.; Petrova, L. T. 2793

Some algorithms for analyzing oriented graphs. (Russian)
Vyčisl. Sistemy No. 11 (1964), 101–113.

Harary, Frank; Prins, Geert 2794

Enumeration of bicolourable graphs.
Canad. J. Math. **15** (1963), 237–248.

This paper corrects and extends results contained in the first author's paper [Pacific J. Math. **8** (1958), 743–755; MR **21** #2598]. The counting series for bicoloured graphs without isolated points and for connected bicoloured graphs are obtained. Corresponding results are found for bicolourable graphs, i.e., graphs whose points can be coloured with two colours so as to obtain a bicoloured graph. *J. W. Moon* (Edmonton, Alta.)

Edmonds, Jack 2795

Existence of k-edge connected ordinary graphs with prescribed degrees.
J. Res. Nat. Bur. Standards Sect. B **68B** (1964), 73–74.

An ordinary graph G is a graph without loops or multiple edges; G is k-edge connected if every cut set of edges has at least k members. It is shown that, for $k>1$, the integers d_i, $i=1,2,\cdots,n$ $(n>1)$, are the respective degrees of the vertices of some k-edge connected ordinary graph if and only if (1) they are the degrees of the vertices of some ordinary graph, and (2) all $d_i\geqq k$. (For $k=1$, conditions (1) and (2) must be supplemented with (3) $\sum_{i=1}^{n} d_i\geqq 2(n-1)$.) The proof of sufficiency for $k>1$ proceeds inductively by starting with an h-edge connected ordinary graph H, $1\leqq h<k$, having degrees $d_i\geqq k$, and showing how to construct a new ordinary graph H' whose vertices have degrees d_i, which is at least h-edge connected, and which has fewer cuts with h members than H does.

D. R. Fulkerson (Santa Monica, Calif.)

Klee, Victor 2796

Diameters of polyhedral graphs.
Canad. J. Math. **16** (1964), 602–614.

Let P be a d-polytope, that is, a d-dimensional convex polytope. Then P has two associated graphs $\Gamma(P)$ and $\Pi(P)$. The first is made up of the edges and vertices of P. The second has edges and vertices corresponding to the $(d-2)$-faces and $(d-1)$-faces of P, respectively. The

author denotes the diameters of $\Gamma(P)$ and $\Pi(P)$ by $\delta(P)$ and $\phi(P)$, respectively.

P is called "simple" if each vertex is incident with exactly d edges, and "simplicial" if each $(d-1)$-face is incident with exactly d $(d-2)$-faces. The classes of d-polytopes, simple d-polytopes and simplicial d-polytopes are denoted by $\mathbf{P}_d$, $\mathbf{P}_d{}^v$ and $\mathbf{P}_d{}^f$, respectively. The author defines $f_s(P)$ as the number of s-faces of P and writes

$$\Delta_s(d, n) = \max\{\delta(P) : P \in \mathbf{P}_d \text{ and } f_s(P) \leqq n\},$$

$$\Phi_s(d, n) = \max\{\phi(P) : P \in \mathbf{P}_d \text{ and } f_s(P) \leqq n\}.$$

The expressions $\Delta_s{}^f(d, n)$, $\Delta_s{}^v(d, n)$, $\Phi_s{}^f(d, n)$ and $\Phi_s{}^v(d, n)$ denote corresponding maxima over $\mathbf{P}_d{}^f$ or $\mathbf{P}_d{}^v$. By polarity we have $\Phi_s = \Delta_{d-1-s}$, $\Phi_s{}^v = \Delta^f_{d-1-s}$ and $\Phi_s{}^f = \Delta^v_{d-1-s}$.

The author obtains the following results. (1) $\Delta_0 = \Delta_0{}^f$ and $\Delta_{d-1} = \Delta^v_{d-1}$; (2) $\Delta_0{}^v(3, 2k) = \Delta_0(3, 2k)$; (3) $\Delta_0{}^v(d, n) \geqq (d-1)[(n-2)/(2^d-2)]+1$ $(n \geqq 2^d)$; (4) $\Delta^f_{d-1}(d, n) \geqq [(n-2d)/(2^d-2)]+2$; (5) $\Delta_{d-1}(d, n) \geqq (d-1)[n/d]-d+2$.

The second result disproves, in the special case $d=3$, a conjecture of Grünbaum and Motzkin to the effect that $\Delta_0{}^v$ is significantly less than Δ_0. The author improves the fifth result to (6) $\Delta_{d-1}(d, n) = [n(d-1)/d]-d+2$ in the special cases $d \leqq 3$ and $n \leqq d+4$, and he conjectures that (6) is true whenever $n > d > 1$.

The author concludes by discussing the bearing of his results on the "m-step" conjecture of linear programming.

W. T. Tutte (Waterloo, Ont.)

Erdős, Paul; Sachs, Horst 2797

Reguläre Graphen gegebener Taillenweite mit minimaler Knotenzahl.

Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe **12** (1963), 251–257.

The authors define $f(k, l)$ as the least integer n such that there exists a regular graph of n vertices whose valency is k and whose girth is l. Such a graph with $f(k, l)$ vertices is called "minimal". In the first part of the paper the authors establish upper and lower bounds for $f(k, l)$ and list all the cases in which $f(k, l)$ is known exactly.

In the second part the authors prove some theorems about minimal graphs. For example, the diameter of a minimal graph is at most l, and if $k=2h$, then the number of edges whose ends are both at distance l from a given vertex x is less than h. *W. T. Tutte* (Waterloo, Ont.)

Fox, R. H.; Smythe, N. 2798

An ideal class invariant of knots.

Proc. Amer. Math. Soc. **15** (1964), 707–709.

Let M denote an $m \times n$ matrix with entries in a commutative integral domain R. Suppose M has rank r. Let $M^{(r)}$ denote the $\binom{m}{r}$-by-$\binom{n}{r}$ matrix of $r \times r$ minors with rows and columns written in lexicographic order. Recall that two ideals I, J in R are equivalent if $PI = QJ$, where P and Q are principal.

The equivalence class containing the ideal generated by the elements in a row [column] of $M^{(r)}$ is independent of the row [column] chosen, and the authors prove that this row [column] class is an invariant of the Alexander matrix of a knot group, where the ring in question is the group ring of the infinite cyclic group modulo any irreducible factor of the Alexander polynomial. Using the ideal class as an invariant, the authors distinguish two knots which have the same elementary ideals, torsion numbers, linking invariants and Minkowski units.

L. Neuwirth (Princeton, N.J.)

Milnor, J. 2799

Most knots are wild.

Fund. Math. **54** (1964), 335–338.

In the space of homeomorphisms of S^1 into R^n there is a dense G_δ of wild (and therefore knotted) imbeddings if $n=3$, whereas if $n \geqq 4$, there is a dense G_δ of unknotted (and therefore tame) imbeddings. The first of these results is one interpretation of the (intentionally) ambiguous title. Another possible interpretation (in terms of probability) is discussed briefly, along with other open questions that occur naturally. The reviewer would like to remark that recently problems in this area have been raised by chemists and biologists.

R. H. Fox (Princeton, N.J.)

Baĭdosov, V. A. 2800

Topologizing groups of singular integral chains. (Russian)

Sibirsk. Mat. Ž. **2** (1961), 331–340.

Since a singular q-simplex is a map $t : s_q \to R$, the (ordered or oriented) singular q-simplexes in a metric space R form a metric space with metric $\rho(t_1, t_2) = \sup_{x \in s_q} \rho(t_1 x, t_2 x)$. This metric leads, by the procedure of Mal'cev [Izv. Akad. Nauk SSSR Ser. Mat. **21** (1957), 171–198; MR **20** #5249], to a topology in the free abelian group $C_q(R)$ of singular chains. The boundary operator is a continuous homomorphism $C_q \to C_{q-1}$; hence the q-cycles form a closed subgroup Z_q of C_q. If R is locally arcwise connected, the boundary operator $C_1 \to C_0$ is an open map, and the group B_0 of 0-boundaries is an open, and hence closed, subgroup of C_0. *C. H. Dowker* (London)

Tsuchida, Kisuke; Ando, Hideo 2801

On the generalized cohomology suspension.

Sci. Rep. Fac. Lit. Sci. Hirosaki Univ. **10** (1963), 35–45.

In this paper, the authors generalise the cohomology suspension $H^q(B) \to H^{q-1}(\Omega B)$ for a 1-connected space B by the method of Eckmann and Hilton, and obtain an exact sequence involving it.

S.-T. Hu (Los Angeles, Calif.)

Izuho, Shigeru 2802

Groupoid and cohomology with values in a sheaf of groupoids.

J. Sci. Hiroshima Univ. Ser. A-I Math. **27** (1963), 61–72.

$\mathfrak{P}$ étant un faisceau de groupoïdes transitifs (connexes) sur un espace X, A. Haefliger a montré [Comment. Math. Helv. **32** (1958), 248–329, Chap. II, Prop. 7; MR **20** #6702] qu'à tout $f \in Z^1(\mathfrak{U}, \mathfrak{P})$, $\mathfrak{U}$ un recouvrement ouvert de X, correspondent (a) un faisceau $\mathfrak{G}^f$ de groupes sur X; (b) une bijection $\varphi_f : H^1(X, \mathfrak{P}) \to H^1(X, \mathfrak{G}^f)$ envoyant la classe $\{f\}$ sur la classe neutre. Partant d'un faisceau $\mathfrak{H}$ de groupes sur X, l'auteur construit par un procédé assez trivial un faisceau $\mathfrak{P}$ et un élément $f \in Z^1(\mathfrak{U}, \mathfrak{P})$ tel que $\mathfrak{G}^f$ soit isomorphe à $\mathfrak{H}$. Il utilise une notion de sous-groupoïde normal qui, bien que restreinte, n'est cependant

pas sans intérêt et intervient, par exemple, dans un article de J. Dixmier et A. Douady [Bull. Soc. Math. France **91** (1963), 227–284; MR **29** #485]. Cette notion est associée à un type simplifié de suite exacte courte de groupoïdes donnant lieu à une suite exacte de cohomologie en dimensions zéro et un, alors qu'il est connu des spécialistes que cette suite pose des problèmes si l'on utilise la notion la plus générale de sous-groupoïde normal.

P. Dedecker (Rhode-St.-Genèse)

Mukhopadhyay, Amiya Gopal 2803

On complexes of higher order. I.

Bull. Calcutta Math. Soc. **54** (1962), 47–57.

A complex of order r is a sequence of abelian groups $\{C^p\}$ and homomorphisms $\delta^p: C^p \to C^{p+1}$ such that $\delta^{p+r-1} \cdots \delta^{p+1}\delta^p = 0$. A group, called the pth cohomology group of the complex of order r, is defined in terms of kernels and cokernels of combinations of the δ's. It is then shown that this group is isomorphic to the pth cohomology group of a certain complex of order $r-1$.

F. Raymond (Ann Arbor, Mich.)

Jussila, Olli 2804

On homology theories in locally connected spaces.

Ann. Acad. Sci. Fenn. Ser. A I No. 340 (1964), 15 *pp.*

The reviewer has proved that for Hausdorff paracompact spaces X the natural homomorphisms t_p of the singular homology groups $H_p(X)$ into the Čech homology groups $\check{H}_p(X)$ are isomorphisms provided X is lc_s^{n-1} (locally connected in the sense of singular homology up to dimension $n-1$) and semi-n-lc_s (semi-locally connected in dimension n) and $p < n+1$ [the reviewer, Michigan Math. J. **6** (1959), 151–166; MR **21** #4415]. The aim of the present paper is to give a new proof of this theorem in a somewhat more general setting.

Let C be a fixed complex on a space X; it is defined as a covariant functor from the category of subsets of X to the category of augmented chain complexes over a ground ring R. With every covering α of X the author associates an augmented double chain complex $\overset{\alpha}{C} = \{\overset{\alpha}{C}_{p,q};\ \partial',\ \partial''\}$, where $\overset{\alpha}{C}_{-1,q} = C_q(X)$ and $\overset{\alpha}{C}_{p,q} = \sum C_q(A_{i_0} \cap \cdots \cap A_{i_p})$, $A_{i_j} \in \alpha$, for $p > -1$. The boundary operator $\partial': \overset{\alpha}{C}_{p,q} \to \overset{\alpha}{C}_{p-1,q}$ is defined as usual on the nerve of a covering, while $\partial'': \overset{\alpha}{C}_{p,q} \to \overset{\alpha}{C}_{p,q-1}$ is induced by the boundary operators in the chain complexes $C(A_{i_0} \cap \cdots \cap A_{i_p})$. $H_p(\partial' \overset{\alpha}{C}_{*,-1})$ is the homology (with coefficients in R) of the nerve of α and is denoted by $H_p(\alpha)$. A net N is a directed system of coverings α, and $H_p(N)$ denotes the inverse limit of modules $H_p(\alpha)$, $\alpha \in N$. If N is the net $\mathrm{Cov}(X)$ of all open coverings of X, then $H_p(N)$ becomes the usual Čech module $\check{H}(X)$ with coefficients in R. $H_p(\partial'' \overset{\alpha}{C}_{-1,*})$ is the homology of $C(X)$ and is denoted by $H_p(X)$. If C is the singular complex on X, then $H_p(X)$ is the usual singular homology module. Next the author requires that C be fine with respect to all $\alpha \in N$, i.e., that $\partial' \overset{\alpha}{C}_{*,q}$ be acyclic for $q > -1$. In this case one obtains canonical homomorphisms $h_p^{\alpha}: H_p(X) \to H_p(\alpha)$ which induce homomorphisms $t_p: H_p(X) \to H_p(N)$ [cf. the author, Ann. Acad. Sci. Fenn. Ser. A I No. 326 (1962); MR **26** #5567].

If X is paracompact and C is fine with respect to $\mathrm{Cov}(X)$, then it is shown that $t_p: H_p(X) \to \check{H}_p(X)$ is an isomorphism provided X is lc^{n-1} and semi-n-lc (in the sense of the homology of the complex C) and $p < n+1$. Consequently, the reviewer's theorem remains true if singular homology is replaced by the homology of any fine complex C.

S. Mardešić (Zagreb)

Kuz'minov, V. 2805

On V^n continua. (Russian)

Dokl. Akad. Nauk SSSR **139** (1961), 24–27.

Each n-dimensional bicompact space X contains an n-dimensional subspace Y which is not only a Cantor manifold but even a V^n-continuum, that is, for each pair G, H of non-intersecting non-empty open sets of Y there is a cover ω of Y such that no closed set C separating G from H has an ω-map into any $(n-2)$-polyhedron.

Indeed, if B is any closed set of X for which $H^n(X, B) \neq 0$ and e is any non-zero element of $H^n(X, B)$, by the continuity of the cohomology groups there is a minimal closed set Y for which $i^*e \neq 0$, where $i^*: H^n(X, B) \to H^n(Y, Y \cap B)$. A cover ω of Y can be found such that, if C separates any non-empty open sets G, H in Y, then the homomorphism $H^{n-1}(N_{\omega|C}, N_{\omega|C\cap B}) \to H^{n-1}(C, C \cap B)$ is non-trivial, where $N_{\omega|C}$ is the nerve of the restriction of ω to C. Hence Y is a V^n-continuum.

The continuum Y has other interesting properties. Let $A = Y \cap B$ and let a be an arbitrary point of $Y \setminus A$. For each neighbourhood U of a, smaller than $X \setminus A$, the homomorphism $H^n(X, X \setminus U) \to H^n(X, A)$ is not trivial. For each neighbourhood W of a in Y for which $\overline{W} \cap A = \varnothing$, the homomorphism $H^{n-1}(\overline{W}) \to H^{n-1}(\dot{W})$ is not surjective. ($\overline{W}$ is the closure and $\dot{W}$ the boundary of W.)

C. H. Dowker (London)

James, I. M. 2806

Quasigroups and topology.

Math. Z. **84** (1964), 329–342.

From the author's introduction: "This paper is partly an attempt to lay some foundations for a theory of H-spaces, avoiding the usual restriction to pathwise-connected spaces. Our definitions, and some of our theorems, are placed in a categorical setting, where the notion of quasigroup arises most naturally. An H-space, by our definition, is the topological representative of a quasigroup in the homotopy category. Some alternative definitions are considered and the situation in related categories is briefly investigated. Most, if not all, of the contents of this paper can be dualized."

A. H. Clark (Providence, R.I.)

Hilton, P. J. 2807

Remark on loop spaces.

Proc. Amer. Math. Soc. **15** (1964), 596–600.

This paper is the dual of a previous paper by I. Berstein and the author [Topology **2** (1963), 73–82; MR **27** #762] and contains the following two results. Theorem C: If (X, μ) is a space with multiplication, if X is $(q-1)$-connected, $q > 1$, and $\pi_n(X) = 0$ for $n \geqq 3q$, then (X, μ) is

equivalent to a loop-space structure. Theorem D: If $\pi_n(A)=0$, $n \geqq 3q-1$, and B is $(q-1)$-connected, $q>1$, then every homomorphism $\Omega B \to \Omega A$ is homotopic to a loop map. *A. H. Clark* (Providence, R.I.)

Shen, Xin-yao [Shen, Hsin-yao] 2808
On natural systems and homotopy types of $A_n{}^2$-polyhedra.
Acta Math. Sinica **13** (1963), 462–470 (*Chinese*); *translated as Chinese Math.* **4** (1964), 501–510.

In this paper, the author gives a direct proof of the fact that the Postnikov natural systems of two $A_n{}^2$-polyhedra K and L are isomorphic if and only if their normal forms (in the sense of S. C. Chang) are identical. Discussions about mutual dependence of the natural system, Betti numbers, torsions and secondary torsions of $A_n{}^2$-polyhedra are also given. *S.-T. Hu* (Los Angeles, Calif.)

Gitler, Samuel 2809
Spaces fibered by H-spaces. (Spanish)
Bol. Soc. Mat. Mexicana (2) **7** (1962), 71–84.

Let A be a connected graded differential algebra over a field K of characteristic $p \neq 0, 2$. Let $\bar{B}(A)$ be the co-algebra obtained by applying the bar construction to A. If the multiplication in A is homotopic to an algebra map $A \otimes A \to A$ such that $H(A)$ is an associative commutative Hopf algebra, then $\bar{B}(A)$ is a Hopf algebra and the terms E^r, $r>1$, of the Moore spectral sequence for $\bar{B}(A)$ in homology are associative commutative Hopf algebras such that the differentials d^r are derivations of Hopf algebras. Moreover, $E^1 \cong \bar{B}(HA)$, $E^2 \cong \mathrm{Tor}^{H(A)}(K,K)$ and these are isomorphisms of Hopf algebras if $K=Z_p$; and in E^2 the primitive elements are of filtration $\leqq 2$ and are indecomposable.

Now let $\sigma_*: H_q(A) \to H_{q+1}(\bar{B}(A))$ be the suspension. Then σ_* maps the indecomposable elements of $H_q(A)$, $Q_q(HA)$, to the primitive elements of $H_{q+1}(\bar{B}A)$, $P_{q+1}(H\bar{B}A)$. By studying the homology and cohomology spectral sequences of $\bar{B}A$ and applying results of Moore, the author shows that if $\pi: E \to X$ is a fibration of an acyclic space E with fibre M, such that both X and M are H-spaces, then the geometric suspension k_*: $H(M) \to H(X)$ coincides with the suspension $\sigma_*: H(M) \to \mathrm{Tor}^{C(M)}(Z_p, Z_p)$; and thence the main theorem (also proved by W. Browder):

$$k_*: Q_q(H(\Omega X; Z_p)) \to P_{q+1}(H(X; Z_p)),$$

for X an H-space, is an isomorphism if q is even and $\not\equiv -2 \bmod 2p$, or if q is odd and $\not\equiv 1 \bmod 2p$.

The author remarks that, having obtained in the case when $H^*(X; Z_p)$ has no generators of infinite height necessary and sufficient conditions on the differentials of the spectral sequence for $H_*(\Omega X; Z_p)$ to have no generators of odd degree, it should be possible to demonstrate Bott's theorem on the absence of torsion in the loop space of a Lie group. *P. J. Hilton* (Ithaca, N.Y.)

Gitler, Samuel 2810
On the Cartan suspension. (Spanish)
Bol. Soc. Mat. Mexicana (2) **7** (1962), 85–91.

Cartan [Séminaire Henri Cartan, 1954/55, Secrétariat mathématique, Paris, 1956; MR **19**, 439] defined an operation which he called transpotence in the homology of Eilenberg-MacLane spaces, thus

$$\phi_p: H_{2q}(\pi, n; Z_p) \to H_{2pq+2}(\pi, n+1; Z_p).$$

Then ϕ_p is zero on decomposables and, for $p \neq 2$ (assumed henceforth), it is additive and takes values in the space of primitive homology classes. The author generalizes the transpotence as follows. Let X be a 1-connected H-space and let $K^r(\Omega X)$ in $H_*(\Omega X; Z_p)$ consist of elements with vanishing p^r-power. Now the space ΣX in $H_*(X; Z_p)$ of suspension elements is a subspace of the space PX of primitive elements, and the author defines an operation

$$\phi_{p^r}: K_{2q}{}^r(X) \to (P(X)/\Sigma(X))_{2p^rq+2}$$

which is additive, vanishes on products in K^r and is natural with respect to maps $X \to Y$. The definition of ϕ_{p^r} proceeds by means of explicit formulae in the bar construction for a differential associative algebra with commutative homology algebra [see #2809 above]. The basic properties of the operation are established and used to obtain a complete description of the even-dimensional part, $P_e(X)$, of PX (Theorem 4.1). As a consequence, the author proves that if X is a 1-connected H-space with homology of finite type and if $H_*(\Omega X)$ is without p-torsion, then the whole of P_e is obtained by transpotence.
P. J. Hilton (Ithaca, N.Y.)

Tognoli, A. 2811
Osservazioni sulle applicazioni continue fra varietà topologiche.
Ann. Scuola Norm. Sup. Pisa (3) **18** (1964), 67–85.

The author considers modification mappings $f:(V,S)\to(W,A)$. V, W are n-dimensional orientable manifolds, f is a continuous compact (i.e., proper) mapping $V \to W$, A a closed subset of W, $S=f^{-1}A$, $f|(V-S)$ a homeomorphism, and $n>\dim S>\dim A$ ($A \neq \varnothing$). Borel-Moore homology groups $H_m{}'$ with coefficients in a field K and cohomology groups $H_c{}^m$ of singular cochains with compact carriers (over K) are used. $f|S$ induces $f_m{}': H_m{}'(S)\to H_m{}'(A)$. Main results: (a) $\operatorname{Ker} f_q{}' \neq 0$ implies $H_c{}^{n-q}(S)\neq 0$ and hence $\dim_R S \geqq \frac{1}{2}n$; (b) V compact Kählerian, S and A complex spaces, f' is holomorphic and has a fiber containing an analytic set P with $\dim_R P=2p$ implies $\dim_R S \geqq \max(2p, n-2p)$; (c) $V_0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} V_{i+1} \xrightarrow{f_{i+1}} \cdots$ is a sequence of modification mappings (definition as above except that $\dim S=\dim A$ is allowed),

$$\sum_m \dim_K H_m{}'(V_0; K) = r < \infty$$

implies there exist indexes $i_0, i_1, \cdots, i_r$ such that $(f_i{}')_q$: $H_q{}'(S_i)\to H_q{}'(A_{i+1})$ are isomorphisms for $i \notin (i_0, \cdots, i_r)$ and for all q. The given statements are analogous to—or would follow immediately from—some of the reviewer's results [Comment. Math. Helv. **31** (1957), 219–301; MR **22** #7151] in the compact combinatorial case. (c) together with additional information implies dimensional conditions in the case of a sequence of holomorphic modification mappings. Such conditions are worked out in the paper.
A. Aeppli (Minneapolis, Minn.)

TOPOLOGY AND GEOMETRY OF DIFFERENTIABLE MANIFOLDS

See also 2323, 2738, 2811.

Suzuki, Haruo 2812

On spheres imbedded in compact differentiable manifolds.

Mem. Fac. Sci. Kyushu Univ. Ser. A **17** (1963), 192–199.

L'auteur considère le problème suivant: A quelles conditions une classe d'homologie s d'une variété différentiable compacte M peut être représentée par une sphère plongée différentiablement. Il montre que l'annulation de certaines opérations cohomologiques, appliquées à la classe duale de s, est nécessaire. Il donne quelques exemples de classes non réalisables par une sphère plongée, mais il n'est même pas clair qu'elles appartiennent à l'image de l'homomorphisme de Hurewicz.

A. Haefliger (Geneva)

Cerf, J. 2813

Invariants des paires d'espaces. Applications à la topologie différentielle.

Topologia Differenziale (*Centro Internaz. Mat. Estivo*, 1° *Ciclo, Urbino*, 1962), *Lezione* 1, 24 *pp. Edizioni Cremonese, Rome*, 1963.

A topological pair (A, B) is a pair of topological spaces A, B with $B \subset A$ such that the identity map $j: B \to A$ is continuous. If the sets A and B coincide, then the pair (A, B) is called a bitopological space. For $i \geqq 0$ and x a point in the bitopological space (E', E), the author defines two groups, a local invariant $\lambda_i(E', E; x)$, and an invariant $\mu_i(E', E; x)$, which participate in an important exact sequence:

$$\pi_{i+1}(E, x) \to \mu_{i+1}(E', E; x) \to \lambda_i(E', E; x) \to \pi_i(E, x) \quad \text{for } i \geqq 0.$$

If (E', E) is "almost n-locally connected" for all $n \geqq 0$, then $\lambda_i(E', E; x) = \pi_i(E', x)$ and the sequence simplifies. Many other pertinent properties of topological pairs are obtained.

The author's interest in bitopological spaces comes from differential topology, in particular, from the study of spaces like the set of C^∞-imbeddings of a manifold into another, with the C^0- and the C^∞-topology [the author, Bull. Soc. Math. France **89** (1961), 227–380; MR **25** #3543]. He applies the theory and obtains, inter alia, the result that for n obeying certain conditions which (as he says) probably are true for $n \geqq 8$, where G' and G are the spaces of C^∞-diffeomorphisms of S^n in the C^0- and C^∞-topology, respectively,

$$\pi_{i+n}(G', \mathrm{SO}(n+1)) \approx \pi_i(G, \mathrm{SO}(n+1)) \quad \text{for } i \geqq 1,$$

and

$$\pi_n(G', \mathrm{SO}(n+1)) \approx \Gamma^n.$$

The application uses a "cascade" of Serre fibre maps of spaces of imbeddings, each with the C^0, C^∞ pair of topologies.

N. H. Kuiper (Bennekom)

Sanderson, B. J. 2814

Immersions and embeddings of projective spaces.

Proc. London Math. Soc. (3) **14** (1964), 137–153.

The set of vector bundles on a space X can be broken up into equivalence classes of bundles by insisting that the bundle ξ should be equivalent to $\xi \oplus \varepsilon$, where ε is a trivial bundle. Each class is called a "stable vector bundle". The geometric dimension of a stable vector bundle is the smallest dimension of a bundle in that equivalence class.

This paper investigates the consequences of a theorem of M. W. Hirsch [Trans. Amer. Math. Soc. **93** (1959), 242–276; MR **22** #9980], which says that a manifold M^n can be immersed in Euclidean space R^{n+k} if and only if the geometric dimension of the stable normal bundle does not exceed k. The author conducts his investigations by using the K-theory of Atiyah and Hirzebruch [Proc. Sympos. Pure Math., Vol. III, pp. 7–38, Amer. Math. Soc., Providence, R.I., 1961; MR **25** #2617].

In § 3 the author calculates the rings of (real and complex) vector bundles (on real, complex and quaternionic projective spaces). In § 4 these calculations are applied to give immersions of real projective space of dimension n. The immersions are best possible when $n \leqq 8$.

The following useful theorem is proved (Corollary 4.8): If $\pi: E \to B$ is the projection of a principal fibre bundle and both E and B are manifolds, and if B can be immersed in $R^{\dim B+k}$, then E can be immersed in $R^{\dim E+k}$. This theorem, together with standard fibrations involving the various projective spaces, enables the author to deduce, when $n > 8$, that RP_n immerses in R^{2n-3} if n is odd, in R^{2n-6} if $n = 4k+3$, and in R^{2n-8} if $n = 4k+3$ and k is not a power of 2.

Another method used by the author runs as follows. Let M^n be a manifold and ξ^s a bundle over M. We immerse ξ^s in R^{n+k+s}, which shows that ξ^s is a sub-bundle of a $(k+s)$-dimensional normal bundle of M. We then embed M in R^{n+k+s}, and this automatically embeds ξ if k is large enough. If the total space of the sphere bundle associated with ξ happens to be a sphere, then by adding a cone to the disc bundle associated with ξ, a certain manifold is embedded in $R^{n+k+s+1}$. By this method the author proves the theorem (independently proved by J. Levine) that CP_n embeds in R^{4n-2} if $n \neq 3$ and quaternionic projective space HP_n embeds in R^{8n-4}.

D. B. A. Epstein (Cambridge, England)

Robertson, S. A. 2815

Generalised constant width for manifolds.

Michigan Math. J. **11** (1964), 97–105.

From the author's introduction: "Let V denote a smooth (that is, C^∞) connected n-manifold without boundary, smoothly imbedded in R^m as a closed subset for some $m > n$. We write $\nu(p)$ for the $(m-n)$-plane in R^m normal to V at $p \in V$, and we say that V is transnormal in R^m if, for each pair $p, q \in V$, the relation $q \in \nu(p)$ implies that $\nu(p) = \nu(q)$. Thus transnormality generalises constant width. It is easy to show that the map ν from a transnormal manifold V to the space of normal $(m-n)$-planes of V is a covering map. We say that V has order r or is r-transnormal if ν is an r-fold. The main result is as follows. Theorem 1.1: Any transnormal n-manifold of order 2 in R^m is diffeomorphic to the cartesian product $V_1 \times V_2$ of differential manifolds V_1, V_2, where V_1 is homeomorphic to S^j and V_2 is homeomorphic to R^{n-j} $(0 < j \leqq n)$."

The author also proves other results, gives examples and states open problems. The proof consists mainly in the study of the geometric properties of the function

$\Lambda_q: V \to R$, where $\Lambda_q(x)$ is the square of the distance from x to $q \in V$. *A. Haefliger* (Geneva)

Watabe, T.; Hirose, T. 2816

On the immersibility of almost parallelizable manifolds.

Sci. Rep. Niigata Univ. Ser. A No. 1 (1964), 1–3.

This paper contains the detailed proofs of the results of an earlier note by the authors [Proc. Japan Acad. **39** (1963), 715–716; MR **29** #1648].

R. H. Szczarba (Princeton, N.J.)

Tomonaga, Yasurô 2817

A-genus and indecomposability of differentiable manifolds.

Proc. Amer. Math. Soc. **15** (1964), 586–589.

Let X be a compact orientable differentiable sub-manifold of $P_{2m+1}(C)$ of co-dimension 2. The author computes the A-genus $A(X)$ and shows that, if $m=2^s$ and if the class dual to X in $H^2(P_{2m+1}(C), Z)$ is an odd multiple of the generator, then $A(X) \equiv 2 \bmod 4$. Since the A-genus is always even, the author deduces that such an X cannot decompose as a product. *M. F. Atiyah* (Oxford)

Kupka, Ivan 2818a

Contribution à la théorie des champs génériques.

Contributions to Differential Equations **2** (1963), 457–484.

Smale, S. 2818b

Stable manifolds for differential equations and diffeomorphisms.

Ann. Scuola Norm. Sup. Pisa (3) **17** (1963), 97–116.

Theorem 1 below is proved in both papers. Let M be an n-dimensional compact differentiable manifold, and call $\mathscr{B}$ the space of all (contravariant) vector fields on M^n with the C^r topology, $r \geqq n$. A generic singularity of a field is one with no eigenvalue with zero real part, and a generic closed orbit is one such that the corresponding Poincaré transformation has linear part with no eigenvalue with absolute value 1. Now to each generic singularity there is associated a stable manifold, the set of trajectories entering the singularity for increasing time, and an unstable manifold, the set of trajectories leaving the singularity for increasing time. To each generic closed orbit there is associated in an analogous way a stable and an unstable manifold. Stable and unstable manifolds, in general, are not submanifolds of M^n, for they may not be properly immersed. The theorem below exhibits a large class of systems for which singularities, closed orbits, and the corresponding stable and unstable manifolds have a nice behaviour. Precisely, the following is proved. Let Σ be the subset of $\mathscr{B}$ formed by all fields χ such that: (i) all singularities of χ are generic; (ii) all closed orbits of χ are generic; (iii) when a stable manifold of χ intersects an unstable manifold of χ, they do so transversally. Theorem 1: Σ is a Baire set in $\mathscr{B}$, i.e., a countable intersection of sets which are open and dense in $\mathscr{B}$. Most of the difficulty in the proof comes from the fact that while condition (i) implies that there is only a finite number of singularities, condition (ii) is perfectly compatible with a countable infinity of closed orbits. For $n=2$ the above theorem is contained in a result of the reviewer [Topology **1** (1962), 101–120; MR **26** #426].

In Smale's paper, besides the above-mentioned theorem, one finds also the following analogue of it stated for diffeomorphisms. Let $\mathscr{D}$ be the space of all diffeomorphisms of M with the C^r topology, $r \geqq n$. For a diffeomorphism one defines in an analogous way what it means to say that a periodic point is generic, and to each such point one associates stable and unstable manifolds. Call Σ' the subset of $\mathscr{D}$ for which periodic points are generic and the stable and unstable manifolds are transversal. Theorem 2: Σ' is a Baire set in $\mathscr{D}$. Smale's proof of Theorem 1 is based on his proof of Theorem 2, whereas Kupka proves Theorem 1 directly. Both proofs are heavily dependent on Thom's transversality methods.

M. M. Peixoto (Providence, R.I.)

Smale, S. 2819

Stable manifolds for differential equations and diffeomorphisms.

Topologia Differenziale (Centro Internaz. Mat. Estivo, 1° *Ciclo, Urbino,* 1962), *Lezione* 4, 31 *pp. Edizioni Cremonese, Rome,* 1963.

This paper is identical in title and content with another paper of the author [see #2818b above].

K.-T. Chen (New Brunswick, N.J.)

Smale, S. 2820

Morse theory and a non-linear generalization of the Dirichlet problem.

Ann. of Math. (2) **80** (1964), 382–396.

This paper supplies details for part of an earlier announcement [Bull. Amer. Math. Soc. **70** (1964), 165–172; MR **28** #1634]. It is well known that the linear Dirichlet problem (say, for sections of a vector bundle ξ over a smooth compact manifold) can be solved by following lines of steepest descent [e.g., J. L. Lions, *Équations différentielles opérationnelles et problèmes aux limites*, Springer, Berlin, 1961; MR **27** #3935], which are (roughly speaking) the gradient lines of a quadratic (or sesquilinear) function J on a Hilbert space H. The present work is devoted to a study of conditions on a C^2 function J on H sufficient to insure that its gradient lines still lead to solutions (weak solutions; differentiability of these solutions can as yet be established only in very special cases). For that we need: (1) estimates on the integrand J, given as a smooth function on $C^k(\xi)$ (the space of C^k sections of ξ) to insure that J can be extended to a C^2 function on the Sobolev completion $H^k(\xi)$ of $C^k(\xi)$; this is an important new technical step; (2) estimates to insure that J-bounded sets are bounded in $H^k(\xi)$ (Poincaré's inequality); (3) estimates of the Legendre (or Gårding) type to insure a form of ellipticity as suggested by the quadratic case. The paper includes some discussion (with proofs) of an abstract Morse theory for such functions. *J. Eells* (Ithaca, N.Y.)

Novikov, S. P. 2821

Foliations of co-dimension 1 on manifolds. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 1010–1013.

Un feuilletage de codimension un sur une variété différentiable M est la donnée d'un champ complètement intégrable de $(n-1)$-plans sur M. Les feuilles sont les

variétés intégrales connexes maximales de ce champ. Un champ de vecteurs sur M, complémentaires en tout point aux feuilles, détermine une orientation du feuilletage, qui est dit orientable.

Dans cette note, l'auteur annonce, entre autres, le résultat remarquable suivant. Soit M une variété compacte de dimension 3, munie d'un feuilletage orientable de codimension un, et dont le groupe fondamental est fini. Il existe alors sur M une feuille compacte, difféomorphe à un tore, et qui borde un tore plein $D^2 \times S^1$ dans lequel le feuilletage est homéomorphe au feuilletage classique de Reeb [*Sur certaines propriétés topologiques des variétés feuilletées*, Actualités Sci. Indust., No. 1183, Hermann, Paris, 1952; MR **14**, 1113]. Il en déduit, comme corollaire, que tout champ de vecteurs transversal aux feuilles possède une solution périodique.

L'auteur ne donne que de très brèves indications sur la démonstration. Une des idées essentielles est de montrer l'existence, sur une feuille A, d'une courbe fermée régulière qui n'est pas un cycle limite d'un certain côté, qui n'est pas homotope à zéro dans A, mais qui, poussée dans les feuilles voisines de ce côté, devient homotope à zéro dans ces feuilles.

Dans la première partie de cette note, l'auteur remarque que, dans un feuilletage orienté, l'ensemble des feuilles est muni d'une structure de préordre naturelle: si A et B sont des feuilles, alors $A \leqq B$ s'il existe une courbe transversale aux feuilles menant d'un point de A à un point de B dans le sens donné par l'orientation, ou si $A = B$. Une composante connexe du feuilletage contenant A est la réunion des feuilles B telles que $A \geqq B$ et $B \geqq A$. L'auteur montre qu'une composante connexe peut être soit une feuille fermée, soit un ouvert de M dont la fermeture est une variété dont chaque composante connexe du bord est une feuille fermée, soit toute la variété. *A. Haefliger* (Geneva)

Kodaira, K. 2822

On the structure of compact complex analytic surfaces. I, II.

Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 218–221; *ibid.* 51 (1964), 1100–1104.

These two notes extend the previous work of the author [Ann. of Math. (2) **71** (1960), 111–152; MR **24** #A2396; ibid. (2) **77** (1963), 563–626; ibid. (2) **78** (1963), 1–40] on the classification of compact complex manifolds of dimension 2 (briefly: surfaces). To a certain degree this extension depends on the recently proved Riemann-Roch Theorem of the reviewer and I. M. Singer [Bull. Amer. Math. Soc. **69** (1963), 422–433; MR **28** #626]. The main results are summarized in the following theorem. Surfaces free from exceptional curves can be classified into the following seven classes: (I) the class of algebraic surfaces free from exceptional curves with $p_g = 0$; (II) the class of K_3 surfaces; (III) the class of complex tori (of complex dimension 2); (IV) the class of elliptic surfaces with $b_1 \equiv 0$ (2), $p_g > 0$, and $c_1^2 = 0$; (V) the class of algebraic surfaces free from exceptional curves with $p_g > 0$ and $c_1^2 > 0$; (VI) the class of elliptic surfaces with $b_1 \equiv 1$ (2), $p_g > 0$, and $c_1^2 = 0$; and (VII) the class of surfaces free from exceptional curves with $b_1 = 1$ and $p_g = 0$. Here c_1 is the first Chern class, b_1 the first Betti number and p_g is the geometric genus (i.e., the dimension of the space of holomorphic 2-forms). An elliptic surface is one fibred by elliptic curves (degeneracies being allowed). Finally, a K_3-surface is defined as a regular surface (i.e., $h^{0,1} = 0$) with trivial canonical bundle.

Deformation questions are also studied and the following results established. (1) A surface is a deformation of an algebraic surface if and only if b_1 is even. (2) A K_3-surface is a deformation of a non-singular quartic surface in projective 3-space. (2) had been conjectured by Weil and Andreotti. In fact, the conjectures are more precise and concern the precise structure of the space of moduli of K_3-surfaces. (2) asserts that this space is at least connected. *M. F. Atiyah* (Oxford)

PROBABILITY

See also 2193, 2272, 2335, 2343, 2350, 2357, 2359, 2360, 2491, 2536, 2879, 2913, 2937, 3242, 3250, 3268, 3276, 3286.

Kac, Mark 2823

Probability.

Sci. Amer. **211** (1964), 92–108.

An expository article on the nature of probability.

Marshall, Albert W.; Olkin, Ingram 2824

Inclusion theorems for eigenvalues from probability inequalities.

Numer. Math. **7** (1964), 98–102.

It is observed that if A is a normal matrix with eigenvalues $\theta_1, \cdots, \theta_m$, and z a unit vector, then $\mu_{ij} = zA^i A^{*j} z^*$ has the form $\sum p_k \theta_k{}^i \bar{\theta}_k{}^j$ with $p_k = P\{\Theta = \theta_k\}$ representing a probability distribution. Thus by application of standard probability inequalities, derivations are made of inclusion theorems such as those given by Bauer and the reviewer [Numer. Math. **2** (1960), 42–53; MR **22** #1070], Kato [J. Phys. Soc. Japan **4** (1949), 334–339] and others. *A. S. Householder* (Oak Ridge, Tenn.)

Fisz, Marek 2825

On an inequality due to Gatti and Birnbaum.

Metron **22** (1962), 110–112.

The author proves that the inequality

$$(*) \qquad (E|X-m|)^r \geqq 2^{r-1} \sum_i p_i{}^r |x_i - m|^r$$

holds for any $r \geqq 1$ and any discrete random variable X with $E|X| < \infty$, $E(X) = m$ and $P[X = x_i] = p_i$, $i = 1, 2, \cdots$. This inequality has previously been proved by S. Gatti [Metron **18** (1956), no. 1–2, 181–188; MR **18**, 683] for the case $r = 2$ and $p_i = 1/n$, $i = 1, 2, \cdots, n$, and by Z. W. Birnbaum [ibid. **19** (1958), no. 1–2, 243–244; MR **20** #6742] for the case $r > 2$ and $p_i = 1/n$, $i = 1, 2, \cdots, n$. In the present paper, the author shows by counter-example that an extension of this inequality to continuous random variables does not hold. *R. Pyke* (Seattle, Wash.)

Békéssy, A. [Békéssy, András] 2826

On classical occupancy problems. I. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 59–71.

The author solves more completely than anybody else has ever done some problems of limit distributions related to the "classical occupancy problem" defined as the problem of determining the probability distribution of the number of cells containing exactly m balls when N balls are thrown independently in n cells, the probability of a single ball falling in a single cell being $1/n$. These problems have been discussed most recently by A. Rényi, whose results [same Közl. **7** (1962), 203–214; MR **26** #5603] the author generalizes.

The results are well presented and elegantly proved.

L. Törnqvist (Helsinki)

Békéssy, András 2827

A new proof of a theorem concerning a distribution problem. (Hungarian)

Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **12** (1962), 329–334.

Let N balls be distributed at random in n boxes; let $\zeta_{n,N}$ denote the number of boxes which remain empty. Recently the reviewer [Magyar Tud. Akad. Mat. Kutató Int. Közl. **7** (1962), 203–214; MR **26** #5603] has shown that the distribution of $\zeta_{n,N}$ is normal in the limit in the following three cases: (a) $N/n \to c > 0$, (b) $N/n \to 0$ and $N^2/n \to +\infty$, (c) $N/n \to +\infty$ and $e^{N/n}/n \to 0$; for each of these three cases a different method of proof has been applied. The present paper gives a unified treatment of the cases (a), (b), (c) by the saddle-point method. The proof makes use of the identity

$$\sum_{n=0}^{\infty} \phi_{n,N}(t) \frac{(nz)^N}{N!} = (e^z + e^{it} - 1)^n,$$

where $\phi_{n,N}(t)$ is the characteristic function of $\zeta_{n,N}$.

A. Rényi (Budapest)

Rhodes, F. 2828a

Convexity in Wald's statistical metric spaces.

J. London Math. Soc. **39** (1964), 117–128.

Kingman, J. F. C. 2828b

Metrics for Wald spaces.

J. London Math. Soc. **39** (1964), 129–130.

A statistical metric space in the sense of A. Wald [Proc. Nat. Acad. Sci. U.S.A. **29** (1943), 196–197; MR **4**, 220], i.e., a Wald space, is an ordered pair $(S, \mathscr{F})$, where S is a set and $\mathscr{F}$ a mapping from $S \times S$ into the set of one-dimensional distribution functions. For any p, q in S, the distribution function $\mathscr{F}(p, q)$ is denoted by F_{pq}, and $F_{pq}(x)$ is interpreted as the probability that the "distance" between p and q is less than x. The functions F_{pq} are assumed to satisfy: (I) $F_{pq}(x) = 1$ for all $x > 0$ if and only if $p = q$; (II) $F_{pq}(0) = 0$; (III) $F_{pq} = F_{qp}$; (IV) $F_{pr}(x) \geqq [F_{pq} * F_{qr}](x)$, for all $x \geqq 0$, where $F_{pq} * F_{qr}$ is the convolution of F_{pq} and F_{qr}, i.e.,

$$[F_{pq} * F_{qr}](x) = \int_0^x F_{pq}(x-y)\, dF_{qr}(y).$$

Conditions (I), (II) and (III) are straightforward generalizations of the corresponding identity, positivity and symmetry conditions imposed on ordinary metrics. Condition (IV) is Wald's version of the generalized triangle inequality. In his paper Wald also introduced a notion of betweenness. He defined a point q to lie between the points p and r if and only if the points are distinct and $F_{pr} = F_{pq} * F_{qr}$, and he showed that this relation had all the usual properties of betweenness in metric spaces.

A neighborhood of a point p in a Wald space may be defined via $N_p(\varepsilon, \lambda) = \{q\,;\, F_{pq}(\varepsilon) > 1 - \lambda,\ \varepsilon > 0,\ \lambda > 0\}$ and it can be shown that this system of neighborhoods is the basis for a topology on S [the reviewer and A. Sklar, Pacific J. Math. **10** (1960), 313–334; MR **22** #A5955]. This topology is metrizable [the reviewer, A. Sklar and E. Thorp, ibid. **10** (1960), 673–675; MR **22** #A5956].

Using the notion of betweenness and the topology defined above, Rhodes studies convexity in Wald spaces. He first shows that a subspace of a Wald space consisting of a pair of points p, r and all points between them is closed. He then defines (metric) convexity in the usual way, says that the distribution functions of a Wald space are regular at ε if there exists a number $\varepsilon > 0$ such that every distribution function F_{pq} is continuous at ε and $F_{pq}(x) < F_{pq}(\varepsilon)$ for all $x < \varepsilon$, and proves that a convex, compact Wald space whose distribution functions are regular at ε contains with every pair of points a segment joining them. His next aim is to metrize the given Wald space in such a way that these segments become metric segments. This he achieves by means of an explicit construction of the metric, but only for those convex, compact Wald spaces which, in addition to being regular at ε, satisfy the condition that $F_{pq}(\varepsilon) = F_{rs}(\varepsilon)$ if and only if $F_{pq}(x) = F_{rs}(x)$ for all $x < \varepsilon$.

Motivated by Rhodes's paper, Kingman employs an elegant "Kunstgriff" to show that Rhodes's results are valid in any Wald space. More precisely, he shows that if S is any Wald space and if d is the two-place function on S defined by $d(p, q) = -\log \int_0^\infty e^{-x}\, dF_{pq}(x)$, then d is a metric on S and the convexity structure (determined by Wald's definition of betweenness), and the topological structure (determined by the ε-, λ-neighborhoods) are exactly those which arise from the metric d. From this he concludes that "it is impossible to regard the theory of convexity and betweenness on a Wald space as a non-trivial generalization of that on a metric space".

{Reviewer's comments: It has for some time been the reviewer's opinion that Wald's generalized triangle inequality for statistical metric spaces is too strong: it forces these spaces to be too similar to metric spaces. The results obtained by Rhodes and Kingman corroborate this opinion. As regards triangle inequalities, the reviewer prefers the one introduced by Menger in his original paper [Proc. Nat. Acad. Sci. U.S.A. **28** (1942), 535–537; MR **4**, 163]. The reviewer also feels that the objects studied in probabilistic geometry should, whenever possible, be probabilistic. Thus, rather than ordinary betweenness and convexity, one should attempt to define and study probable betweenness, probable convexity, etc.}

B. Schweizer (Tucson, Ariz.)

Wittenberg, Helen 2829

Limiting distributions of random sums of independent random variables.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **3**, 7–18 (1964).

If X and Y are random variables with distributions F and G, respectively, define $\rho(X, Y) = \sup_x |F(x) - G(x)|$.

Let $\{X_n, n=1, 2, \cdots\}$ be a sequence of independent identically distributed random variables, and for each positive integer k let $S(k)=\sum_{i=1}^{k} X_i$. Let $\{\tau_n, n=1, 2, \cdots\}$ be a sequence of positive integer-valued random variables and $\{c_n, n=1, 2, \cdots\}$ be a sequence of positive constants. Suppose τ_n/c_n converges in probability to a positive random variable φ. The author investigates the question of when $S([\varphi c_n])$ is close to $S(\tau_n)$ for large n, in the sense of the above distance.

Several theorems are proved. A typical one states that if (a) X_i is symmetric about the origin, (b) τ_n converges to infinity in probability, (c) c_n is a median of τ_n, then $\rho\{S(\tau_n), S([\varphi c_n])\}\rightarrow 0$.

The author's results include known results in the case when $S(n)$ has a limiting distribution when appropriately normed. *J. R. Blum* (Albuquerque, N.M.)

Linnik, Yu. V. [Linnik, Ju. V.] 2830

Polynomial statistics and polynomial ideals.

Calcutta Math. Soc. Golden Jubilee Commemoration Vol. (1958/59), *Part I, pp.* 95–98. *Calcutta Math. Soc., Calcutta*, 1963.

Let $X=(x_1, \cdots, x_n)$ be a finite set of random variables; let us suppose that all their moments exist and

$$(1) \qquad E(x_j^r) = O(e^{\sigma(r \log r)})$$

for $r\rightarrow +\infty$ and for $j=1, \cdots, n$. Let $Q_1(X)=Q_1$ and $Q_2(X)=Q_2$ be two polynomials in the variables $x_1, \cdots, x_n$. It is proved that there exist constants M_1 and M_2 such that $Q_1(X)$ and $Q_2(X)$ are independent if and only if Q_1^r and Q_2^s are uncorrelated for $r, s=1, 2, \cdots, M_1$; further, Q_1 and Q_2 are equally distributed if and only if the moments of order $r\leqq M_2$ of Q_1 and Q_2 coincide. The proofs are based on the theory of polynomial ideals. The author calls attention to the open problem of replacing the very restrictive condition (1) by a weaker one.

A. Rényi (Budapest)

Letac, Gérard 2831

Une propriété de fluctuation des processus de Poisson composés croissants.

C. R. Acad. Sci. Paris **258** (1964), 1700–1703.

Let $\{Y_i; i=1, 2, \cdots\}$ be a sequence of positive independent random variables with identical distribution, and let $N(t)$ be a Poisson process which is independent of $\{Y_i\}$. Put $X(t)=Y_1+Y_2+\cdots+Y_{N(t)}$ $(t>0)$. Then we have a process $X(t)$ with non-decreasing sample paths. The author proves that if $E(Y)<\infty$, then

$$\operatorname{Prob}\left[\sup_t \frac{X(t)}{t} \leqq x\right] = 1-\frac{\nu E(Y)}{x} \quad \text{if } x > \nu E(Y),$$
$$= 0 \qquad \text{otherwise},$$

where $E(Y)$ denotes the common mean value of $E(Y_i)$ $(i=1, 2, \cdots)$ and ν denotes the mean parameter of $N(t)$. Moreover, the author makes three remarks. The second one is as follows. Let $\{X_i; i=1, 2, \cdots\}$ be a sequence of independent random variables whose distribution is the identical exponential type distribution with mean value $1/\nu$. Then the distribution of the random variable $\inf_n(X_1+X_2+\cdots+X_n)/(Y_1+Y_2+\cdots+Y_n)$ is the uniform distribution on $[0, 1/\nu E(Y)]$. *T. Sirao* (Nagoya)

Kremser, H. 2832

Über Transformationen von Laplace-Stieltjes-Transformierten.

Österreich. Ing.-Arch. **18** (1964/65), 74–78.

The author considers distribution functions $G(x)$ with left extremity zero [i.e., $G(-0)=0$] and their Laplace-Stieltjes transforms $g(s)$. He studies functions which operate on these transforms and proves the following theorem: Suppose that the first moment α of $G(x)$ exists and that $\alpha\lambda<1$ (λ a positive constant); then

$$(*) \qquad f(s) = \frac{1-\lambda\alpha}{1+\lambda s^{-1}[g(s)-1]}$$

is the Laplace-Stieltjes transform of an infinitely divisible distribution $F(x)$ such that $F(-0)=0$. The formula (*) occurs in the theory of queues, $F(x)$ is the limiting probability of the distribution of virtual waiting times in a single-server queue with Poisson input (parameter λ) and general service time [$g(s)$ is the Laplace-Stieltjes transform of the service-time distribution]. Formula (*) is derived independently of queueing theory. Its proof is based on a transformation which was given earlier and independently by V. M. Zolotarev [see Ju. V. Linnik, *Decompositions of probability laws* (Russian), p. 43, Izdat. Leningrad. Univ., Leningrad, 1960; MR **23** #A2900]. *E. Lukacs* (Washington, D.C.)

Barlow, Richard E.; Marshall, Albert W. 2833

Bounds for distributions with monotone hazard rate. I.

Ann. Math. Statist. **35** (1964), 1234–1257.

The hazard rate $q(x)=f(x)/[1+F(x)]$ corresponding to a probability distribution F with density f is increasing [decreasing] in $x\geqq 0$ if and only if $\log[1-F(x)]$ is concave [convex] in $x\geqq 0$. The main purpose of this paper is to improve the Markov inequality (which states that if F is a probability distribution such that $F(0-)=0$ and $\int_0^\infty x^r\,dF(x)=\mu_r<\infty$, and if $r, t>0$, then $0\leqq 1-F(t)\leqq \mu_r/t^r$ for $t\geqq\mu_r$, and $1-F(t)\leqq 1$ for $t\leqq\mu_r$) under the additional assumption that F has a monotone hazard rate. As in the case of the Markov inequality, the various bounds obtained are sharp in the sense that there exist distributions for which equality holds. For example, Theorem 3.8 states that if F has increasing hazard rate, $F(0)=0$, $r\geqq 1$ and $\int_0^\infty x^r\,dF(x)=\mu_r$, then

$$1-F(t) \geqq \exp[t/\lambda_r^{1/r}], \qquad t \leqq \mu_r^{1/r},$$
$$\geqq 0, \qquad t > \mu_r^{1/r},$$

where $\lambda_r=\mu_r/\Gamma(r+1)$; and Theorem 3.15 states that if F has increasing hazard rate, $F(0)=0$, $s\neq 0$ and $\int_0^\infty e^{-sx}\,dF(x)=f^*(x)<0$, then

$$1-F(t) \leqq 1, \qquad t \leqq -s^{-1}\log f^*(s),$$
$$\leqq e^{-a_0 t}, \qquad t > -s^{-1}\log f^*(s),$$

where a_0 is the unique solution of $f^*(s)=[s/(s+a)]e^{-(s+a)t}+a/(s+a)$.

Bounds for $1-F$ are also obtained in terms of bounds on the hazard rate. For example, Theorem 4.3 states that if $F(0)=0$, $q(x)\leqq\beta<\infty$ for all $x\geqq 0$ and $\int_0^\infty xf(x)\,dx=\mu$, then

$$1-F(t) \leqq e^{-\beta z_0}, \qquad t > \mu-\beta^{-1},$$
$$\leqq 1, \qquad t \leqq \mu-\beta^{-1},$$

where z_0 is the unique solution of $(t-z)e^{-\beta z}=\mu-\beta^{-1}$ satisfying $0\leqq z_0\leqq t$;

$$1-F(t) \geqq e^{-\beta t}.$$

The bounds obtained under the assumption of monotone hazard rate are compared numerically with the Markov bounds and bounds for other classes of distributions (unimodal, decreasing density, Pólya frequency function of order 2).

Monotone hazard rates occur in applications to life testing, reliability, and in actuarial science, where F is interpreted as a life distribution and $q(x)$ as the conditional probability of death, given survival to age x.

G. Berman (Waterloo, Ont.)

Barlow, Richard E.; Marshall, Albert W. 2834

Bounds for distributions with monotone hazard rate. II.

Ann. Math. Statist. **35** (1964), 1258–1274.

In this paper the authors derive inequalities for distributions analogous to the Chebyshev inequality which states that if F is a probability distribution such that $F(0-)=0$ and $\int_0^\infty x^r\,dF(x)=\mu_r$, $r=1, 2$ and $\mu_1=1$, then

$$\begin{aligned} 1-F(t) &\leqq 1, & 0 &\leqq t\leqq 1,\\ &\leqq t^{-1}, & 1 &< t < \mu_2,\\ &\leqq (\mu_2-1)/[\mu_2-1+(t-1)^2], & t &\geqq \mu_2;\\ 1-F(t) &\geqq (1-t)^2/[\mu_2-1+(1-t)^2], & 0 &\leqq t\leqq 1,\\ &\geqq 0, & t &\geqq 1. \end{aligned}$$

The inequalities derived may be regarded as improvements on the Chebyshev inequality made possible by the additional assumption of a monotone hazard rate [see review of Part I; #2833 above].

The method of proof differs from that employed in Part I in deriving inequalities analogous to the Markov inequality. In this paper the proofs are obtained by associating with the class of distributions $\mathfrak{F}$ under consideration a subclass $\mathfrak{G}$ which is extremal for $\mathfrak{F}$ on T, i.e., for each t in T, F in $\mathfrak{F}$, there exists a G in $\mathfrak{G}$ such that $F(t)=G(t)$. The inequalities follow from the observation that if $\mathfrak{G}$ is extremal for $\mathfrak{F}$ and F is in $\mathfrak{F}$, then $\inf_{\mathfrak{G}} G(t)\leqq F(t)\leqq \sup_{\mathfrak{G}} G(t)$.

The distributions G usually employed in the proofs consist of two exponential pieces and have the same moments μ_1, μ_2 as F. For example, in Section 4 it is assumed that

$$\begin{aligned} 1-G_{T:\alpha}(x) &= \alpha e^{-a_1 x}, & 0 &< x\leqq T,\\ &= \alpha e^{-a_2 x+(a_2-a_1)T}, & x &\geqq T, \end{aligned}$$

for all T, $0\leqq T\leqq\infty$, where

$$(1)\qquad 1 = \int_0^\infty [1-G_{T:\alpha}(x)]\,dx,$$

$$(2)\qquad \frac{1}{2}\mu_2 = \int_0^\infty x[1-G_{T:\alpha}(x)]\,dx.$$

These functions G are then used to prove Theorem 4.1, which states in part that if F is a distribution with decreasing hazard rate, $F(0)=0$, and if F has first and second moments $\mu_1=1$ and μ_2, then

$$\begin{aligned} 1-F(t+) &\geqq 2\mu_2^{-1}, & t &= 0,\\ &\geqq e^{-a_1 t}, & t &> 0, \end{aligned}$$

where a_1 is determined by (1) and (2) with $\beta=1$ and $T=t$.

Families of extremal distributions are also employed to prove Markov type inequalities (see Part I) under the assumption that F has a density f which is a Pólya function of order 2 (i.e., $\log f(x)$ is concave on the support of F).

G. Berman (Waterloo, Ont.)

Kerstan, Johannes 2835

Verallgemeinerung eines Satzes von Prochorow und Le Cam.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2**, 173–179 (1964).

Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent random variables assuming the values $0, 1, 2, \cdots$. Let ξ be their sum and P_ξ their distribution. Let π_a be the Poisson law with expectation a. The aim of the paper is to give an estimate of the distance $\|P_\xi, \pi_a\|=\sum_{k=0}^\infty |P_\xi(k)-(a^k/k!)e^{-a}|$ for suitable a. The author proves the following inequalities:

$$\|P_\xi, \pi_{\sum a_i}\| \leqq 5.4\frac{\sum d_i}{\sum a_i}\qquad (\max a_i \leqq 1/4),$$

$$\|P_\xi, \pi_{\sum a_i}\| \leqq 2\sum d_i,$$

where $a_i=E(\xi_i)$, $b_i=E(\xi_i(\xi_i-1))$,

$$d_i = 2\left(\left(1-a_i+\frac{b_i}{2}-P_{\xi_i}(0)\right)\right.$$
$$\left.+\left(1-a_i+\frac{a_i^2}{2}-e^{-a_i}\right)+|e^{-a_i}-P_{\xi_i}(0)|\right).$$

This is a generalization and improvement of results obtained by Prochorow and LeCam.

J. E. Cigler (Vienna)

Petrov, V. V. 2836

An inequality for concentration functions. (Russian. Lithuanian and English summaries)

Litovsk. Mat. Sb. **3** (1963), no. 1, 195–197.

Let $X_1, \cdots, X_n$ be independent random variables with means $a_1, \cdots, a_n$, variances $\sigma_1^2, \cdots, \sigma_n^2$ (at least one of which is nonzero) and absolute $(2+\delta)$th moments about the mean $\beta_{2+\delta,1}, \cdots, \beta_{2+\delta,n}$ (where $0<\delta\leqq 1$), and let Φ be the standard normal distribution function. Then for $x>0$, letting $B_n^2=\sum_{j=1}^n \sigma_j^2$, we have

$$\sup_{-\infty<y<\infty} \mathsf{P}\left\{y\leqq \sum_{i=1}^n X_i\leqq y+x\right\}$$
$$\leqq C_\delta\frac{1}{B_n^{2+\delta}}\sum_{j=1}^n \beta_{2+\delta,j}+2\Phi\left(\frac{x}{2B_n}\right)-1,$$

where C_δ is a positive constant.

For identically distributed random variables

$$\sup_{-\infty<y<\infty} \mathsf{P}\left\{y\leqq \sum_{i=1}^n X_i\leqq y+x\right\}$$
$$\leqq \frac{4.062E|X_1-a_1|^3}{\sigma_1^3\sqrt{n}}+2\Phi\left(\frac{x}{2\sigma_1\sqrt{n}}\right)-1,$$

which is nontrivial for $n>16$, whereas the previous result of Offord is nontrivial only for $n>10^{12}$.

Judah Rosenblatt (Albuquerque, N.M.)

Šakovič, G. N. 2837

Strictly stable multidimensional Gaussian distributions with non-zero mathematical expectation. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 166–169.

Author's summary: "Strictly stable laws of distribution with respect to a group of normalizing matricial factors are defined as in his earlier paper [Teor. Verojatnost. i Primenen. **5** (1960), 254]. In the present note the author establishes the existence of these Gaussian laws and shows how to determine all such laws."

Sytaja, G. N. 2838

Limiting distribution of a certain class of functionals of a sequence of sums of independent random variables. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 53–55.

The author states a theorem of the Donsker type: If $S_{nk}=n^{-1/2}\sum_{i=1}^{k}\xi_i$ (ξ_i independent identically distributed reduced random variables), and if Φ_n is a non-negative measurable function on the plane, then

$$\eta_n = \sum_{j,k=0}^{n-1}\sum \Phi_n(S_{nj}, S_{nk})$$

converges, under certain hypotheses, to a functional of Brownian motion. Three lemmas required for the proof are also stated. *K. Balagangadharan* (Bombay)

Silvey, S. D. 2839

On a measure of association.

Ann. Math. Statist. **35** (1964), 1157–1166.

Let ξ and η be two random variables. Let $r(x, y)$ denote the density function of their joint distribution (which is supposed to be absolutely continuous), and let $p(x)$ and $q(y)$ denote the density functions of ξ and η, respectively. Put $\varphi(x, y)=r(x, y)/p(x)q(y)$ and define $\Delta=\Delta(\xi, \eta)$ by

$$\Delta = \iint_E (r(x, y)-p(x)q(y))\,dxdy,$$

where E is the set of those points (x, y) for which $\varphi(x, y)>1$. The author proposes Δ as a measure of the association between the random variables ξ and η. It is easy to see that $0\leq\Delta\leq 1$; $\Delta(\xi, \eta)=\Delta(\eta, \xi)$; $\Delta=0$ if and only if ξ and η are independent; further, if $f(x)$ and $g(x)$ are Borel-measurable one-to-one mappings of the real line into itself, then $\Delta(f(\xi), g(\eta))=\Delta(\xi, \eta)$. It is shown also that if the joint distribution of the random variables ξ and η is normal, $\Delta(\xi, \eta)$ is a monotonic increasing function of the correlation coefficient $\rho=\rho(\xi, \eta)$ if $\rho>0$. The author finally suggests also that in case ξ is a statistic whose distribution depends on a parameter θ which itself is a random variable with a known distribution, then $\Delta(\xi, \theta)$ may be taken as an alternative measure of the amount of information in the experiment yielding ξ about the parameter θ [see D. V. Lindley, same Ann. **27** (1956), 986–1005; MR **18**, 783]. *A. Rényi* (Budapest)

Nadaraja, È. A. 2840

On estimating the density of random variables. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **32** (1963), 277–280.

The author investigates the convergence to the density of a class of estimates introduced by Parzen. Under certain conditions on h, where

$$f_n(x) = \frac{1}{nh(n)}\sum_{i=1}^{n} K\left(\frac{x-x_i}{h(n)}\right),$$

he shows uniform convergence of f_n to f and under others that $\int_{-\infty}^{\infty}|f_n(x)-f(x)|\varphi(x)\,dx\to 0$, with probability one. *Judah Rosenblatt* (Albuquerque, N.M.)

Hole, Njål 2841

Ein Wahrscheinlichkeitsproblem aus der Theorie der Zähler.

Norske Vid. Selsk. Forh. (*Trondheim*) **36** (1963), 150–153.

The classical problem of a counter of type II is discussed and a new proof for the well-known formula [see, e.g., the reviewer, *Wahrscheinlichkeitsrechnung*, pp. 203–204, Exercise 45(c), VEB Deutscher Verlag Wiss., Berlin, 1962; MR **26** #5597] for the distribution function of the time interval between two consecutive counted events is given. (For the bibliography of the problem see A. T. Bharucha-Reid [*Elements of the theory of Markov processes and their applications*, pp. 331–333, McGraw-Hill, New York, 1960; MR **22** #3032].) *A. Rényi* (Budapest)

Medgyessy, Pál 2842

On the interconnection between the representation theorems of characteristic functions of unimodal distribution functions and of convex characteristic functions. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 425–430 (1964).

A. Ja. Hinčin [Izv. Naučn.-Issled. Inst. Mat. Meh. Tomsk Gos. Univ. **2** (1938), no. 2, 1–7] has proven the following theorem. (A) The function $\varphi(t)$ is the characteristic function of a unimodal distribution if and only if it can be represented in the form $\varphi(t)=(1/t)\int_0^t\psi(u)\,du$, where $\psi(u)$ is some characteristic function. Dugué and Girault [Publ. Inst. Statist. Univ. Paris **4** (1955), 3–10; MR **17**, 47] obtained the following result: (B) The function $\psi(t)$ is a Pólya type (convex) characteristic function if and only if it can be represented in the form $\psi(t)=\int_t^\infty(1-t/x)\,dG(x)$ for $t>0$, $\psi(t)=\psi(-t)$. Here $G(x)$ is a distribution function for which $G(-0)=G(+0)=0$.

The author investigates the connection between these two theorems and first deduces (B) from (A) and then (A) from (B). *E. Lukacs* (Washington, D.C.)

Pergel, József 2843

On a "big deviations" problem. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 303–308 (1964).

Let $X_1, X_2, \cdots, X_n$ be independent random variables with a common distribution function $F(x)$ of the form

$$(*)\qquad F(x) = \int_a^{a+5}|x|^{-v}\,dG(v)+O(|x|^{-4a-5-\varepsilon})\qquad (x\to-\infty),$$

with a similar expression for $1-F(x)$ when $x\to+\infty$, where G is a function of bounded variation and a is an integer

greater than 3 and $0<\varepsilon<1$. Assume F has a density function. In this paper the author proves that

$$F_n(x)\left\{(2\pi)^{-1/2}\int_{-\infty}^{x}\exp(-u^2/2)\,du+R(x,n^{1/2})\right\}^{-1}\to 1$$

as $n\to\infty$, uniformly in $x\leqq -1$, where F_n denotes the distribution function of the standardized sample mean and where $R(x,k)$ is a function which may be written as $R(x,k)=\int_c^d k^{-v}x^{1-v}q(x,v)\,dK(v)$, where $q(x,v)$ is a rational function of x, bounded as $x\to\infty$ and continuous in v, and where K is a function of bounded variation. It is also proved that $1-F_n(x)\sim n[1-F(xn^{1/2})]$ whenever $x>n^{3/2+a^{-1}+\varepsilon}$. These results generalize those of Linnik [Proc. Fourth Berkeley Sympos. Math. Statist. and Prob., Vol. II, pp. 289–306, Univ. California Press, Berkeley, Calif., 1961; MR **25** #598] which concerned the case in which G is a degenerate distribution function.

R. Pyke (Seattle, Wash.)

Slobodenjuk, M. P. [Slobodenjuk, N. P.] 2844

A limit theorem for additive functionals of a sequence of sums of independent random variables. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 698–701.

Author's summary: "Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent identically distributed random variables, $M\xi_n=0$, $D\xi_n=1$, $S_{n0}=0$, $S_{nk}=(1/\sqrt{n})(\xi_1+\xi_2+\cdots+\xi_n)$, and $f_n(x,y)$ be a sequence of measurable functions for which $\sup|f_n(x,y)|\to 0$. A limit theorem for $\eta_n=\sum_{k=0}^{n-1}f_n(S_{nk},S_{nk+1})$ is obtained."

J. Wolfowitz (Ithaca, N.Y.)

Jaglom, A. M. 2845

Spectral representations for various classes of random functions. (Russian)

Proc. 4th All-Union Math. Congr. (Leningrad, 1961), Vol. I, pp. 250–273. Izdat. Akad. Nauk SSSR, Leningrad, 1963.

This is an expository paper, aiming at presenting the main known results and some open problems concerning the spectral representation of stochastic processes in the language of functional analysis, in order to make this important field of research accessible also for those who are not familiar with probability theory. A bibliography containing 62 items is given. *A. Rényi* (Budapest)

Gusak, D. V. 2846

On the asymptotic behaviour of the distribution of extrema of a centered monotone homogeneous process with independent increments. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1287–1292.

Gusak, D. V. 2847

The asymptotic behavior of the distribution of the first passage time of a homogeneous process with independent increments. (Russian)

Ukrain. Mat. Ž. **16** (1964), 463–474.

Let $\xi(t)$ be a homogeneous process with independent increments and finite second moments,

$$\zeta_\lambda(t)=\lambda^{-1/2}[\xi(\lambda t)-M\xi(\lambda t)],$$

$$\tau(\lambda;z_-,z_+)=\sup\{t:z_-(\tau)<\zeta_\lambda(\tau)<z_+(\tau),\,0\leqq\tau\leqq t\},$$

$$a_\lambda(t,x)=\frac{\partial}{\partial x}P\{\zeta_\lambda(t)<x,\,\tau(\lambda;z_-,z_+)<t\}.$$

The main result (Theorem 8) is an asymptotic expansion of $a_\lambda(t,x)$ as $\lambda\to\infty$. The details are too involved to be reproduced here. *W. Hoeffding* (Chapel Hill, N.C.)

Kočetkov, E. S. 2848

Estimates for the mathematical expectation of a stochastic process. (Russian)

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (Liblice, 1962), pp. 417–428. Publ. House Czech. Acad. Sci., Prague, 1964.

Let $Y(t)$ be a weakly stationary process with zero mean, and let $X(t)=m(t)+Y(t)$, where $m(t)$ is an unknown function. The author considers the problem of estimating $m(t)$ for $0\leqq t\leqq T$ from a realization $\{x(t)\}$ of the X-process in that range. The estimators used are of the form $m^*(t)=\int_0^T p(t,\tau)x(\tau)\,d\tau$ for a given "weight function" p required to be such that $m^*(t)$ is unbiased, for each t. Various asymptotic properties of the estimator are obtained. In the case where $m(t)=\sum_{k=1}^n a_k\phi_k(t)$ (ϕ_k being known functions and a_k unknown constants), least squares, and minimum variance estimators are considered and shown to have the same asymptotic variance.

M. R. Leadbetter (Durham, N.C.)

Blumenthal, R. M.; Getoor, R. K. 2849

Local times for Markov processes.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **3**, 50–74 (1964).

Consider a reasonable Markov process with sample paths $t\to x(t)$, and let a be an accessible state, i.e., if T is the hitting time to a, then $P_\cdot(T<\infty)\not\equiv 0$. Supposing that $P_a(T=0)=1$, the authors construct a local time at a; this is a continuous additive functional $A(t)$ that grows exactly on the visiting set $Z=(t:x(t)=a)$. Under an extra condition, $E_\cdot(\int_0^\infty e^{-\lambda t}\,dA)$ is proportional to the kernel for λ-potentials evaluated at $(\cdot,a)$. A^{-1}, the inverse function of A, is a homogeneous differential process. Applications to the Hausdorff-Besicovitch dimension of Z are given; some of these are new, others are reminiscent of the results of Itô and the reviewer [*Diffusion processes*, Springer, Berlin, 1964 (in press)] and of C. Stone [Illinois J. Math. **7** (1963), 631–637; MR **28** #1662]. Previous work on local time at a point was confined to the 1-dimensional case; see, for example, E. S. Boylan [ibid. **8** (1964), 19–39; MR **28** #1657], D. B. Ray [ibid. **7** (1963), 615–630; MR **27** #6306], H. F. Trotter [ibid. **2** (1958), 425–433; MR **20** #2795], and the references cited above.

H. P. McKean, Jr. (Cambridge, Mass.)

Rosenblatt, M. 2850

Some nonlinear problems arising in the study of random processes.

J. Res. Nat. Bur. Standards Sect. D **68D** (1964), 933–936.

This is an interesting expository article dealing with two non-linear problems in stationary stochastic processes. The first problem concerns the representation of a discrete-time parameter process as a one-sided function (and its

shifts) of independent random variables. A few comments are made about representations when the time parameter is continuous. The second problem is concerned with the harmonic resolution of the moments of stationary random processes. *D. L. Hanson* (Columbia, Mo.)

Kingman, J. F. C. 2851

Recurrence properties of processes with stationary independent increments.

J. Austral. Math. Soc. **4** (1964), 223–228.

The starting point of the paper is the result of Chung and Fuchs [Mem. Amer. Math. Soc. No. 6 (1951); MR **12**, 722] concerning necessary and sufficient conditions for the recurrence of a random walk generated by independent and identically distributed random variables. Let $Z(t)$ $(t \geqq 0)$ be a separable and measurable version of a process with stationary independent increments, and let us define the random set $S(a) = \{t \geqq 0, |Z(t)| < a\}$. For any measurable subset A of R_1 we denote by $\|A\|$ the Lebesgue measure of A. The author proves that for the propositions: (I) $P(S(a)$ is unbounded$)=1$, (II) $P(\|S(a)\| = \infty) = 1$, (III) $\int_0^\infty P(|Z(t)| < 2a)\,dt = \infty$, (IV) for some $h > 0$, $\sum_{n=1}^\infty P(|Z(nh)| < 2a) = \infty$, (V) for some $h > 0$ the random walk $\{Z(nh);\ n = 1, 2, \cdots\}$ is recurrent in the sense of Chung and Fuchs, the chain of implications I$\Rightarrow$II$\Rightarrow$III$\Rightarrow$IV$\Rightarrow$V$\Rightarrow$I holds. Further, he gives a necessary and sufficient condition in terms of the characteristic function of $Z(t)$ for the recurrence which is the continuous-time analogue of a result of Feller and Orey [J. Math. Mech. **10** (1961), 619–624; MR **24** #A581].

I. Vincze (Budapest)

Bailey, Norman T. J. 2852

★The elements of stochastic processes with applications to the natural sciences.

John Wiley & Sons, Inc., New York-London-Sydney, 1964. xi + 249 *pp.* \$7.95.

This book has just the right coverage and is written at just the right level to meet the needs of a wide circle of readers. The book is unfortunately mistitled; it contains no information about stationary time-series, but gives a very useful account of those processes, loosely called "evolutionary", which have been intensively studied by such writers as Armitage, Bartlett, McKendrick, Moran, and the author himself, principally in connexion with biological applications. Thus there are chapters on branching processes, birth-and-death processes, mutations, queues, epidemics, competition, predation, and so forth. These are preceded by enough preliminary general information to make the text accessible to a reader with at least some familiarity with the unstarred sections of Volume I of the book by W. Feller [*An introduction to probability theory and its applications*, second edition, Wiley, New York, 1957; MR **19**, 466]. Very little general theory is used or mentioned (the reviewer does not think there is an Ω in the book); however, in the quest for explicit formulae, the reader must take in his stride the use of, for example, Lagrange's reversion of series. For a biologist equipped with the mathematical knowledge of the average physicist, this book is ideal.

Most of the content of the book will be familiar ground to the user of stochastic process, but he will probably find food for thought in Chapter 15 ("Approximations to stochastic processes"), where the author touches on one of the growing points of the subject.

Many examples are worked out in detail, and each chapter closes with problems (solutions being provided).

D. G. Kendall (Cambridge, England)

Šatašvili, A. D. 2853

Non-linear transformations of functional integrals by Gaussian measures. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 717–719.

Author's summary: "The author considers transformations of measures corresponding to Gaussian processes with certain nonlinear transformations of these processes. The measures corresponding to Gaussian processes pass into measures that are absolutely continuous with respect to the initial measures. In addition, the densities of the measures obtained after the transformation are calculated relative to the initial one, and vice versa."

Bui Trong Lieu; Carton, D. A. 2854

Sur la loi du χ^2 dans certains processus de Markov.

Studia Math. **24** (1964), 25–35.

The authors show that certain random variables which are functions of the sample paths of a stationary Markov process X_t, $t \in T$, asymptotically have the χ^2 distribution. Thus, given a sample path from a stochastic process Y_t, $t \in T$, one can test whether Y_t is stationary Markov. Different random variables are given, depending on whether the indexing set T is discrete or, say, all $t \geqq 0$. To give some idea of the nature of the random variables involved, we state a simplified form of the authors' Proposition 2. Let $T = \{0, 1, \cdots\}$ and $X = \{1, 2, \cdots, r\}$. Suppose $X_t \in X$ for all $t \in T$. Let $\pi_i = P(X = i)$ and $P_{ij} = P(X_0 = i, X = j)$ be the initial distribution and one-step transition probabilities of X_t. Let $N_n(x, A)$ be the number of times t that $X_t = x_1$ and $X_{t+1} \in A$, where $t < n$, $t \in T$, $x \in X$ and $A \subset X$. If all the P_{ij} are non-zero, then

$$\sum_{1 \leqq i, j \leqq r} \frac{[N_n(i, j) - N_n(i, X)P_{ij}]^2}{n\pi_i P_{ij}}$$

asymptotically has the χ^2 distribution.

E. S. Boylan (New Brunswick, N.J.)

Chover, J. 2855

Nonabsorption probability for a Gaussian process in the Karhunen representation.

Duke Math. J. **31** (1964), 427–443.

Let $X(t)$ be a real Gaussian random process (r.p.) defined on $[0, T]$ with continuous covariance $r(\cdot, \cdot)$ and zero mean. It is desired to obtain the "nonabsorption probability"

$$(*) \qquad P[a(t) \leqq X(t) \leqq b(t), \quad 0 \leqq t \leqq T]$$

in terms of the eigenfunctions $\{\varphi_n\}$ and eigenvalues $\{\lambda_n\}$ of the covariance. In particular, assume that the Karhunen representation [see Loève, *Probability theory*, second edition, Van Nostrand, Princeton, N.J., 1960; MR **23** #A670] of $X(t)$ consists only of a finite number of terms (say N), and that $a(\cdot)$ and $b(\cdot)$ can be expanded in terms of the same N eigenfunctions. Then (*) is given as an N-fold integral of products of Gaussian densities with

means $\lambda_1, \lambda_2, \cdots, \lambda_N$ integrated over a convex region determined by the eigenfunctions. Under given conditions, which include the assumption that the boundaries can be expressed as fixed finite sums of the φ_n, (*) can be approximated (with error bounds tending toward zero) by the absorption probabilities for $X_N(t)$, the latter being obtained as the N-term truncation of the expansion for $X(t)$.

Estimated also is $P[Z(t)+X(t) \leqq b(t)]$, where $X(t)$ again has an N-term Karhunen expansion, and $Z(t)$ is an r.p. independent of $X(t)$. More explicit forms are secured when $Z(t)$ is a Brownian motion process and $b(t)=0$. In the final section, the author uses linear space arguments (without the Karhunen representation) to prove results on nonabsorption probabilities with boundaries centered on the "most probable path"; these are new proofs of known theorems [Slepian, Proc. Sympos. Time Series Anal. (Brown Univ., 1962), pp. 104–115, Wiley, New York, 1963; MR **26** #5636].

F. J. Beutler (Ann Arbor, Mich.)

Vincze, István 2856

On a Gaussian stochastic process. (Hungarian)

Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **12** (1962), 35–46.

The paper is mainly of expository character. It deals with the Wiener process $\zeta(t)$ and with the Gaussian process $\xi(t)$ $(0 \leqq t \leqq 1)$ having the correlation function $E(\xi(s)\xi(t)) = s(1-t)$ $(0 \leqq s \leqq t \leqq 1)$ and satisfying the conditions $\xi(0)=\xi(1)=0$, $E(\xi(t))=0$. In § 2, utilising an idea of P. Turán, a new rapidly convergent expansion is given for the density function of the random variable τ defined as the value of t for which $|\xi(t)|$ takes on its maximum in $(0, 1)$. In § 4 an integral equation is given for the density function of the abscissa of the first intersection of the curve $\zeta(t)$ with a given continuous curve $g(t)$ for which $g(0)>0$.

A. Rényi (Budapest)

Veršik, A. M. 2857

Some characteristic properties of Gaussian stochastic processes. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 390–394.

In this paper the author defines spherically invariant processes, and he gives a representation of the characteristic functions associated with such processes. It is shown that the spherical invariance is necessary for every extrapolation problem to have a linear solution and for every functional transformation which leaves the correlation function of the process invariant to preserve its measure in the space of realizations. Further, it is shown that if a spherically invariant process is stationary and ergodic, then it is Gaussian.

A. T. Bharucha-Reid (Detroit, Mich.)

Sudakov, V. N.; Halfin, L. A. 2858

A statistical approach to the correctness of the problems of mathematical physics. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 1058–1060.

The note concerns the continuous dependence of solutions of differential equations on initial conditions. The authors explain the statistical approach on the example of the Cauchy problem for the Laplace equation in the upper half-plane with initial data

$$u(x, 0) = 0, \qquad u_y'(x, 0) = \varphi(x),$$

where $\varphi(x)$ is a periodic function. They suppose that $\varphi(x)$ is the realisation of a periodic stochastic process and examine in what sense a sequence of processes $\varphi_n(x)$ has to tend to zero in order to obtain the convergence to zero of the corresponding solutions. Their condition for convergence involves eigenvalues and eigenfunctions of the kernels $r_n(x, y) = \mathbf{E}\varphi_n(x)\varphi_n(y)$.

Petr Mandl (Prague)

Rozenblat-Rot, Millu [Rosenblatt-Roth, Millu] 2859

Some problems related to a non-homogeneous Markov chain of random variables. (Russian)

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (Liblice, 1962), *pp.* 587–597. *Publ. House Czech. Acad. Sci., Prague,* 1964.

A law (weak and strong) is obtained for some classes of non-homogeneous Markov chains. The results are slightly different from those the author obtained earlier [e.g., Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 1 (1962/63), 433–445; MR **27** #4262].

C. V. Stanojevic (Detroit, Mich.)

Cigler, J.; Schmetterer, L. 2860

Über die Summe Markowscher Ketten auf endlichen Gruppen.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (Liblice, 1962), *pp.* 45–53. *Publ. House Czech. Acad. Sci., Prague,* 1964.

Gegeben sei eine zeitlich homogene Markoffsche Kette $(x_n; n=1, 2, \cdots)$, deren Zustandsraum eine nicht notwendig kommutative Gruppe G einer endlichen Ordnung k ist. Die Arbeit beschäftigt sich mit der Markoffschen Kette $((z_m, x_m); m=1, 2, \cdots)$ im Zustandsraum $S=G\times G$, wobei $z_m = x_1\cdots x_m$. Es sei G irreduzibel, d.h., ohne transiente Zustände und aus nur einer ergodischen oder zyklischen Klasse bestehend, und l die Periode von G. Dann zerfällt S in a irreduzible Klassen mit k^2/a Elementen, wobei $a|k$, und diese Klassen haben dieselbe Periode ml mit $m \geqq 1$. Ist $g\in G$, so bezeichnen die Verfasser als Invarianzgruppe $H(g)$ die Menge aller $h\in G$, für die ein Übergang von (s, g) nach (sh, g) möglich ist, und als Periodengruppe $H_1(g)$ die Menge der h, für die ein solcher Übergang in einer Anzahl von Schritten, die ein Vielfaches von ml ist, erfolgen kann. $H_1(g)$ ist eine Untergruppe von $H(g)$ einer von g unabhängigen Ordnung b_1, alle $H(g)$ sind zueinander konjugiert, also von derselben Ordnung b, und es gilt $m=b/b_1$. Bildet G eine reguläre Klasse und stimmen Invarianz- und Periodengruppe mit G überein, so konvergiert die Wahrscheinlichkeitsverteilung $P\{z_m=h|z_0, z_1\}$ unabhängig von z_0 und z_1 gegen das Haarsche Maß auf G. Die Voraussetzung der Regularität kann nicht dahin abgeschwächt werden, daß G eine ergodische Klasse bildet und S keine zyklischen Elemente enthält. Für den Fall einer zyklischen Gruppe G hatte Koutský [Trans. Second Prague Conf. Information Theory, pp. 263–278, Publ. House Czech. Acad. Sci., Prague, 1960; MR **23** #A2934] einen großen Teil der beschriebenen Resultate bewiesen.

K. Krickeberg (New York)

Singer, Arnold 2861

The steady state probabilities of a Markov chain as a function of the transition probabilities.

Operations Res. **12** (1964), 498–499.

The author calculates the limiting matrix for the transition matrix of an ergodic Markov process with finitely many states. *D. Austin* (Evanston, Ill.)

Williams, David 2862

The process extended to the boundary.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2**, 332–339 (1964).

The results of this paper, which are too involved to report here, show that the results of Neveu [Proc. Fourth Berkeley Sympos. Math. Statist. and Prob., Vol. II, pp. 347–391, Univ. California Press, Berkeley, Calif., 1961; MR **25** #2636] on absolute dominance can be used to obtain the general solution of the Kolmogorov equations as given by Feller [Ann. of Math. (2) **65** (1957), 527–570; MR **19**, 892]. *A. T. Bharucha-Reid* (Detroit, Mich.)

Ney, P. E. 2863

Ratio limit theorems for cascade processes.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **3**, 32–49 (1964).

The process investigated in this paper can be described as follows. Suppose that a parent particle of "energy", X_0, splits at time T into N particles of energies $X_1, \cdots, X_N$. Each offspring particle then behaves like a new parent particle, and this behavior depends only on its energy. The quantities $T, N, X_1, \cdots, X_N$ are random variables. Let $p_n(x, t)$ be the probability that a single parent particle of energy 1 gives rise to n particles at time t having energy $\geqq x$.

The main purpose of the paper is to investigate $p_n(x, t)$ for large t under the assumption that no new energy can be created, i.e., $P(X_1 + \cdots + X_N \leqq X_0) = 1$. The main result is that for $n > m$, $p_n(x, t)/p_m(x, t) \to 0$ as $t \to \infty$. *S. C. Port* (Santa Monica, Calif.)

Iglehart, Donald L. 2864

Reversible competition processes.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2**, 314–331 (1964).

Let $p_{ij}(t)$ be the transition probabilities of a Markov chain, i, j ranging over the denumerable state space E, $0 < t < \infty$, and let q_{ij} be the derivative $p_{ij}'(0)$. Chains with state space $E = \{(m, n) : m, n = 0, 1, \cdots\}$, each q_{ij} finite, $\sum_j q_{ij} = 0$, and $q_{ij} = 0$ unless $j = i$ or j is immediately above, below, to the right, or to the left of i, the author calls competition processes. These chains are slightly more special than the competition processes of Reuter [Proc. Fourth Berkeley Sympos. Math. Statist. and Prob., Vol. II, pp. 421–430, Univ. California Press, Berkeley, Calif., 1961; MR **26** #812]. A chain is reversible if there exist positive constants $\{\pi_i\}$, $i \in E$, such that $\pi_i p_{ij}(t) = \pi_j p_{ji}(t)$ for all t. The principal result is an integral representation for reversible competition processes similar in nature to that obtained by Karlin and McGregor [Trans. Amer. Math. Soc. **85** (1957), 489–546; MR **19**, 989] for unilateral birth and death processes and Pruitt [Appl. Math. Statist. Lab. Stanford Univ. Tech. Rep. No. 22 (1960)] for bilateral birth and death processes. All of these representations are instances of the representation obtained by Kendall [Proc. London Math. Soc. (3) **9** (1959), 417–431; MR **22** #7178], but they are more explicit than what can be obtained from Kendall's results.

The matrix (q_{ij}) is called regular if it uniquely determines $(p_{ij}(t))$. The paper also considers the following topics for competition processes: regularity of (q_{ij}), ergodic and recurrence properties, reversibility criteria. *S. Orey* (Minneapolis, Minn.)

Ibragimov, I. A.; Tovstik, T. M. 2865

An estimate for spectral functions of a class of stationary random sequences. (Russian. English summary)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **19** (1964), *no.* 1, 42–57.

In a study of the asymptotic distribution of the estimate

$$F_N^*(\lambda) = (2\pi N)^{-1} \int_0^\lambda \left| \sum_{j=1}^N x_j e^{ij\lambda} \right|^2 d\lambda$$

of the absolutely continuous spectral function $F(\lambda)$ (with spectral density $f(\lambda)$) of a real stationary Gaussian process $\{x_n\}$, Ibragimov [Teor. Verojatnost. i Primenen. **8** (1963), 391–430; MR **28** #3488] proved that the measure P_N induced on $C[0, \pi]$ by the process $\zeta_N(\lambda) = N^{1/2}(F_N^*(\lambda) - F(\lambda))$ converges weakly to the measure induced by the Gaussian process $\zeta(\lambda)$, with $\zeta(0) = 0$, $E\zeta(\lambda) \equiv 0$ and

$$E\zeta(\lambda)\zeta(\mu) = 2\pi \int_0^{\min(\lambda,\mu)} (f(t))^2 \, dt,$$

$0 \leqq \lambda$, $\mu \leqq \pi$, provided that (i) $\int_a^b f(t)\,dt > 0$ for every $(a, b) \subset [-\pi, \pi]$, and (ii) $\int_{-\pi}^{\pi} (f(t))^{2+\delta}\,dt < \infty$ for some $\delta > 0$. The present authors prove a similar convergence theorem under the following modified hypotheses: (a) The limiting Gaussian process $\zeta(\lambda)$ is now assumed to be such that

$$E\zeta(\lambda)\zeta(\mu) = (E|\xi|_0{}^4 - 3)F(\lambda)F(\mu) + 2\pi \int_0^{\min(\lambda,\mu)} (f(t))^2 \, dt;$$

(b) Condition (i) above is replaced by $E|\xi|_0{}^a < \infty$, $a > 4 + 8\delta^{-1}$. Here ξ_0 is the random variable which appears in the two-way moving average representation for $x_n : x_n = \sum_{k=-\infty}^{\infty} a_{k-n}\xi_k$. Use is again made of the convergence theorem of Prohorov. *K. Balagangadharan* (Bombay)

Verbickaja, I. N. 2866

Conditions of applicability to stationary processes in the wide sense of the strong law of large numbers. (Russian. French summary)

Teor. Verojatnost. i Primenen. **9** (1964), 358–365.

Let $\xi(t)$ be a wide-sense stationary and continuous stochastic process with $E\xi(t) = 0$. Let $R(t)$ be the covariance function of the process,

$$\bar{R}(t) = t^{-1} \int_0^t R(s)\,ds.$$

The author shows that the conditions

$$\int_1^\infty t^{-1} \ln^2 t |\bar{R}(t)| \, dt < \infty \quad \text{or} \quad \int_1^\infty t^{-1} |\bar{R}(t)| \, dt < \infty$$

and $\xi(t)$ uniformly bounded are sufficient for the convergence of $T^{-1}\int_0^T \xi(t)\,dt$ to zero with probability one for $T\to\infty$. A corresponding result holds also for stationary sequences. *Petr Mandl* (Prague)

Maljutov, M. B. 2867

Brownian motion with reflection and the problem of the inclined derivative. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 1285–1287.

From the author's summary: "Suppose in the closure of the smooth plane region D we are given a continuous function $f(z)$, and on its boundary ∂D a smooth vector field $l(z)$. Assume for simplicity that the vector $l(z)$ is tangent to ∂D only on a set M consisting of a finite number of points. We seek bounded functions $F(z)$ satisfying the equation $\Delta F=f$ in D (Δ is the Laplace operator) which are continuous outside the set M and satisfy the equation $dF/dl=0$ on $\partial D\setminus M$. This problem turns out to be clearly related to the study of the behavior of the trajectory of a certain diffusion process. In comparison with the thoroughly studied problem of finding everywhere-continuous solutions, called the inclined derivative problem, this formulation of the problem has the advantage that it does not involve topological difficulties. Therefore it appears useful to consider it as a first stage in the study of various generalized inclined derivative problems, for example, the n-dimensional case of the problem. The subspace of continuous solutions of the problem is separated out from the space of bounded solutions with the help of considerations of a topological nature."

Judah Rosenblatt (Albuquerque, N.M.)

Farrell, R. H. 2868

Limit theorems for stopped random walks.

Ann. Math. Statist. **35** (1964), 1332–1343.

Let $X_i=(X_i^1,\cdots,X_i^k)$, $i\geqq 1$, be a sequence of independent identically distributed random vectors and suppose that $EX_1^j=\mu_j>0$ for $1\leqq j\leqq k$. One of the theorems the author obtains is that if $N(c)$ denotes the first n for which $\min_{1\leqq j\leqq k}(\sum_{i=1}^n X_i^j)>c$, then, as $c\to\infty$, $EN(c)\sim c/\mu$, where $\mu=\min(\mu_1,\cdots,\mu_k)$. This result is used to obtain similar asymptotic results for some other stopping rules. When $k=1$ and h is a continuous, strictly increasing, convex function on $[0,\infty)$ satisfying $h(0)=0$, $h(x)/x\to\infty$ as $x\to\infty$ and $Eh(|X_1|)\leqq a$ the author is able to show that $c\leqq \mu EN(c)\leqq c+h^{-1}(aEN(c))$, which gives a kind of uniform bound on $EN(c)$. For $k>1$ a similar uniform bound is obtained when $X_1^2,\cdots,X_1^k$ are positive random variables and $EX_1^1\leqq EX_1^j$, $j=2,\cdots,k$.

J. Sacks (Evanston, Ill.)

Gangolli, Ramesh 2869

Brownian transition functions and random walks on certain homogeneous spaces.

J. Math. Mech. **13** (1964), 603–615.

Let G be a compact Lie group and K a closed subgroup. This paper concerns itself primarily with the relation between Brownian transition functions on $M=G/K$ and the probability measures on another compact metric space Q. A Markov transition function $P(t,m,B)$ on M is said to be Brownian if $P(t,m,B)=P(t,xm,xB)$ for each $x\in G$, and if for each $m\in M$ and U a neighborhood of M, $P(t,m,M-U)=O(t)$ as $t\to 0+$. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$, where $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of G and K, and $\mathrm{Ad}_G(K)\mathfrak{m}\subset\mathfrak{m}$. $\mathfrak{m}$ itself is identified with the tangent space of $M=G/K$ at the coset K. There is a G invariant Riemannian structure on M and such a structure is fixed once and for all. In this structure $\mathfrak{s}$ is used to designate the unit sphere in $\mathfrak{m}$. Further, it is assumed that $\mathrm{Ad}_G(K)$ acts freely on $\mathfrak{s}$ in $\mathfrak{m}$, and Q is used to designate the orbit space of $\mathfrak{s}$ under the action of $\mathrm{Ad}_G(K)$.

The author shows that for each probability measure μ on Q there corresponds a Brownian transition function P_μ, and conversely. The general outline of the major ideas is as follows. The measure $dk\times d\mu$ is made to give rise by translation to a measure $(dk\times d\mu)_p$ on the unit sphere of the tangent space at $p\in M$. Consider the random walk x_t in which a particle starts at p at $t=0$ and chooses a direction on s_p in M_p in accordance with the measure $(dk\times d\mu)_p$. Let $\alpha>0$, and let the particle follow the geodesic in the chosen direction at speed $1/\alpha$ for time α^2 and then begin the game anew. For each $f\in C(M)$ let $T_{t,\alpha}f$ be the expectation of $f(x_t)$. It is then shown that for each $f\in C(M)$, $(T_tf)(p)=\lim_{\alpha\to 0}(T_{t,\alpha}f)(p)$ exists and the operators T_t defined in this manner form a semi-group of contraction operators on $C(M)$ which is strongly continuous at $t=0$. If E is the infinitesimal generator of this semi-group, then on the intersection of its domain with $C^\infty(M)$ it agrees with an elliptic differential operator E_2 on M. Results of Hunt and Yosida are appealed to, asserting that E_2 gives rise to a Brownian transition function on M. The author remarks that if G is a noncompact Lie group with finite center and K a maximal compact subgroup, then the above results still hold. The paper is concluded with a theorem which shows how to recover μ from a Brownian transition function.

F. Hahn (New Haven, Conn.)

Gangolli, Ramesh 2870

On the construction of certain diffusions on a differentiable manifold.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2**, 406–419 (1964).

The author considers the following idea: Let $\mathfrak{M}$ be a differentiable manifold with an affine connection ∇. In the tangent space at each point of $\mathfrak{M}$ one is given a skew Brownian motion. He then shows that if this assignment of these Brownian motions is sufficiently smooth, then we may use the exponential map to induce a diffusion process on $\mathfrak{M}$.

These are the major ideas involved. Let $b(t,w)$, $0\leqq t<+\infty$, be the Brownian motion in R^n. If V is an n-dimensional vector space, and Q a linear transformation of R^n to V, and q an element of V then $y(t,w)=Q(b(t,w))+tq$ is a skew Brownian motion on V. By a locally Lipschitzian field of skew Brownian motions the author means a collection $\{y(p):p\in\mathfrak{M}\}$, where each $y(p)$ is a skew Brownian motion on the tangent space above p. The term "locally Lipschitzian" denotes explicit Lipschitz conditions which are imposed on the matrix representations of $Q(p)$ and $q(p)$.

The next step is to attack the original problem locally. He supposes that he is given a locally Lipschitzian field of skew Brownian motions which vanishes outside a coordinate neighborhood. For each non-negative integer k this field is used to construct polygonal curves $z_k(s)$ in $\mathfrak{M}$.

The construction of the $z_k(s)$ is essentially the following. Let p be a point in $\mathfrak{M}$ and $w(t)$ a curve in the Brownian motion $b(t, w)$. For $0 \leq s < 2^{-k}$ let

$$z_k(s) = \mathrm{Exp}_p\,[Q(p) * w(s) + q(p)s];$$

for $2^{-k} \leq s < 2 \cdot 2^{-k}$ let

$$z_k(s) = \mathrm{Exp}_{z_k(2^{-k})}[Q(z_k(2^{-k})) * [w(s) - w(2^{-k})] + q(z_k(2^{-k})) \cdot [s - 2^{-k}]].$$

The above process is used to extend the curve $z_k(s)$ over the entire parameter domain $0 \leq s < +\infty$. The next two sections of the paper are devoted to showing that there is a continuous curve $z(s)$ in $\mathfrak{M}$ to which $z_k(s)$ converges with probability 1, the convergence being uniform on compact sets as $k \to +\infty$. This completes the local arguments.

In the next section the author extends the local arguments to global arguments. The result he obtains is that for each $p \in \mathfrak{M}$ there is a mapping of almost all (Wiener measure) continuous curves in R^n with initial point the origin to the set of all continuous curves on $\mathfrak{M}$ with initial point p. A measure is defined on the image space in a natural manner.

The next section concerns the computation of the "infinitesimal generator" of the process constructed. The author then discusses a "standard field" of Brownian motions on a Riemannian manifold and points out that its backward operator is the Laplace-Beltrami operator.

F. Hahn (New Haven, Conn.)

Stephens, M. A. 2871

Random walk on a circle.

Biometrika **50** (1963), 385–390.

Consider a density function f on the surface of a d-dimensional unit sphere ($d \geq 1$) describing the distribution of a unit vector, and demand that the maximum likelihood estimate of its direction be the direction of the (vector) sum of the observations in a sample; in the 2-dimensional case, $f = \exp[l \cos\theta]/2\pi I_0(l)$ and this can be closely fitted to the density function of a particle in circular Brownian motion. Roberts and Ursell [Philos. Trans. Roy. Soc. London Ser. A **252** (1960), 317–356; MR **22** #8569] have carried out the corresponding estimation for the 3-dimensional case.

H. P. McKean, Jr. (Cambridge, Mass.)

Suzuki, Takeji 2872

Batch-arrival queueing problem.

J. Operations Res. Soc. Japan **5** (1962/63), 137–148.

Customers arrive at a counter in batches of random size in accordance with a recurrent process and are served singly by a single server in order of arrival. The batch sizes are mutually independent, identically distributed random variables taking on a finite number of possible values, and the service times are mutually independent random variables having a common exponential distribution. The arrival times, the batch sizes and the service times are also independent. The author finds the generating function of the limiting distribution of the queue size and the Laplace-Stieltjes transform of the limiting distribution of the waiting time. His results are generalizations of some of the results of F. G. Foster [Acta Math. Acad. Sci. Hungar. **12** (1961), 1–10; MR **24** #A579] and the reviewer [Ann. Math. Statist. **32** (1961), 1286–1298; MR **25** #628], who considered the case of constant batch size.

L. Takács (New York)

Suzuki, Takeji 2873

Two queues in series.

J. Operations Res. Soc. Japan **5** (1962/63), 149–155.

Customers arrive at a counter in accordance with a Poisson process and are served by a single server in order of arrival. The service times are mutually independent, identically distributed random variables with distribution function $H_1(x)$ and independent of the arrival times. After being served, each customer goes to a second counter where the customers are served by a single server in order of arrival, and the service times are mutually independent random variables with distribution function $H_2(x) = 1 - e^{-\mu_2 x}$ ($x \geq 0$) and independent of the arrival times. The author is interested in finding the stationary distribution of the queue size and the stationary distribution of the waiting time at the second counter for arbitrary $H_1(x)$. The case $H_1(x) = 1 - e^{-\mu_1 x}$ ($x \geq 0$) was known previously. However, the author's results are incorrect, because his proof is based on the false assumption that the queue sizes immediately before arrivals in the second queue form a Markov chain.

L. Takács (New York)

Meissl, P. 2874

Über ein Wartezeitenproblem an einem Schalter mit regelmässigen Öffnungs- und Sperrzeiten.

Österreich. Ing.-Arch. **18** (1964/65), 79–93.

In the time interval $(0, \infty)$ customers arrive at a counter in accordance with a Poisson process of density λ and are served by a single server in order of arrival. The server is available for service only periodically during the time intervals $(nr + (n-1)g, nr + ng)$, $n = 1, 2, 3, \cdots$. The service times are mutually independent, identically distributed random variables with expectation α and independent of the arrival times. Denote by η_n the virtual waiting time at time $t = (r+g)n$, $n = 0, 1, \cdots$, i.e., the time that a customer would have to wait if he arrived at time $t = (r+g)n$. The author proves that if $\lambda\alpha(r+g)/g < 1$, then the limiting distribution $\lim_{n\to\infty} P\{\eta_n \leq x\} = W(x)$ exists and is independent of the distribution of η_0; he also finds the Laplace-Stieltjes transform of $W(x)$.

L. Takács (New York)

Foster, F. G. 2875

Batched queuing processes.

Operations Res. **12** (1964), 441–449.

Author's summary: "General single-server queues are considered with (i) arrivals in single units and batched departures, (ii) batched arrivals and departures in single units. Queue sizes at the following instants are compared: (1) just before arrivals; (2) just after departures; (3) through time. It is shown how formulas for batched queues are related to those for certain associated unit queues."

L. Takács (New York)

Loynes, R. M. 2876

The stability of a system of queues in series.

Proc. Cambridge Philos. Soc. **60** (1964), 569–574.

The author completes the discussion in his previous work

[same Proc. 58 (1962), 497–520; MR 25 #4581] on giving a necessary and sufficient condition for a system of queues in series to be substable (i.e., each subsequence of waiting-time vectors has a further subsequence which converges in distribution) when the service-time random variables and interarrival-time random variables form a stationary metrically transitive sequence. *J. Sacks* (Evanston, Ill.)

Runnenburg, J. Th. 2877

Renewal theory for Markov-dependent random variables. (French summary)

Bull. Inst. Internat. Statist. **39** (1962), *livraison* 2, 317–327.

Let y_n be positive integer-valued random variables, $n=0, 1, \cdots$, $U_r=\sum_{n=0}^{\infty} P[\sum_{k=0}^{n} y_k=r]$. When the y_n are independent and identically distributed, the asymptotic behavior of U_r as $r\to\infty$ is described by the renewal theorem. Here analogous results are established under the hypothesis that the y_n form a Markov chain satisfying certain natural conditions. It is stated that results without the restriction to integer-valued random variables appear in the author's thesis [Univ. Amsterdam, Amsterdam, 1960; MR 26 #5649]. Similar results, some in more general settings, are given by Anselone [Duke Math. J. 27 (1960), 33–40; MR 22 #12567] and the reviewer [Pacific J. Math. 9 (1959), 805–827; MR 23 #A2931; Trans. Amer. Math. Soc. 99 (1961), 384–397; MR 26 #3116].

S. Orey (Minneapolis, Minn.)

Smith, Walter L. 2878

On the elementary renewal theorem for non-identically distributed variables.

Pacific J. Math. **14** (1964), 673–699.

Let $\{X_n\}$ be a sequence of independent random variables and let us write $S_n=\sum_1^n X_k$. The renewal theorems deal with the asymptotic behavior of $\sum_1^\infty G_n(x)$, where $G_n(x)=P(S_n\leqq x)$. In the ordinary renewal theorems it is usually assumed that X_n are non-negative identically distributed random variables with $0<EX_n<\infty$. The reviewer [J. Math. Soc. Japan 8 (1956), 118–126; MR 18, 75] handled the general case of non-identically distributed random variables, and H. Hatori [Kōdai Math. Sem. Rep. 11 (1959), 139–146; MR 22 #1001; ibid. 12 (1960), 28–37; MR 22 #10031] dealt with the more general sum $\sum_1^\infty a_nG_n(x)$, but they needed rather severe conditions about the behavior of distributions of X_n at $-\infty$. The author of the present paper deals mainly with the asymptotic behavior of $\sum_1^\infty a_nG_n(x)$, where he proves that if $(1/n)\sum_1^n EX_k\to\mu$, $0<\mu<\infty$,

$$\int_{n\varepsilon}^{\infty}\frac{1}{n}\sum_{k=1}^{n}\{1-F_k(x)\}\,dx\to 0$$

as $n\to\infty$ for every $\varepsilon>0$, where $F_k(x)$ is the distribution function of X_k, and a_n is such that

$$\sum_1^\infty a_nx^n \sim \alpha(1-x)^{-\gamma}L((1-x)^{-1})$$

as $x\to 1-0$ for some $\alpha>0$, $\gamma\geqq 0$, where L is some non-negative function of slow growth and $\sum_1^\infty a_n$ diverges, then $\sum_1^\infty a_nG_n(x)\sim\alpha L(x)/\Gamma(1+\gamma)\cdot(x/\mu)^\gamma$ as $x\to\infty$, under the condition that the X_n are non-negative or under certain conditions on the properties of the distributions of X_n at $-\infty$, which are not reproduced here but are substantially more general than the reviewer's and Hatori's. The theorem itself is much more general than those of the authors mentioned above.

T. Kawata (Washington, D.C.)

STATISTICS

See also 2828a–b, 2833, 2834, 2839, 2871, 2964, 3273, 3275.

Cramér, Harald 2879

★Random variables and probability distributions.

Second edition. Cambridge Tracts in Mathematics and Mathematical Physics, No. 39.

Cambridge University Press, New York, 1962. viii+119 *pp.* $4.00.

Except for minor corrections, this Tract, including the bibliography, is in the main the same as the 1937 edition. However, the statement and proof of Theorem 11 have been corrected, and the contents of Chapter IV, § 4, have been rearranged in order to permit a simplification in the development of asymptotic expansions in Chapter VII.

Hacking, Ian 2880

On the foundations of statistics.

British J. Philos. Sci. **15** (1964/65), 1–26.

★The Annals of Mathematical Statistics. 2881

Indexes to Volumes 1–31, 1930–1960.

Compiled under the editorship of J. Arthur Greenwood, Ingram Olkin, and I. Richard Savage.

North Central Publishing Co., St. Paul, Minn., 1962. xiii+658 *pp.*

These indexes are arranged in five parts: (1) Author index and citations; (2) Subject index; (3) Index of tables; (4) Citations to other journals; (5) News items.

In the first of the parts, references to a given author's paper are given in eight reviewing or abstracting journals whenever available, along with citations in papers appearing in 44 major publications.

Anderson, T. W.; Walker, A. M. 2882

On the asymptotic distribution of the autocorrelations of a sample from a linear stochastic process.

Ann. Math. Statist. **35** (1964), 1296–1303.

The authors prove the following theorem: Let $\{x_t\}$ be a linear stochastic process defined by

$$x_t-\mu = \sum_{i=-\infty}^{\infty}\gamma_i\varepsilon_{t-i} \qquad (t=0, \pm 1, \pm 2, \cdots),$$

where $\sum_{i=-\infty}^{\infty}|\gamma_i|<\infty$ and $\sum_{i=-\infty}^{\infty}|i|\gamma_i^2<\infty$, and $\{\varepsilon_t\}$ is a set of independently and identically distributed random variables with means 0 and variances finite. Let $r_i=C_i/C_0$ $(i=1, 2, \cdots, n-1)$, where $C_i=\sum_{t=1}^{n-i}(x_t-\mu)(x_{t+i}-\mu)/(n-i)$ $(i=0, 1, \cdots, n-1)$. Then the joint distribution of $n^{1/2}(r_i-\rho_i)$, $1\leqq i\leqq s$, where $\rho_i=\operatorname{cov}(x_t, x_{t+i})/\operatorname{var} x_t$, tends to $N(0, W)$ when $n\to\infty$, where $W=(w_{ij})$ is given by

$$w_{ij} = \sum_{v=-\infty}^{\infty}(\rho_v\rho_{v+i-j}+\rho_v\rho_{v+i+j}+2\rho_i\rho_j\rho_v^2 -2\rho_i\rho_v\rho_{v+j}-2\rho_j\rho_v\rho_{v+i}).$$

A similar result holds if, in the C_i, μ is replaced by its estimator $\bar{x}=\sum_{i=1}^{n} x_i/n$.

This theorem improves previous results of Mann and Wald [Econometrica **11** (1943), 173–220; MR **5**, 129], Bartlett [Suppl. J. Roy. Statist. Soc. **8** (1946), 27–41; MR **8**, 283], Hoeffding and Robbins [Duke Math. J. **15** (1948), 773–780; MR **10**, 200], the reviewer [Proc. Cambridge Philos. Soc. **49** (1953), 239–246; MR **14**, 771], Walker [ibid. **50** (1954), 60–64; MR **15**, 331] and Anderson [Ann. Math. Statist. **30** (1959), 676–687; MR **21** #6072].

P. H. Diananda (Singapore)

Singh, S. N. 2883

Inflated Poisson distribution.

J. Sci. Res. Banaras Hindu Univ. **13** (1962/63), 317–326.

The author re-examines a problem which is essentially that dealt with earlier by the reviewer [Biometrics **16** (1960), 446–450; MR **22** #12627]. The content of this paper appears to be almost identical with that of another paper by the same author [J. Indian Statist. Assoc. **1** (1963), 140–144; MR **28** #5508].

A. C. Cohen (Athens, Ga.)

Kale, B. K. 2884

Maximum likelihood estimation for truncated exponential family.

J. Indian Statist. Assoc. **1** (1963), 86–90.

Let X be a random variable with range R, density belonging to the exponential family, $f(x, \theta)=\beta(\theta)\exp(x\theta)$ with respect to a σ-finite measure μ. The parameter space $\Omega=(\underline{\theta}, \bar{\theta})$ is the set of values of θ for which $\int_R \exp(x\theta)\,d\mu$ is finite. The likelihood equation has been shown by Girshick and Savage [Proc. Second Berkeley Sympos. Math. Statist. and Prob., pp. 53–73, Univ. California Press, Berkeley, Calif., 1951; MR **13**, 571] to be $x=E[X]=h(\theta)$. Denote by X_A the random variable truncated to a proper subset $A\subset R$. Let the parameter space in the truncated case be Ω_T. Then X_A has density belonging to the exponential family and $\Omega_T\supseteq\Omega$. If $\Omega_T=\Omega$, then a sufficient condition that the likelihood equation $x=h_A(\theta)=E[X|X\in A]$ admit a unique maximum likelihood estimator a.e. is that $P[X\in(h_A(\underline{\theta}), h_A(\bar{\theta}))|X\in A]=1$, for all $\theta\in\Omega$. Several examples are given.

M. Katz (Detroit, Mich.)

Shah, B. K.; Khatri, C. G. 2885

Further investigation in fitting the regression curve of the type $y=\alpha+\delta x+\beta\rho^x$.

J. Indian Statist. Assoc. **1** (1963), 202–214.

This paper is concerned with a comparison of Patterson's "quadratic estimate" of ρ [Biometrika **45** (1958), 389–400; MR **21** #3076] with estimates of this parameter obtained using a modification of Hartley's method as suggested in an earlier paper by the authors [J. Maharaja Sayajirao Univ. Baroda **8** (1959), no. 2, 33–40]. The relative efficiencies and biases of these estimates are considered for $n=5, 6, 7$ and 8 equally-spaced ordinates, where the observed y's are independently and normally distributed about their expected values with constant variance. The authors conclude that on the whole, the modified Hartley method leads to improved estimates.

A. C. Cohen (Athens, Ga.)

Lord, Frederic M. 2886

Biserial estimates of correlation.

Psychometrika **28** (1963), 81–85.

Author's summary: "Brogden's coefficient of selective efficiency and Clemans' λ are more efficient than biserial r when the correlation is high. Some empirical sampling studies are reported comparing the standard errors of these statistics."

Ibramhalilov, I. Š. 2887

Some methods of determining estimates for parameters. (Russian. Azerbaijani summary)

Akad. Nauk Azerbaĭdžan. SSR Dokl. **20** (1964), *no.* 3, 9–15.

Let $\xi_1, \cdots, \xi_n$ be independent random variables with common distribution function (df) $F(x, \alpha)$, where α is a real parameter. Under appropriate regularity conditions the consistency and asymptotic normality of estimators of α are proved which are obtained by minimizing with respect to α certain functionals of the difference between the empirical df and $F(x, \alpha)$.

W. Hoeffding (Chapel Hill, N.C.)

Raja Rao, B. 2888

A further note on the geometry of linear estimation.

Biometrika **50** (1963), 540–541.

Given that t_1 and t_2 are any two linear unbiased estimators of a parameter μ of the form $t_1=\sum\lambda_j x_j$ and $t_2=\sum\lambda_j' x_j$ with efficiencies e_1 and e_2, respectively, and with correlation coefficient ρ, the author presents a geometrical proof that

$$\rho = \sqrt{(e_1e_2)}+\cos\varphi\cdot\sqrt{[(1-e_1)(1-e_2)]},$$

where φ is the angle $P_1\hat{Q}P_2$ in which P_1 and P_2 are points in the hyperplane $\sum\lambda_j=1$ corresponding to t_1 and t_2, and Q is the point corresponding to the most efficient estimator of μ. *A. C. Cohen* (Athens, Ga.)

Furukawa, Nagata 2889

Estimability and completeness in a mixed model.

Mem. Fac. Sci. Kyushu Univ. Ser. A **18** (1964), 56–73.

Chen, Xi-ru [Chen, Hsi-ju] 2890

Minimax estimates of parameter vectors of translation.

Acta Math. Sinica **14** (1964), 276–290 (*Chinese*); *translated as Chinese Math.* **5** (1964), 300–315.

The reviewer's paper [Ann. Math. Statist. **28** (1957), 573–601; MR **19**, 1097] does not treat the question of existence of minimax procedures, instead proving a generalization of the Hunt-Stein theorem which holds without regard to this question. Also, a semicontinuity assumption which yields the measurability developments was stated [op. cit., p. 579] without details of the latter. The author therefore carries out existence and detailed measurability proofs in the special case of nonsequential estimation of a vector translation parameter for continuous loss functions which are unbounded at ∞ and nondecreasing on all rays. {Such results, not referred to in the paper, have also been proved by Kudō [Natur. Sci. Rep. Ochanomizu Univ. **6** (1955), 31–73; MR **17**, 1103] and Wesler [Ann. Math. Statist. **30** (1959), 1–20; MR **21** #5263]; general existence and measurability techniques are considered by LeCam

[ibid. **26** (1955), 69–81; MR **16**, 730]; the fixed-point invariant game method of Karlin [Ann. of Math. (2) **58** (1953), 371–401; MR **16**, 1135] applies in such problems with a minimum of such measurability considerations.}

J. Kiefer (Ithaca, N.Y.)

Konijn, H. S. 2891

On a theorem of Halmos concerning unbiased estimation of moments.

J. Austral. Math. Soc. **4** (1964), 229–232.

This paper is an extension of some earlier results of Halmos [Ann. Math. Statist. **17** (1946), 34–43; MR **7**, 483] concerning a class of distribution functions $\mathfrak{D}$ over a given (Borel) subset E of the real line, where F is a function over $\mathfrak{D}$. Halmos demonstrated that on the basis of a sample of size n one can always obtain an estimate of the first moment which is unbiased in $\mathfrak{D}$ and that the central moments $\bar{F}_m$ of order $m \geqq 2$ have estimates which are unbiased in $\mathfrak{D}$ if and only if $n \geqq m$, provided that $\mathfrak{D}$ satisfies the following properties: $\bar{F}_m$ exists and is finite for all distributions in $\mathfrak{D}$, and $\mathfrak{D}$ includes all distributions which assign probability 1 to a finite number of points of E. He also found that symmetric estimates which are unbiased on $\mathfrak{D}$ are unique and have smaller variances on $\mathfrak{D}$ than nonsymmetric unbiased estimates. In the present paper, the author proves that if $\mathfrak{D}$ is the class of all non-degenerate univariate normal distributions, then, on the basis of a sample of size n, an estimate of the first moment which is unbiased over $\mathfrak{D}$ exists and is unique when $n=1$; and a central moment of order $2r \geqq 2$ has estimates which are unbiased over $\mathfrak{D}$ if and only if $n \geqq 2$, and has a unique symmetric unbiased estimate when $n=2$, but not when $n>2$.

A. C. Cohen (Athens, Ga.)

Romanovskaja, I. L. 2892

On the Fisher-Welch-Wald test. **(Russian)**

Dokl. Akad. Nauk SSSR **156** (1964), 513–514.

The author extends to the case of unequal sample sizes the result of Linnik, which shows that there is no similar (relative to $n_2\sigma_1^2/n_1\sigma_2^2$) nonrandomized test of equality of means of two samples from normal populations with unknown variances. He also shows the nonexistence of certain types of randomized similar tests for this problem.

Judah Rosenblatt (Albuquerque, N.M.)

Patil, S. A. 2893

The maximum likelihood estimation of the parameter of the truncated censored Poisson distribution.

J. Indian Soc. Agric. Statist. **14** (1962), 177–187.

The author presents straightforward derivations of maximum likelihood estimators of the Poisson parameter from samples that are truncated over one interval of the random variable and censored over another interval. Six different combinations of truncated and censored intervals are considered. Asymptotic variances of the estimators are given and an illustrative example is included.

A. C. Cohen (Athens, Ga.)

Cucconi, Odoardo 2894

Intorno ad una semplice relazione tra il numero di gradi di libertà e i valori critici della t di Student.

Atti Mem. Accad. Patavina Sci. Lett. Arti **74** (1961/62), 179–187.

The author suggests an approximation formula for the determination of the critical value ${}_nt_\alpha$ of Student's distribution with n degrees of freedom and significance level α. The value of ${}_nt_\alpha$ is expressed in terms of n and the critical value of the normal distribution at level α.

E. Lukacs (Washington, D.C.)

Li, Ši-lin' [Li, Shih-lin] 2895

On the asymptotic power of a non-parametric test analogous to the χ^2 statistic. **(Ukrainian. Russian and English summaries)**

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 856–861.

Author's summary: "In this note the author considers the asymptotic power of agreement criteria analogous to χ^2 with an unlimited simultaneous increase of the number of observations and the grouping intervals for an alternative of the form $\bar{f}(x)=f_0(x)+(c/N^\delta)g(x')$ and

$$\bar{f}(x,\theta)+(\bar{c}/N^\delta)\bar{g}(x,\theta), \qquad 0<\delta<\tfrac{1}{2}."$$

Li, Ši-lin' [Li, Shih-lin] 2896

A non-parametric criterion of homogeneity for k samples similar to the χ^2. **(Ukrainian. Russian and English summaries)**

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 42–46.

Author's summary: "The author considers the asymptotic property of the statistic $X_k(r)$. To test the hypothesis of homogeneity of k samples the statistic $\max_{1\leq r\leq m} X_k(r)$ or $X_k(n)$ is to be taken. The asymptotic power of the criterion based on the statistic $\max_{1\leq r\leq m}|X_2(r)|$ or $X_2(n)$ for near alternatives is studied in this note with certain limitations."

$X_k(r)$ is a χ^2-like statistic; r is the number of classes into which the observed chance variables can fall.

J. Wolfowitz (Ithaca, N.Y.)

Taylor, Wilson L. 2897

Correcting the average rank correlation coefficient for ties in rankings.

J. Amer. Statist. Assoc. **59** (1964), 872–876.

Author's summary: "This report supplements a previous article by Taylor and Fong [same J. **58** (1963), 756–769; MR **27** #2035] on the computation and distribution of the 'average rank correlation coefficient', $\bar{\rho}$, by specifying methods of computing, and of assessing the null-case probabilities of, the 'average rank correlation coefficient corrected for ties in ranks', $\bar{\rho}_t$, and the 'average rank correlation coefficient corrected both for ties and for continuity', $\bar{\rho}_{t,c}$."

Taha, Mohamed Abdel-Hamid 2898

Über die verallgemeinerten Teste von Kolmogorov und Smirnov für unstetige Verteilungen. **(French, English, and Italian summaries)**

Mitt. Verein. Schweiz. Versich.-Math. **64** (1964), 145–174.

Author's summary: "The distribution of the Kolmogorov criteria is obtained for discontinuous distributions if the values of the jumps are estimated. The distribution of Smirnov's two-sample criteria is also got for discontinuous distributions. Lastly, the distribution of

Smirnov's criteria over a finite interval $(-\infty < x_1 < x_2 < +\infty)$ is given. In all the cases, both the one-sided and the two-sided tests are treated." *R. Pyke* (Seattle, Wash.)

Taga, Yasushi 2899

On unbiased ratio and regression estimators. (Japanese. English summary)

Proc. Inst. Statist. Math. **11** (1963/64), 1–6.

Author's summary: "Various studies have been made on unbiased ratio and regression type estimators, since an unbiased ratio-type estimator was proposed by Hartley and Ross [see, e.g., H. O. Hartley and A. Ross, Nature **174** (1954), 270–271; L. A. Goodman and H. O. Hartley, J. Amer. Statist. Assoc. **53** (1958), 491–508; MR **20** #1380]. In this report, these methods are surveyed and an unbiased regression type estimator is proposed which is applicable even when the regression line in the population is nonlinear."

Ladd, George W. 2900

Regression analysis of seasonal data.

J. Amer. Statist. Assoc. **59** (1964), 402–421.

If monthly or quarterly data is used for the purpose of regression analysis, one has to take care of the seasonal variation in the data. Normally one uses either seasonally adjusted data or allows for seasonal variation through dummy variables in the model. The bias of the coefficient estimates is considered in each case. If the coefficients do not vary seasonally, the latter method yields unbiased estimates. Standard errors of the coefficient estimates still remain to be compared. *A. L. Nagar* (Delhi)

Kôno, Kazumasa 2901

Corrections to: "Optimum design for quadratic regression of k-cube".

Mem. Fac. Sci. Kyushu Univ. Ser. A **18** (1964), 120.

Correction of misprints, together with a revised proof of inequality (2.11) of an earlier paper [same Mem. **16** (1962), 114–122; MR **27** #3059].

Ruben, Harold 2902

An asymptotic expansion for the multivariate normal distribution and Mills' ratio.

J. Res. Nat. Bur. Standards Sect. B **68B** (1964), 3–11.

An asymptotic and enveloping series is obtained for the multivariate normal distribution and the corresponding analogue of Mills' ratio provided that the cut-off point is interior to a certain polyhedral half-cone determined by the set of regression planes. The result is specialised to the particular case of equicorrelated normal variates, and possible extensions considered. *P. A. P. Moran* (Canberra)

Kshirsagar, A. M. 2903

Distributions of the direction and collinearity factors in discriminant analysis.

Proc. Cambridge Philos. Soc. **60** (1964), 217–225.

Author's summary: "Exact tests devised by Bartlett [Ann. Eugenics **16** (1951), 199–214; MR **13**, 666] and Williams [Biometrika **42** (1955), 360–381; MR **17**, 381] for the adequacy of a single discriminant function in the case of several groups are applicable to a wide range of problems in multivariate analysis. The appropriate likelihood criterion is factorized into direction and collinearity factors; and the exact distribution of these factors under the condition of only one nonzero canonical correlation was first given by Bartlett in the general case of any number of dependent and independent variables, making use of geometrical arguments. In this paper the factors are expressed in terms of the matrices occurring in the multivariate analysis of the variance table; and the results are derived by an analytical method making use of matrix transformations and rectangular coordinates." *K. G. Jöreskog* (Princeton, N.J.)

Bickel, Peter J. 2904

On some alternative estimates for shift in the p-variate one sample problem.

Ann. Math. Statist. **35** (1964), 1079–1090.

The author investigates the vector of medians M_n and the vector of medians of averages of pairs W_n as competitors of the vector mean $\bar{X}_n$ in estimating the location parameters in the p-variate one-sample problem. These estimates are found to be asymptotically normal and unbiased, and necessary and sufficient conditions are given for the degeneracy of the asymptotic distribution of M_n and W_n. For W_n, in the case $p=2$, these reduce to the condition that one coordinate variable be a monotone function of the other. Sufficient symmetric conditions are given for the asymptotic independence of the coordinates of these estimates. When compared to $\bar{X}_n$ in terms of the Wilks generalized variance, W_n and M_n are demonstrated to be robust in the case of asymptotically independent coordinates. For $p \geqq 3$, however, they can have arbitrarily small efficiency even in the non-singular p-variate normal case if the underlying distribution is permitted to approach a suitable degenerate distribution arbitrarily closely. For $p=2$, in the normal case, W_n is highly efficient, although M_n can have arbitrarily small efficiency. However, W_n is also shown to have arbitrarily small efficiency for a suitable highly correlated family of distributions, even in the case $p=2$. On the other hand, W_n becomes infinitely more efficient than $\bar{X}_n$ as a given fixed distribution is mixed with an increasingly heavy gross error distribution. The behavior of these estimates is also considered for other non-normal families. *A. C. Cohen* (Athens, Ga.)

McDonald, Roderick P. 2905

A general approach to nonlinear factor analysis.

Psychometrika **27** (1962), 397–415.

Author's summary: "The method presented attempts to allow for nonlinear, possibly non-monotonic relations between manifest and latent variates. An attempt is made to provide a workable criterion for choosing between alternative models on the basis of observable data as well as for constructing the appropriate function. An idealized numerical example is given."

Mijares, Tito A. 2906

On elementary symmetric functions of the roots of a multivariate matrix: Distributions.

Ann. Math. Statist. **35** (1964), 1186–1198.

This paper presents the joint distribution of the e.s.f. (elementary symmetric function) of the roots of a multivariate matrix under the null hypothesis. It then deals with the moments of the e.s.f. Notation is introduced and previous pertinent results are stated. Then a proof by construction for determinantal expressions for moments and product moments (of the e.s.f.) is presented and the results are applied, for illustration, to the third moment of the second e.s.f. and to the product moments of the first, second, and third e.s.f. An evaluating formula for the determinantal expression is given.

P. S. Dwyer (Ann Arbor, Mich.)

Des Raj 2907

A note on the variance of the ratio estimate.

J. Amer. Statist. Assoc. **59** (1964), 895–898.

Author's summary: "An examination is made of the accuracy of the usual approximation used for the variance of the ratio estimate. It is shown that, under certain conditions, the usual approximation understates the true variance by an amount depending upon the distribution of the sample mean of the auxiliary variate."

A. C. Cohen (Athens, Ga.)

Glasser, M. 2908

Linear regression analysis with missing observations among the independent variables.

J. Amer. Statist. Assoc. **59** (1964), 834–844.

Author's summary: "An intuitive estimate is presented for estimating linear regression coefficients when an unspecified number and pattern of the independent variables for some individuals are not observed. It is shown that the efficiency of this approach depends upon the correlation among the independent variables as well as the proportion of observations which are missing."

A. C. Cohen (Athens, Ga.)

Watson, F. R. 2909

A new method for solving simultaneous linear equations associated with mutivariate analysis.

Psychometrika **29** (1964), 75–86.

Author's summary: "A method is outlined whereby the equations resulting from multivariate analyses may be solved. Only the upper or lower triangular coefficient matrix need be employed. All solutions involving any combination(s) of the variables represented in the coefficient matrix are readily obtainable without permutation of the matrix. All solutions are given immediately without the need for back solving. One variable is added to or subtracted from the regression equation with each application of the method. The method may be applied as repeatedly as needed for any particular solution."

P. S. Dwyer (Ann Arbor, Mich.)

Jöreskog, K. G. 2910

On the statistical treatment of residuals in factor analysis.

Psychometrika **27** (1962), 335–354.

Author's summary: "A method for estimation in factor analysis is presented. The method is based on the assumption that the residual (specific and error) variances are proportional to the reciprocal values of the diagonal elements of the inverted covariance (correlation) matrix. The estimation is performed by a modification of Whittle's least squares technique. The method is independent of the unit of scoring in the tests. Applications are given in the form of nine reanalyses of data of various kinds found in earlier literature."

P. S. Dwyer (Ann Arbor, Mich.)

Dwyer, P. S. 2911

Properties of polykays of deviates.

Ann. Math. Statist. **35** (1964), 1167–1173.

The author presents some formulae for polykays of deviates, which are called d-statistics, by general combinatorial methods. This is a generalization of work by Tukey [same Ann. **27** (1956), 37–54; MR **17**, 868].

C. Hayashi (Tokyo)

Dwyer, P. S.; Tracy, D. S. 2912

A combinatorial method for products of two polykays with some general formulae.

Ann. Math. Statist. **35** (1964), 1174–1185.

The authors present systematically some general formulae for expected value of products of polykays by a general combinatorial method. This paper is closely related with the works of Wishart [Biometrika **39** (1952), 1–13; MR **14**, 296; Trabajos Estadíst. **3** (1952), 13–26] and Tukey [J. Amer. Statist. Assoc. **45** (1950), 501–519; MR **12**, 725; Ann. Math. Statist. **27** (1956), 37–54; MR **17**, 868].

C. Hayashi (Tokyo)

Čibisov, D. M. 2913

Limit distributions for order statistics. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 159–165.

The author investigates the limiting distribution of the kth order statistic from a sample of n independent observations with common distribution, where $k=k(n)$ satisfies $\lim_{n\to\infty}(\sqrt{k(n+r)}-\sqrt{k(n)})=\frac{1}{2}\alpha l\nu$ with $r=r(n)$ an integer satisfying $\lim_{n\to\infty} r(n)/n^{1-\alpha/2}\to\nu$ and $\alpha\in(0,1)$, $l>0$ and ν being constants. He shows that there are only three possible limiting distributions, $G_1^\beta(x)=\Phi(\beta\ln x)$, $x>0$, $G_2^\beta(x)=\Phi(-\beta\ln|x|)$, $x<0$, and $G_3(x)=\Phi(x)$, the normal distribution, and derives necessary and sufficient conditions for convergence to each.

Judah Rosenblatt (Albuquerque, N.M.)

Yen, Elizabeth H. 2914

On two-stage non-parametric estimation.

Ann. Math. Statist. **35** (1964), 1099–1114.

Let $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ be samples from distributions F and G, and let $\theta=\theta(F,G)$ be an estimable functional as (F,G) ranges over all continuous distributions, the best unbiased estimator being a U-statistic $U[m,n]$. For fixed total sample size $N=m+n$, the best allocation $[m(F,G),n(F,G)]$ depends on F and G. The author proposes two-stage estimators, with the allocation in the second part of the sample depending on the observations of the first stage, which are asymptotically normal and as efficient as $U[m(F,G),n(F,G)]$ as $N\to\infty$.

E. L. Lehmann (Berkeley, Calif.)

Agrawal, Hiralal 2915

On the bounds of the number of common treatments between blocks of semi-regular group divisible designs.

J. Amer. Statist. Assoc. **59** (1964), 867–871.

Author's summary: "The bounds for the number of common treatments between any two blocks of a semi-regular group divisible design are derived using the characteristic roots of NN', where N is the incidence matrix of the design. It is further shown that these bounds are superior to the existing ones. Some new results are obtained and known ones are established in an elegant way." *D. A. Sprott* (Waterloo, Ont.)

Romier, Guy 2916

Automorphismes et algèbre d'un schéma d'association. Caractérisation algébrique des correspondances partiellement équilibrées.

C. R. Acad. Sci. Paris **258** (1964), 5345–5348.

From the author's summary: "L'algèbre d'un schéma [d'association] est interprétée à partir de l'algèbre du groupe d'automorphismes. . . . Une caractérisation algébrique des correspondances . . . (PBIBD) en est déduite. . . . Ces résultats . . . sont utilisés pour l'étude des configurations 'transitives' . . ." *H. L. Seal* (New Haven, Conn.)

Heuzé, Guy 2917

Équivalences dans les schémas d'association.

C. R. Acad. Sci. Paris **258** (1964), 5349–5351.

A theorem relates certain PBIB designs with m associate classes to others with $m-1$ associate classes.

H. L. Seal (New Haven, Conn.)

Vartak, M. N. 2918

Connectedness of Kronecker product designs.

J. Indian Statist. Assoc. **1** (1963), 215–218.

A design whose incidence matrix is the Kronecker product of the incidence matrices of two given designs is connected if and only if the given designs are connected.

J. Sacks (Evanston, Ill.)

Shah, S. M. 2919

On the upper bound for the number of blocks in balanced incomplete block designs having a given number of treatments common with a given block.

J. Indian Statist. Assoc. **1** (1963), 219–220.

An upper bound, as described in the title, is obtained.

J. Sacks (Evanston, Ill.)

Shah, K. R. 2920

Use of inter-block information to obtain uniformly better estimators.

Ann. Math. Statist. **35** (1964), 1064–1078.

The author presents new methods which are designed to make use of inter-block information for the purpose of estimating treatment contrasts in incomplete block designs where the block effects are random. For some classes of designs (including BIB's with more than five treatments and simple lattice designs with at least sixteen treatments) the author shows that his methods will yield unbiased estimators having smaller variance than estimators based only on intra-block information for all values of the unknown error variance and unknown block variance. The author's estimators coincide with the customary Yates-Rao estimators only for linked block designs.

J. Sacks (Evanston, Ill.)

Thaker, P. J. 2921

Some infinite series of second order rotatable designs.

J. Indian Soc. Agric. Statist. **14** (1962), 110–120.

In this paper the author utilizes cyclical groups of elements to construct second-order rotatable designs. Infinite series of second-order rotatable designs with four, five and six factors are obtained, and a general method for constructing such designs for any number of factors is indicated. A simplified procedure for use with an even number of factors is also described. The author concludes the paper by presenting a method for constructing infinite series of second-order rotatable designs with $k=3k'$, $4k'$ and $5k'$ factors, where $k'=2, 3, \cdots$.

S. Addelman (Durham, N.C.)

Neymann, J. [Neyman, J.] 2922

Two breakthroughs in the theory of statistical decision making. (Spanish)

Estadíst. Española No. 18 (1963), 5–28.

Translation of an article which appeared earlier in English [Rev. Inst. Internat. Statist. **30** (1962), 11–27; MR **28** #4646].

Arribat, Paul 2923

Sur l'estimation de la valeur probable d'une variable aléatoire normale Y au moyen d'une variable aléatoire normale auxiliaire X, liée à Y, de valeur probable EX connue.

C. R. Acad. Sci. Paris **258** (1964), 4914–4916.

This paper deals with the regression estimates of $E(Y)$ when using a related normal X with $E(X)$ known. The first part of the paper presents notation and states known results which include, for a random sample and under the hypothesis specified, a formula for a quantity z which is an unbiased estimate of $E(Y)$. The paper provides confidence intervals of the form $z-c'\leqq E(Y)\leqq z+c'$ and compares them with the usual confidence intervals $\bar{y}-c\leqq E(Y)\leqq \bar{y}+c$ in which the related variable, X, is not used. A formula for c'/c is derived, and it is evident therefrom that, except for very small values of n, this ratio is approximately $\sqrt{(1-r^2)}$. Hence $c'<c$ for all $r\neq 0$ and the ratio decreases with increasing $|r|$. A numerical illustration is given. *P. S. Dwyer* (Ann Arbor, Mich.)

Cramer, Elliot M. 2924

Some comparisons of methods of fitting the dosage response curve for small samples.

J. Amer. Statist. Assoc. **59** (1964), 779–793.

Author's summary: "In recent years there has been much discussion of the relative merits of several methods of fitting the normal integral dosage response curve. For a three-dose experiment with ten observations at each dose and for a five-dose experiment with four observations, three methods are compared with respect to their root mean square error in estimating m, σ, and β. Essentially

the total population of samples is used. The main result is that the maximum likelihood method is superior to the minimum normit method in estimating m and σ, but for β the minimum normit method is superior."

Madansky, Albert 2925
Instrumental variables in factor analysis.
Psychometrika **29** (1964), 105–113.
Author's summary: "The factor analysis model is rewritten as a system of linear structural relations with errors in variables. The method of instrumental variables is applied to this revised form of the model to obtain estimates of the factor loading matrix. The relation between this method and interbattery analysis, proportional profile analysis, and canonical factor analysis is pointed out. In addition, an estimation procedure based on replicated sampling different from proportional profile analysis is given." *J. Neyman* (Berkeley, Calif.)

Basavarajappa, K. G.; Ramachandran, K. V. 2926
A note on testing interviewer difference in a stratified sampling design.
J. Indian Statist. Assoc. **1** (1963), 32–39.
The paper gives statistical tools to a method for the detection of the investigator's bias and error in a sample survey. The method, suggested by P. C. Mahalanobis [Philos. Trans. Roy. Soc. London Ser. B **231** (1944), 329–451], is the following. Having to survey k areas we send two investigators to each area to collect half of the sample units, which are selected randomly. Assuming that the random variables considered are normally distributed with common variances and with expectations having different values on the different areas, the hypothesis is tested that the means of the first half samples are the same as those of the corresponding second half samples. A procedure is given with a certain optimum property.
I. Vincze (Budapest)

Hayashi, Chikio 2927
Estimation of the size of universe by matching in K samples from K lists and estimation of the mean value of that universe. (Japanese. English summary)
Proc. Inst. Statist. Math. **11** (1963/34), 49–61.

Anderson, T. W. 2928
The use of factor analysis in the statistical analysis of multiple time series.
Psychometrika **28** (1963), 1–25.
Author's summary: "Factor analysis has been proposed and used as a method of statistical analysis of several measurements made on one individual repeatedly over time. This paper discusses some difficulties in applying factor analysis to multiple time series and attempts to indicate to what extent such methods can accomplish the goals of time series analysis. Some other methods are suggested." *E. J. Hannan* (Baltimore, Md.)

Parzen, Emanuel 2929
An approach to empirical time series analysis.
J. Res. Nat. Bur. Standards Sect. D **68D** (1964), 937–951.
Author's summary: "This paper attempts to develop a philosophy for empirical time series analysis, involving the routine use of four data handling procedures (covariance estimation, spectral estimation, autoregressive model fitting and spectral estimation, and trend elimination and estimation) embodied in a computer program. The cross-spectral analysis of a pair of time series, each consisting of 4000 observations, requires approximately 10 minutes of [an IBM] 7090, including computation of covariances. Several examples of empirical time series analysis are given."

Vincze, István 2930
Some questions on the probabilistic concept of information. (Hungarian)
Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **12** (1962), 7–14.
Let $X_1, X_2, \cdots, X_n$ be a sample of n independent observations with the common density function $f(x, a)$ depending on the unknown parameter a. Let $\alpha = \alpha(x_1, x_2, \cdots, x_n)$ be a statistic and consider the quantity

$$I = \int f(x, a) \log \frac{f(x, a)}{f(x, \alpha)}\, dx.$$

Let S_α denote the set of those values of a for which the inequality $I < \lambda E(I)$ holds (where $\lambda > 0$ is a constant and $E(I)$ denotes the expectation of I computed under the supposition that a is the true value of the parameter). Three examples are given in which S_α is a confidence interval for a. *A. Rényi* (Budapest)

NUMERICAL METHODS

See also 2143, 2435, 2436, 2610, 2652, 3211, 3249, 3252.

Collatz, Lothar 2931
★Funktionalanalysis und numerische Mathematik.
Die Grundlehren der mathematischen Wissenschaften, Band 120.
Springer-Verlag, Berlin, 1964. xvi + 371 *pp.* (1 *plate*) *DM* 58.00.
It seems strange that this book should be the first of its kind, since it hardly needs to be said that "numerical mathematics" must draw heavily from functional analysis. Nevertheless, in an unnecessarily modest preface, the author disclaims any intention of writing a textbook on either functional analysis or numerical mathematics, offering the book instead as illustrating the artificiality of any separation of applied from pure mathematics.

In spite of the disclaimer, the book could be used as a basis for a rather extensive course on functional analysis for numerical analysts, and the footnotes and the bibliography provide for a considerable amount of collateral reading.

There are three chapters: Foundations of functional analysis with application; Iterative procedures; and Monotonic inequalities and other topics (in free translation). The first of these deals with types of spaces; convergence, completeness, and compactness; operators and characteristic values; and vector and matrix norms. The second chapter starts with "The fixed-point theorem for the general iterative procedure in pseudometric

spaces", and later devotes nearly 50 pages to Newton's method and regula-falsi. The final chapter applies the Schauder fixed-point theorem in several ways, and concludes with Chebyshev approximation, both finite and continuous. A fair number of well-chosen exercises scattered through the book enhance its value as a text.

A. S. Householder (Oak Ridge, Tenn.)

Singer, James 2932

★Elements of numerical analysis.

Academic Press Textbooks in Mathematics.

Academic Press, New York-London, 1964. x + 395 *pp.* $8.75.

This book, of nearly 400 pages, "for junior undergraduate through first year graduate students in mathematics", covers the following topics. Chapter 1, Numbers and errors (significant figures, relative error, accuracy and precision, computational error); Chapter 2, The approximating polynomial—Approximation at a point (power series, asymptotic series, Euler's summation formula, Bernoulli numbers and polynomials, some discussion of error terms); Chapter 3, Approximation in an interval (various forms of the Lagrangian interpolation formula, finite-difference representation, error terms, subtabulation, inverse interpolation); Chapter 4, Solution of equations in one unknown—Geometrical methods (graphical methods, nomograms in some theoretical and practical detail); Chapter 5, Arithmetic methods (methods of Horner and Graeffe for polynomials, Newton and others (excluding Bairstow) for iteration for general functions); Chapter 6, Simultaneous algebraic and transcendental equations (natural extension of Chapters 4 and 5, but very little on linear equations); Chapter 7, Numerical differentiation and integration (Lagrange and finite-difference formulae, with error terms, orthogonal polynomials and Gauss integration); Chapter 8, Solution of ordinary differential equations (graphical methods, also Picard's method, power-series, with and without step change of origin, Runge-Kutta formulae, finite-difference and predictor-corrector methods, some discussion of error but not of stability); Chapter 9, Curve fitting (mainly by straight-lines and polynomials with method of least squares).

There are many worked examples and exercises for students, with answers to some of them, a short bibliography of books for supplementary reading, and a somewhat terse index. A reasonable attempt, at this level, to answer the question "why" as well as "how", but probably more appealing to the scientist than the mathematician.

L. Fox (Oxford)

Tornheim, Leonard 2933

Convergence of multipoint iterative methods.

J. Assoc. Comput. Mach. **11** (1964), 210–220.

An equation, or system of equations, of the form $f_j(X)=0$, $j=1,\cdots,n$, where X is a point in n-dimensional space, is considered. An r-point iterative method is defined by a function F, which generates a sequence of points $\{X_i\}$, $i=r+1,r+2,\cdots$, by means of the relation $X_i=F(X_{i-1},\cdots,X_{i-r})$, the points $X_1,\cdots,X_r$ being given initially. Conditions are established for the convergence of the sequence $\{X_i\}$ to a solution X^* of the system (provided $X_1,\cdots,X_r$ are sufficiently close to X^*), and the order of convergence is shown to be bounded from below by the positive root of an equation of the form $X^r=a_1X^{r-1}+\cdots+a_r$, with $a_i\geq 0$ for $i=1,\cdots,r$.

Particular attention is given to the secant method, in which X_i is taken as the common zero of the set of linear functions $g_1,\cdots,g_n$ defined by the condition

$$g_j(X_{i-k})=f_j(X_{i-k})\quad\text{for } j=1,\cdots,n,\ k=1,\cdots,n+1.$$

Difficulties arise with this method if $n+1$ successive points lie within, or close to, a common hyperplane. The avoidance of this situation is, in effect, one of the conditions of a theorem that is proved about convergence with this method.

The paper is essentially self-contained, the proofs being based on elementary concepts.

W. M. Kincaid (Ann Arbor, Mich.)

Jansson, Birger 2934

Autocorrelations between pseudo-random numbers.

Nordisk Tidskr. Informations-Behandling **4** (1964), 6–27.

Author's summary: "In this paper some generating methods of the multiplicative and mixed congruential types for obtaining pseudo-random numbers are reviewed. Procedures to determine autocorrelations are demonstrated and a few numerical results are also given."

R. R. Coveyou (Oak Ridge, Tenn.)

Shah, B. V.; Buehler, R. J.; Kempthorne, O. 2935

Some algorithms for minimizing a function of several variables.

J. Soc. Indust. Appl. Math. **12** (1964), 74–92.

The authors deal with the problem of locating the minimum of a function ϕ of n variables by use of successive approximation methods. They propose two procedures (called "partan" and "steepest descent partan") which they show lead to the minimum of ϕ, when ϕ is a positive definite quadratic function, in less than $2n$ steps, where each step involves minimizing ϕ on certain straight lines. The results also hold when ϕ is a monotone function of a positive definite quadratic function. For "non-quadratic" ϕ's the authors suggest some modifications of their procedures but do not have any results on the behavior of the procedures when ϕ is non-quadratic except for some computations in a particular example.

J. Sacks (Evanston, Ill.)

Schmidt, Jochen W. 2936

Extremwertermittlung mit Funktionswerten.

Wiss. Z. Techn. Univ. Dresden **12** (1963), 1601–1605.

Let x_n, x_{n-1}, x_{n-2} be three approximations for an extremum of the real function f; then the abscissa x_{n+1} of the extremum of the interpolation parabola through $(x_i, f(x_i))$ $(i=n,n-1,n-2)$ is generally an improved approximation. A modified method uses only two starting approximations x_n, x_{n-1} and the value $\alpha x_n+(1-\alpha)x_{n-1}$ (with $0<\alpha<1$, α not too close to 0 or 1) as third abscissa. Under the assumption that f is sufficiently often differentiable and that $f''(x)\neq 0$ at each zero x of f', it is shown that the first method has a superlinear convergence rate of about 1.32 . . . and the second one of about 1.41. . . . Without the assumption about f'', the convergence is only linear. Specific convergence theorems are then proven which assure the continuation of the iteration, the superlinear convergence, and the existence of a solution, and which

give error estimates. The form of the theorems, as well as the proofs, is patterned after the author's generalized method of false position [Z. Angew. Math. Mech. **43** (1963), 1–8; MR **26** #5442; ibid. **43** (1963), 97–110; MR **27** #616]. The paper concludes with some numerical examples. *W. C. Rheinboldt* (Silver Spring, Md.)

Gurin, L. S. 2937

Optimization in stochastic models. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 367–370.

The author considers a stochastic process depending on one or several parameters $x, y, \cdots$, which can be simulated on an electronic computer, obtaining independent realizations of it. Let a certain functional be associated with this process such that its values form a random variable $\xi(x, y, \cdots)$ which depends on the parameters $x, y, \cdots$. Denote by $f(x, y, \cdots)$ the mean value of this random variable. The problem is to determine the maximum of the function $f(x, y, \cdots)$ using a minimum number of realizations of the considered process. In order to solve this problem the author uses an iteration method. For the sake of simplicity he considers the case of two parameters (x, y). He starts from an initial point (x_0, y_0) and finds a new point (x_1, y_1) using the formulas

$$x_1 = x_0+\lambda \frac{\sqrt{N}}{A\sigma}\left[f_N\left(x_0+\frac{A\sigma}{\sqrt{N}}, y_0\right)-f_N(x_0, y_0)\right],$$

$$y_1 = y_0+\lambda \frac{\sqrt{N}}{A\sigma}\left[f_N\left(x_0, y_0+\frac{A\sigma}{\sqrt{N}}\right)-f_N(x_0, y_0)\right],$$

where N is the number of realizations used for obtaining the value of $f(x, y)$, σ^2 is the variance of ξ, while A and λ are certain constants. If the inequality (1) $\Delta f=f_N(x_0, y_0)-f_N(x_1, y_1)<k\sigma/N^{1/2}$ is satisfied, then the point (x_0, y_0) will be replaced by (x_1, y_1) and this process will be repeated. If (1) is not true, then one checks if the inequality (2) $a\sigma^2N^{-1}+(1-a)(\Delta f)^2<\varepsilon$ is fulfilled or not. If this is realized, then the calculation is completed and he states that for (x_1, y_1) the considered function attains its maximum. If (2) is not satisfied, then one increases N n times and one repeats this process of calculation. No demonstration is given for the validity of this method for finding the maximum value of $f(x, y)$.

He discusses in detail the case when $\xi(x, y)=f(x, y)+\xi_1(x, y)$, where $f(x, y)=ax^2+2bxy+cy^2$ and $\xi_1(x, y)$ is a random variable having a normal distribution. For this case a numerical illustration is given. *D. D. Stancu* (Cluj)

Gershinsky, Morris; Levine, David A. 2938

Aitken-Hermite interpolation.

J. Assoc. Comput. Mach. **11** (1964), 352–356.

Authors' summary: "To employ Aitken interpolation in the construction of Hermite interpolating polynomials, a fundamental theorem for Aitken interpolation is given which permits the development of an Aitken interpolation algorithm for a wide class of constraints." *T. Fort* (Columbia, S.C.)

Salzer, Herbert E. 2939

Divided differences for functions of two variables for irregularly spaced arguments.

Numer. Math. **6** (1964), 68–77.

Newton's divided-difference formula effects interpolation with that polynomial of degree $n-1$ which assumes a given value at each of n values of a single argument. In the two-variable case, a straightforward extension covers the situation where the points are the points of intersection of an irregular rectangular mesh; the author here extends the formula to deal with the more formidable problem with completely irregular spacing. The effect of coincidence of one or more points is discussed in detail. *C. W. Clenshaw* (Teddington)

Topoljanskiĭ, D. B. 2940

Quasi-Hermitian interpolation polynomials for two variables. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 30–34.

Author's summary: "A two-dimensional formula of polynomial interpolation with an arbitrary arrangement of the points is derived; at each point the interpolational conditions determine the values not only of the polynomial but, generally speaking, of a certain number of successive partial derivatives as well."

Ling, Chih-Bing; Tsai, Chen-Peng 2941

Evaluation at half periods of Weierstrass' elliptic function with rhombic primitive period-parallelogram.

Math. Comp. **18** (1964), 433–440.

This paper contains tables to 16D of the three quantities $e_1=\wp(\omega_1)$, $e_2=\wp(\omega_2)$, $e_3=\wp(\omega_3)$, with the periods $2\omega_1=1$, $2\omega_2=\frac{1}{2}+ci$, and of the coefficients of the cubic equation having e_1, e_2, e_3, as roots, for

$$c = 0.05(0.05)0.5(0.1)1(0.25)2(0.5)3(1)6 \text{ and } \infty.$$

Tables to 21D of $e^{\pi a}$ and $e^{-\pi a}$ for $a=0.1(0.1)10$ are also included, together with a brief description of the methods of computation used. *W. M. Kincaid* (Ann Arbor, Mich.)

Savenkov, V. M. [Savenkov, V. N.] 2942

Calculation of the Schwarz-Christoffel constants for an arbitrary simply connected quadrilateral. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 574–576.

Author's summary: "Hypergeometric series are used to solve the problem of the determination of the constants of the Schwarz-Christoffel integral for an arbitrary simply connected quadrangle. Twelve concrete examples of quadrangles are considered for which the values of the constant k are calculated to six significant digits. The results are presented in a table."

Karpenko, P. D. 2943

On a numerical method of mapping a polygon onto a circle. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 177–180.

Author's summary: "The author considers an approximate method of mapping a polygon on a circle and determining the constants of the Christoffel-Schwarz integral by the method of successive conformal mapping [P. F. Fil'čakov, Ukrain. Mat. Ž. **10** (1958), 340–344; MR **20** #5855]. The mapping of the outer region of a broken

line (degenerate tetragon) on the outside of a circle is considered as an example. The results of the numerical example are presented in the table."

Ugodčikov, A. G. 2944

The construction of conformal mappings by electric simulation and by Lagrange interpolation polynomials. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1424–1427.

Author's summary: "The problem of constructing interpolation polynomials (1) $z=\omega_n(\zeta)=\sum_{k=1}^{n} C_k\zeta^k$ for conformal mapping functions $z=\omega(\zeta)$ is presented in this paper. The function (1) is Lagrange's polynomial which coincides with the function $z=\omega(\zeta)$ at points $\zeta=0$ and $\zeta=\zeta_j=e^{i\theta_j}$, where $\theta_j=2\pi j/m$ $(j=1,\cdots,m)$. The coefficients of the polynomial (1) are calculated by means of the formula $C_k=m^{-1}\sum_{j=1}^{m} z_j e^{-ik\theta_j}$. The points z_j of the boundary z of the domain S corresponding to the points $\zeta=\zeta=e^{i\theta_j}$, are determined by means of electric analogy to a certain conformal mapping."

Altman, M. 2945

A feasible direction method for solving the non-linear programming problem.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 43–50.

Author's introduction: "A study of the non-linear programming problem is contained in the monograph of G. Zoutendijk [*Methods of feasible directions*, Elsevier, Amsterdam, 1960; MR **23** #B2156]. The procedures proposed there are called feasible direction methods. This approach seems to be only natural; it makes systematic use of the idea of steepest descent, which generalizes the gradient method for maximizing a function without constraints. However, there is one disadvantage of these methods: In order to ensure the convergence of the procedures, Zoutendijk introduces the so-called anti-zigzagging precautions. But these precautions are of such a kind that they make the method very complicated and much too laborious on the computational side. In fact, to construct the optimum direction of motion to the next step so as to keep himself in the constraint set and to magnify the value of the objective function at the same time, one has to make several trials. This means that one has to solve a couple of linear (or quadratic) programming problems for one step only. The purpose of this paper is to show how to overcome these difficulties. In other words, we shall propose an iterative feasible direction method for solving the non-linear programming problem. The advantage of this method is that the construction of the optimum direction for each iteration step requires one or at most two solutions of the corresponding linear (or quadratic) programming problem without additional trials."

Ky Fan (Evanston, Ill.)

Steck, Richard 2946

A dynamic programming strategy for the two machine problem.

Math. Comp. **18** (1964), 285–291.

Tables of computations are presented giving strategies for a version of the "two-armed bandit" problem recently considered by the reviewer [Information and Control **5** (1962), 268–275; MR **26** #4810]: Given two Bernoulli machines with unknown fixed probabilities p_1, p_2 of scoring a 1, and $1-p_1$, $1-p_2$ of scoring a 0, and given a priori information on the results of previous trials with each machine, which machine should be chosen at the next trial so as to maximize the total expected discounted score over the next N trials? Strategies are given up to $N=25$. They seem intuitively reasonable in general; exceptions given in footnotes to the tables suggest the presence of cumulative round-off errors. *B. Gluss* (Berkeley, Calif.)

White, Paul A.; Brown, Robert R. 2947

A comparison of methods for computing the eigenvalues and eigenvectors of a real symmetric matrix.

Math. Comp. **18** (1964), 457–463.

The authors give comparative experience with three methods for calculating the eigenvalues and eigenvectors of a real symmetric matrix. The methods are (i) the classical method of Jacobi, (ii) the method of Givens, (iii) the method of Householder. In the last two methods the eigenvalues were found by bisection and the eigenvectors allegedly by a method involving inverse iteration recommended by the reviewer. The methods were used on 5 matrices of order 32 generated in a simple manner from 2×2 matrices, a matrix of order 8 devised by Rosser and a matrix of order 21 used by the reviewer. A tabular comparison is given of the results obtained. The results quoted for the eigenvectors of the matrix of order 21 would suggest that the code used is not a correct version of the method recommended by the reviewer.

James H. Wilkinson (Teddington)

Parlett, Beresford 2948

Laguerre's method applied to the matrix eigenvalue problem.

Math. Comp. **18** (1964), 464–485.

The author describes the application of Laguerre's method to the calculation of the eigenvalues of a general real matrix. The matrix A is first reduced to Hessenberg form for H by elementary similarity (non-unitary) transformations. The zeros of $f(z)=\det(H-zI)$ are then determined by Laguerre's method, the necessary values of $f(z)$, $f'(z)$ and $f''(z)$ being derived directly from H using an extension of Hyman's method. When p roots $r_1, r_2, \cdots, r_p$ have been accepted, Laguerre's method is applied to

$$g(z)=f(z)/(z-r_1)\cdots(z-r_p)$$

using $g'/g=f'/f-\sum(z-r_i)^{-1}$ and $g''/g-(g'/g)^2=f''/f-(f'/f)^2+\sum(z-r_i)^2$. The difficult problems of dealing efficiently with complex-conjugate roots, the criterion for accepting a root, choosing starting values and scaling are all considered in some detail. The paper concludes with ALGOL 60 procedures and a number of numerical examples. The examples show that on the average three iterations per root were required for matrices of orders up to 100, an impressive result. There is no doubt that this is one of the most effective procedures for the general eigenvalue problem. *James H. Wilkinson* (Teddington)

Forsythe, George E. 2949

Tests of Parlett's ALGOL eigenvalue procedure Eig 3.

Math. Comp. **18** (1964), 486–487.

The author describes his experience with the procedures described by Parlett [#2948 above]. The procedure was converted to extended ALGOL on the Burroughs B5000 and then was used to compute the eigenvalues of some of the matrices solved by Parlett and also the eigenvalues of 100 matrices of order 10 with random elements. The accuracy of Parlett's procedure was confirmed.
James H. Wilkinson (Teddington)

Wang, H. H.; Gregory, R. T. 2950
On the reduction of an arbitrary real square matrix to tridiagonal form.
Math. Comp. **18** (1964), 501–505.
The title refers to a method proposed by La Budde [Math. Comp. **17** (1963), 433–437; MR **27** #6378] and using matrices of the form $I-xy^H$ with x and y vectors. By means of a counterexample, the authors refute La Budde's assertion that parameters can be "theoretically" chosen always so as to allow the transformation to be carried to completion. For a fuller discussion of this and related methods see the paper by Bauer [J. Soc. Indust. Appl. Math. **7** (1959), 107–113; MR **20** #6778].
A. S. Householder (Oak Ridge, Tenn.)

Parlett, Beresford 2951
A note on La Budde's algorithm.
Math. Comp. **18** (1964), 505–506.
The algorithm is that discussed in the preceding review [#2950]. It is shown that when applied to a Hessenberg matrix the algorithm is identical with the standard one, and that for such a matrix the algorithm can break down.
A. S. Householder (Oak Ridge, Tenn.)

Rigal, J. 2952
Nombre de conditions et de signification en analyse matricielle.
Deux. Congr. Assoc. Française Calcul et Traitement Information (*Paris*, 1961), *pp.* 47–53. *Gauthier-Villars, Paris*, 1962.
The author discusses the effects of errors in the data and of rounding errors made in the course of the calculation of the inverse of a matrix. If A is the given matrix, four classes of matrices are defined: matrices a such that $\|A-a\|<\varepsilon$, matrices b such that b^{-1} is an a, matrices c consisting of those obtained by the algorithm in question by inverting the matrices a, and matrices d such that d^{-1} is a matrix c. The stability number, significance number and condition number are then defined in terms of measures of these classes. Similar considerations apply to the analysis of a wide range of problems.
James H. Wilkinson (Teddington)

Franck, P. 2953
Sur la distance minimale d'une matrice régulière donnée au lieu des matrices singulières.
Deux. Congr. Assoc. Française Calcul et Traitement Information (*Paris*, 1961), *pp.* 55–60. *Gauthier-Villars, Paris*, 1962.
Mirsky has shown that in terms of the spectral norm the minimum distance δ of a non-singular matrix A from the space of singular matrices is given by $\delta^{-1}=\|A^{-1}\|_2$. In this paper the same problem is solved for a number of different classes of norms. Values (or bounds) for δ are given for each of the classes of norms considered, and brief sketches of the proofs are included. The quantity δ is involved in the determination of the "nombre de signification" discussed by Rigal [#2952 above].
James H. Wilkinson (Teddington)

Schmidt, J. W. 2954
Ausgangsvektoren für monotone Iterationen bei linearen Gleichungssystemen.
Numer. Math. **6** (1964), 78–88.
Zur Lösung eines linearen Gleichungssystems $u=Mu+s$ sind Iterationsverfahren für Folgen $\{x^n\}$, $\{y^n\}$ angegeben worden, welche die Lösung einschliessen, falls (*) $x^0\leq y^0$, $x^0\leq x^1$, $y^1\leq y^0$ gilt. Ausgangsvektoren x^0, y^0 mit der Eigenschaft (*) existieren, wenn der Spektralradius der Matrix $(|m_{ik}|)$ kleiner als 1 ist. {Siehe den Referent [Math. Comp. **16** (1962), 323–337; MR **26** #7147] und das Literaturverzeichnis jener Arbeit. Dort wurde auch das Problem behandelt, x^0 und y^0 praktisch zu berechnen.} Macht man den Ansatz $x^0=(\xi,\xi,\xi,\cdots)$, $y^0=(\eta,\eta,\eta,\cdots)$, so kommt man im Falle eines Systems mit überwiegender Hauptdiagonale zu meist sehr groben Ausgangsvektoren. Die zugehörigen Formeln für ξ und η sind das Hauptergebnis dieses Aufsatzes. Der Verfasser bringt ferner einige Modifikationen (u.a. Iteration bei fehlerhaften Elementen und eine direkte Übertragung auf lineare Integralgleichungen) sowie als Beispiele zwei Systeme mit jeweils drei Unbekannten und eine einfache Randwertaufgabe.
J. Schröder (Seattle, Wash.)

Heinrich, H. 2955
Bemerkung zu einem Konditionsmass für lineare Gleichungssysteme.
Z. Angew. Math. Mech. **43** (1963), 568.
A condition number proposed by Focke [Wiss. Z. Karl-Marx-Univ. Leipzig Math.-Nat. Reihe **11** (1962), 41–43; MR **25** #745] satisfies $0\leq K(A)\leq 1$, vanishing if and only if A is singular. It is shown that for a diagonal matrix P, $K(PA)$ is maximum when the rows of A are equal in norm, the value of the maximum being the ratio of the modulus of the determinant to the product of the norms of the rows.
A. S. Householder (Oak Ridge, Tenn.)

Bergmann, Wilfried 2956
Ein Algorithmus zur Formulierung von Iterationsverfahren in der linearen Algebra.
Wiss. Z. Techn. Univ. Dresden **12** (1963), 114–116.
The solution y of $Ay=a$ is an equilibrium point for the system of differential equations $Ay(t)-a=-C^{-1}\,dy/dt$, stable when C is positive definite. Hence difference approximations to the differential system should provide iterative methods for solving the algebraic system. However, admittedly this approach provides only particular iterations and gives no information concerning convergence.
A. S. Householder (Oak Ridge, Tenn.)

Kublanovskaja, V. N. 2957
Reduction of an arbitrary matrix to tridiagonal form. (Russian)
Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 544.

It is observed that the transformations that reduce a matrix to lower Hessenberg form reduce an upper Hessenberg matrix to tridiagonal form.

A. S. Householder (Oak Ridge, Tenn.)

Golub, Gene H. 2958

Bounds for the round-off errors in the Richardson second order method.

Nordisk Tidskr. Informations-Behandling **2** (1962), 212–223.

For solving $(I-B)x=g$ by the scheme

$$x_{k+1} = \omega(Bx_k+g-x_{k-1})+x_{k-1},$$
$$\omega = 2/[1+(1-\rho^2)^{1/2}],$$

and $\rho<1$ is the spectral radius of B, with x_0 and x_1 given arbitrarily, if at the kth step the machine operations (with rounding) deviate by δ_k from the result of the strict arithmetic operations performed in that step, then the accumulated deviation is a_k, where

$$\|a_k\| \leqq \delta\{(1-r^k)/(1-r)^2-kr^{k-1}/(1-r)\},$$

and $\|\delta_k\|\leqq\delta$, $r=(\omega-1)^{1/2}$. A statistical estimate is also given and compared with some experimental results.

A. S. Householder (Oak Ridge, Tenn.)

Davidenko, D. F. 2959

Application of the method of variation of parameters to the approximate calculation of exponential functions of a real matrix. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 158–163.

Author's summary: "The method of variation of parameters is applied to the calculation of exponential functions of a real matrix, the elements of which are functions of the parameter λ, assuming given values on some finite interval. Differential equations are also constructed for the determination in the given interval of elements of a union matrix for an exponential function of the given matrix and its determinant, without having recourse to calculation of the exponential function of the given matrix."

Adamskaja, I. A.; Godunov, S. K. 2960

The method of spherical harmonics in the problem of critical parameters. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 473–484.

The critical parameters are those that determine the criticality of a spherical reactor. On using $2n$ harmonics and m groups, the equations can be written in matrix form $P\,dy/dr+r^{-1}Qy+\lambda Vy=\rho Dy$, the matrices being of order $2mn$ and of quite special form. The method proposed for finding λ the authors call the "method of trials", and is based upon taking a set of vector solutions satisfying the boundary conditions at the origin, and forming a determinant which is to vanish as a function of λ at the outer boundary. Reference is made to a previous paper by Godunov [Uspehi Mat. Nauk **16** (1961), no. 3 (99), 171–174; MR **25** #4640].

A. S. Householder (Oak Ridge, Tenn.)

Samokiš, B. A. 2961

On the stability of an abstract Galerkin method. (Russian. English summary)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **19** (1964), *no.* 1, 160–162.

For solving $Ax=y$ by the method of Galerkin, one chooses a finite system of functions $\{\varphi_k\}$ and of functionals $\{l_i\}$, and solves $l_iAx^2=l_iy$ for $x^2=\sum c_k\varphi_k$. Estimates are given for the Euclidean condition number for the resulting finite matrix in several special cases.

A. S. Householder (Oak Ridge, Tenn.)

Zurmühl, Rudolf 2962

★Matrizen und ihre technischen Anwendungen.

Vierte neubearbeitete Auflage.

Springer-Verlag, Berlin, 1964. xii+452 *pp.* *DM* 36.00.

This new edition, though several pages shorter than the third edition of 1961 [MR **24** #B1752] includes a few new features. The most superficially obvious change is the use of boldface instead of Fraktur for vectors and matrices, and the next most obvious is the occasional use of ALGOL. The principal theoretical innovation is the use of norms and the introduction of the "Hadamard condition number" of a matrix, defined as the ratio of the modulus of its determinant to the product of the Euclidean norms of the rows. The major novelty in the techniques is the inclusion of Rutishauser's LR transformation.

The continued demand for new editions attests to the book's general usefulness and high quality.

A. S. Householder (Oak Ridge, Tenn.)

Hollingsworth, C. A.; Schumacher, David P. 2963

Concerning the roots of a polynomial which contains a perturbation parameter.

J. Chem. Phys. **41** (1964), 1109–1111.

Authors' summary: "Methods are presented for the calculation of the roots of a polynomial equation in terms of a perturbation parameter. The roots are expressed in the form $\lambda_i=\lambda_{i0}+\lambda_{i1}\xi+\lambda_{i2}\xi^2+\cdots$, where λ_i is the ith root, and ξ is the parameter. The methods use recurrence relationships by which $\lambda_{i(k+1)}$ can be obtained from λ_{ik}."

Mond, B. 2964

Multivariate polynomial approximation for equidistant data.

Math. Comp. **18** (1964), 298–301.

The standard theory of least squares polynomial approximation for equidistant data is extended to multivariate polynomial approximation. It is also shown how univariate tables can be used in the multivariate case.

J. R. Rice (W. Lafayette, Ind.)

Börsch-Supan, W. 2965

A posteriori error bounds for the zeros of polynomials.

Numer. Math. **5** (1963), 380–398.

The author considers the problem of improving the computed zeros λ_i of a polynomial $F(x)$ with the true zeros x_i and at the same time providing rigorous bounds for the improved zeros. By differentiating $F(x)=$

$\sum \ln(x - x_i)$, the solutions x_i are shown to be the solutions of the non-linear equations

$$x_j - \lambda_j = \alpha_j/(1 + \alpha_j \sum' (\lambda_j - x_i)^{-1}), \text{where } \alpha_j = -F(\lambda_j)/F'(\lambda_j).$$

The convergence of the iterative procedure for solving these equations is considered using Brouwer's fixed-point theorem. Since the process must itself be carried out numerically, the effect of the rounding error is discussed. After considering more refined iterates for simple roots, multiple roots and root clusters are discussed. The problem of finding bounds for certain roots, when only rough approximations to the others are known, is also treated and numerical examples are given. This paper is of considerable interest to all who regard the accuracy of computed solutions as a matter of importance.

James H. Wilkinson (Teddington)

Savenko, S. S. 2966

An iterative method for solving algebraic and transcendental equations. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 738–744.

New methods are expounded for deriving iterative formulas for solving $f(x) = 0$. Among others, the special formula

$$x_{k+1} = x_k + \frac{f(x_k)}{[\frac{3}{2} f(x_k) f'(x_k) f''(x_k) - f'^3(x_k) - \frac{1}{2} f^2(x_k) f'''(x_k)]^{1/3}}$$

is useful when $f(x)$ has a maximum between x_k and the root.

J. L. Brenner (Palo Alto, Calif.)

Hummer, David G. 2967

Expansion of Dawson's function in a series of Chebyshev polynomials.

Math. Comp. **18** (1964), 317–319.

In this short paper is given the Chebyshev series for Dawson's function (the error function of imaginary argument) given by $e^{-x^2} \int_0^x e^{t^2} dt$, for the range $-k \leq x \leq k$. The coefficients a_n of $T_n(x)$ are given analytically in terms of k, and also numerically in the case $k = 5$ to an accuracy of 16D, of which 14D have been checked. For this accuracy, some 33 terms of the series are required.

C. W. Clenshaw (Teddington)

Bulirsch, Roland 2968

Bemerkungen zur Romberg-Integration.

Numer. Math. **6** (1964), 6–16.

The Romberg quadrature [see Bauer, Rutishauser and Stiefel, Proc. Sympos. Appl. Math., Vol. XV, pp. 199–218, Amer. Math. Soc., Providence, R.I., 1963] can be generalized by applying the trapezoidal rule to subdivisions of the full interval into $n_0, n_1, \cdots$ equal parts and then extrapolating to $n = \infty$ (Romberg used $n_k = 2^k$). The present author investigates the convergence of the method for the sequences $n_k = k+1$ and $n_k' = \{1, 2, 3, 4, 6, 8, 12, \cdots\}$. For the first of these he obtains a strict error estimate

$$\int_0^1 F(x)\,dx - T_m = -\frac{|B_{2m+2}|}{[(m+1)!]^2} \cdot \frac{f^{(2m+2)}(\xi)}{(2m+2)!}, \quad 0 < \xi < 1,$$

which establishes the convergence for a wide class of integrands. However, it tends to produce numerical instability, and therefore the sequence n_k' is proposed for numerical purposes. Its convergence follows trivially from Toeplitz's theorem, but a remainder formula like that above can be established only experimentally up to a certain m. Nevertheless, numerical experiments show the superiority of the n'-sequence over $n_k = k+1$ and 2^k. An ALGOL program for the n'-sequence is given.

H. Rutishauser (Zürich)

Sag, T. W.; Szekeres, G. 2969

Numerical evaluation of high-dimensional integrals.

Math. Comp. **18** (1964), 245–253.

The authors propose a method of numerical estimation of integrals by determining a formula for the n-ball S and then finding transformations mapping another region R into S in such manner that the Jacobian of the transformation vanishes at the boundary together with a certain number of higher derivatives. Such transformations are discussed for the n-ball, n-cube, and n-simplex.

Computations are tabulated to show errors in spaces up to 15 dimensions. The particular method is novel in the establishment of transformations. The difficulty of establishing transformations is perhaps one of the most important limiting factors in numerical integration problems for functions of more than one variable.

P. C. Hammer (Madison, Wis.)

Ionesku, D. V. [Ionescu, D. V.] 2970

A generalization of the quadrature formulae of Simpson, Newton and Milne. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **34** (1964), 11–18.

This is a study of the remainder terms R_n, R_n' in quadrature formulas of the form

$$\int_{x_0}^{x_n} f(x)\,dx = A_n \sum_{k=0}^{n} f(x_k) + R_n,$$

$$\int_{x_0}^{x_n} f(x)\,dx = A_n'[f(x_0) + f(x_n)] + B_n' \sum_{k=1}^{n-1} f(x_k) + R_n',$$

where $x_k = x_0 + kh$ ($k = 0, 1, \cdots, n$), and in analogous quadrature formulas of the open type. It is shown that the Peano kernels φ_n and ψ_n in

$$R_n = \int_{x_0}^{x_n} \varphi_n(x) f''(x)\,dx, \qquad R_n' = \int_{x_0}^{x_n} \psi_n(x) f^{(4)}(x)\,dx$$

have constant signs in $[x_0, x_n]$ for each n. Therefore $R_n = -[(b-a)^3/12n] f''(\xi_n)$, $R_n' = -[(b-a)^5/720n^2] f^{(4)}(\xi_n')$ with ξ_n, ξ_n' in (x_0, x_n). Similar results hold for the corresponding open formulas.

Walter Gautschi (Lafayette, Ind.)

Roman, Irwin 2971

Controlled quadrature.

Math. Comp. **18** (1964), 254–263.

This paper develops an algorithm for numerical quadrature which maintains an upper and lower estimate of deviation step by step. The estimates are applied to adjust the result. The actual quadrature method used is Simpson's rule and the estimates of error are based on the remainders. The method is illustrated with simple examples.

P. C. Hammer (Madison, Wis.)

Butcher, J. C. 2972
On Runge-Kutta processes of high order.
J. Austral. Math. Soc. **4** (1964), 179–194.

It is well known that a Runge-Kutta process of order r requires at least r function evaluations per step. For $r=1, 2, 3, 4$ there exist processes of order r requiring exactly r evaluations per step. For $r=5$ there is a process requiring 6, and for $r=6$ there are processes requiring 8 evaluations per step.

In this paper the author shows that for $r=5$ there are no processes involving only 5 evaluations per step. The method of proof involves consideration of a large number of special cases, and showing that each leads to a contradiction unless at least 6 evaluations are allowed.

By exhibiting examples of special processes it is also shown that there are processes requiring only 7 evaluations when $r=6$. The author believes that his methods might lead to a proof that 6 evaluations could not be enough when $r=6$. *T. E. Hull* (Toronto, Ont.)

Butcher, J. C. 2973
Integration processes based on Radau quadrature formulas.
Math. Comp. **18** (1964), 233–244.

This is a sequel to earlier results on implicit Runge-Kutta processes obtained by the same author [J. Austral. Math. Soc. **3** (1963), 185–201; MR **27** #2109; Math. Comp. 18 (1964), 50–64; MR **28** #2641].

The class of implicit methods considered in this paper is based on Radau quadrature formulas. Existence and uniqueness theorems are given for three distinct classes of methods, the point being that there are methods of relatively high order of accuracy. Formulas are exhibited for methods involving only a few stages; others will be given in a later paper.

Results for the principal error function are given, and some trivial numerical examples are used to show what happens to the error in a few special circumstances. There is no convincing evidence that the implicit nature of the processes, and their complexity, are worth the extra trouble in practice. *T. E. Hull* (Toronto, Ont.)

Reimer, Manfred 2974
★Stabilität bei numerischer Integration von Differentialgleichungen.
Inaugural-Dissertation zur Erlangung des Grades eines Doktors der Naturwissenschaften der Mathematisch-naturwissenschaftlichen Fakultät der Eberhard-Karls-Universität zu Tübingen, Tübingen, 1963. iii + 55 pp.

Stability investigations for multipoint integration methods

$$(*)\qquad \sum_{j=0}^{k} a_j y_{n+j} = h \sum_{j=0}^{k} b_j f(x_{n+j}, y_{n+j})$$

for solving $y'=f(x,y)$ usually use the classical stability concept of Dahlquist [*Stability and error bounds in the numerical integration of ordinary differential equations,* Inaugural Dissertation, Univ. Stockholm, Almqvist & Wiksells, Uppsala, 1958; MR **20** #7391]. However, this concept does not always meet the practical requirements; for instance, it is still possible that a method is stable in the classical sense and yet may considerably amplify small perturbations for some time until they finally die out. In order to distinguish that kind of behaviour, the author introduces the concept of "absolute stability" which essentially requires that for a stable differential equation every solution η_j of the variational difference equation induced by the method (*) must be such that for any $j>k$:

$$|\eta_j| \leqq |\eta_i| \quad \text{for } i = j-1, j-2, \cdots, j-k.$$

Obviously such a method is also stable. It turns out that if a_k (the highest coefficient in (*)) is normalized to -1, then $a_i \geqq 0$ for $i=0, 1, \cdots, k-1$ is necessary and $a_i>0$ for $i=0, 1, \cdots, k-1$ is sufficient for absolute stability of the method (*). Most interesting, the following analogue to Dahlquist's theorem holds: An absolutely stable method (*) has at most degree $p=k+1$.

It is to be expected that absolutely stable methods produce smoother results than just stable ones. This is established by the investigations of Chapter III, where optimal methods are looked for. For closed 3-point formulae of degree 3 which can be represented as linear combinations of the Adams-Moulton 3-point formula and the Simpson rule, the optimal method is always "between" the two and approaches the Simpson rule if the tolerated error approaches zero. (However, the Simpson rule is not absolutely stable and even produces weak instability in the sense of Dahlquist [loc. cit.].) The numerical experiments clearly show the superiority of the absolutely stable methods of this class over those which are just stable. Similar investigations are carried out for closed 4-point formulae. *H. Rutishauser* (Zürich)

Aliev, R. M. 2975
An application of Galerkin's method to the solution of equations with retarded argument. (Russian. Azerbaijani summary)
Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Tehn. Mat. Nauk **1964**, *no.* 1, 3–6.

The author investigates periodic approximate solutions of the non-linear differential equation

$$x''(t) = f(x(t-h(t)), x'(t-h(t)), t),$$

where f is a continuous function of all the arguments and has period 2π in t. Galerkin's method is used with the coordinate functions

$$1, \cos t, \sin t, \cdots, \cos nt, \sin nt.$$

The author establishes a sufficient condition under which a sequence of approximate solutions contains a subsequence converging in norm (in the space $C_{2\pi}^{(1)}$) to the exact solution. *P. K. Ghosh* (Calcutta)

Witty, W. H. 2976
A new method of numerical integration of differential equations.
Math. Comp. **18** (1964), 497–500.

The author describes a new method of numerical integration of the initial-value problem $y'=f(x,y)$, $y(x_0)=y_0$, which shares with a method described by the reviewer [J. Assoc. Comput. Mach. **3** (1956), 208–211; MR **18**, 338] the property that in advancing the numerical integration by one step it requires only one evaluation of the function f, in contrast to the Heun (modified Euler) method which requires two evaluations of the frequently cumbersome expression for f. Whereas in the reviewer's

method linear extrapolation is used to determine $y_{n+1/2}$ from y_n and y_{n-1}, in the method discussed here linear extrapolation is utilized to calculate y'_{n+1} from $y'_{n+1/2}$ and y_n'. An analysis of the respective error terms of the three methods shows them all to be of the form $-(1/24)h^3 \times C(y'', y''')$, with the function C for the new method possessing somewhat smaller numerical coefficients than for the other two. Thus there may be situations in which the new method is more accurate than the other two, a fact that is illustrated by means of the example $y' = (1+y^2)^{-1}$, $y(0) = 0$. *M. Lotkin* (Merchantville, N.J.)

Bellman, Richard; Kalaba, Robert; Kotkin, Bella 2977

Differential approximation applied to the solution of convolution equations.

Math. Comp. **18** (1964), 487–491.

Differential approximation is defined as follows. Suppose the function of interest satisfies a differential equation $f^{(m)} = h(f, f', \cdots, f^{(m-1)}, t)$. One determines a linear differential equation with constant coefficients which approximates the given one:

$$(*) \qquad f^{(N)} + a_1 f^{(N-1)} + \cdots + a_N f = 0.$$

The coefficients are found by minimizing the expression $\int_0^T (f^{(N)} + \cdots + a_N f)^2\,dt$, where f is determined from the original differential equation $f^{(m)} = h$. The approximation to f will then be the solution of

$$u^{(N)} + a_1 u^{(N-1)} + \cdots + a_N u = 0$$

with initial conditions yet to be specified.

A first approach suggested is to take $u^{(i)}(0) = f^{(i)}(0)$, and the authors find this choice to be remarkably successful in specific examples. More generally, it is suggested to proceed as follows: Every solution can be written in the form $u(t) = \sum_{i=1}^N c_i u_i$. The scalars c_i are found by minimizing the expression $\int_0^T (f - \sum_{i=1}^N c_i u_i)^2\,dt$. The method is illustrated by two examples: a differential equation for $u = e^{-t^2}$ and an integral equation of the renewal type. Using third-order differential approximations, the agreement is better than four significant figures.

H. Oser (Washington, D.C.)

Rickman, Seppo 2978

A method for construction of the solution of ordinary differential equations with the aid of the complex plane.

Acta Polytech. Scand. Math. and Comput. Mach. Ser. No. 7 (1962), 28 *pp.*

Author's summary: "The principle of the method to be presented is that the coefficients of the differential equation and its right-hand member are assumed to be constant over short intervals. We may then determine the Laplace transform for the solution of each equation corresponding to a given interval, and these transforms will obviously be rational functions. The poles and 'zeros' (which are true zeros only if the right-hand member of the respective differential equation is zero) we plot in the complex plane, which shall be called the s-plane in the following. The points of the s-plane defined in this manner possess a fairly simple relationship with the quotient of the function that is being constructed and its derivative, and the desired function is easy to plot on the basis of this ratio. The idea proper of our procedure is that transition from one interval to another is accomplished by studying the changes of the poles and zeros in the s-plane. In their determination we may make efficient use in the actual calculations of the so-called spirule. (Regarding its use, reference can be made, e.g., to W. R. Evans [*Control-system dynamics*, McGraw-Hill, New York, 1954].) It is to be admitted that this use of the s-plane as an essential part of the solving procedure implies comparatively cumbersome calculations, particularly in the case of differential equations of higher order. But if the problem is associated, for instance, with the analysis of a control system, the consideration of the said poles and zeros may be useful in that one finds not only a certain solution but also an easily comprehensible explanation for it." *J. T. Day* (Zürich)

Gendžojan, G. V. 2979

Two-sided Chaplygin approximations to the solution of a two-point boundary-value problem. (Russian. Armenian summary)

Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk **17** (1964), *no.* 3, 21–27.

The nonlinear boundary-value problem

$$(1) \qquad P(y) = -y'' + f(x, y, y') = 0, \quad 0 < x < 1,\ y(0) = y(1) = 0,$$

is studied under the assumption that in the region $0 \leqq x \leqq 1$, $y^2 + y'^2 < \infty$, $f(x, y, y')$ is a continuous function with continuous derivatives $f_y, f_{y'}$ which satisfy $0 \leqq f_y \leqq M$, $|f_{y'}| \leqq M$. The author constructs two monotone sequences of approximations $u_0(x) \leqq u_1(x) \leqq \cdots \leqq u_n(x) \leqq \cdots \leqq v_n(x) \leqq \cdots \leqq v_1(x) \leqq v_0(x)$ by means of an algorithm of Chaplygin type, and proves that both these sequences converge uniformly to the unique solution of (1).

M. Zlámal (College Park, Md.)

Hull, T. E.; Johnston, R. L. 2980

Optimum Runge-Kutta methods.

Math. Comp. **18** (1964), 306–310.

Les auteurs considèrent l'erreur sur un pas dans la méthode de Runge-Kutta d'ordre 4 comme fonction des abscisses des substitutions intermédiaires. Quelque soit la mesure choisie pour l'erreur (somme des carrés, somme des valeurs absolues des coefficients des divers termes d'erreur), on trouve toujours des valeurs voisines de 0,35 et 0,45 pour les abscisses des substitutions intermédiaires. Quelques exemples simples semblent confirmer ces vues.

Le rapporteur n'est pas d'accord avec l'auteur sur l'affirmation qu'un des termes d'erreur disparaîtrait pour les équations linéaires. *J. Kuntzman* (Grenoble)

Aziz, A. K.; Hubbard, B. E. 2981

Bounds for the solution of the Sturm-Liouville problem with application to finite difference methods.

J. Soc. Indust. Appl. Math. **12** (1964), 163–178.

Die Verfasser betrachten Differenzenverfahren zweiter (und z.T. auch vierter Ordnung) zur Lösung der ersten und dritten Randwertaufgabe für die Differentialgleichung $-(py')' + qy = f$ $(p > 0,\ q \geqq 0)$. Es werden Fehlerschranken für die Näherungen angegeben, welche im Falle der ersten Randwertaufgabe aus bekannten Daten

(wie p, q, f und Ableitungen dieser Funktionen) durch Integration und Berechnung von Maxima ermittelt werden können. Im Falle der dritten Randwertaufgabe wird ausserdem ein Eigenwert benötigt. Bei den numerischen Beispielen mit der einfachen Differentialgleichung $-y''+y=4xe^x$ waren die Fehlerschranken 7-mal bzw. 15-mal bzw. (im Falle eines Differenzenverfahrens vierter Ordnung) 350-mal so gross wie die wahren Fehler.
J. Schröder (Seattle, Wash.)

Bramble, James H.; Hubbard, Bert E. 2982
New monotone type approximations for elliptic problems.
Math. Comp. **18** (1964), 349–367.

Es werden Kriterien dafür aufgestellt, daß eine reelle Matrix "monoton" ist, d.h., daß ihre Inverse nur nichtnegative Elemente hat; mit ihrer Hilfe kann die Monotonie gewisser Matrizen bewiesen werden, bei denen die Elemente außerhalb der Hauptdiagonale verschiedenes Vorzeichen haben. Eine solche Matrix gehört unter anderen zu einer bekannten aus dem Differenzenverfahren höherer Approximation für die Potentialgleichung entstehenden 9 Punkte verwendenden Formel (Koeffizienten in einer Zeile proportional: $-1, 16, -60, 16, -1$).

Der "Diskretisierungsfehler" ε zwischen Lösung der Differenzengleichungen und Lösung der 1. Randwertaufgabe hat dann bei passender Erfassung der Randbedingung die Größenordnung $\varepsilon=O(h^4)$, wobei h die Maschenweite bedeutet. Das wird zunächst für einen Rechtecksbereich und dann für einen allgemeinen ebenen Bereich gezeigt. Für den Rechtecksbereich wird eine 13-Punktformel mit $\varepsilon=O(h^9)$ aufgestellt, und für die 1. Randwertaufgabe bei der Gleichung $u_{xx}+u_{xy}+u_{yy}=-f$ eine Formel mit $\varepsilon=O(h^2)$. *L. Collatz* (Hamburg)

Sobolev, S. L. 2983
Équations aux dérivées partielles pour les fonctions extrémales des problèmes du calcul numérique à plusieurs variables indépendantes.
Les Équations aux Dérivées Partielles (Paris, 1962), pp. 197–206. *Éditions du Centre National de la Recherche Scientifique, Paris,* 1963.

Expository article. *B. Frank Jones, Jr.* (Houston, Tex.)

Mozok, G. M. 2984
On the solution of a class of problems of mathematical physics by the method of nets. (Ukrainian. Russian and English summaries)
Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1570–1573.

Author's summary: "The author establishes the convergence and stability in P. S. Bondarenko's sense of constructed computational algorithms of solutions of the mixed problem and Neumann's problem for Poisson's equation for regions bounded by a convex piecewise-smooth curve having at least one axis of symmetry. An estimate is given of the error of the real approximate solution of the boundary-value problems under consideration."

Mozok, G. M. 2985
On solving biharmonic problems by the method of nets. (Ukrainian. Russian and English summaries)
Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 163–166.

Author's summary: "The author establishes the convergence and the stability in P. S. Bondarenko's sense of plotted calculational algorithms of the solution of boundary-value problems for biharmonic equations and equations of the theory of thin plates—the boundary conditions of which contain a derivative with respect to the normal—for regions limited by a convex piecewise smooth curve having at least one axis of symmetry. An estimate is given for the error of the real approximate solution."

Karimberdieva, S. 2986
An error bound for the solution of Poisson's equation by various methods. (Russian. Uzbek summary)
Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk **1963**, *no.* 6, 7–13.

The paper is an exercise in trigonometric identities and computation. The author considers a Poisson equation

$$\nabla^2 H = -(\mu^2+\gamma^2)D\cos(\mu x+\gamma y).$$

The Laplacian is replaced by three different second-order difference schemes, and the resulting difference equations are solved with the initial values taken from a particular integral $H=D\cos(\mu x+\gamma y)$. The results are compared with the values of this exact solution for various values of μ and γ. The exercise is repeated with a somewhat more complicated partial differential equation whose exact solution is known. *Č. Masaitis* (Aberdeen, Md.)

Omarov, E. O. 2987
An approximate solution by the method of lines of an elliptic partial differential equation of type 1. (Russian)
Ž. Vyčisl. Mat. i Mat. Fiz. 4 (1964), 585–592.

The problem $\Delta^2 u+a\Delta u+bu=f(x,y)$, $u|_s=g_1(s)$, $\partial u/\partial n|_s=g_2(s)$, is solved for a rectangular region in the xy-plane by the method of lines. This solution generalizes the equation but specializes the domain in the problem of Langenbah [Vestnik Leningrad. Univ. **11** (1956), no. **13**, 119–123; MR **18**, 475]. *R. N. Goss* (San Diego, Calif.)

Buravs'kiĭ, È. S. 2988
Two-sided approximations to the eigenvalues of a problem associated with a pair of quasi-differential operators with the parameter in the boundary conditions. (Ukrainian. Russian and English summaries)
Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 726–728.

Author's summary: "The author proves formulae for the upper and lower estimates of eigenvalues of a boundary problem connected with a pair of quasidifferential operators. The problem under consideration contains the parameter both in the given equation and in the boundary conditions."

Buravs'kiĭ, È. S. 2989
A method of approximate solution of a class of boundary-value problems with a parameter. (Ukrainian. Russian and English summaries)
Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1163–1167.

Author's summary: "A method is proposed for the

approximate calculation of the eigenvalues and the derivation of some approximate expressions for the eigenfunctions of a boundary problem of the second order. An example is presented and the results are compared."

Serdjukova, S. I. 2990

A study of stability in C of explicit difference schemes with constant real coefficients which are stable in l_2. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **3** (1963), 365–370.

The author considers an explicit difference equation with constant real coefficients: $u_j^{n+1}=\sum_{|l|\le k} a_l u_{j+l}^n$, with given initial values $u_j^0=r_j$. If $f(e^{i\varphi})=\sum_{|l|\le k} a_l e^{il\varphi}$, it is assumed that $f(1)=1$ and $|f(e^{i\varphi})|<1$, $0<\varphi<2\pi$. This implies that the difference equation is stable in the l_2 sense, and the author examines the question of stability in C, which means that there exists a constant c such that

$$\max_j |u_j^n| \leqq c \max_j |u_j^0|, \qquad 0 \leqq n.$$

Both a necessary condition and a sufficient condition are given for stability in C in terms of a certain representation for f. *B. Frank Jones, Jr.* (Houston, Tex.)

Volkov, E. A. 2991

Application of the Lagrange interpolation polynomial for solving the Dirichlet problem for the Poisson equation by the method of nets. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 466–472.

The author obtains by means of a Lagrange interpolation polynomial, with coefficients alternating in signs, a system of difference equations in boundary grid points. The method for solving these equations and the estimation of errors for an approximate solution are given, and the uniqueness of the solution is proved.

J. Albrycht (Poznań)

Kreiss, Heinz-Otto 2992

Über die Stabilitätsdefinition für Differenzengleichungen die partielle Differentialgleichungen approximieren. (English summary)

Nordisk Tidskr. Informations-Behandling **2** (1962), 153–181.

This paper is a landmark in the theory of partial difference equations with constant coefficients. The deepest result is an analogue for powers of matrices of the author's theorem on matrix exponentials [Math. Scand. **7** (1959), 71–80; MR **22** #1820]. It asserts that for a set of $m\times m$ matrices A, the following are equivalent: (i) $|A^n|<$ const for $n\geqq 0$ and all A; (ii) $|(sA-I)^{-1}|\leqq \text{const}/(1-|s|)$ for all $|s|<1$ and all A; (iii) $H-A^*HA$ non-negative definite for some positive definite $H(A)$, with $|H|\,|H^{-1}|\leqq$ const for all A. The third condition, in which a stable family is renormed to be strongly stable, prepares the way for a successful extension to variable coefficient systems. The relationship of the constants to each other and to m is another important question on which limited progress has recently been made.

The author also includes examples of both differential and difference systems, first-order in t but second-order in x, for which either the system is well-posed for each fixed x_0 but ill-posed globally, or vice versa. A weaker stability definition which allows polynomial growth of $|A^n|$ is shown to be equivalent to the von Neumann condition, but unstable with respect to zero-order perturbations $A\to A+B/n$. {To avoid the publication of corrections, we note that a factor $\bar{\kappa}_1$, mislaid in deriving (4.3.3), is easily handled.}

W. G. Strang (Cambridge, Mass.)

Veidinger, L. 2993

On finite-difference approximations to solutions of quasilinear hyperbolic systems.

Acta Math. Acad. Sci. Hungar. **15** (1964), 211–225.

Consider the Cauchy problem for a hyperbolic quasilinear system of the first order:

$$(*)\quad u_t = \sum_{i=1}^{n} A^i(x,t,u)u_{x^i}+b(x,t,u), \quad u(x,0) = u^0(x)$$

in a time interval $0\leqq t\leqq T$. The author solves this problem approximately with the help of a consistent approximation

$$(**)\quad v(x,t+h) = \sum_{j=1}^{m} C^j(x,t,v(x,t))v(x+hr_j,t) + hb(x,t,v(x,t)).$$

Defining stability of $(**)$ in the appropriate way, the author shows that $\max_{x,0\leq t\leq T}|u(x,t)-v(x,t)|=O(h)$ provided that $(**)$ is stable, the coefficients of $(*)$ and $(**)$ are sufficiently smooth and that $(*)$ has a sufficiently smooth solution. *H.-O. Kreiss* (Göteborg)

Bobkov, V. V. 2994

On the solution of the Goursat problem by the method of integral relations. (Russian)

Vesci Akad. Navuk BSSR Ser. Fiz.-Tèhn. Navuk **1963**, *no.* 4, 14–24.

The author considers the system

$$\partial^2 u_i/\partial x\partial y = \Phi_i(x,y,u_1,\cdots,u_m), \qquad i = 1,2,\cdots,m,$$

with the boundary conditions $u_i(x,0)=\varphi_i(x)$, $u_i(0,y)=\psi_i(y)$, $\varphi_i(0)=\psi_i(0)$ in a rectangular domain D. He assumes that some approximation of u_i on $k-1$ equally spaced characteristics (parallel to the x-axis) is known. Polynomial extrapolation of Φ_i and subsequent numerical integration of Φ yields a system of ordinary differential equations with constant coefficients. The bound of the error of this integration is expressed in terms of the $(k+3)$rd partial derivatives of u_i. An unwieldy and numerically useless bound of the differences between the exact solution and the solution of the resulting ordinary equations is given. The paper looks like a complex exercise in polynomial extrapolation.

Č. Masaitis (Aberdeen, Md.)

Bobkov, V. V. 2995

On the convergence of the method of integral relations. (Russian)

Dokl. Akad. Nauk BSSR **8** (1964), 5–9.

The author observes that no general discussion of convergence exists for Dorodnitsyn's method of integral relations, recently much used in fluid mechanics, which replaces partial differential equations by an N-fold greater number of ordinary differential equations. He proves convergence as $N\to\infty$ for the Cauchy problem

involving a second-order linear hyperbolic equation in canonical form, and gives an error estimate. He remarks that the extension to systems of quasilinear equations is not difficult. *M. D. Van Dyke* (Stanford, Calif.)

Krylov, V. I.; Liskovec, O. A. 2996

The method of lines for non-stationary mixed problems and a bound for the mean square error. (Russian)

Dokl. Akad. Nauk BSSR **8** (1964), 353–356.

The main result of this paper is to obtain a bound for the mean square error for the numerical solution, by a difference scheme, of the problem

$$\partial^k u(x,t)/\partial t^k = L(u)+f(x,t), \qquad x\in V,\ 0\leq t\leq T,$$
$$\partial^i u(x,0)/\partial t^i = \phi_i(x), \qquad x\in \bar V,\ 0\leq i\leq k-1,$$
$$D(u)|_\Gamma = \psi(\Gamma,t), \qquad 0\leq t\leq T,$$

where $k\geqq 1$, $x\equiv(x_1,\cdots,x_r)$; V is a region of the space variables with boundary Γ; $L(u)$ and $D(u)$ are linear operators of the space variables and $L(u)$ is elliptic of the second order. It is assumed that the solution u is unique and that the derivatives $\partial^{s+1}u/\partial t^{s+1}$ $(s\geqq k)$ are continuous with respect to t $(0\leqq t\leqq T)$ and belong to the space $L_2(V)$. *A. H. Stroud* (Lawrence, Kans.)

Getmancev, V. D. 2997

Some applications of the abstract theory of the method of alternating upper and lower approximations. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 718–722.

Author's summary: "Cauchy's problem is considered for systems of ordinary nonlinear differential equations. No limitations are imposed on the right sides of the systems except the conditions of Cauchy's theorem on the existence and uniqueness of the solution. A formula is deduced giving alternately the upper and the lower approximations to the solution sought."

Törnig, W. 2998

Über Differenzenverfahren in Rechteckgittern zur numerischen Lösung quasilinearer hyperbolischer Differentialgleichungen.

Numer. Math. **5** (1963), 353–370.

Let

$$\sum_{i=1}^n a^{ij}(u_y{}^i+c^ju_x{}^i)-b^j = 0, \qquad j=1,2,\cdots,n,$$

be a hyperbolic system of n quasilinear differential equations in two independent variables x, y. There exist several finite-difference methods for the approximate calculation of the solution of this system determined by the initial values

$$u^j(x_i,0) = g^j(x), \qquad a\leqq x\leqq b,\ j=1,\cdots,n,$$

notably those proposed and analyzed by Courant, Isaacson and Rees [Comm. Pure Appl. Math. **5** (1952), 243–255; MR **14**, 756]. The present paper contains a convergence and stability proof for a general class of explicit finite-difference schemes in rectangular grids that contains those previously discussed as special cases. If h is the mesh length in the x-direction, the discretization error of the method of Courant, Isaacson and Rees is $O(h)$. The author's generalization includes methods for which the order of approximation is of any order $O(h^\sigma)$, provided the differential equation can be formally approximated to this order by a difference expression satisfying certain additional hypotheses. *W. Wasow* (Madison, Wis.)

Bondarenko, P. S. 2999

Numerical algorithms for approximating the solution of operator equations. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 295–299.

Author's summary: "Definitions are given of computational, real computational, converging, regularly converging, as well as stable and real stable computational algorithms of the approximate solution of operator equations. Sufficient criteria for such algorithms are established."

Bandemer, Hans 3000

Anwendung des QD-Algorithmus von Rutishauser zur Bestimmung der Eigenwerte von Integralgleichungen.

Wiss. Z. Techn. Univ. Dresden **12** (1963), 112–114.

The author reports how a combination of the Lanczos method and the quotient-difference algorithm [the reviewer, Mitt. Inst. Angew. Math. Zürich, No. **7** (1957), see p. 49; MR **19**, 686] can be used to compute eigenvalues of an integral kernel. A numerical example is given and the conditions stated under which the convergence of the method can be proved. *H. Rutishauser* (Zürich)

Yoshida, Midori 3001

A proof of a theorem of Altman on iterative approximations. (Japanese. English summary)

Mem. Osaka Univ. Lib. Arts Ed. Ser. B No. 12 (1963), 11–13.

Author's summary: "Antosiewicz and Rheinboldt wrote in *Survey of numerical analysis*, pp. 485–517 [McGraw-Hill, New York, 1962; MR **25** #1632] as follows: Altman showed that the Newton iteration for $F(x)=\|Ax-v\|$ is identical with the iteration by the steepest descent for solving the linear equation $Ax=v$. The author, who does not have available Altman's paper [Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 1031–1036; MR **19**, 984], gives a proof in an elementary manner."

Mosolov, B. G. 3002

Approximate solution of operator equations by the method of Ju. D. Sokolov. (Russian. Uzbek summary)

Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk **1963**, *no.* 5, 26–29.

This paper discusses the topic announced in the title in $L_2=L_2(d\omega)$ spaces. Given $f\in L_2$ and a bounded linear transformation P of L_2 into itself, find $u\in L_2$ such that $u=f+Pu$. For a given set $\{\varphi_i\}\subset L_2$, a first approximation is obtained as $u_1=f+\sum\alpha_iP\varphi_i$; the α's are chosen so as to minimize the quadratic expression $\int(\sum\alpha_iT\varphi_i-Pf)^2\,d\omega$; this is extended by induction to $u_2,\cdots,u_n,\cdots$, and estimates of the rate of convergence $u_n\to u$ are obtained under certain hypotheses. This generalizes work by Ju. M. Molokovič [Izv. Vysš. Učebn. Zaved Matematika **1959**,

no. 5 (12), 164–170; MR **27** #3098] and the author [Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk **1961**, no. 5, 29–34]. The method used is that of Ju. D. Sokolov [Dopovidi Akad. Nauk Ukraïn. RSR **1955**, 107–111; MR **17**, 196]. The Uzbek summary is totally inadequate.

G. K. Kalisch (Minneapolis, Minn.)

Petryshyn, W. V. 3003

The generalized overrelaxation method for the approximate solution of operator equations in Hilbert space.

J. Soc. Indust. Appl. Math. **10** (1962), 675–690.

Let H be a complex Hilbert space and let A be an operator which is K positive definite, i.e., there exists an operator K and a positive constant β such that $(Au, Ku) \geqq \beta \|Ku\|^2$ for all u in H. The author considers a generalized successive overrelaxation method (go-method) for solving the equation $Au=f$, where f is in H. The following assumptions are made: A is of the form $D-S-Q$, where (Du, Ku) is real for all u in H; the operator $(2-\omega)\omega^{-1}D-S+Q^*$, where Q^* is the adjoint of Q and $\omega>0$, is K positive definite; for every g in H, $(D-\omega S)u=g$ is easily and uniquely solvable for u. The go-method is defined by $(D-\omega S)u_{n+1}=((1-\omega)D+\omega Q)u_n+\omega f$. The author proves the convergence of the go-method under certain conditions, and he gives estimates for $\|u_{n+1}-u^*\|$, where u^* is the exact solution, in terms of $\|u_{n+1}-u_n\|$. By the use of special choices of K, D, S, Q, and ω these results are applied to several methods, including the Gauss-Seidel method, the ordinary successive overrelaxation method, and a method considered by Keller ["Numerical Methods", Lecture Notes, Chapter 2, New York Univ., New York, 1959]. For the case where $A \geqq 0$ and where certain other assumptions are satisfied, the author shows that the go-method converges if and only if the equation $Au=f$ is solvable.

The author studies the problem of determining the optimum value of ω. His analysis in the case where A is K positive definite and $A=D-S-S^*$ indicates that for each ω the eigenvalues of the go-method operator $T=(D-\omega S)^{-1}((1-\omega)D+\omega S^*)$ may be arbitrarily close to one. He concludes that for the determination of the optimum ω additional assumptions are needed. By assuming $AS=SA$ and by making certain other assumptions, he derives a formula for the optimum ω. Since the formula appears to be very difficult to apply in actual cases, it seems to be of theoretical rather than practical interest. Moreover, the author uses as his criteria for the optimum ω the minimization of the spectral norm of T rather than the spectral radius, as is customary. As a matter of fact, it is well known [see, for instance, Sheldon, J. Assoc. Comput. Mach. **6** (1959), 494–505; MR **23** #B1647] that there are cases where if ω_b, the optimum ω based on the spectral radius criterion, is used, the spectral norm would exceed unity. Since the use of ω_b in an important class of problems arising from the finite-difference solution of certain elliptic partial differential equations actually turns out to be optimal in practice, it would appear that the spectral norm criterion is not ideal. It should also be mentioned that the assumption $AS=SA$ does not hold in many problems arising from the solution of partial differential equations. Indeed, the author does not give any practical examples where this assumption does hold.

The exposition is confusing in a number of places. For instance, in the statement of Theorem 1 the author assumes A to be K positive definite. Later he makes an assumption on K. It is not clear whether the assumption must hold for all K having the properties specified. On page 676, in the remark after (c), it is by no means obvious to the reviewer how the symmetry of the operators A and D follows. In Theorem 1 it is tacitly assumed that a unique solution u^* of $Au=f$ exists. However, at the end of the proof of the theorem it is stated that the existence of u^* follows from the assumptions of the theorem. Reference is made to the author's thesis [Ph.D. Dissertation, Columbia Univ., New York, 1961]. Just exactly what assumptions are sufficient for the existence of u^* is not stated. In the statement of Theorem 2, the meaning of the phrase "in the latter case" can only be deduced from the proof of the theorem.

D. M. Young, Jr. (Austin, Tex.)

Ljaščenko, M. Ja. [Ljaščenko, N. Ja.] 3004

On the numerical solution of non-linear integral equations. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1139–1144.

Author's summary: "A scheme is presented for the numerical solution of nonlinear integral equations consisting of a combination of three methods—mechanical quadrature, differentiation with respect to the parameter, and Newton's method. The solution of a system of differential equations is used as an initial approximation in Newton's method. The scheme is applicable to an extensive class of problems for integro-differential equations with a variable upper limit."

A. H. Stroud (Lawrence, Kans.)

Ljaščenko, M. Ja. [Ljaščenko, N. Ja.] 3005

On the numerical solution of a class of non-linear integro-differential equations. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 3–7.

A method is given for the numerical solution of nonlinear integral equations of the form

$$y(x) = f(x) + \int_a^b K(x, s)F[x, s, y(s), y'(s), \cdots, y^{n-1}(s)]\,ds.$$

This method involves mechanical quadratures, differentiation with respect to a parameter, and Newton's method. The method can readily be realized on modern computing machines. It is applicable to the solution of Cauchy's problem and to boundary-value problems for integro-differential equations.

H. P. Thielman (Alexandria, Va.)

Kadner, Horst 3006

Untersuchungen zu Quadraturformelmethoden für lineare Integralgleichungen 2. Art auf der Grundlage der Kollokation.

Wiss. Z. Techn. Univ. Dresden **12** (1963), 119–120.

Author's summary: "Der Problemkreis der Quadraturformelmethoden und deren Approximationsverhalten wird vom Blickwinkel der Kollokation bearbeitet. Bekannte und allgemein gebräuchliche Verfahren—wie z.B. das Verfahren von Gauss-Nyström u.a.—ergeben sich als spezielle Kollokationen einschließlich Konvergenzeigenschaften."

L. Collatz (Hamburg)

COMPUTING MACHINES

See also 2149, 2188, 2214, 2929, 3025, 3319, 3320, 3324.

Ulam, Stanislaw M. 3007

Computers.

Sci. Amer. **211** (1964), 202–216.

Expository article.

Thurnau, D. H.; Johnson, R. E.; Ham, R. J. 3008

★ALGOL programming: A basic approach.

Big Mountain Press, Denver, Colo., 1964. 158 *pp.* \$2.50.

The authors are quite explicit about the kind of book they intended to produce. It is a text on the ALGOL 60 language, as it is used on the Burroughs B-5000 computer. Although the B-5000 ALGOL goes beyond the official ALGOL language, the book is based on that part of the language which is in the intersection of the two. There is not very much motivation in the way of examples, and the typography of the book gives one a rather crowded impression, but there are several very interesting features in it. The authors make very good use of the so-called Backus normal form, which expresses the syntax of a language, in describing their language. This is the first time this reviewer has seen the syntax forms used as an integral part of the exposition, rather than a device for making the syntax precise. There is also a very nice example of the use of the syntax forms in the definition of a telephone number, as an introduction to the notation. This book would be very useful in a course on B-5000 ALGOL, a little less useful for a different version of ALGOL, and "interesting" to other people. *B. A. Galler* (Ann Arbor, Mich.)

Sobel'man, V. I.; Šura-Bura, M. R. 3009

Realization of recursive procedures in the ALGOL-60 language. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **2** (1962), 303–316.

The general concept of procedure in ALGOL-60 leads to difficulties in the translation of algorithms written in ALGOL-60 into specific machine languages, particularly when producing compilers. The use of recursive procedures aggravates these difficulties, while at the same time presenting advantages in the compactness of representation of algorithms. The recursive representation of the process of translation is particularly convenient when the syntax of the language being translated from is recursively defined, as in the case of ALGOL-60. The authors discuss specific problems encountered in ALGOL-60 and a method for their solution. They base their work on the methods of P. Z. Ingerman [Comm. ACM **4** (1961), 55–58] and of E. T. Irons and W. Feurzeig [ibid. **4** (1961), 65–69]. Interested readers should consult the review of the present article by R. N. Goss in Comput. Rev. **3** (1962), #3121. {A mediocre translation of the article under review has been produced in the series "Foreign Developments in Machine Translation and Information Processing, No. 99" (JPRS: 14164, 19 June 1962) of the Joint Publications Research Service of the Office of Technical Services, U.S. Department of Commerce.}

R. See (Washington, D.C.)

Böhm, Corrado; Santolini, Adelina 3010

A quasi-decision algorithm for the *P*-equivalence of two matrices.

ICC Bull. **3** (1964), 57–69.

This paper presents an algorithm which gives in most cases a decision procedure for the problem of determining whether two arrays are equal up to some permutations of rows and/or columns. The algorithm should prove useful for this problem, since in those cases in which the result is inconclusive this situation is recognized. One of the most striking features of this paper is the presentation of the complete algorithm in ALGOL. In other words, the algorithm is so completely and precisely stated that anyone who can read the ALGOL language can immediately carry out the algorithm or produce a computer program that can. There is no doubt that many more useful algorithms would be successfully employed by readers of mathematical papers if they were as clearly and precisely communicated as this one.

B. A. Galler (Ann Arbor, Mich.)

Ginsburg, Seymour; Hibbard, Thomas N. 3011

Solvability of machine mappings of regular sets to regular sets.

J. Assoc. Comput. Mach. **11** (1964), 302–312.

Ginsburg and the reviewer [same J. **10** (1963), 29–47; MR **28** #748] showed that the following decision problems involving arbitrary context-free languages U and V are (recursively) unsolvable. (1) Is there a complete sequential machine that maps U into V? (2) Is there a generalized sequential machine that maps U onto an infinite subset of V? (3) Is there a complete sequential machine that maps U onto V? (4) Is there a generalized sequential machine that maps U onto V? (Regarding complete sequential machines (csm) and generalized sequential machines (gsm), see Ginsburg [*An introduction to mathematical machine theory*, Addison-Wesley, Reading, Mass., 1962; MR **26** #3222].)

In the present paper, it is shown that if U and V are regular sets, then (1), (2) and (3) are solvable. The solvability of (4) remains an open question. (A regular set is the set of words accepted by some finite automaton [Rabin and Scott, IBM J. Res. Develop. **3** (1959), 114–125; MR **21** #2559].)

The construction of algorithms is based on three noteworthy lemmas concerning arbitrary regular sets U and V accepted by automata of m and n states, respectively: (*) If there is a complete sequential mapping of U into V, then there is a csm with at most mn states that maps U into V. (**) If there is a gsm mapping U onto an infinite subset of V, then there is a gsm with at most mn states and with outputs of length at most n^n that maps U onto an infinite subset of V. (***) If there is a complete sequential mapping of U onto V, then there is a csm that maps U onto V. (The complete sequential mappings are the infinite-state analogues of the complete sequential machines.)

From (*) and (**), algorithms for (1) and (2) follow directly. Less obviously, (***) leads to a computation procedure that yields either a csm that maps U onto V (if there is such a csm) or an integer k such that no csm maps the subset of U with length at most k onto the subset of V with length at most k (otherwise).

G. F. Rose (Pacific Palisades, Calif.)

Riesel, Hans 3012

In which order are different conditions to be examined? *Nordisk Tidskr. Informations-Behandling* **3** (1963), 255–256.

By use of a computer it is required to test whether or not all the conditions ϑ_i, $i = 1, 2, \cdots, n$, are satisfied. Let t_i be the mean time for examining the condition ϑ_i and s_i the average of all cases that it is fulfilled. By a straightforward argument the author shows that if

$$\frac{t_1}{1-s_1} \leqq \frac{t_2}{1-s_2} \leqq \cdots \leqq \frac{t_n}{1-s_n},$$

then the computing time is minimized if the conditions are examined in the order $\vartheta_1, \vartheta_2, \cdots, \vartheta_n$.

M. S. Cheema (Tucson, Ariz.)

Varšavskiĭ, V. I. 3013

Ternary majority logic. (Russian. English summary) *Avtomat. i Telemeh.* **25** (1964), 673–684.

Author's summary: "The design of ternary computing and control circuits consisting of ternary majority elements is considered. The synthesis method is proposed for the circuits. As a ternary majority element there can be used a three-stable parametron or twin."

Lawler, Eugene L. 3014

An approach to multilevel Boolean minimization. *J. Assoc. Comput. Mach.* **11** (1964), 283–295.

While much has been written on minimizing for two-level Boolean expressions, little attention has been given to multilevel minimization. This paper makes a serious attack on the problem. The author first generalizes prime implicants to the multilevel case (and presents a "reasonable efficient" algorithm for determining them). He then shows that multilevel minimization, and thus absolute minimization, can be obtained from these generalized prime implicants. Some conditions are given which ensure absolute minimization. The results are then applied to several examples.

S. Ginsburg (Van Nuys, Calif.)

Popov, V. N.; Stepanov, V. A.; Stiševa, A. G.; Travnikova, N. A. 3015

A programming program. (Russian) *Ž. Vyčisl. Mat. i Mat. Fiz.* **4** (1964), 78–95.

One gives some details about a programming program (PP) for computers with three addresses and a big internal memory. The programs obtained by means of the PP are 1.5–2.5 times longer than the programs accomplished by hand, and the time for solving a given problem is 1.5–5 times longer. This fact strongly depends of the quality of procedures. The input language is ALGOL 60 with some modifications.

We can remark that the formation of the variable addresses is not so economical, and the dependence is linear. The authors propose to ameliorate these procedures and also the processing of the description of the blocks.

P. Constantinescu (Bucharest)

Popov, V. N. 3016

A formation system in standard programs. (Russian) *Ž. Vyčisl. Mat. i Mat. Fiz.* **4** (1964), 189–192.

The author proposes a new method for the formation of variable addresses in the standard programs (SP) by considering two parts of the SP: SP_1, the standard program without the formation part, and SP_2, the information part for the calculus of variable addresses.

The method is advantageous in the case of a complicated dependence of the variable addresses as functions of various parameters.

P. Constantinescu (Bucharest)

Evreinov, È. V. 3017

On the microstructure of the elementary machines of a computing system. (Russian) *Vyčisl. Sistemy No.* 4 (1962), 5–28.

Puhov, G. È. 3018

A linear algebraic transformer. (Ukrainian. Russian and English summaries) *Dopovidi Akad. Nauk Ukraïn. RSR* **1963**, 1006–1009.

Author's summary: "A computer is proposed for the transformation of an n-dimensional vector y into an m-dimensional vector $x (m \gtrless n)$, corresponding to the system of equations $Ax + By = 0$, where A is a square matrix, B is a rectangular matrix. The vectors x and y are simulated by means of electrical voltages. It is proved that the process of the transformation of the vectors for any matrix A will be convergent. The computer circuit is very simple. Each coefficient of the matrix A is simulated with one conductivity only."

Zagoruĭko, N. G. 3019

The exchange of verbal information between a person and computing systems. (Russian) *Vyčisl. Sistemy No.* 10 (1964), 3–12.

Zagoruĭko, N. G. 3020

Procedures for estimating the informational effectiveness of independent parameters of a speech signal. (Russian) *Vyčisl. Sistemy No.* 10 (1964), 13–19.

Zagoruĭko, N. G. 3021

Computational errors of the energy and enveloping speech signal on an electronic computer. (Russian) *Vyčisl. Sistemy No.* 10 (1964), 41–45.

Vološin, G. Ja. 3022

Spectral analysis of speech signals by means of electronic computers. (Russian) *Vyčisl. Sistemy No.* 10 (1964), 20–40.

GENERAL APPLIED MATHEMATICS

See also 2498, 2624, 2682, 2858.

Almer, Bertil 3023

Boundary values of analytical functions in relation to distribution functions—a contribution to the modern mathematics of electronics. *Skand. Aktuarietidskr.* **1961**, 145–170 (1963).

MECHANICS OF PARTICLES AND SYSTEMS

See also 2496, 2692.

Capra, Vincenzo 3024

Sulla determinazione delle soluzioni periodiche della equazione delle vibrazioni libere di un sistema meccanico ad un grado di libertà.

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **97** (1962/63), 1287–1308.

The validity of Newcomb's method for the determination of the periodic solution of the free pseudoharmonic oscillator is demonstrated. The period of the solution is shown to be an analytic function of the initial value of the coordinate q_0. The convergence of the solution as a power series in q_0 is discussed.

N. D. SenGupta (Bombay)

Verlan', A. F. 3025

Determination of the frequencies of eigenvalues of vibrations on a computer. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 733–735.

Struble, R. A. 3026

On the oscillations of a pendulum under parametric excitation.

Quart. Appl. Math. **22** (1964), 157–159.

In this note the author corroborates, through the application of a new mapping theorem due to Moser [Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II **1962**, 1–20; MR **26** #5255], the general behaviour of the oscillations of a pendulum under parametric excitation, which was studied by him by use of a formal asymptotic method [Quart. Appl. Math. **21** (1963), 121–131; MR **27** #3118].

N. D. SenGupta (Bombay)

Osipov, P. M. 3027

Revolution of a body of variable mass in the case when $M_c^{(a)}=0$ and $M_c^{(p)}=0$. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1587–1591.

Author's summary: "A generalization is given of Euler's equation in diad form for a body of variable mass under the assumption that the body revolves with an angular velocity of $\boldsymbol{\Omega}$, the centre of inertia C moves relatively to the configuration of the body with a velocity v_c, and the axes of coordinates C_{xyz} rotate relatively to the body with an angular velocity $\boldsymbol{\omega}$. The revolution of a body of variable mass is considered for the case when the moments of external and reactive forces relative to the centre of inertia are equal to zero. As a result three equations are obtained. Assuming that in these equations the vector of the angular velocity $\boldsymbol{\omega}$ is equal to zero and the moments of inertia are constant, we obtain Euler's equation for a body of constant mass."

ELASTICITY, PLASTICITY

See also 2422, 2460, 2985, 3078.

Dryden, H. L.; von Kármán, Th. (Editors) 3028

★Advances in Applied Mechanics. Vol. 8.

Academic Press, New York-London, 1964. x + 298 *pp.* $12.00.

The papers of mathematical interest will be reviewed individually. Volume 7 appeared in 1962 [cf. MR **26** #7204].

Finzi, Leo 3029

On the uniqueness of solution in statics of continua with general differential stress-strain relations.

Second-order Effects in Elasticity, Plasticity and Fluid Dynamics (Internat. Sympos., Haifa, 1962), *pp.* 715–728. *Jerusalem Academic Press, Jerusalem; Pergamon, Oxford*, 1964.

Most of the paper is an unilluminating account of known uniqueness proofs for Green elastic and Hencky plastic solids. [Omitted references: the reviewer, J. Mech. Phys. Solids 5 (1957), 229–241; MR **19**, 1103; ibid. 5 (1956), 66–74; MR **18**, 431.] The author makes an unfounded claim of originality for his equation (36) in connection with a general differential constitutive law, not necessarily possessing a strain-rate potential; it actually appeared some years ago as equation (5) in reference [10] of the paper, which also incorporates geometric effects. The final part of the paper is a roundabout proof of uniqueness for the Prandtl/Reuss solid, for which a strain-rate potential does exist; a succinct proof appears in the 1956 paper mentioned above. The author also omits to mention the reviewer's sufficiency criterion for uniqueness in the form of an integral inequality, which is stronger than the classical criterion in terms of the definiteness of an integrand.

R. Hill (Cambridge, England)

Deev, V. M. 3030

Representation of the general solution of a three-dimensional problem of elasticity theory by particular solutions of Lamé's equations. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1464–1467.

Author's summary: "In this paper a special solution u_2 is obtained for Lamé's equations. It satisfies $\nabla\cdot u\neq 0$, $\nabla\times u=0$. Such a solution in the static problem of elasticity theory has not hitherto been known. Using the well-known solution u_1, which satisfies conditions $\nabla\cdot u=0$, $\nabla\times u\neq 0$, the general solution of the space problem is constructed in the form $u_{\text{com}}=c_1u_1+c_2u_2$, where c_1 and c_2 are arbitrary constants. The article also shows the process of finding general solutions of Lamé's equations, expressed through harmonic functions and possessing prescribed properties."

Martinenko, M. D. 3031

Some boundary-value problems in the theory of elasticity for a region with slits. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 983–986.

Author's summary: "The author considers the problem of finding solutions of Lamé's system in an infinite three-dimensional space for a given displacement on two sides

of the slit. This problem is reduced to regular integral equations, solvable by the method of successive approximations."

Martinenko, M. D. 3032

On the second boundary-value problem of elasticity theory for domains with slits. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1151–1156.

Author's summary: "The second boundary-value problem of the theory of elasticity is considered for the case of stress applied on both sides of a plane slit. The solution of the problem is reduced to integral equations which prove to be explicitly solvable."

Martinenko, M. D. 3033

The second boundary-value problem of the theory of elasticity for domains with slits. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 707–711.

Author's summary: "The second boundary-value problem of the theory of elasticity (the problem with stresses) is considered for the case of given boundary conditions on two sides of an unclosed surface of Liapunov type with a smooth boundary on which one supplementary condition is imposed—the normal to it must not intersect this surface in two or more points. Using a potential representation of the solution, this problem is reduced to regular integral equations, the solvability of which is investigated."

Bramble, J. H.; Payne, L. E. 3034

Some mean value theorems in elastostatics.

J. Soc. Indust. Appl. Math. **12** (1964), 105–114.

Suppose an elastic medium occupies some 3-dimensional domain D and u_i $(i=1, 2, 3)$ represent displacement components. If Σ_R is the boundary of a sphere of radius R contained in D, and $h(0)$ denotes the value of the function h at the center of the sphere, then

(1) $u_i(0) = [2(3+\alpha)]^{-1}[2(1+2\alpha)I_n(R)-(2-\alpha)I_t(R)]$,

(2) $u_i(0) = I_n(R)+2(\alpha-2)(3\alpha-1)^{-1}J_n(R)$,

(3) $u_i(0) = -\frac{1}{2}I_t(R)+2(1+2\alpha)(3\alpha-1)^{-1}J_t(R)$,

(4) $u_i(0) = I_n(R)+2(\alpha-2)(3\alpha-1)^{-1}J_t(R)$,

(5) $u_i(0) = -\frac{1}{2}I_t(R)+2(1+2\alpha)(3\alpha-1)^{-1}J_n(R)$,

where α depends on Poisson's ratio, and I_n and I_t involve only normal and tangential displacement components on Σ_R, respectively, while J_n and J_t involve only normal and tangential components of surface traction, respectively. Equation (1) was obtained by Aquaro and Synge. The remaining mean-value expressions, as well as converses of (1)–(5), are established in the present paper.

J. Nowinski (Newark, Del.)

Bramble, J. H.; Payne, L. E. 3035

Error bounds in the pointwise approximation of solutions of elastic plate problems.

J. Res. Nat. Bur. Standards Sect. B **67B** (1963), 145–155.

Pointwise bounds for the deflection and its first three derivatives are obtained for the cases where (i) the deflection and bending moment are prescribed on the boundary, and (ii) where the bending moment and normal reaction are prescribed. The work is a continuation of that described for the first boundary-value problem for elastic plates [J. Math. and Phys. **42** (1963), 278–286; MR **28** #2353]. The inequalities which are obtained bound an arbitrary (sufficiently smooth) function at a point in a region in terms of the data of the given problem. The trial functions need only possess piecewise continuous fourth derivatives and are not required to satisfy the governing differential equations or the boundary conditions.

No numerical applications are presented. It is stated that a further paper will deal with mixed boundary-value problems [the authors, J. Res. Nat. Bur. Standards Sect. B **68B** (1964), 75–92]. *L. S. D. Morley* (Farnborough)

Truesdell, C. 3036

Exact theory of self-expanding piston rings.

Ing.-Arch. **30** (1961), 77–87.

The problem is studied of selecting the unloaded form of a piston ring, so that when inserted into a cylinder it closes and exerts uniform pressure on the cylinder wall. Bernoulli-Euler beam theory is used. The fact that the bending moment due to uniform normal pressure along an arc is equal to that due to the same pressure along its chord leads to a concise expression for the difference in curvature caused by the load, and to a complex line integral for the unloaded shape. Simplifying approximations of first and second order in deflection are given with error bounds. The particular case of an elliptical cylinder is presented, with approximations for small eccentricity, and small eccentricity combined with small deflection. A numerical example illustrates the accuracy of the approximations. To avoid the restriction of the symmetry of the ellipse, the solution is also given for a small general perturbation about a circular cylindrical form.

E. H. Lee (Stanford, Calif.)

Truesdell, C. 3037

Second-order theory of wave propagation in isotropic elastic materials.

Second-order Effects in Elasticity, Plasticity and Fluid Dynamics (Internat. Sympos., Haifa, 1962), pp. 187–199. *Jerusalem Academic Press, Jerusalem; Pergamon, Oxford*, 1964.

A material is isotropically elastic if the Cauchy stress and deformation tensors T and C are related by

$$T = \alpha_0 I+\alpha_1 C+\alpha_2 C^{-1},$$

where the α_i are scalar functions of the invariants of C^{-1}. If such a material also possesses a stored energy function, it is said to be hyperelastic. Correct to first powers of the principal extensions, the constitutive equations are unaltered; correct to second powers, the equations for hyperelastic materials contain one less constant. Consequently there is one relation between the constants of the elastic material if it is to be hyperelastic (correct to second powers of the extensions). The object of this paper is to express this relation in terms of measurable quantities. The final result is an equation between the speeds, correct to second order, of principal longitudinal and transverse waves. *D. R. Bland* (Manchester)

Fersht, S. 3038
Examination of quasi-linear elasticity.
Second-order Effects in Elasticity, Plasticity and Fluid Dynamics (Internat. Sympos., Haifa, 1962), pp. 252–260. *Jerusalem Academic Press, Jerusalem; Pergamon, Oxford,* 1964.

Author's summary: "It is shown that in the quasi-linear stress-strain relation the volume strain ε_v must be introduced in the first term when the displacement gradient is finite." *M. Hayes* (Newcastle upon Tyne)

Theocaris, Pericles S. 3039
The method of isostatics applied to rectangular bars with uniform loading. (French, German, Italian, and Russian summaries)
Internat. J. Engrg. Sci. **2** (1964), 1–19.

Author's summary: "A new method which involves the determination of a function of the complex variable representing the field of isostatics is used for the solution of stress distribution in rectangular bars compressed by uniformly distributed normal loads. This function is determined to have a convenient form for the expression of boundary conditions by simple relations. The Schwarz-Christoffel transformation is used for the definition of the function mapping the isostatics from the half-space to the rectangular bars. Conformal mapping introduces normal stresses to the longitudinal boundaries of the bar. Elimination of these stresses is obtained by using a method of multiple Fourier series in a part of the bar. Stresses are calculated on the boundaries, the axis of symmetry and five sections at the neighbourhood of the load. Four cases are considered with various ratios of the loading interval to the width of the bar."

Tul'čiĭ, V. I. 3040
On a method of solving the two-dimensional problems of the theory of elasticity. (Ukrainian. Russian and English summaries)
Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 884–887.

Author's summary: "A method is presented for solving the plane stress and plane strain problems. The method is based on the fact that, near the contour which bounds the plate, an elastic ring is formed. The basic equations of this method are obtained from the conditions of the interaction of the ring and plate."

Rokva, Ž. P. 3041
Approximation of the solutions of integral equations in two-dimensional elasticity theory. (Georgian. Russian summary)
Soobšč. Akad. Nauk Gruzin. SSR **35** (1964), 45–50.

The integral equation

$$(1)\qquad \varphi_0'(\sigma_0)-\lambda\int_\gamma K(\sigma,\sigma_0)\overline{\varphi_0'(\sigma)}\,d\sigma = B(\sigma_0),$$

where the kernel is of the form

$$K(\sigma,\sigma_0)=\frac{-1}{2\pi i}\frac{\partial}{\partial\sigma_0}\left\{\frac{\omega(\sigma)-\omega(\sigma_0)}{(\sigma-\sigma_0)\overline{\omega'(\sigma)}}\right\},$$

was shown by N. I. Mushelišvili [*Some fundamental problems of the mathematical theory of elasticity* (Russian), fourth edition, Izdat. Akad. Nauk SSSR, Moscow, 1954; MR **16**, 1067; English transl. of the third edition, Noordhoff, Groningen, 1953; MR **15**, 370] to be basic in certain two-dimensional problems. Whenever the domain in question is nearly circular, the author gives a sufficient condition of a simple type for the convergence of the method of successive approximations, together with an upper bound for the resolvent of the kernel.

This bound is used in conjunction with certain results of Kantorovič [cf., e.g., L. V. Kantorovič and V. I. Krylov, *Approximate methods of higher analysis* (Russian), fifth corrected edition, Fizmatgiz, Moscow, 1962; MR **27** #4338] to find effective error estimates for the solution when the kernel is replaced by an approximation corresponding to an approximate conformal mapping of the domain onto the unit circle.

Smith, S. F. 3042
On a flat punch indenting an elastic layer in plane strain.
Quart. J. Math. Oxford Ser. (2) **15** (1964), 223–237.

The author considers the problem of a punch of half-width b indenting an elastic layer of thickness a in plane strain when both the punch and the support for the elastic layer are smooth, rigid and flat. He uses transform techniques to reduce the problem to one in interpolation theory, which is solved by making use of the known solution of the punch problem for $b/a\to\infty$. The solution obtained is thus suitable for numerical computation when b/a is large, and graphs of the stress on both surfaces of the elastic layer and of the deformation of the free surface are presented for various values of b/a. The paper is a useful contribution to research on this problem, as most previous work on it has been concerned with obtaining approximate solutions for small, as against large, values of b/a. *W. D. Collins* (Manchester)

Berezovs'kiĭ, A. A.; Šuležko, L. F. 3043
On the parametric resonance of a plate in a non-linear setting. (Ukrainian. Russian and English summaries)
Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 989–993.

Authors' summary: "The authors investigate the nonlinear integro-differential equations of the problem. In the first approximation one ordinary nonlinear differential equation was obtained for three cases of fixing the plate. The equation was investigated by the method of asymptotic integration of N. N. Bogolyubov and Y. A. Mitropolsky."

Berezovs'kiĭ, A. A. 3044
On large deflections of a freely supported circular plate. (Ukrainian. Russian and English summaries)
Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1295–1299.

Author's summary: "The author investigates the nonlinear integro-differential equations of the problem. The kernel of the system is approximated as degenerate by the Bubnov-Galerkin method, which reduces the solution of the problem to the solution of a cubic equation having a single real root. The solutions are obtained in the form of a sum of the linear solution and the correction for nonlinearity."

Şuhubi, Erdoğan 3045

On bending of rectangular plates of varying thickness. (Turkish summary)

İstanbul Tek. Üniv. Bül. **16** (1963), 1–6.

Author's summary: "In this paper a problem which had been studied in a previous paper [same Bül. **15** (1962), 63–70] has been solved for a more general loading. One assumes a load function in the form $q(x, y) = f(x)\varphi(y)$, and the resulting non-homogeneous differential equation has been worked out by means of the kernel function. The result has been applied to a plate subjected to hydrostatic pressure."

Venkatraman, B.; Sankaranarayanan, R. 3046

Collapse loads of orthotropic cylindrical shells under radial pressures.

J. Franklin Inst. **278** (1964), 183–194.

Authors' summary: "The present paper investigates the load carrying capacities of circular cylindrical shells made of an orthotropic material (with yield stresses different in tension and compression under radial pressures). Based on the Tresca yield condition, equations for the interaction curves are first derived. Use is then made of linear approximations to these curves to analyze shells under external and internal pressures. The analysis leads to closed form solutions for the collapse loads. A graphical presentation of collapse loads versus shell parameters for various values of the yield stresses concludes the paper."

Koiter, W. T. 3047

On the dynamic boundary conditions in the theory of thin shells.

Nederl. Akad. Wetensch. Proc. Ser. B **67** (1964), 117–126.

The author proves that the Kirchhoff boundary conditions in the theory of plates and shells are exact. They are incomplete, in the sense that higher approximation theories (those which include effects of transverse shear and normal strain, for instance) may call for additional boundary conditions, but the Kirchhoff boundary conditions are correct, even for the exact three-dimensional problem. *J. L. Sanders, Jr.* (Cambridge, Mass.)

Manevič, A. I. 3048

Optimal design of a reinforced cylindrical shell under uniform external pressure. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 875–879.

Author's summary: "This paper deals with the problem of designing a cylindrical shell reinforced by elastic rings under external pressure. The matrix method is used to determine the rigidity of the rings. Two arrangements of the rings are proposed for securing equistability of the construction."

Goodier, J. N.; McIvor, I. K. 3049

The elastic cylindrical shell under nearly uniform radial impulse.

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 259–266.

Free vibrations of a cylindrical shell under impulsive radial pressure are considered. For perfectly uniform pressure, the vibration is a "breathing" mode with the cross-section remaining circular at every instant. Small imperfections in the uniformity are found to excite flexural motion through the interaction between circumferential stresses and flexural curvature. In particular, flexural vibrations having a frequency one-half of the breathing mode can be strongly excited, and almost all energy can be transferred from breathing to flexure.

Both small and large deflection theories are employed with some additional simplifications, e.g., in the nonlinear equations the deviations from the circular mode are assumed to be inextensional (within a linearized theory extensional terms are computed and their relative smallness is shown to be a consequence of the thinness of the shell). The velocity amplitudes are found to satisfy Mathieu equations, which are solved numerically on an analog computer for a variety of parameters.

H. J. Weinitschke (Hamburg)

Hrustalev, A. F. 3050

An axisymmetric mixed problem in elasticity theory for a transversally isotropic cylinder. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 1 (38), 139–143.

The boundary conditions of the title problem are zero traction over the surface of the infinite cylinder except for a finite length upon which the tangential traction is zero and the normal displacement is prescribed. It is shown that the dual integral equations which result from these boundary conditions can be reduced to a Fredholm integral equation of the second kind.

F. J. Lockett (Sevenoaks)

Bhargava, R. D.; Radhakrishna, H. C. 3051

Elliptic inclusion in orthotropic medium.

J. Phys. Soc. Japan **19** (1964), 396–405.

Authors' summary: "The problems of circular and elliptic inclusions tending to undergo spontaneous dimensional changes in infinite isotropic elastic continuous unstressed or stressed material (the matrix) have been considered by the authors in previous papers. In these problems the elastic properties of the inclusions were taken to be different from those of the matrix. In this paper the exact analytical solution to such a problem when the elliptic inclusion and matrix are of different orthotropic materials has been obtained. The solution has been obtained by the applications of the minimum strain energy principle first suggested by one of the authors (R.D.B.) coupled with a semi-inverse method and complex variable technique. Continuity of normal and shearing stresses at the interface is demonstrated, providing a useful check on the analysis." *R. M. Morris* (Cardiff)

Kondo, Kazuo 3052

On the analytical and physical foundations of the theory of dislocations and yielding by the differential geometry of continua. (French, German, Italian, and Russian summaries)

Internat. J. Engrg. Sci. **2** (1964), 219–251.

The author summarizes distorted lattice and continuum theory by means of Riemannian and non-Riemannian

geometry, emphasizing his view that Riemannian curvature is important as opposed to the view of Bilby that the relevant spaces have torsion, but show distant parallelism. *F. R. N. Nabarro* (Cleveland, Ohio)

Dimaggio, Frank L. 3053
Longitudinal vibration of a prolate ellipsoid.
Quart. Appl. Math. **20** (1962/63), 182–183.
Author's summary: "In this paper, the mode shapes and frequency parameters for longitudinal vibration of a bar whose cross-sectional area varies parabolically along its axis are obtained as single tabulated prolate spheroidal angle functions and their eigenvalues."
I. Marx (Lafayette, Ind.)

Kaliski, Sylwester 3054
Self-excited vibration of a system of oscillators moving on the surface of an elastic semi-space. (Polish and Russian summaries)
Proc. Vibration Problems **5** (1964), 3–18.
From the author's summary: "In addition to its direct practical interest for mechanics the present paper constitutes a certain preliminary stage for a study of more complex mechanical-electronic problems, such as the problem of self-excited vibrations of a stream of particles moving over the surface of an elastic conductor in a magnetic field with velocities of the order of the magneto-acoustic velocity in the elastic conductor. The solution obtained in the present paper is transferred, according to an analogous procedure, to the problem of motion of a system of oscillators along a beam resting on an elastic foundation which is of considerable practical importance."

Seidel, B. S.; Erdélyi, E. A. 3055
On the vibration of a thick ring in its own plane.
Trans. ASME Ser. B. J. Engrg. Indust. **86** (1964), 240–244.
An elementary derivation is given of the governing equations for the motions of a thick ring. The analysis is highly physically motivated and is essentially treating the ring as a Timoshenko beam. An energy method is used to find the natural frequencies. A comparison is given in which experimental and computed frequencies agree within a few percent up to $h/R = 0.7$.
P. R. Paslay (Houston, Tex.)

Rajappa, Nanguneri R. 3056
On the exact solutions of vibration problems of rectangular orthotropic plates. (German summary)
Z. Angew. Math. Phys. **15** (1964), 201–205.
The author uses the fact that the frequency equation for flexural vibrations of a thin plate of isotropic material is a special case of that equation for orthotropic material in order to determine certain simple affine relations between the defining parameters of the two cases. Using these relations and assuming that the frequency for the isotropic plate is given, he asserts that one can calculate the corresponding frequency for the orthotropic plate with little effort. Illustrations of the method are given for rectangular plates with two opposite edges simply-supported and several boundary conditions for the other two edges. The reviewer considers that the method is too limited for practical analysis of modern structural vibration problems. *W. H. Hoppmann, II* (Troy, N.Y.)

Chadwick, P.; Windle, D. W. 3057
Propagation of Rayleigh waves along isothermal and insulated boundaries.
Proc. Roy. Soc. Ser. A **280** (1964), 47–71.
Authors' summary: "In this paper the effects of heat conduction upon the propagation of Rayleigh surface waves in a semi-infinite elastic solid are studied theoretically in two special cases: (i) when the surface of the solid is maintained at constant temperature (case 1); and (ii) when the surface is thermally insulated (case 2). The investigation is carried out within the framework of the linear theory of thermoelasticity, a principal objective being the clarification of the relation between so-called thermoelastic Rayleigh waves and the Rayleigh waves of classical elastokinetics.

"The secular equation for thermoelastic Rayleigh waves is shown to define a many-valued algebraic function $\mu(\chi)$ (χ being a dimensionless frequency) the branches of which represent possible modes of surface wave propagation. Two different types of surface mode are recognised: E-modes, which resemble classical Rayleigh waves but are subject to damping and dispersion; and T-modes, which are essentially diffusive in character. Necessary and sufficient conditions for the existence of a surface wave are formulated, and questions of the existence and multiplicity of Rayleigh E- and T-modes in particular situations are resolved by submitting the branches of $\mu(\chi)$ to these requirements.

"The algebraic function $\mu(\chi)$ has singular points at $\chi = 0$ and $\chi = \infty$, and approximations, valid at sufficiently low or at sufficiently high frequencies, to the speed of propagation v and the attenuation coefficient q of a given surface mode are obtainable from series representations of the appropriate branch of $\mu(\chi)$ in neighbourhoods of these singularities. The singular point $\chi = 0$ is associated with adiabatic deformations of the solid, and hence with classical Rayleigh waves, and the singularity at $\chi = \infty$ with isothermal deformations.

"Particular attention is devoted to the Rayleigh E-modes and the main conclusions reached are as follows. In case 1 there exists a single E-mode (mode 2) at low frequencies and two distinct E-modes (modes 1 and 2) at high frequencies. For mode 2, $v/v_R = 1 + O(\chi^{1/2})$, $q = O(\chi^{3/2})$ as $\chi \to 0$ (v_R being the speed of propagation of classical Rayleigh waves), and for both modes v and q approach finite limits as $\chi \to \infty$. In case 2 the converse situation applies, there being two distinct E-modes (modes 1 and 2) at low frequencies and only one (mode 1) at high frequencies. For both modes, $v/v_R = 1 + O(\chi^{3/2})$, $q = O(\chi^2)$ as $\chi \to 0$, and for mode 1, v and q approach finite limits as $\chi \to \infty$.

"Detailed numerical results referring to a medium of worked pure copper at a reference temperature of 20° C are given. In particular, the frequency dependence of the speeds of propagation and attenuation coefficients of the various E-modes are exhibited, and the frequencies at which mode 1 appears in case 1 and at which mode 2 disappears in case 2 are determined."
M. Hayes (Newcastle upon Tyne)

Nigul, U. K. 3058

The application of the three-dimensional theory of elasticity to the analysis of flexural waves in a semi-infinite plate acted on by a short-time boundary loading.

Prikl. Mat. Meh. **27** (1963), 1044–1056 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1602–1620.

On the basis of the three-dimensional theory of elasticity, the waves excited in a semi-infinite plate by suddenly applied boundary stresses linearly distributed over the thickness of the plate are considered. By means of a two-fold integral transform method, a formal solution is established in the form of an infinite sum of one-dimensional integrals of the Laplace transform. Their approximate inversion by means of the saddle-point method is examined, and numerical results for the first six integrals are presented. The error in the Kirchhoff theory and in the Timoshenko-type theory are then studied. *B. A. Boley* (New York)

Hunter, S. C. 3059

The Hertz problem for a rigid spherical indenter and a viscoelastic half-space.

J. Mech. Phys. Solids **8** (1960), 219–234.

The problem of contact between viscoelastic bodies involves boundary conditions which can change in type for a particular boundary point as the contact region spreads, being a prescribed displacement in the contact region and zero surface traction outside. This situation precludes the standard approach of the Laplace transform generating a well-set elastic problem. The solution for a rigid spherical indenter pressed on a viscoelastic half-space is obtained by investing the transformed elastic solution for arbitrary axi-symmetric surface pressure in terms of integrals of Bessel functions, and so representing the problem as dual integral equations, which are solved for monotonically increasing contact radius, and for the radius attaining a single maximum and then decreasing. The former recovers a known result. An approximate solution for bouncing impact is given for a body which is nearly elastic with a small viscoelastic term. Constant Poisson's ratio is assumed, but this is stated not to be a necessary feature of the method. *E. H. Lee* (Stanford, Calif.)

Hunter, S. C. 3060

Tentative equations for the propagation of stress, strain and temperature fields in viscoelastic solids.

J. Mech. Phys. Solids **9** (1961), 39–51.

The linear viscoelastic laws are stated and functional expressions for the elastic energy stored and rate of dissipation are given. Thermo-mechanical coupling is introduced by assuming the free energy to be the elastic stored energy plus a function of temperature, leading to an expression for the internal energy. The heat balance equation is written to second-order terms, and includes a convective derivative even though infinitesimal displacement theory is used. Plane waves are analyzed, and the thermo-mechanical coupling assumed determines a reduction in the isothermal attenuation, although not so for propagation along thin rods. The range of validity of the approximations involved in the analysis is assessed using measured material constants.

E. H. Lee (Stanford, Calif.)

Kric'ka, S. S. [Krickaja, S. S.] 3061

On the impact of a visco-elastic rod of variable cross-section. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1455–1460.

Author's summary: "An analysis is presented of the function $u(x, t)$ of displacement of cross-sections of an elastoviscous homogeneous rod of variable cross-section, the left end of which is linked with a load, while the right end is subjected to impact by a definite load which subsequently remains in contact with the rod."

Tokuoka, Tatsuo 3062

Fundamental relations between Prandtl-Reuss type plastic body and Maxwell type viscoelastic body.

Bull. JSME **7** (1964), 17–20.

A generalized Maxwell-type viscoelastic material is characterized by the constitutive relation

$$\dot{e}_{ij} = \frac{d}{dt}\left(\frac{\partial \Phi}{\partial \sigma_{ij}}\right) + \frac{\partial \Psi}{\partial \sigma_{ij}} \qquad (i, j = 1, 2, 3),$$

where Φ and Ψ are the elastic and dissipative energy per unit volume, respectively. It is shown that the Prandtl-Reuss plastic material is a special case of this material.

F. J. Lockett (Sevenoaks)

Hoff, N. J. 3063

Reversed creep: A remark to the creep buckling theory of Rabotnov and Shesterikov.

J. Mech. Phys. Solids **12** (1964), 113–123.

A detailed analysis of the creep buckling theory of Rabotnov and Shesterikov [same J. **6** (1957), 27–34; MR **19**, 1214] is presented to explain why the theory predicts instability for sufficiently small creep straining whereas rapid creep straining or the complete absence of creep leads to stability. The theory of Rabotnov and Shesterikov is based upon assuming a mechanical equation of state for the material and examining the stability of the system by the classical dynamic method, i.e., by determining the boundedness of oscillations of the system. The equation of state that describes the stress, strain, strain rate relation is also assumed to govern the dynamic problem. It is shown in the present paper that under rapid creep conditions the oscillations would cause reversals of creep strains and thereby result in stability. The critical condition is reached when the creep strain rate is sufficiently small so that the oscillations do not cause a change in the creep strain rate of the system.

S. R. Bodner (Providence, R.I.)

Ting, T. C. T.; Symonds, P. S. 3064

Longitudinal impact on viscoplastic rods—linear stress-strain rate law.

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 199–207.

The problem considered is that of longitudinal impact on a rod, the material of the rod being rigid-plastic with a yield stress which depends on strain-rate. The basic equations reduce to the heat conduction equation. For both semi-infinite and finite rods two types of loading are considered, and numerical results are presented in graphical form. *G. Eason* (Glasgow)

Cozzarelli, F. A.; Patel, Sharad A. 3065

Creep deformations in membrane shells.

J. Franklin Inst. **278** (1964), 45–60.

Using the creep law suggested by Hoff, the authors develop small-deflection membrane theory for circumferentially closed shells of revolution and shells with one principal curvature infinite, with boundary conditions such that stresses can be determined from equilibrium conditions only. Applications are given to cones and hemispheres under an antisymmetric loading and a cylindrical vault with cycloidal cross-section under dead load. As is too often the case in mathematical discussions of "membrane" theory, no consideration is given to whether this approximation is adequate in the cases studied. In the reviewer's opinion it would not be safe to ignore flexure in the applications described. *L. H. Donnell* (Chesterton, Ind.)

Bhattacharyya, Shankar Prasad 3066

Thermal stresses due to prescribed flux of heat on the surface of a thick plate.

J. Sci. Engrg. Res. **7** (1963), 334–342.

The object of this paper is to find the exact solution of the thermo-elastic problem of a thick plate of infinite radius of an isotropic material, with stress free edges subjected to two different temperature distributions, namely, in the first case, a constant flux of heat within a circular region of exposure, the exterior of the circular region being free of any flux of heat; secondly, a paraboloidal distribution of temperature within the circular region, the exterior being insulated. Numerical calculations for the stresses at free surface have also been obtained in the second case. *B. A. Boley* (New York)

FLUID MECHANICS, ACOUSTICS

See also 2452, 2520, 2995, 3028, 3117.

Golovkin, K. K. 3067

On the non-uniqueness of solutions of certain boundary-value problems for the equations of hydromechanics. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 773–775.

Non-linear partial differential equations of the same form as Euler's equations are examined for uniqueness, and it is shown that in certain circumstances the solution is not unique. Detailed consideration is given to the problem of two-dimensional steady viscous flow in an annulus, with prescribed normal velocity and vorticity on the boundaries and a distribution of sources throughout the field. It is shown that two solutions can exist, both having the same constant (non-zero) source strength, the same radial velocity on the boundaries, and zero vorticity on the boundaries. *J. N. Newman* (Washington, D.C.)

Lipovoĭ, G. S. 3068

Electric simulation of flow problems with a break in the stream. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1292–1295.

Author's summary: "The mapping of a region occupied by a stream on a unit semicircle is discussed in the paper. A correspondence is found between the planes z and ζ by means of an electric analogy. An analytical expression is found for the velocity of the stream."

Mei, C. C.; Wu, T. Yao-tsu 3069

Gravity waves due to a point disturbance in a plane free surface flow of stratified fluids.

Phys. Fluids **7** (1964), 1117–1133.

Under certain circumstances, hydraulic jumps exist on a liquid free surface. This phenomenon has also been observed on the interface between two fluids of similar density. It appears likely that this accounts for the so-called dead-water phenomenon frequently encountered by ships moving in a layer of fresh water on top of heavier salt water. In the present paper, the authors attempt to generalize the dead-water problem to the case of fluids with arbitrary density stratification. Solutions are found for the disturbance caused by a point singularity in the steady flow of a stratified fluid. The equations are linearized and a search is made for subcritical and supercritical flows with respect to internal waves. This linear investigation cannot, unfortunately, account for wave resistance due to the breaking of internal waves of finite amplitude. *R. R. Long* (Baltimore, Md.)

Jeffrey, Alan 3070

The breaking of waves on a sloping beach. (German summary)

Z. Angew. Math. Phys. **15** (1964), 97–106.

This paper concerns the breaking of a periodic wave propagating into water at rest over a sloping beach. The nonlinear shallow-water equations as derived by Friedrichs are used to study the breaking of the wave. The method applied is based upon the fact established by Lax that if the initial values possess a Lipschitz discontinuity across the waterfront, the breaking of the wave will take place on the waterfront. However, as the author also notes in the paper, in general, waves may break behind the wavefront. A transformation of the independent variables is introduced, and the critical time at which the wave breaks is determined by setting the Jacobian of the transformation on the wavefront equal to zero. Numerical computations of the critical time and distance for a prescribed wave profile $\bar{y}(0, t) = A \sin \omega t$ over a uniformly sloping beach are given. The method may be applicable to the case of beaches of non-uniform slope. *M. C. Shen* (New York)

Carter, D. S. 3071

Local behavior of plane gravity flows. II. Infinite jets on fixed boundaries.

J. Math. Mech. **13** (1964), 329–351.

In the complex z-plane a steady irrotational flow of an incompressible inviscid fluid under gravity is considered, where the flow boundary contains a free boundary arc Γ_f and an analytic fixed arc Γ_a. In a previous paper [same J. **10** (1961), 441–449; MR **23** #B2278], the author dealt with the local flow behavior near a finite common endpoint of Γ_a and Γ_f. In the present paper, he considers the case in which Γ_a and Γ_f meet at infinity and form the boundaries of a jet of fluid having a source or sink at infinity where the flow velocity becomes infinite. Denote by $z = f(w)$ the

inverse velocity potential function, so that the conjugate velocity is given by $u(w) = 1/f'(w)$. The main purpose is to establish asymptotic expansions of $f(w)$ and $u(w)$. Suppose that the singularity at infinity is a sink and the flow domain is on the left side of Γ_a when Γ_a is oriented toward infinity. Let g be the absolute specific force and α the inclination of gravity measured from the asymptotic direction of Γ_a. Then it is shown that α satisfies $|\alpha| < \pi/2$ and that $f(w)$ and $u(w)$ have infinite asymptotic expansions of the form

$$f(w) = f_0 + \left(\frac{9w^2}{8g\cos\alpha}\right)^{1/3}[1+P(w)],$$

$$u(w) = (3gw\cos\alpha)^{1/3}[1+Q(w)].$$

Here f_0 is a real constant and P, Q are infinite series in powers of $w^{-1/3}$ and $w^{-1/3}\log w$ such that both P and Q are $O(w^{-1}\log w)$. If Γ_a is straight, P and Q are in powers of w^{-1} and $w^{-1}\log w$, and if $\alpha = 0$, the logarithmic terms are absent. *Y. Komatu* (Tokyo)

Yanowitch, M. 3072

Gravity waves in a heterogeneous incompressible fluid.

Comm. Pure Appl. Math. **15** (1962), 45–61.

The author uses the variational principal to show, among other things, the main results that the velocity c of gravity waves in a stratified incompressible fluid decreases with wave number k, and that the velocity of the nth wave cannot exceed $(gh/n)^{1/2}$ if g is the gravitational acceleration and h the depth of the fluid.

That c decreases with k for a continuously stratified incompressible fluid between two horizontal planes was shown by Groen and the reviewer. The same result in the general case, in which density discontinuities (including a free surface) are allowed, as treated in this paper, can also be obtained by Bôcher's extensions of the Sturm-Liouville theory, as presented in a paper entitled "Some results on gravity waves in a stratified fluid", by this reviewer, to the Conference in Boulder, Colorado, on Fluid Dynamics in Geophysics, in 1962. As to the upper bounds, the author shows in a special case that they can be attained only under impossible conditions, and concludes that therefore the bounds are best possible.

{The reviewer is unable to see this reasoning. Consider the wave of the mode with two zeros for the vertical velocity in a continuously stratified fluid with a free surface. If the density gradient is constant and denoted by β, and the surface density by a, the Sturm-Liouville theory furnishes the bound $(g\beta/a)^{1/2}(h/\pi)$, which, if βh is much less than a, is much less than $(gh/2)^{1/2}$. The bounds $(gh/n)^{1/2}$ are therefore not the best possible ones.}

Chia-Shun Yih (Ann Arbor, Mich.)

Förste, J. 3073

Die instationäre Strömung durch ein gerades Rohr mit kreisringförmigem Querschnitt.

Monatsb. Deutsch. Akad. Wiss. Berlin **6** (1964), 241–245.

The instationary, incompressible, viscous flow is computed for the case that the pressure distribution $dp/dx = -f(t)$ is given. The following special cases are investigated: $f(t) = c = \text{const}$; $f(t) = ct^n$ for $0 \leq t \leq 1$ with $n = 1, 2, \cdots$ and $f(t) = c$ for $t > 1$; $f(t) = \sin \omega t$; and finally $f(t) = e^{-a^2t}\sin \omega t$.

I. Flügge-Lotz (Stanford, Calif.)

Harlow, Francis H.; Fromm, Jacob E. 3074

Dynamics and heat transfer in the von Kármán wake of a rectangular cylinder.

Phys. Fluids **7** (1964), 1147–1156.

Authors' summary: "A numerical method has been used to solve completely the full equations for the motion of a viscous, heat-conducting, incompressible fluid flowing through a channel past a rectangular cylinder. The dynamics of the unsteady flow were made visible by means of a number of different types of contour plots, and selected ones are presented and discussed to illustrate various newly discovered features of the von Kármán vortex street. In addition, the heat transfer was measured from the heated rear of the cylinder to the colder fluid, with data presented in the form of Nusselt number correlated with Reynolds number. Corrections are derived for blockage, and Prandtl number scaling is discussed."

Blottner, F. G.; Flügge-Lotz, I. 3075

Finite-difference computation of the boundary layer with displacement thickness interaction.

J. Mécanique **2** (1963), 397–423.

The conventional equations for a laminar compressible boundary layer are, after application of the Howarth-Dorodnitsyn transformation, replaced by a linearized implicit finite-difference scheme, suitable for electronic computation. The utility of the method is verified by continuing the self-similar solution for a flat plate at Mach number 9.6. Strong interaction with the outer inviscid flow is included by iterating for the pressure gradient at each step using the tangent-wedge approximation. Calculations for the flat plate agree with existing theories and experiment.

M. D. Van Dyke (Stanford, Calif.)

Toomre, Alar; Rott, Nicholas 3076

On the pressure induced by the boundary layer on a flat plate in shear flow.

J. Fluid Mech. **19** (1964), 1–10.

The bow wave of a body in supersonic flight generates vorticity that interacts with the boundary layer; as a model of this phenomenon Li [J. Aero. Sci. **22** (1955), 651–652; MR **17**, 99] studied the flat plate in uniform shear. However, a prolonged controversy ensued as to the validity of his pressure gradient along the plate. That issue is clarified here by subjecting the plate to a shear flow that is symmetrical (to avoid displacement effects) and of finite lateral extent, beyond which the stream is uniform. The linearized flow past a boundary layer of parabolic thickness is calculated using the Fourier integral. Li's pressure gradient is found to exist near the leading edge, but to disappear at distances downstream which are large compared with the width of the shear flow. The pressure level even near the leading edge depends strongly upon that width. *M. D. Van Dyke* (Stanford, Calif.)

Szablewski, W. 3077

Ringwirbel im ruhenden Medium innerhalb eines Zylinders.

Monatsb. Deutsch. Akad. Wiss. Berlin **6** (1964), 161–166.

The linearized Navier-Stokes equations are solved for the interior of an infinitely long cylinder with circular cross-section. A disturbance periodic in the direction of the axis of the cylinder produces ring vortices (the plane of the ring is normal to the cylinder axis). Frequency and damping of the produced motion are computed. A figure of the motion reminds the reviewer of similar figures of secondary motions between two circular cylinders (see references to G. I. Taylor's work in Schlichting [*Boundary layer theory*, fourth edition, p. 439, McGraw-Hill, New York, 1960; MR **22** #12948]).

I. Flügge-Lotz (Stanford, Calif.)

Schade, H. 3078

Contribution to the nonlinear stability theory of inviscid shear layers.

Phys. Fluids **7** (1964), 623–628.

The author applies J. T. Stuart's nonlinear stability theory to two-dimensional neutral disturbances of a basic flow with the velocity profile $u_0 = \tanh y$ at infinite Reynolds number, and succeeds in calculating the equilibrium amplitude of this disturbance by calculating the so-called second Landau constant. The paper is interesting and the arguments neat.

Almost all nonlinear stability theories, and certainly Stuart's theory applied by the author, are subject to the following criticism: In these theories the linear theory is first applied to a basic disturbance with wave number α. The higher "harmonics" associated with the wave numbers 0, 2α, 3α, 4α, etc., are then calculated from the nonlinear equations of motion. But in the linear theory other values for the complex wave velocity c are associated with these wave numbers, which are in general different from the c associated with the wave number α. Stuart and many others ignore this, and assume in effect that the higher "harmonics" do not independently exist and are exclusively produced by the basic one. Future work on this subject may well be directed toward meeting this criticism. *Chia-Shun Yih* (Ann Arbor, Mich.)

Corrsin, Stanley 3079

Further generalization of Onsager's cascade model for turbulent spectra.

Phys. Fluids **7** (1964), 1156–1159.

Author's summary: "Onsager's conservative cascade model used to deduce the turbulent energy spectrum in the isotropic 'inertial range' is generalized in three different ways, with examples taken from the turbulent mixing problem. (a) Extension to a nonconservative cascade, mixing with a first-order isothermal chemical reaction, is presented more compactly than an earlier treatment and is applied to a new case. (b) Application to mixing at wavenumbers in the viscous range is indicated briefly. (c) Extension is made to a dual cascade, a model for turbulent mixing with a second-order isothermal reaction. The nonlinearity of the reaction causes spectral transfer."

Betchov, R. 3080

Measure of the intricacy of turbulence.

Phys. Fluids **7** (1964), 1160–1162.

Author's summary: "A quantity is defined which measures the degree of complication of a turbulent flow. It indicates the amount of information necessary per unit volume to describe the turbulence with an accuracy comparable to thermal agitation. It represents a small supplement to the entropy of the fluid. It is also found that maximum intricacy corresponds to a maximum rate of energy dissipation."

Kraichnan, Robert H. 3081

Decay of isotropic turbulence in the direct-interaction approximation.

Phys. Fluids **7** (1964), 1030–1048.

By using the author's direct-interaction stochastic model for the third-order moments, the equations for the decay of isotropic turbulence are integrated numerically with several initial spectrum shapes. Results are presented for the evolution of the energy spectrum, total energy, dissipation, energy transfer spectrum, and skewness factor. Most of the results compare favorably with those obtained experimentally. The computed spectra show similarity in the high-wave-number region. The transfer spectra show, in most cases, a transfer of energy from low to high wave numbers, although there is, in some cases, a small reverse transfer at low wave numbers. The reviewer notes that this small reverse transfer is similar to that shown in his own results with a velocity gradient present and triple correlations neglected [Phys. Fluids **4** (1961), 1187–1198; MR **23** #B2291].

The author notes that direct-interaction approximation is most accurate at low Reynolds numbers. It would be of interest to carry calculations to large times to see if they approach those for the final period of decay.

R. G. Deissler (Cleveland, Ohio)

Smith, P. 3082

Some intrinsic properties of spatial gas flows.

J. Math. Mech. **12** (1963), 27–32.

Author's introduction: "The steady rotational flow of an inviscid, thermally nonconducting gas possesses certain intrinsic properties which are independent of any particular solution. In the present paper it is shown that the normals to the surfaces of constant pressure, entropy, density and velocity magnitude—these quantities being interpreted in the reduced sense given by the substitution principle—are coplanar when drawn at a point in the flow. The normal to this plane lies in the Lamb plane at the point. Furthermore, it is shown that the cross-ratio of this pencil of straight lines through a point is constant along surfaces of constant pressure for a gas having a separable equation of state. In particular, for a perfect gas the cross-ratio is constant throughout the fluid and equal to the ratio of the specific heats. The vector-lines of the normal to the plane of normals to the surfaces of constant pressure, reduced entropy, density and velocity magnitude form a solenoidal field and, in addition, all the flow quantities are constant along these curves. This means that, through the curves of intersection of two surfaces over which two of the flow quantities are respectively constant, we can draw further surfaces over which the other flow quantities are constant. The major exception is the vorticity magnitude which is, in general, not constant along these curves." *H. Özoklav* (Istanbul)

Sun, E. Y. C. 3083
Nicht-angestellte Deltaflügel mit Unterschall- und Schallvorderkanten.
J. Mécanique **3** (1964), 141–163.
This paper is concerned with the steady supersonic flow past a delta wing of small thickness-chord ratio at zero incidence. The method is essentially to expand both the flow variables and space coordinates in terms of three-dimensional characteristic coordinates. Postulating small disturbances, the flow equations are linearized, yielding explicit formulae for the velocity field. The position of the bow shock is obtained as the envelope of characteristic surfaces. The resulting pressure distribution over typical delta wings agrees closely with that derived by acoustic theory, provided the leading edge is subsonic; however, significant departures occur for sonic leading edges, due to the acoustic theory becoming less accurate. It is not indicated whether the theory also applies to supersonic leading edges. *P. Mandl* (Ottawa, Ont.)

Schincke, Erich 3084
Stetige schallnahe Potentialströmungen um eine Familie symmetrischer Profile mit abgerundeter Nase und ihre Grenzlinieneigenschaften.
Ber. Verh. Sächs. Akad. Wiss. Leipzig Math.-Natur. Kl. **105**, *no.* 3, 73 *pp.* (1963).
Using a variation of the techniques of Tomotika and Tamada [Quart. Appl. Math. **9** (1951), 129–147; MR **13**, 179] the author finds numerically some steady transonic irrotational circulationless flows past two-dimensional profiles. He also formulates conditions on the solution in the hodograph plane for obtaining flows past profiles. One calculated profile is cusped aft like a Joukowski profile. There are some theorems on the occurrence of limiting lines and a lengthy review of the subject. Finally, there is a calculated smooth flow for a profile and a perturbation on this profile that is properly chosen.
C. S. Morawetz (New York)

Clarke, Joseph H. 3085
Reverse-flow integral methods for second-order supersonic flow theory.
J. Fluid Mech. **18** (1964), 209–224.
The results of classical linearized supersonic flow theory can in principle be improved by iterating to find a second approximation. Except for plane and axisymmetric flows, however, this attractive idea has been stymied for some years by the analytical complications associated with the non-homogeneous terms in the second-order wave equation. Now this important paper shows that those difficulties can be by-passed using reverse-flow relations. The second-order drag of a quasi-cylinder is expressed simply by surface and volume integrals of two first-order solutions, for forward flow past the body and reverse flow past a strategically related body. Analogous results are given for lift and moment. The second-order surface pressure can likewise be found by inversion of an integral equation; details of application to a wide delta wing are given in the companion paper reviewed below [#3086]. It may be necessary in any specific case to adjust the results for defects arising from non-uniform validity of the formal perturbation scheme. *M. D. Van Dyke* (Stanford, Calif.)

Clarke, Joseph H.; Wallace, James 3086
Uniform second-order solution for supersonic flow over delta wing using reverse-flow integral method.
J. Fluid Mech. **18** (1964), 225–238.
It is the achievement of this paper to present in closed analytic form the uniformly valid second-order solution for surface pressures on a flat-plate triangular wing having supersonic leading edges. This remarkable result, sought in vain by previous investigators, is found using the reverse-flow integrals developed in the paper reviewed above [#3085]. The appropriate reverse flow is the linearized one for a delta wing with an arbitrarily swept straight supersonic leading edge. Reverse-flow methods yield a singular integral equation which, after manipulation, is inverted to yield the second-order pressure. This has a spurious square-root singularity at the Mach cone, which is removed by applying a shifting rule that is a generalization of Lighthill's technique of straining coordinates to achieve uniform validity. Comparison with an earlier numerical calculation demonstrates the improvement over linearized theory. *M. D. Van Dyke* (Stanford, Calif.)

Belocerkovskiĭ, O. M.; Dušin, V. K. 3087
Supersonic flow of a non-equilibrium gas around blunt bodies. (Russian)
Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 61–77.
Belocerkovskiĭ and Čuškin [same Ž. **2** (1962), 731–759; MR **27** #3090] and Belocerkovskiĭ [ibid. **2** (1962), 1062–1085; MR **27** #3091] have previously described in general terms the application of Dorodnitsyn's method of integral relations to the problem of inviscid supersonic flow past a blunt body, and introduced Scheme II with strips normal to the shock wave and body surface. Here complete details of that procedure are given for the second approximation in axisymmetric flow. Application is made to spheres of various sizes at Mach number 10 in dissociating oxygen at two different pressures, and flow quantities are exhibited graphically and in tables.
M. D. Van Dyke (Stanford, Calif.)

Antonov, A. M.; Stecenko, A. G. 3088
Application of the small-parameter method for calculating the flow of a hypersonic gas flow around planar bodies. (Russian)
Approximate methods of solving differential equations, *pp.* 3–10. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, 1963.
The small-disturbance theory of inviscid hypersonic flow is carried to the second approximation for plane motion. The explicit solution is given for the initial departures from the wedge solution near the tip of a sharp-nosed curved airfoil. *M. D. Van Dyke* (Stanford, Calif.)

Hamel, Bernard B. 3089
A model for the transition regime in hypersonic rarefied gasdynamics.
AIAA J. **2** (1964), 1047–1054.
If a solid body (for example, a rocket) moves with a high velocity through a rarefied and cold atmosphere, an observer on the body may think that the body is in a high-speed molecular beam. Molecules reflected at the wall of the body, which are called wall particles, rarely collide with incoming molecules which are called cold particles. The present theory, which considers the interaction between cold particles and wall particles, is valid

for Kn ≧ 1, where Kn denotes the Knudsen number ([mean free path of wall particles]/[dimension of solid body]). The main assumptions are: (i) the distribution of cold particles is represented by an expansion of the delta function and its derivatives, according to Obermann; (ii) the wall particles are distributed according to the Maxwell function with the temperature of the wall which is very low as compared with the energy of the beam — in view of the low temperature, the distribution function may be approximated by the delta function, according to Baker and Charwat; (iii) after the collision between a cold particle and a wall particle, the scattered particles take the same distribution as the cold particles; (iv) neither cold particle-cold particle collisions nor wall particle-wall particle collisions are considered. According to these assumptions, momentum equations are derived and numerically solved for a one-dimension hypersonic compression problem.

{The reviewer's questions are: (a) To make the theory valid for Kn ≧ 1, it seems necessary to consider wall particle-wall particle collisions to some extent; (b) Is assumption (iii) feasible?} *T. Koga* (Farmingdale, N.Y.)

Mirels, H.; Mullen, J. F. 3090

Small perturbation theory for shock-tube attenuation and nonuniformity.

Phys. Fluids **7** (1964), 1208–1218.

Authors' summary: "Closed-form expressions are derived, based on the concept of self-similarity, for evaluating small perturbations of shock tube flow due to the wall boundary layer. Ideal diaphragm rupture is assumed. Extensive numerical results are obtained for the perturbations induced by the boundary layer in the driven gas, assuming that the boundary layer is wholly laminar or wholly turbulent. The driven gas is taken to be air. Various driver gases are considered. Attenuation results are obtained for shock Mach numbers up to 20. It is shown that the effect of the driver gas boundary layer on perturbations in the driven gas is small for large diaphragm pressure ratios. The results indicate that a shock tube with an efficient driver, e.g., helium, should have less attenuation for a given shock Mach number than would an inefficient driver, e.g., air. The latter result disagrees with existing experimental data. The source of the discrepancy is not known."

Chang, Tien Sun; Laporte, Otto 3091

Reflection of strong blast waves.

Phys. Fluids **7** (1964), 1225–1232.

Authors' summary: "The reflection of strong point-source blast waves is studied using the continuum concept of flow of ideal gases. Methods of obtaining the upper and lower bounds, as well as a Taylor series expansion of the position of the reflected shock, are considered. The procedure of studying the flow variables near the reflected shock is also described."

Selezov, I. T. 3092

On the reduction of a non-linear problem of hydro-elasticity to the solution of a system of linear differential equations. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 185–188.

Author's summary: "The author investigates the dynamic interaction of liquid and an elastic shell of arbitrary curvature. The solution of the nonlinear problem of the hydro-elasticity is sought in the form of expansion of a small parameter in the case when the rate of motion of the liquid is less than the velocity of sound in the liquid. As a result the problem is reduced to a series of linear approximations. The first approximation is acoustic while the following ones take into account the nonlinear hydrodynamic effects."

Coleman, Bernard D.; Noll, Walter 3093

Recent results in the continuum theory of viscoelastic fluids.

Ann. New York Acad. Sci. **89**, 672–714 (1961).

This article summarizes results in the mechanics of viscoelastic fluids, emphasizing configurations used in the measurement of material properties. It is written to present material which has mainly appeared in mathematical journals in a form appropriate for polymer physicists. It provides a readable description of the theory of simple fluids, flow solutions (including simple shearing flow), Poiseuille rectilinear pipe flow, Couette flow between rotating cylinders (contrasting the consequences of the simple fluid definition), the Reiner-Rivlin fluids and the Rivlin-Ericksen fluids. A precise approximation theorem for slow motions is given by means of a normed function space and a Fréchet differentiable functional, showing the zero and first order limits of ideal fluid theory and Newtonian viscosity, again contrasting the behavior for the special simple fluids mentioned above. References are given to the complete mathematical developments and proofs. *E. H. Lee* (Stanford, Calif.)

Hashin, Zvi 3094

Bounds for viscosity coefficients of fluid mixtures by variational methods.

Second-order Effects in Elasticity, Plasticity and Fluid Dynamics (Internat. Sympos., Haifa, 1962), pp. 434–449. Jerusalem Academic Press, Jerusalem; Pergamon, Oxford, 1964.

From the author's summary: "A variational principle for steady Stokes flow (neglecting inertia terms) of isotropic non-homogeneous fluids is established. This is applied to the derivation of lower and upper bounds for the effective viscosity coefficients of mixtures of different fluids for the initial Newtonian and the ultimate non-Newtonian stages of flow." *M. N. L. Narasimhan* (Bombay)

Truesdell, C. 3095

The natural time of a viscoelastic fluid: Its significance and measurement.

Phys. Fluids **7** (1964), 1134–1142.

An important concept in non-Newtonian hydrodynamics is that of a viscometric flow, defined as a velocity field having contravariant components $(v_x, v_y, v_z) = (v(y), 0, 0)$ in some appropriate orthogonal coordinate system x, y, z. It has been shown that the dynamics of any such flow can be expressed in terms of three material functions $\mu(\ )$, $\sigma_1(\ )$, $\sigma_2(\ )$. For simple shearing, viz., $v(y) = ky$, these functions are related to the stress system according to

$$t_{xy} = k\mu(k), \quad t_{yy} - t_{zz} = \sigma_1(k), \quad t_{xx} - t_{zz} = \sigma_2(k).$$

In the paper under review, the author proposes for consideration a time-function

$$\tau(k) =_{\text{def}} \{[\sigma_1(k)]^2+[\sigma_2(k)]^2\}^{1/2}/k^2\mu(k),$$

from which a natural time τ can be defined for the fluid according to $\tau =_{\text{def}} \lim_{k\to 0} \tau(k)$. Under this definition, a Newtonian fluid (or any other fluid which fails to exhibit normal stress effects) has a natural time of zero. The fluid also possesses a natural viscosity $\mu =_{\text{def}} \mu(0)$, the "Newtonian viscosity" of the rheologist. The Pi-theorem then leads to a dimensionally invariant description of viscometric flows:

$$t_{xy} = \mu k V(\tau k),$$
$$t_{yy}-t_{zz} = \tau\mu k^2 N_1(\tau k),$$
$$t_{xx}-t_{zz} = \tau\mu k^2 N_2(\tau k),$$

where V, N_1, N_2 are dimensionless functions of their dimensionless arguments.

Under the assumptions that $\mu(k)$ is continuously differentiable at $k=0$ and positive definite, and that $\sigma_1(k)$, $\sigma_2(k)$ are twice continuously differentiable at $k=0$, the author investigates limiting conditions under which the behavior of viscoelastic fluids in viscometric flow is asymptotic to that of a Newtonian fluid. It is well known that this is the case if slower and slower flows are considered for a given fluid. The present paper explores an alternate possibility: holding fixed the flow, the natural viscosity, and dimensionless viscometric functions, but considering smaller and smaller natural times. If one extends this view further, the Newtonian theory of viscometry can, under certain conditions, be applied in the limit of infinitely rapid shearing. The author points out, for example, that this is the case if $k\to\infty$ but $\tau\to 0$ so fast that in the sequence of shearings postulated $\tau k^2\to 0$ (if τk fails to approach zero in this sequence, the stated conclusion entails certain assumptions on the asymptotic behavior of the dimensionless viscometric functions, not explicitly stated by the author but easily discerned in context).

Before leaving the subject of limit processes in which the natural viscosity and dimensionless viscometric functions are held fixed, but τ and k are both permitted to vary, the author calls attention to the important fact that infinitesimally slow shearing does not necessarily lead to the Newtonian limit if the product τk approaches a finite value. The author's statement "... in general, a fluid with arbitrarily large natural time does not obey the Navier-Stokes theory even approximately for slow shearing" should be interpreted in this context, for it refers not to a fixed fluid but to a sequence of fluids with longer and longer natural times.

Special attention is paid to the flow of a "simple fluid" which is characterized by the dimensionally invariant constitutive equation

$$\mathbf{T}+p\mathbf{1} = (\mu/\tau)\, \underset{\sigma=0}{\overset{\infty}{\mathfrak{R}}}\, [\mathbf{G}(\sigma\tau)],$$

where $\mathfrak{R}$ is a dimensionless and isotropic functional and $\mathbf{1}+\mathbf{G}$ is the right Cauchy-Green tensor. Pursuing the main theme of the paper, the author investigates the asymptotics of a sequence of viscometric flow situations for such fields, with μ, k and the functional $\mathfrak{R}$ held fixed, but with τ, the natural time, approaching zero. Mathematically, this is almost the same thing as considering a sequence of retarded flows, but it offers the reader an alternate—in some ways simpler—route to the now-familiar theories of "slow viscoelastic flow".

In order to illuminate further the nature of the natural time, the author concludes with a treatment of the unsteady flow generated in a semi-infinite mass of fluid by sinusoidal oscillation of a plane boundary.

W. Langlois (San Jose, Calif.)

Ferraro, V. C. A. 3096

Generalità sulla magnetofluidodinamica.

Univ. e Politec. Torino Rend. Sem. Mat. **22** (1962/63), 91–107 (1964).

A general and expository article.

Culick, Fred E. C. 3097

Compressible magnetogasdynamic channel flow. (German summary)

Z. Angew. Math. Phys. **15** (1964), 126–143.

The stationary flow of a compressible, electrically conducting fluid in a nonuniform channel, subjected to a nonuniform applied field, may be studied as a quasi-one-dimensional problem with scalar electrical conductivity. The solution may be represented by integral curves in a velocity-Mach number plane. A numerical example is given in which the cross-sectional area and magnetic field vary in such a way that the electrical field strength remains constant.

R. M. Gundersen (Milwaukee, Wis.)

Viskanta, Raymond 3098

Effect of transverse magnetic field on heat transfer to an electrically conducting and thermal radiating fluid flowing in a parallel-plate channel. (German summary)

Z. Angew. Math. Phys. **14** (1963), 353–368.

The author considers an electrically conducting, radiating, viscous, incompressible fluid flowing between two infinite parallel plates under the action of a constant pressure gradient and subjected to an external magnetic field of constant strength in the direction perpendicular to the plates and the direction of flow. This extends the work of the reviewer by including the effect of radiative heat transfer [Nigam and Singh, Quart. J. Mech. Appl. Math. **13** (1960), 85–97; MR **23** #B149]. It is concluded that the effect of magnetic field on heat transfer in an electrically conducting and thermally radiating fluid is to increase heat transfer by conduction as well as radiation; however, this increase is very moderate. The heat transfer by conduction is a larger fraction of the total heat transfer rate for the case of cooling for the same temperature difference between the wall temperature and the ambient temperature.

S. D. Nigam (Madras)

Mallick, D. D. 3099

Instability of a surface of discontinuity of velocity in a parallel uniform magnetic field.

J. Fluid Mech. **16** (1963), 187–196.

The problem described by the title is investigated when the magnetic field is uniform and parallel to the velocity on the two sides of a surface of discontinuity of velocity in an electrically conducting inviscid fluid. The secular equation depends on two parameters β and N, where β

is the ratio of magnetic Reynolds number to dimensionless wave number and N is the ratio of the magnetic to the kinetic energy of the fluid. It is found that the flow is unstable for all values of β and N.

S. D. Nigam (Madras)

D'jačenko, V. F.; Imšennik, V. S. 3100

A converging cylindrical shock wave in a plasma with account taken of the structure of the front. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **3** (1963), 915–926.

The structure of a cylindrical shock wave in a plasma with converging cylindrical adiabatic boundary is investigated numerically under the assumptions that (1) the plasma consists of an electron ideal gas and an ion ideal gas; (2) the thermal conductivity of the ion gas and the viscosity of the electron gas are negligible; (3) other transport coefficients and the coefficient of energy exchange between two species obey simple power-laws with respect to temperatures. A system of difference equations for 200 points on the radius is solved to give five stages of evolution with special reference to the first reflection from the axis. One of the remarkable results is the rapid attainment of uniform temperature by the electron gas in contrast to the ion temperature which yields a sharp maximum in the front near the axis.

H. Hasimoto (Baltimore, Md.)

Cordey, J. G. 3101

Solitary waves in a collision-free plasma with an isotropic pressure.

Phys. Fluids **7** (1964), 778–782.

Longitudinal, non-linear plasma waves can propagate along a magnetic field with a speed related to the Alfvén speed a and sound speed C_s ahead of the wave. Using a two-fluid model and assuming the plasma pressure is isotropic and follows an adiabatic equation of state, the author finds two distinct classes of solitary wave type solutions. The ranges of possible speeds of the two classes are differently related to the ratio C_s/a. The two classes are further distinguished by the profile of the perturbed component of transverse magnetic field. The effect of isotropic plasma pressure is to shorten the width of that wave class corresponding to the zero pressure wave.

J. D. Jukes (Culham)

Sen, Amiya K. 3102

Stability of hydromagnetic Kelvin-Helmholtz discontinuity.

Phys. Fluids **6** (1963), 1154–1163.

The most general form of hydromagnetic Kelvin-Helmholtz discontinuity in ideal incompressible fluids has been considered. The presence of surface tension on the surface of discontinuity and gravity is also assumed. The stability of the hydromagnetic flow of two uniform infinite streams with planar interface exhibiting this type of discontinuity has been analyzed in detail. Two necessary and sufficient criteria for its stability have been obtained, in contrast to only one that is usually given. The stabilizing effect of magnetic field is strongly dependent on its discontinuity at the interface and its orientation relative to that of streaming. The consideration of the stability to a single mode of perturbation leads to the conclusion that this is dependent on the above-mentioned factors, as well as the following ones. The effect of surface tension and gravity is dependent on the wavelength, while that of the magnetic field is dependent on the direction of propagation of perturbation.

S. D. Nigam (Madras)

Sen, Amiya K. 3103

Effect of compressibility on Kelvin-Helmholtz instability in a plasma.

Phys. Fluids **7** (1964), 1293–1298.

Author's summary: "The problem of stability of the most general type of Kelvin-Helmholtz discontinuity in a compressible plasma has been formulated. It is found that unlike surface tension, gravity and magnetic field, compressibility can have either a stabilizing or destabilizing effect, depending on the stationary state of motion. In general, this may lead to two domains of stability, in contrast to only one that exists in the case of incompressible plasma. Compressibility will have a stabilizing or destabilizing effect if the streaming speed is sufficiently larger or smaller, respectively, than the magnetoacoustic speed."

Stewartson, K.; Wilson, D. H. 3104

Dual solutions of the Greenspan-Carrier equations. II.

J. Fluid Mech. **18** (1964), 337–349.

In the study of flow of a highly conducting, almost inviscid, incompressible fluid past a thin flat plate in presence of magnetic field (which is uniform at infinity and parallel to the stream), Carrier and Greenspan in an earlier paper [same J. **6** (1959), 77–96; MR **21** #6907] showed that the velocity and magnetic fields in the boundary layer on either side of the thin plate depend on the solution of the pair of coupled ordinary differential equations

$$(1)\qquad \begin{aligned} f''' + ff'' - \beta gg'' &= 0,\\ g'' + 4\pi\sigma\nu(fg' - gf') &= 0, \end{aligned}$$

subject to the boundary conditions

$$(2)\quad f(0) = f'(0) = g(0) = 0, \qquad f'(\infty) = g'(\infty) = 2,$$

where $u = \frac{1}{2}Uf'(\eta)$, $H_x = \frac{1}{2}H_0 g'(\eta)$ and $\eta = \frac{1}{2}y(U/\nu x)^{1/2}$, U being the undisturbed flow-velocity, H_0 the strength of undisturbed magnetic field, ν the kinematic viscosity, σ the conductivity of the fluid of permeability unity. Here y measures the distance from the plate, x the distance along the plate from the leading edge, while u and H_x give the x-components of the velocity and magnetic fields, respectively.

The solutions of (1) subject to the above boundary conditions (2) have been studied by Carrier and Greenspan [loc. cit.], Reuter and Stewartson [Phys. Fluids **4** (1961), 276–277] and Meksyn [J. Aerospace Sci. **29** (1962), 662–665; MR **25** #1758], and it has been shown that there are no solutions for all $\varepsilon\,(\equiv 4\pi\sigma\nu) > 0$ if $\beta\,(\equiv \frac{1}{8}\pi^{-1}H_0^2/\frac{1}{2}\rho U^2) > 1$, ρ being the density of the fluid, while Glauert [J. Fluid Mech. **10** (1961), 276–288; MR **22** #10505] has examined these solutions when $\varepsilon \gg 1$ and $\varepsilon \ll 1$. In Part I of the present series Wilson [ibid. **18** (1964), 161–166; MR **28** #2738] has shown numerically that if $0 < \varepsilon < 1$, the solution of the problem is not unique for a range of values of β. In the present paper the authors give some analytic arguments in support of this work. Further, the authors solve numerically the equations (1) which are also associated with flow past a solid boundary in which the magnetic

field is zero and where the boundary conditions (2) are modified as $f(0)=f'(0)=g'(0)=0$, $f'(\infty)=g'(\infty)=2$.
K. M. Ghosh (Belgharia)

Szabó, J. 3105
Magnetohydrodynamic shock waves. (Russian summary)
Acta Phys. Acad. Sci. Hungar. **17** (1964), 75–89.
Author's summary: "Starting from the fundamental equations of magnetohydrodynamics we derive the conservation laws for the case of an ideal plasma. Based on these laws the fundamental equations—that govern the behaviour of magnetohydrodynamic (mhd) shock waves along an arbitrary continuous surface of discontinuity—are obtained. It is shown that the mhd shock waves—similar to those of gas dynamics—are compression waves. Finally the properties of different kinds of mhd shock waves are discussed."

Terrill, R. M. 3106
Laminar flow in a uniformly porous channel.
Aero. Quart. **15** (1964), 299–310.
Author's summary: "The flow in a two-dimensional channel with porous walls through which fluid is uniformly injected or extracted is considered. Following Berman [J. Appl. Physics **24** (1953), 1232–1235; MR **15**, 573], a solution is obtained giving a fourth-order non-linear differential equation which depends on a suction Reynolds number R. Numerical solutions of this equation have been obtained. Series solutions of this equation for small and large Reynolds number are given and are shown to give good agreement with the numerical solutions."

Wooding, R. A. 3107
Mixing-layer flows in a saturated porous medium.
J. Fluid Mech. **19** (1964), 103–112. (1 *plate*)
From the author's summary: "The steady two-dimensional seepage flow by gravitational convection of a fluid of density ρ_1 surrounded by a fluid of density ρ_0 ($\neq \rho_1$) at rest leads to a potential problem from which the shape of the interface can be determined. When the two fluids are slightly miscible, the interface is replaced by a mixing layer and it is shown that the first-order Prandtl equations for the flow in the layer possess an exact similarity solution. The profile across the layer is of the same form as the profile of the laminar incompressible half-jet with one fluid at rest, and a formula is obtained for the scale of the mixing layer thickness as a function of distance downstream. Three examples are discussed."

Good agreement with corresponding experiments is obtained. *H. Brenner* (New York)

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 2519, 3209.

Vassails, Gérard 3108
Sur la conjugaison en optique géométrique.
Cahiers de Phys. **17** (1962/63), 500–508.
A method of studying the theory of conjugation in geometrical optics is proposed. It involves a convention which appears to simplify the treatment of virtual images and also to facilitate the study of combinations of systems. Reflecting systems are included by using the concept of negative refractive indices.
E. W. Marchand (Rochester, N.Y.)

Rainwater, James 3109
Generalization of the Abbe sine law in geometric optics.
Amer. J. Phys. **32** (1964), 626–631.
In the case of an axially symmetrical optical system it is known that the Herschel condition and the Abbe sine condition are special cases of the cosine law of T. Smith. The present paper derives some more general conditions related to the cosine law, formulae which are not strictly new but which are not well known by optical workers.
E. W. Marchand (Rochester, N.Y.)

Barakat, Richard 3110
Application of the sampling theorem to optical diffraction theory.
J. Opt. Soc. Amer. **54** (1964), 920–930.
The sampling theorem from information theory is applied to optical diffraction theory as a computational tool. Formulas involving sampled values are derived for spread functions, transfer functions, total illuminance and cumulative line spread functions. A number of special cases of optical importance are considered.

For the case of a slit aperture, the question is considered: Given the real part of the incoherent transfer function, find its imaginary point, and vice versa. The solution is based on conjugate Fourier series and finite Hilbert transforms. *E. W. Marchand* (Rochester, N.Y.)

Wolf, E.; Marchand, E. W. 3111
Comparison of the Kirchhoff and the Rayleigh-Sommerfeld theories of diffraction at an aperture.
J. Opt. Soc. Amer. **54** (1964), 587–594.
Authors' summary: "With a view to elucidating the effect of a well-known mathematical inconsistency in Kirchhoff's diffraction theory, a comparison is made of the predictions relating to the field diffracted at an aperture, based on Kirchhoff's theory (U_K) and on formulas due to Rayleigh and Sommerfeld (U_R). It is shown that, when the incident wave is plane or spherical, the difference $\delta = U_K - U_R$ represents a boundary wave, i.e., a wave which may be thought of as originating at each point of the edge of the aperture. It is shown further that, when the linear dimensions of the aperture are large compared with the wavelength, the boundary values of δ in the plane of the aperture change very rapidly and almost periodically from point to point, with the mean period close to the wavelength of the incident radiation. This result is shown to imply that if the linear dimensions of the aperture are large compared with the wavelength, the two theories predict essentially the same behavior for the diffracted field in the far zone, at moderate angles of diffraction." *W. Franzen* (Boston, Mass.)

Marchand, E. W. 3112
Derivation of the point spread function from the line spread function.
J. Opt. Soc. Amer. **54** (1964), 915–919.

Author's summary: "In dealing with optical imaging systems, it is more feasible experimentally to measure line spread functions than point spread functions. When the intensity distribution is known to possess rotational symmetry, the point spread function can be obtained mathematically from the corresponding line spread function by solving an integral equation. A direct solution of this equation is given which represents a procedure that is simpler for practical use than the usual one involving Fourier transforms."

W. Franzen (Boston, Mass.)

Rädler, K.-H. 3113

Elektromagnetische Induktion in rotierenden Medien.

Monatsb. Deutsch. Akad. Wiss. Berlin **6** (1964), 5–8.

The magneto-hydrodynamic approximation to the field and current distribution is formulated for an isotropic medium which is rotating about one common and fixed axis. The angular momentum as well as the conductivity are of spherical symmetry but are arbitrary functions of distance from the point of symmetry on the axis of rotation and of time. The permeability is a constant while the permittivity is an arbitrary function of space and time. In addition, an arbitrary distribution of impressed time-varying electric currents is admitted.

In essence via auxiliary functions the impressed current is expanded in terms of tesseral harmonics. The coefficients of this expansion are functions of radius and time. An analogous expansion is written for two auxiliary functions to represent the magnetic induction. Independent and linear partial differential equations are found for each of the coefficients of this expansion. For stationary problems they reduce to ordinary differential equations of second order. These equations may in principle be integrated for specified impressed currents and boundary and initial conditions. For some special cases an exact integration is possible. *H. G. Unger* (Braunschweig)

Rädler, K.-H. 3114

Rotation einer elektrisch leitenden Kugel in leitender Umgebung unter dem Einfluss eines magnetischen Potentialfeldes.

Monatsb. Deutsch. Akad. Wiss. Berlin **6** (1964), 8–10.

The magnetic induction at an electrically conducting sphere rotating uniformly in an electrically conducting medium at rest is calculated by a method which for a more general problem has been developed elsewhere [#3113]. The sphere and the outside medium are each homogeneous, and an arbitrary magnetic potential field is applied. By representing the applied potential field at infinity, as well as the total magnetic induction, in terms of tesseral harmonics, second-order differential equations for the coefficients of the total field presentation are found, and the boundary conditions are formulated in terms of the coefficients of the applied field representation. Integration leads to explicit expressions for the total field functions. *H. G. Unger* (Braunschweig)

Lebedev, D. S.; Levitin, L. B. 3115

The maximum amount of information transmissible by an electromagnetic field.

Dokl. Akad. Nauk SSSR **149** (1963), 1299–1302 (*Russian*); *translated as Soviet Physics Dokl.* **8** (1963), 377–379.

The authors compute the maximum rate at which information may be transmitted by an electromagnetic signal of given power embedded in an (additive) noise field of specified temperature. The result is shown to be in agreement with expressions previously obtained by other workers for various specific signal and noise conditions.

The result is obtained by computing the "entropy defect", or the difference between the maximum entropy corresponding to the total energy (signal plus noise) present in the receiver and the actual entropy due to noise alone (i.e., assuming that the signal is ideally transmitted without contributing to an increase in entropy). The authors also indicate how the maximum information rate may be obtained from probability considerations rather than by a thermodynamic argument.

A. Ksienski (Culver City, Calif.)

Nakagami, Minoru 3116

On the intensity distribution

$$\frac{2R}{\sqrt{(\alpha\beta)}}\exp\left[-\frac{R^2}{2}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\right]I_0\left(\frac{R^2}{2}\left[\frac{1}{\beta}-\frac{1}{\alpha}\right]\right)$$

and its application to signal statistics.

J. Res. Nat. Bur. Standards Sect. D **68D** (1964), 995–1003.

Author's summary: "A few derivation methods of the distribution in the title are discussed from some different viewpoints. It is found that this type of fading may be caused by the interferences of two correlated waves. A transitional aspect of the distribution due to the change of a parameter is illustrated in detail. Some applications of the distribution to signal statistics are discussed. The functional similarity is found between the characteristic function and the error probability of signals. Using this basic relation some error probabilities are estimated for various cases."

Cade, R. 3117

Hydrostatic electrostriction and the Clausius-Mossotti law. (Italian summary)

Nuovo Cimento (10) **32** (1964), 895–904.

The author assumes, following Brown [Amer. J. Phys. **19** (1951), 290–304], that in a polarized fluid the stress components $T_{\mu\nu}$ ($\mu=1, 2, 3$) do not form a tensor, though the trace of the matrix $T_{\nu\nu}=-\frac{1}{3}q$ is a scalar. Furthermore, he stipulates that the equation of state and the polarization P_ν be given by the relations $q=q(\rho, E)$, $P_\nu=\chi(\rho, E)E_\nu$, ρ being the density, E_ν the electric field and χ the susceptibility. Using these assumptions in the mechanical equilibrium conditions for a fluid, the author finds that the integrability conditions for this equation lead to the Clausius-Mossotti formula as a relation between the dielectric constant and density, provided the dielectric is linear, $\chi=\chi(\rho)$, and the equation of state field-independent, $q=q(\rho)$. *N. L. Balazs* (Stony Brook, N.Y.)

Zucker, Francis J. 3118

Current topics in the stochastic theory of radiation.

J. Res. Nat. Bur. Standards Sect. D **68D** (1964), 989–993.

Author's summary: "The stochastic properties of fluctuating electromagnetic fields are defined in terms of the joint moments of the probability distribution. Their

physical interpretations (coherence, higher order correlations) are briefly described, and the connection is indicated between the complete set of correlations and the quantum theory of radiation."

Bremmer, H. 3119

Random volume scattering.

J. Res. Nat. Bur. Standards Sect. D **68D** (1964), 967–981.

Author's summary: "The disturbances produced by a slightly inhomogeneous random medium on a passing wave can be classified into contributions depending on an increasing number of successive scatterings. The individual contributions appear in an expansion of the solution of an integral equation. The first term, the Born approximation, only accounts for a single scattering. Convenient expressions for this approximation result from a saddle point treatment for short distances, and from a Fraunhofer approximation for larger distances. The evaluation of the higher-order contributions, describing plural-scattering effects, leads to mathematical difficulties which are evaded by considering the scattering mechanism as a Markovian process. The corresponding theory can be developed with the aid of an integro-differential diffusion equation; the latter refers to the joint probability density of the lateral and angular deviations suffered by the trajectory of the passing wave. The equation in question can be solved with the aid of four-dimensional operational calculus; it reduces to the simple differential equation of Fokker-Planck under special conditions. The application of the general equation to tropospheric point-to-point radio communication is worked out. It is shown that the far-distance field, associated in this case with multiple scattering, does decrease proportionally to the second or third power of the inverse distance."

Millar, R. F. 3120

Plane wave spectra in grating theory. III. Scattering by a semiinfinite grating of identical cylinders.

Canad. J. Phys. **42** (1964), 1149–1184.

Part II appeared in same J. **41** (1963), 2135–2154. This paper examines the problem of the diffraction of a time harmonic plane wave by a semi-infinite grating of identical cylinders. The scattered field is expressed as an angular spectrum of plane waves with an unknown scattering amplitude function. By decomposing the grating into two bodies (namely, the first N elements and the remaining infinity of elements), integral equations are obtained for the unknown amplitude function. These integral equations are not of a type which can be solved explicitly, but the author shows that the singularities of this function in the complex planes of the angles of incidence and observation may be located and classified. This information is then employed to determine the general form of the field far from the end element of the grating.

W. E. Williams (Liverpool)

Radlow, James 3121

Diffraction by a right-angled dielectric wedge. (French, **German, Italian, and Russian summaries)**

Internat. J. Engrg. Sci. **2** (1964), 275–290.

Author's summary: "The problem considered is that of determining the field of a plane wave of transverse magnetic type in the presence of a right-angled dielectric wedge. An explicit solution is derived and shown to be unique. Field behavior at the corner of the wedge is discussed. The analysis, an extension and refinement of earlier work of the author, is in essence a generalization of the function-theoretical method of Wiener and Hopf from one to two complex variables."

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See also 2460, 2539, 2540, 2546, 2547, 3074, 3098, 3192.

Krastev, R. 3122

Sur une relation entre la "masse de la chaleur", le "centre de gravité de la chaleur" et le "moment d'inertie de la chaleur" dans un cône tronqué.

C. R. Acad. Bulgare Sci. **17** (1964), 17–18.

Sidljar, M. M. 3123

An application of the variational method to the solution of non-stationary problems of heat conduction. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 846–849.

Author's summary: "An approximate method for solving the problem of a nonstationary temperature field is given for certain bodies under the assumption that the coefficient of relative heat loss is a function of the time. The investigation is based on the variational method. The solution of the problem is reduced to a consideration of certain differential equations. For the case of sufficiently small perturbations of this coefficient we obtain simpler systems while the temperature is determined by a certain series."

Boley, Bruno A. 3124

Upper and lower bounds in problems of melting or solidifying slabs.

Quart. J. Mech. Appl. Math. **17** (1964), 253–269.

From the author's summary: "The problem studied is that of a slab, heated in an arbitrary manner on one face and insulated on the other, which melts (or solidifies), the material being allowed to remain stationary after change of phase. Variable material properties are taken into account. After preliminary general considerations, it is shown that the solution to the stated problem is unique. It is then proved that higher rates of melting and higher temperatures will result from certain combinations of the magnitude of the applied heat input and of a fictitious heat source travelling with the solid-liquid interface. From this result a method is developed for the construction of upper and lower bounds to the solution of the problem; an example is also presented. It is also shown that, under the same arbitrary heat input, the rate of melting in the present problem is always lower than that in the companion problem in which the material is instantaneously removed after change of phase."

S. J. Citron (Lafayette, Ind.)

QUANTUM MECHANICS

See also 2146, 2149, 2328, 2527, 2653, 2682, 3206, 3214, 3231.

Costa de Beauregard, O. 3125

Translational inertial spin effect with moving particles.

Phys. Rev. (2) **134** (1964), B471–B475.

Essentially the point the author is making in this paper and a previous one on the same topic [same Rev. (2) **129** (1963), 466–471; MR **26** #7391] is that the momentum and velocity of a Dirac particle are different, and this difference should have experimental consequences. This argument is presented in the language of energy momentum tensors with the tetrode asymmetrical tensor T^{ij} corresponding, through the energy momentum vector $P^i = \int T^{ij}\,du_j$, to the momentum, and the so-called longitudinal tensor T^{ji}, through $L^i = \int T^{ji}\,du_j$, to the velocity. Differences between T^{ij} and T^{ji} can only occur through boundary effects as they differ by a divergence term, and in the paper cited above, the author thought he had proposed an experiment which would do this. However, in this paper he acknowledges that the proposed effect would not be observed because the hoped for differences are integrated out. Now he proposes an experiment which is a local rather than an integrated one, and consists of a beam of Dirac particles having momentum components only in the (x, y)-plane, which is bent by an inhomogeneous magnetic field with only a z-component. Surprisingly, the transverse momentum $P^i - L^i$ has a z-component which depends on the sign of the helicity of the particles. The splitting, if observed, would, he claims, confirm the existence of this "inertial spin effect".

C. A. Hurst (Adelaide)

Costa de Beauregard, Olivier 3126

L'effet inertial de spin du photon.

C. R. Acad. Sci. Paris **258** (1964), 6088–6090.

The considerations employed in the preceding paper [#3125 above] have been extended to the case of the de Broglie photon, i.e., a neutral vector meson with small mass. Once again, an inertial spin effect is predicted, although it depends essentially on whether complex wave functions are employed or not, and so whether the classical Poynting vector is used or not. *C. A. Hurst* (Adelaide)

Urbanik, K. 3127

The principle of increase of entropy in quantum mechanics.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 743–764. *Publ. House Czech. Acad. Sci., Prague*, 1964.

This is the continuation of a previous paper by R. S. Ingarden and the author [Acta Phys. Polon. **21** (1962), 281–304; MR **26** #3471] in which the notion of "A-thermodynamics" was introduced, A being any observable (the usual thermodynamics corresponds to $A = H$, the Hamiltonian). A remarkable feature of this generalized thermodynamics is its applicability to quantum systems composed of a single particle. An important notion is a "macrostate with respect to A", which is defined as the set of all states for which A has a given expectation value.

The purpose of the present paper is to discuss the time behavior of the A-entropy $S_A(\Phi)$ of some macrostate Φ. The following theorems are proved. (1) All macrostates with respect to an operator A are invariant under the motion if and only if the operator A commutes with the Hamiltonian of the physical system. (2) Let Φ_0 be a macrostate for which the expectation value of A is neither the least nor the largest proper value of A. If for every t, $\phi(t)$ and $\psi(t)$ belong to the same macrostate of A whenever $\phi(0)$ and $\psi(0)$ belong to Φ_0, then all macrostates with respect to A are invariant under the motion. (3) If A is a spin component operator, then for any motion of the system and for any macrostate Φ the inequality

$$S_A^t(\Phi) \geqq S_A(\Phi) \quad (t \geqq 0)$$

holds. (4) For any operator A, any motion of the physical system and any macrostate Φ we have the inequality

$$S_A^\infty(\Phi) \equiv \limsup_{t\to\infty} S_A^t(\Phi) \geqq S_A(\Phi).$$

(5) The equality $S_A^\infty(\Phi) = S_A(\Phi)$ holds for all macrostates Φ if and only if the operator A commutes with the Hamiltonian. (6) Let A be an operator with a simple spectrum. If all stationary microstates, i.e., proper microstates of the Hamiltonian, are equivalent with respect to A, then for any macrostate Φ we have the equality $S_A^\infty(\Phi) = \log N$, where N is the number of proper values of A and $N < \infty$. (7) Let A be a spin component operator. If the direction of the external magnetic field is orthogonal to the direction of the spin component A, then for all macrostates Φ we have the equality $S_A^\infty(\Phi) = \log(2s+1)$, where s is the largest proper value of A.

A. Peres (Haifa)

Goldberger, Marvin L.; Watson, Kenneth M. 3128

★Collision theory.

John Wiley & Sons, Inc., New York-London-Sydney, 1964. ix + 919 *pp.* $19.95.

The reviewer has just read this book. This is a serious remark, because the book contains 912 pages of closely reasoned text with 3747 numbered equations, and under the immediate impact of such a traumatic experience one's judgment may be significantly distorted. The experience is traumatic because of the combination of one of the book's most characteristic virtues with its only evident defect. The virtue is the authors' manifest relish in uncompromisingly confronting Nature as she really is, that is to say, complete with wavepackets, spins, charges, Coulomb effects, and multichannel reactions; the defect is that not enough attention seems to have been paid to the problem of controlling and rationalizing the resultant proliferation of labels, indices, superfixes, suffixes, and primes, and to make it visually easy for the reader to absorb the contents of the equations from the printed page. In a book of such length this is a serious defect, especially when it is combined, as it is here, with the absence of even a modest list of notations.

It is difficult to convey a correct impression of the nature and scope of this book. The authors declare that they aim not at mathematical rigour but "at mathematical reasonableness and physical rigour". What is most original in their work is that they achieve this aim. For instance, it is a tremendous experience to read their chapter entitled "Formal scattering theory" (pp. 173–220)

after encountering the two preceding chapters (pp. 58–172) which give the most careful and lucid discussion known to the reviewer of scattering processes in terms of wavepackets, so that the elegant manipulations of the formal scattering theory can be, and are, justified at every stage by a brief reference to an adequate physical argument already encountered.

After this generally introductory part (which also contains a fairly brief survey of quantum mechanics and of symmetry operations), there comes a long chapter on "The two-body problem with central forces" (pp. 221–343) which treats also the analytic properties of partial wave amplitudes. Then come "Scattering by noncentral forces" (pp. 344–423), and "The lifetime and decay of virtual states" (pp. 424–519), which is a veritable tour-de-force in its own right, being the most coherent and physically satisfying treatment anywhere of unstable states. After a chapter on "Miscellaneous applications" (chiefly to pion photoproduction and to final state interactions) there follow two final chapters which are really fair-sized monographs in their own right: "Introduction to dispersion theory" (pp. 557–678), essentially a reprint of the first author's famous lecture notes at Princeton in 1959/60 and at Les Houches Summer School in 1960 (also available in Relations de Dispersion et Particules Élémentaires (Grenoble, 1960), pp. 15–157, Hermann, Paris, 1960; MR **22** #11869], and "Scattering by systems of bound particles" (pp. 679–866), which ranges, for example, from deuteron stripping to the refraction of light.

In conclusion, this is, and is likely to remain, a unique book, containing the most definitive and the most exhaustive statement of scattering theory. Perhaps one should emphasize that, insofar as this is possible, it is even more illuminating, physically, than any book by its authors would automatically be expected to be. Apart from Feynman diagrams it contains everything up to, but not including, the currently beginning attack on the three-body problem. Because of its price, it may be almost superfluous to voice the reviewer's fear that, taken neat, it may prove a dangerous book for beginning graduate students who, mistaking it for a textbook, may be intimidated rather than encouraged by such an unsoftened glimpse of the real world. *G. Barton* (Brighton)

Epstein, Saul T. 3129

Hartree-Fock Hamiltonians and separable nonlocal potentials.

J. Chem. Phys. **41** (1964), 1045–1046.

Author's summary: "It is pointed out that there are many Hamiltonians which have the Hartree-Fock wavefunction as an eigenfunction. These can be used to start perturbation calculations, different choices giving different results. The possible usefulness of separable, nonlocal potentials is outlined."

Datzeff, Assène [Dacev, Asen B.] 3130

Sur une généralisation de l'équation de Schrödinger en cas de mouvement non lié des corpuscules.

C. R. Acad. Sci. Paris **258** (1964), 6354–6357.

The author observes that the Schrödinger equation, for a single electron in a given external potential, cannot be applied to a dense beam of electrons without allowing for the extra potential energy of interaction between the particles, and proposes a method equivalent to Hartree's self-consistent fields without consideration of exchange corrections or self-energy effects. *K. J. Le Couteur* (Canberra)

Lonsky, E. S. [Lonskiĭ, E. S.]; Shirokov, Yu. M. [Širokov, Ju. M.] 3131

New types of connections between local operators and the scattering matrix.

Nuclear Phys. **55** (1964), 293–299.

Given a Heisenberg interacting field $A(x)$, the "dynamical momenta" $D_{i_1\cdots i_n}(t)$ $(i_k=1,2,3;\ k=1,2,\cdots,n)$ are defined as certain spatial moments of multiple time derivatives of $A(x)$. Due to the assumed locality, their asymptotic values at $t=\pm\infty$ are connected by the S-operator:

$$(1)\qquad D_{i_1\cdots i_n}(-\infty)S = SD_{i_1\cdots i_n}(+\infty).$$

The authors show how recursion relations for the partial wave components of the elastic scattering matrix elements of S can be obtained from (1) in terms of the assumed known matrix elements of $A(x)$ between two particle states. A certain class of singularities of the dynamical momenta is assumed.

The claim that the scattering matrix can be entirely reconstructed (up to a constant phase factor) from a knowledge of the $\langle pk|A(x)|p'k'\rangle$ seems too strong. At the general energy, where many inelastic channels are open, one would of course need to know all the matrix elements of $A(x)$. Moreover, be it remarked that the simplicity of this method, which yields simple recursion relations for the two particle matrix elements of S only at sufficiently low energies, is lost at the general energy. *R. Ingraham* (University Park, N.M.)

Calogero, F.; Ravenhall, D. G. 3132

A generalization of the phase approach to scattering theory, and some numerical results. (Italian summary)

Nuovo Cimento (10) **32** (1964), 1755–1771.

The object of this paper is to calculate scattering phase shifts by a new method. Instead of the usual integration of the radial wave equation from $r=0$ to the asymptotic region, the new approach is to introduce a function representing the phase accumulation from $r=0$, satisfying a first-order non-linear differential equation.

Generally, two simultaneous equations

$$y_1' = A(x)y_1-[1-B(x)]y_2,\qquad y_2' = [1-C(x)]y_1-A(x)y_2$$

are considered in conjunction with the known solutions of a comparison system

$$\bar y_1' = \bar A(x)\bar y_1-[1-\bar B(x)]\bar y_2,\qquad \bar y_2' = [1-\bar C(x)]\bar y_1-\bar A(x)\bar y_2.$$

The phase shift of the former is related directly to that of the latter by means of a function δ, where δ and $t=\tan\delta$ satisfy non-linear first-order equations.

In paragraph 3, the radial Schrödinger equation is discussed, functions $A, B, C, \bar A, \bar B, \bar C$ being suitably selected. In paragraph 4, the radial Dirac equation is considered, suitable choices of these six coefficients again being made. In this case, the resulting simple forms in the extreme relativistic limit are also discussed.

Finally, numerical comparisons are made between this new method and the results obtained from the direct integration of the radial equations. In the case of the Schrödinger equation, an exponential potential is used, while in the case of the Dirac radial equation a constant square-well potential in the relativistic limit is adopted.
J. Heading (Southampton)

Calogero, F.; Charap, J. M. 3133
Asymptotic behaviour for large imaginary values of angular momentum of the amplitudes for the scattering of a Dirac particle on a central scalar potential.
Nuovo Cimento (10) **32** (1964), 1665–1684.
The authors consider a Dirac particle scattering on a suitably restricted central scalar potential, namely, a superposition of Yukawa functions (equation (3.1)); their object is to show that the partial-wave amplitudes behave as $O(J^{-1/2})$ as $|\operatorname{Im} J|\to\infty$ and $0<\operatorname{Re} J<1$, where J denotes the total angular momentum.

Paragraph 2 reviews the standard theory of scattering by a central potential of a Dirac particle, the appropriate simultaneous equations being written in matrix form (equation (2.11)). In paragraph 4, these differential equations are cast into integral form (equation (4.6)), use being made of various auxiliary functions of the Riccati-Bessel type. These integral equations are then remoulded in terms of the dependent variables denoted by $b_{J\pm}(r)$ and $s_{J\pm}(r)$, in terms of which the scattering amplitudes are given by equation (4.16).

The integral equations are solved in paragraph 5 in series form, recurrence relations being derived for the various terms in the two series; these terms are susceptible to solution, being given explicitly in the form of multiple integrals (equation (5.30)). Finally, in paragraph 6, an interesting series of integrations reduces complication to simplicity, the desired result thereby being obtained. *J. Heading* (Southampton)

Charap, J. M.; Dombey, N. 3134
Singular logarithmic potentials in $g\varphi^4$ theory.
Phys. Lett. **9** (1964), 210–212.
It has recently been shown [A. T. Filippov, Dubna preprint R-1493] that a relativistic field theory with $g\varphi^4$ coupling leads to a quasi-potential $V(r)$ for the radial part of the relativistic one-particle equation [Filippov, Phys. Lett. **9** (1964), 78–80; MR **29** #943] which behaves like $r^{-2}\log r$ when $r\to 0$. The result is obtained using only the contributions from the lowest-order diagrams.

In this note it is shown that if a certain class of higher-order diagrams is taken into account, the potential acquires terms which behave, when $r\to 0$, like $r^{-2}(\log r)^{-n}$. The solution of the Schrödinger equation with a potential $V(r)=r^{-2}(\log r)^{-n}$ is given for $n=1$ and $n=2$. If $n=1$, the solution as a function of the angular momentum has an essential singularity. If $n=2$ instead, the solution is a regular function of λ. *G. F. Dell'Antonio* (Naples)

MacGregor, Malcolm H. 3135
Unitarity corrections to Born amplitudes for nucleon-nucleon scattering.
Phys. Rev. Lett. **12** (1964), 403–406.
The dispersion relation for the amplitude $h_l(T)$, where T is the laboratory kinetic energy, in the Mandelstam representation is given by

$$\operatorname{Re} h_l(T)=\frac{P}{\pi}\int_{-\infty}^{-L}\frac{\operatorname{Im} h_l(T')}{T'-T}\,dT'+\frac{P}{\pi}\int_0^{\infty}\frac{\operatorname{Im} h_l(T')}{T'-T}\,dT'.$$

In the pole-model approximation, the left-hand integral is represented by a sum of Born terms, $B_l(T)$. At threshold, $\operatorname{Re} h_l(T)\propto T^l$. But none of the two integrals have this behaviour and therefore the threshold requirement must be satisfied by a left-right cancellation. However, in the pole approximation of the left-hand integral, since the behaviour at threshold is T^l, the necessary left-right cancellation is destroyed. In this paper it is suggested that one express the left-hand integral by the sum $B_l(T)+F_l(T)$, where $F_l(T)$ represents the remaining part of the left-hand integral and corresponds to the sum of the distant resonances not included in $B_l(T)$ and of continuum (non-resonant) contributions. The satisfying of the threshold condition leads to l additional equations. Calculations have been carried out for the ${}^1P_1$, ${}^1D_2$, and ${}^1F_3$ nucleon-nucleon phase shifts. The important conclusion drawn is that the correction arising from the $F_l(T)$ integral is larger than the right-hand integral, so that approximation schemes using only the right-hand integral as correction to Born terms would give misleading results.
B. P. Nigam (Tempe, Ariz.)

Delbourgo, R. 3136
Renormalization of a CP-invariant theory. (Italian summary)
Nuovo Cimento (10) **32** (1964), 1380–1383.
Author's summary: "With reference to the $\Lambda N\pi$ system, it is confirmed that the well-known Feynman rules may still be applied for computation, provided additional subtractions are performed on the parity-breaking contributions to the fermion propagator and vertex function."
A. Salam (London)

Yamagata, Hideo 3137
The role of mollifiers in Wightman functions.
Proc. Japan Acad. **40** (1964), 200–205.
The author emphasizes the distinction between the following three kinds of "multiplications" of operator-valued distributions: (1) $\langle\varphi\times\psi,h\rangle=\sum c_i\varphi(f_i)\psi(g_i)$ if $h(x,x')=\sum c_if_i(x)g_i(x')$, (2) $\langle\varphi\circ\psi,f\rangle=\varphi(f)\psi(f)$, (3) $\langle\varphi\circledcirc\psi,f\rangle=\lim_{\varepsilon\to 0}\int(\varphi*\rho_\varepsilon)(x)(\psi*\rho_\varepsilon)(x)f(x)\,dx$, where ρ_ε is a specific test function tending to Dirac's δ-function as $\varepsilon\to 0$. It is remarked that (1) is used in the canonical commutation relations, (2) is used in the definition of $\exp i\varphi(f)$, and (3) is preferable for a definition of $\exp i\varphi(x)$. It is also remarked that $(\exp i\varphi(f))\psi(0(k))$ for sufficiently smooth f is in the Fock space, where $\psi(0(k))$ denotes the Fock vacuum. The reviewer could not find any essentially new ideas in these remarks.

The author asserts that the uniqueness theorem in Haag's article [*Lectures in theoretical physics*, Vol. III, pp. 353–381, Interscience, New York, 1961; MR **24** #B1139] can be applied only to free fields and that Wightman's method using Gel'fand's construction can treat only free fields [author's quotation of the paper under review in #3138 reviewed immediately below]. These assertions seem to be based on the author's insufficient understanding of the subject.
H. Araki (Kyoto)

Yamagata, Hideo 3138

Representation of the state vectors by Gelfand's construction.

Proc. Japan Acad. **40** (1964), 253–258.

The author treats the exponential of canonical fields in a finite volume by considering the (complete) infinite direct product of von Neumann. The author defines $\tilde{\psi} = (\exp i\varphi(f))\psi(0(k))$, where $\psi(0(k))$ denotes the Fock vacuum (the infinite product of vectors with the 0 occupation numbers). The author then shows that if f is of class $\mathscr{S}$, $\tilde{\psi}$ is in the Fock space. The author gives an example of more singular f for which $\tilde{\psi}$ can be outside of the Fock space. *H. Araki* (Kyoto)

Ishihara, Tadashige 3139

On the ambiguity of cut-off process in the theory of quantum field.

Proc. Japan Acad. **40** (1964), 19–22.

The author considers the following mathematical scheme as a model of the cut-off procedure in quantum field theory: Distinct eigenstates of a complete set of commuting observables for a single particle are indexed by i. For each i, a Hilbert space H_i with an orthonormal basis Ψ_{n_i}, $n_i = 0, 1, \cdots$, is considered, where Ψ_{n_i} represents the state with n_i particles in i. A physical state of the (interacting) system is, then, represented by a vector in the (complete) infinite direct product $H = \prod \otimes H_i$ of spaces H_i in the sense of J. von Neumann [Compositio Math. **6** (1939), 1–77]. (The author calls H the universal Hilbert space.) The author defines a cut-off operator P_N which maps a vector $\prod_{i=1}^{\infty} \otimes f_i$ to $(\prod_{i=1}^{N} \otimes f_i) \otimes (\prod_{i=N+1}^{\infty} \otimes e_{i0})$, where e_{i0} is the Ψ_{n_i} with $n_i = 0$. The procedure $\langle \Phi | G | \psi \rangle = \lim_{N\to\infty} \langle \Phi | G | P_N \psi \rangle$ is considered as a mathematical formulation of (some of the) cut-off procedures.

The aim of the paper is to show in terms of several examples that the above definition of P_N is actually extremely ambiguous. The ambiguities come from the fact that a tensor product $f \otimes g$ can also be expressed as $\lambda f \otimes \lambda^{-1} g$ or $f \otimes (g-h) + f \otimes h$. *H. Araki* (Kyoto)

Ishihara, Tadashige 3140

On some topologies in the universal Hilbert space.

Proc. Japan Acad. **40** (1964), 23–38.

The author wants to eliminate the ambiguities in the "cut-off procedures" found in the previous paper [#3139] by introducing a fixed orthonormal basis $\varphi_{i\beta}$ (which is Ψ_{n_i} in the previous paper with $\beta = n_i$) and by defining a topology relative to this basis.

The product $c \prod \otimes \varphi_{i\beta(i)}$ is called a standard monomial and $\mathfrak{h}$ denotes the set of all finite linear combinations of standard monomials. The cut-off operator P_N is defined for standard monomials as in the previous paper and for their linear combinations by linearity. A topology τ in $\mathfrak{h}$ is then defined by a set of seminorms $\|\Phi\|_N = \|P_N\Phi\|$. A cut-off procedure is formulated as a limiting process with respect to this topology.

Some features of this topology is studied by considering examples of sequences in $\mathfrak{h}$. The main feature of this topology in connection with its physical application seems to be that the closure of the restriction of $\mathfrak{h}$ to the Fock space contains elements outside the Fock space.

This paper does not contain any discussion of whether this mathematical scheme is actually relevant and useful in the discussion of cut-off procedures in quantum field theory. *H. Araki* (Kyoto)

Albino, E.; Bertero, M.; Viano, G. A. 3141

An analyticity test in the theory of complex angular momentum. (Italian summary)

Nuovo Cimento (10) **32** (1964), 1269–1294.

Authors' summary: "From the solution of the interpolation problem in the theory of complex angular momentum, there derives a system of equations which are satisfied by the values of an analytic function $f(z)$ (regular in the half-plane $\mathrm{Re}\, z > 0$) on a set of equally spaced points. The ability of this system to work as an analyticity test is investigated. A numerical analysis is made on some trial functions and on some available experimental data." *E. Predazzi* (Turin)

Žislin, G. M.; Sigalov, A. G. 3142

On the mixed spectrum of certain higher-dimensional differential operators of quantum mechanics. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 1329–1331.

Exact mathematical results concerning the spectrum of the Schrödinger operator for an n-electron atom, with wave functions on L^2-space, are as yet few indeed. The authors state (without proofs) four theorems for the (non-relativistic) Schrödinger operator containing Coulomb interactions but without explicit spin terms. The major mathematical tool which seems to have been used, apart from Hilbert space theory, is the theory of representations of the permutation groups. The theorems stated appear to deal primarily with inequalities between the spectral properties of the energy spectrum for atoms with $n-1$ and with n electrons. {Although it is difficult to know how results of this nature are to be interpreted in terms of the empirical data of atomic spectroscopy, studies of this type are potentially of high importance for the understanding of atomic structure models.} *E. L. Hill* (Minneapolis, Minn.)

Pike, E. R. 3143

On the related-equation method of asymptotic approximation (W. K. B. or A-A method). II. Direct solutions of wave-penetration problems.

Quart. J. Mech. Appl. Math. **17** (1964), 369–379.

In Part I [same J. **17** (1964), 105–124; MR **28** #3214] a theorem was conjectured whereby a solution of $d^2y/d\xi^2 = \lambda^2 f(\xi)y$, where all variables are real and λ is a large parameter and $f(\xi)$ may have any number of poles and zeros in a range, may be uniformly approximated over the whole range by some solution of a related but simpler equation. In the present paper this conjecture is applied to the problems of the quasi-classical oscillator and free-electron tunnelling, using approximations by Weber functions. *T. M. Cherry* (Melbourne)

Haag, Rudolf; Kastler, Daniel 3144

An algebraic approach to quantum field theory.

J. Mathematical Phys. **5** (1964), 848–861.

The C^*-algebra approach to quantum field theory has been championed for many years by I. E. Segal [Ann. of Math. (2) **48** (1947), 930–948; MR **9**, 241; Les Problèmes

Mathématiques de la Théorie Quantique des Champs (Lille, 1957), pp. 57–103, Centre Nat. Recherche Sci., Paris, 1959; MR 23 #B842]. The aim of the present authors is to give this approach a specific framework similar to that of W^*-algebra approach and discuss a few interesting features which are more or less related to the distinction of global and local quantities.

To each bounded open space-time region B, one associates an abstract C^*-algebra $\mathfrak{A}(B)$ of local observables in B. The norm completion of the union of all $\mathfrak{A}(B)$ is the C^*-algebra $\mathfrak{A}$ of quasilocal observables. A physical state is represented by a positive linear form φ over $\mathfrak{A}$. If one confines one's attention to a concrete representation R of $\mathfrak{A}$ on a Hilbert space H and to the associated set $\Omega(R)$ of positive linear forms determined by density matrices on H, one goes back to an ordinary Hilbert space approach.

A weak neighbourhood of a φ, determined by a finite number of $A_i \in \mathfrak{A}$ and positive ε_i (inaccuracies), is the set of φ' such that $|\varphi(A_i)-\varphi'(A_i)|<\varepsilon_i$. Any experiment is believed to determine only whether the state under observation is in a certain weak neighbourhood. Therefore two representations R_1 and R_2 of $\mathfrak{A}$ are defined to be physically equivalent if every weak neighbourhood of any element of $\Omega(R_1)$ contains an element of $\Omega(R_2)$, and vice versa. This physical equivalence happens to be the same as the weak equivalence of J. M. G. Fell [Trans. Amer. Math. Soc. **94** (1960), 365–403; MR **26** #4201]. The equivalence class is known to be determined by the kernel of the representation.

A heuristic argument is presented to show the existence of unitarily inequivalent irreducible representations of $\mathfrak{A}$, inequivalence being induced by different values (or behaviours) of global quantities. Superselection rules are discussed as a typical case of such phenomena. (Other typical cases are perhaps equilibrium states of gas with zero temperature and various densities.) Lorentz transformations which are defined as automorphisms of $\mathfrak{A}$ are also global operations, and whether they are unitary depends on the representation. Finally, a comparison of this approach with the W^*-algebra approach is given and the concept of local unitary equivalence of two representations R_1 and R_2 is introduced as the unitary equivalence of $R_1(\mathfrak{A}(B))$ and $R_2(\mathfrak{A}(B))$ for every B.

Appendix I is a concise and useful summary on relevant definitions and theorems on C^*-algebras, and Appendix II gives an example of a non-simple algebra.

Recently, Misra ["On the representation of Haag field", a preprint] has proved an important theorem in this direction. He takes a W^*-algebra as $\mathfrak{A}(B)$ and proves under general assumptions that $\mathfrak{A}$ is simple and that all (nontrivial) representations of $\mathfrak{A}$ are mutually physically equivalent (due to simplicity) and locally unitarily equivalent. *H. Araki* (Kyoto)

Finkelstein, R. J. 3145

Electromagnetic interactions of a Yang-Mills field.
Rev. Modern Phys. **36** (1964), 632–640.

In this paper a general formulation of the theory of vector bosons is presented. By means of this formulation, the hypothesis that the photon is the neutral member of a multiplet embracing the charged W-mesons, postulated in weak interactions [A. Salam and J. Ward, Nuovo Cimento (10) **11** (1959), 568–577; MR **21** #3238], can be easily introduced. The consequences of the possible existence of magnetic monopoles can also be investigated. Finally the nonlocal representation of operators due to S. Mandelstam [Ann. Physics **19** (1962), 1–24; MR **25** #5722] is discussed, and the Dirac condition on the magnitude of the elementary magnetic charge is again found. *E. Ferrari* (Rome)

Wyatt, Robert E.; Parr, Robert G. 3146

One-electron perturbations in self-consistent field theory.
J. Chem. Phys. **41** (1964), 514–520.

A version of the Hartree-Fock perturbation theory is presented which shows that for approximate applications certain advantages may arise from a particular specification of the off-diagonal Lagrange parameters.
A. Dalgarno (Belfast)

Gulmanelli, P. 3147

Classical representations of the restricted Poincaré group.
Phys. Lett. **5** (1963), 320–321.

In two previous papers [A. Loinger, Ann. Physics **20** (1962), 132–144; MR **26** #1093; ibid. **23** (1963), 23–27; MR **27** #5391], the notion of "classical representation" of Lie algebra was introduced. It involves, in addition to the Lie algebra itself, operators representing the action of the elements of the Lie algebra on the L^2 space of the phase space of a classical dynamical system. It yields a larger Lie algebra whose invariants can be important in classifying the transformation properties of the classical dynamical system. In the present note this algebra and its invariants are given for the case of the restricted Poincaré group. *A. S. Wightman* (Princeton, N.J.)

Bollini, C. G.; Giambiagi, J. J.; Gonzáles Domínguez, A. 3148

Analytic regularization and the divergences of quantum field theories. (Italian summary)
Nuovo Cimento (10) **31** (1964), 550–561.

We present a method of analytic regularization with which any element of the S-matrix becomes an analytic function of a complex parameter. The usual divergences appear simply as poles at the physical value of the parameter. The subtraction of these poles leads to the usual finite parts. A simple example is discussed, in which the mathematical justification for these subtractions is given. In the consideration of this example we discuss the causal Green's functions of the iterated D'Alembertian. They are constructed from the retarded and advanced solutions introduced by M. Riesz. The application to the self-energy of the electron is explicitly given. An heuristic deduction is then used to convert the problem of the evaluation of the self-energy into the problem of solving a differential equation. The self-energy integral is a formal solution of the latter equation, the finite part (with the pole subtracted) being a rigorous solution.
E. R. Caianiello (Naples)

Olesen, P. 3149

On the description of bound states in quantum field theory.
Phys. Lett. **9** (1964), 277–278.

The paper seeks to criticise the conjecture of Salam [Nuovo Cimento (10) **25** (1962), 224–227] that a self-consistent field theory may be constructed in which there is no distinction between elementary and composite particles under the conditions that the renormalisation constants Z_i vanish and that the self-masses δ_m vanish. These conditions are shown to be self-contradictory for two simple models, for a scalar and for a spinor theory. However, as pointed out by Deshpande and Gerstein [Phys. Lett. **11** (1964), 264], the effects of contact interactions, ignored in this paper, invalidate its arguments.
J. M. Charap (London)

Allcock, G. R. 3150
A geometrized class of Yang-Mills fields.
Nuovo Cimento (10) **28** (1963), 858–864.
Starting with a unit-normalized velocity field $V^\mu(x)$, it is shown to be possible to introduce in a natural way a Yang-Mills field B_μ [C. N. Yang and R. L. Mills, Phys. Rev. (2) **96** (1954), 191–195; MR **16**, 432] associated with the freedom of choice of a triad of vectors orthogonal to V^μ at each point x, which shall form an isospin reference frame at x. A current J^μ is introduced, which satisfies the differential conservation law $\partial_\mu J^\mu + B_\mu \times J^\mu = 0$; since locally B_μ may be transformed away by a gauge transformation, the current J^μ is locally conserved (local conservation of isobaric spin). It is not, however, the source for the velocity field V^μ. This emphasizes the importance of a consideration of the field equations as well as of the gauge group in any attempt to geometrize the electromagnetic field. *J. M. Charap* (London)

Miyatake, Osamu 3151
Note on perturbation method in quantum field theory.
Math. Japon. **8** (1963), 37–45.
The author considers the following problem. The usual Hamiltonian H is split into $H_0 + H'$ for perturbation-type calculations, where H_0 is the unperturbed part and H' is the interaction part. However, the author in a previous paper has shown that the Hilbert spaces associated with H_0 and H' in a simple model are mutually orthogonal, thus invalidating perturbation procedures. Here he splits the total Hamiltonian in three ways and compares the results obtained by perturbation theory with the exact solution of $H\psi = E\psi$, all within the framework of a neutral meson field interacting with fixed sources.
S. F. Tuan (Lafayette, Ind.)

Van Vleck, J. H. 3152
Note on the use of the Dirac vector model in magnetic materials.
Univ. Nac. Tucumán Rev. Ser. A **14** (1962), 189–196.
Author's summary: "The present paper is concerned with when the exchange energy coupling spins can be represented by a formula of the form $A_{12}\mathbf{S}_1 \cdot \mathbf{S}_2$, where $\mathbf{S}_1$ and $\mathbf{S}_2$ are respectively the total spins of two atoms. It is shown that only in the case of half-filled shells this formula is really adequate, and one should expect, in general, coupling terms of anisotropic structure. Use of the conventional vector model is, however, warranted for the isotropic part of the exchange energy."

Zalewski, Kacper 3153
On the quantum-mechanical description of non-isolated systems.
Acta Phys. Polon. **24** (1963), 681–688.
Author's summary: "The Schrödinger equation with a time-dependent Hamiltonian is sometimes applied to the study of non-isolated systems. We show that this is correct (with a proper choice of the Hamiltonian) if and only if the state of the system is initially pure and remains so with increasing time. Further, the problem of describing a non-isolated system by the density matrix of a pure state is discussed." *S. L. Glashow* (Berkeley, Calif.)

Hellman, W. S.; Mullen, G. H.; Coury, F. M. 3154
The canonical property and asymptotic reduction procedures. (Italian summary)
Nuovo Cimento (10) **32** (1964), 609–618.
The authors suggest that if there are derivative couplings, so that the canonical conjugate to the field is no longer the time derivative of the field, then the usual reduction formula of Lehmann, Symanzik and Zimmermann [Nuovo Cimento (10) **6** (1957), 319–333; MR **19**, 1133] (LSZ) should be modified by an extra term; however, they then show that this does not contribute to the S-matrix except as an equal-time commutator (= subtraction constant).

{Reviewer's remarks: The authors are wrong to claim that their treatment allows a generalization of the formulae of LSZ; the authors seem to think that LSZ assume canonical commutation relations for the operator $A^f(t) = -i\int d^3x\Phi(x)\overleftrightarrow{\partial_0}f_\alpha(x)$, where $\Phi(x)$ is the interacting field; this is not the case. Further, the authors postulate that at time $t = t_0$ the field $\Phi(x)$ is equal to a free field, in violation of Haag's theorem. In other words, the present paper is purely heuristic, and has been written in the style prevalent in the early 1950's.}
R. F. Streater (London)

Frank, W. M.; Licht, A. L. 3155
The nature of perturbation expansion in regularized field theories.
Nuovo Cimento (10) **31** (1964), 682–687.
As a preliminary step in the investigation of the convergence characteristics of renormalized perturbation expansions, the analytic properties of perturbation expansions in regularized field theories are under investigation. In the theories considered relativistic fermions and bosons interact via a Yukawa-type interaction, and the bare propagations for these particles are regularized by the introduction of a finite number of auxiliary masses (Pauli-Villars regularization). These are introduced to guarantee that not only are all integrals appearing in the amplitudes finite, but both the fermion and boson propagators are absolutely integrable with invertible Fourier transforms, and the fermion propagator is everywhere finite. Nonlinear counter-terms become unnecessary and are not included. The considerations applicable to theories with auxiliary masses will also be pertinent to theories which are regularized by a nonlocality in the interaction structure. The Pauli-Villars form will, however, be our model. The convergence of such theories is of interest for two reasons: (a) Bogoliubov has shown that the renormalized expansion is an appropriate limit of the

regularized expansion; (b) A regularized structure follows from the introduction of an indefinite metric into the space of states. It has been shown that physical unitarity and perhaps some sort of causality apply to regularized and nonlocal theories. Thus they may even be candidates for a realistic physical theory. *E. R. Caianiello* (Naples)

Eriksson, K. E. 3156

On the connection between renormalization and the renormalization group in quantum electrodynamics. (Italian summary)

Nuovo Cimento (10) **30** (1963), 1423–1433.

The connection between renormalization with a cut-off procedure and the renormalization group in quantum electrodynamics is discussed. A formalism is used in which mass renormalization is multiplicative in character like charge renormalization and electron field renormalization. The cut-off dependence of the charge and mass renormalization constants is discussed.

E. R. Caianiello (Naples)

Johnson, Kenneth; Baker, Marshall; Willey, Raymond S. 3157

Quantum electrodynamics.

Phys. Rev. Lett. **11** (1963), 518–520.

A systematic, non-perturbative approximation scheme for the solution of the unrenormalized Schwinger-Dyson equations for the Green's functions of quantum electrodynamics is presented. The finiteness of the renormalization constants is assumed, so that $[S(p)]^{-1}\to\gamma p+m_0$, $\Gamma_\mu(p+k,p)\to\gamma_\mu$ as $p\to\infty$ and $D(k^2)\to k^{-2}$ as $k\to\infty$. Using these boundary conditions as the starting point for an iterative approximation scheme, and working in the Landau gauge $[D_{\alpha\beta}(k)=(g_{\alpha\beta}-k_\alpha k_\beta/k^2)D(k)]$, it is shown that the equations admit a finite solution if and only if the mechanical mass of the electron, m_0, vanishes. The successive approximations are not gauge-invariant except asymptotically, although the Ward identity is satisfied at every stage. The appearance of a non-zero mass for the physical electron is attributed to the spontaneous breakdown of γ_5 invariance. *J. M. Charap* (London)

Zlatev, I. S.; Nikolov, A. V. 3158

Analytic properties of the Feynman amplitude. I. (Russian. Bulgarian and English summaries)

Annuaire Univ. Sofia Fac. Sci. Phys. Math. Livre 2 *Phys.* **56** (1961/62), 51–87 (1963).

Authors' summary: "In connection with the realization of the work program suggested to us by J. C. Polkinghorne and G. R. Screaton [Nuovo Cimento (10) **15** (1960), 289–300; MR **23** #B869] and J. Tarski ["Spectral representations in perturbation theory, III" (preprint)] first steps have been taken for the determination of the Feynman amplitude singularities, based upon preliminary development of a convenient representation of same amplitude, corresponding to a diagram with negative index. The above representation is a result of a well-grounded technique of integration over the virtual momenta (Appendix), and of a convenient 'diagram technique' based, in part, upon quasi-diagrams introduced in this work. With the help of this representation it was established, for instance, that when the independent virtual momenta number is not less than 3, the technique of the so-called 'Trägheitsformen' cannot serve for determination of the singularities, a thing that is not clear a priori. More exactly, although at first sight it seems the investigation may be carried out using this technique (as a generalization of the technique used above [loc. cit.]), it actually proves we have to find more powerful means, or change the program as a whole. Further investigation in this direction will be given in a forthcoming paper."

Crawford, Frank S., Jr. 3159

The "radial kinetic energy" term in the Schrödinger equation for a central force.

Amer. J. Phys. **32** (1964), 611–614.

This didactic paper shows that the radial term in the kinetic energy of the non-relativistic Schrödinger equation directly obtained by transforming the Laplacian to polar coordinates is more naturally written as $p_r^2/2m$ with

$$p_r \equiv \frac{1}{r}\frac{\hbar}{i}\frac{\partial}{\partial r}r.$$

This has the advantages that (1) p_r is the Hermitian operator of the radial momentum, and (2) in this form the substitution $R(r)\equiv u(r)/r$ produces a simple differential equation for $u(r)$. *R. Ingraham* (University Park, N.M.)

Shale, David; Stinespring, W. Forrest 3160

States of the Clifford algebra.

Ann. of Math. (2) **80** (1964), 365–381.

Representations of the states of the fermion field are discussed in terms of the theory of a Clifford algebra. First a real Hilbert space $\mathscr{H}$ with orthonormal basis $p_1, p_2, \cdots, q_1, q_2, \cdots$ is considered. This space is to be embedded in an algebra with unit so that $xy+yx=2(x,y)$ for all x and y in $\mathscr{H}$ (the Clifford algebra). We can give $\mathscr{H}$ a complex structure so that the action of i takes p_j into q_j. Lorentz transformations induce unitary transformations of $\mathscr{H}$. A state of the fermion field is a normalized linear functional on the Clifford algebra of $\mathscr{H}$. All states that are invariant under the full unitary group of $\mathscr{H}$ are determined. Among the extreme ones are the Fock vacuum, the anti-vacuum, and the central state (the state of greatest randomness). The quantized field is given by mapping $\mathscr{H}$ linearly into operators obeying the anti-commutation relations. A family of quantizations is presented, and it is shown that it yields explicit constructions of the invariant states and Gibbs canonical ensembles for free fermions. *Y. Kato* (Kobe)

Musher, Jeremy I. 3161

Time-dependent perturbation theory by gauge transformations.

Ann. Physics **27** (1964), 167–182.

It is shown that a given time-dependent solution of a one-electron Schrödinger equation for a Hamiltonian H_0 is also the solution to the Schrödinger equation when a time-dependent perturbation H_1 is added to the Hamiltonian for H_0 transformed by a particular gauge transformation. The transformation is obtained from solutions to inhomogeneous partial differential equations. The cases of an adiabatically switched-on perturbation and an oscillating perturbation are investigated.

A. Dalgarno (Belfast)

Nagy, K. L. 3162
Generalized Lippmann-Schwinger equations for theories with multipole ghosts. (Russian summary)
Acta Phys. Acad. Sci. Hungar. **17** (1964), 97–101.
Author's summary: "Generalized Lippmann-Schwinger equations are derived for theories where it is supposed that indefinite metric and multipole ghost states are present. It is shown that the adiabatic S-matrix gives the transition amplitude also in this case."

Weinberg, Steven 3163
Systematic solution of multiparticle scattering problems.
Phys. Rev. (2) **133** (1964), B232–B256.
Author's summary: "Scattering problems for three or more particles cannot be solved by a direct use of those techniques, like the Fredholm or quasiparticle methods, which work for two particles. The trouble is that the kernel $[W-H_0]^{-1}V$ of the Lippmann-Schwinger integral equation is not L^2, even for complex W. In fact, this kernel has a continuous spectrum, giving rise to cuts in the coupling-constant plane for multiparticle scattering amplitudes. We show how to overcome this difficulty, and calculate all Green's functions and scattering amplitudes in a systematic and essentially rigorous manner. The dynamical equations are rewritten as a sequence of linear integral equations for successively larger systems, each with a kernel and inhomogeneous term which can be calculated explicitly from the solutions of the previous equations. The kernels are L^2 because they arise from connected graphs only, so each integral equation can be solved by the Fredholm, quasiparticle, or other methods. The distorted wave approximation appears very naturally in this approach. One minor by-product is an explicit upper bound on the binding energy of any N-particle composite system with square-integrable potentials. A mathematical Appendix on relevant topics in functional analysis is provided."

Weinberg, Steven 3164
Feynman rules for any spin.
Phys. Rev. (2) **133** (1964), B1318–B1332.
The author develops the relativistic theory of higher spin from a point of view midway between that of the classic Lagrangian field theories and the more recent S-matrix approach. The chief aim of the article is to present the explicit Feynman rules for perturbation calculations in a formalism which varies as little as possible from one spin to another.

The treatment is based on three main assumptions: perturbation theory, the Lorentz invariance of the S-matrix, and particle interpretation. Field equations and Lagrangians are not required within this formalism, and it is noted that most of the work presented might be translated into a pure S-matrix theory with unitarity replacing the first and last assumptions.

The explicit Feynman rules are given for massive particles of any spin j, in both a $2j+1$ component and a $2(2j+1)$ component formalism. The propagators involve matrices which transform like symmetric traceless tensors of rank $2j$, the natural generalizations of the 2×2 four-vector σ^μ and 4×4 four vector γ^μ for $j=1/2$.

The above approach is also used to prove the spin-statistics theorem and crossing symmetry, and to discuss T, C, and P transformations. *I. M. Barbour* (Rome)

Pursey, D. L. 3165
A Foldy-Wouthuysen transformation for particles of arbitrary spin.
Nuclear Phys. **53** (1964), 174–176.
The Foldy-Wouthuysen transformation [L. L. Foldy and S. A. Wouthuysen, Phys. Rev. (2) **78** (1950), 29–36] of the Dirac equation, which transforms it to the form $p^0\chi=\beta(\mathbf{p}^2+m^2)^{1/2}\chi$, is generalized to the relativistic wave equations of V. Bargmann and E. P. Wigner [Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 211–223; MR **9**, 553] which define the irreducible states of type P_s of the full inhomogeneous Lorentz group.
R. Ingraham (University Park, N.M.)

Bryden, A. D. 3166
A Foldy-Wouthuysen transformation for particles of spin 3/2.
Nuclear Phys. **53** (1964), 165–173.
The Rarita-Schwinger [W. Rarita and J. Schwinger, Phys. Rev. (2) **60** (1941), 61] equations for free particles of spin 3/2 is transformed to the Foldy-Wouthuysen canonical form $p_0\chi=\beta\omega\chi$, $\omega\equiv(\mathbf{p}^2+m^2)^{1/2}$. This is done by first expressing the wave function in terms of the irreducible states of the representation $D^{1,1/2}\oplus D^{1/2,1}$ of the full homogeneous Lorentz group, then eliminating the spin 1/2 components. *R. Ingraham* (University Park, N.M.)

Morita, M.; Morita, R.; Tsukamoto, T.; Yamada, M. 3167
Clebsch-Gordan coefficients for $j_2=5/2$, 3 and 7/2.
Progr. Theoret. Phys. Suppl. No. 26 (1963), 64–74.

Gotsman, E. 3168
A bootstrap calculation for the Z^- mass. (Italian summary)
Nuovo Cimento (10) **32** (1964), 628–632.
Author's summary: "The mass of the Z^- particle, predicted at the unitary symmetry model, is calculated by using the bootstrap technique. Results obtained are in agreement with the Okubo mass formula, that Z^- should be a bound state of $\bar{K}\Xi$ and not a resonance."
S. L. Glashow (Berkeley, Calif.)

Diu, B.; Rubinstein, H. R. 3169
Note on the "bootstrap" explanation of the $\mathcal{N}$ and $\mathcal{N}_{33}$. (Italian summary)
Nuovo Cimento (10) **32** (1964), 1103–1106.
Authors' summary: "As the relativistic calculations already performed in the π-$\mathcal{N}$ system do not prove that the $\mathcal{N}$ and the $\mathcal{N}_{33}$ are simultaneously composite, the corresponding 'bootstrap' equations are solved in the static model. The solution is found without need of computers, and is reasonable."
S. L. Glashow (Berkeley, Calif.)

Tiktopoulos, George; Treiman, S. B. 3170
Weak-coupling limit for scattering by strongly singular potentials.
Phys. Rev. (2) **134** (1964), B844–B847.
The problem of singular potentials in the non-relativistic

frame is considered. Although the perturbative expansion, as it is well known, is termwise divergent, it is assumed that this is due to an improper expansion, while actually the sum of the series exists and is finite. To prove the validity of this assumption, a cut-off is allowed, and it is shown that by summing the leading terms of the expansion at the limit in which the cut-off goes to infinity, the correct asymptotic behaviour (for small values of the coupling constant) is obtained. These methods for obtaining asymptotic estimates are investigated in the non-relativistic case because of the inherent mathematical difficulties in investigating their legitimacy in the field-theoretical context. *E. Predazzi* (Turin)

Fuščič, V. I. 3171

Analytic properties of generalized loop diagrams. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 7–9.

Author's summary: "A method is proposed for finding the singularity surface for certain diagrams and an equation is obtained for the singularity surface for diagrams in any order of the theory of perturbations. The equation of the singularity surface is obtained, using the integral representation for the vertex of functions [N. Nakanishi, Progr. Theoret. Phys. Suppl. No. 18 (1961), 1–81; MR **24** #B2366; I.-C. Liu and I. T. Todorov, Dokl. Akad. Nauk SSSR **148** (1963), 806–809; MR **27** #4522], as well as Landau's condition."

Domokos, G.; Suranyi, P. 3172

Bound states and analytic properties in angular momentum.

Nuclear Phys. **54** (1964), 529–548.

The aim of this paper is to study Regge trajectories in relativistic field theory. The Bethe-Salpeter equation with a special class of kernels is transformed into an ordinary differential equation for the four-dimensional partial wave radial two-body wave function. The four-dimensional rotational symmetry of this problem arises from the Lorentz group symmetry by rotating the integration path in the time component p^0 into the imaginary axis. Four classes of bound state solutions ($0 \leqq E^2 \leqq 4m^2$, $E_\mu \equiv$ total energy) are distinguished, according to the type of singularity of the equivalent local kernel $V(r)$ at $r=0$ (equivalently: according to the behavior of the original kernel at high momentum transfers). Solutions exist here regardless of the type of singularity, as contrasted with the non-relativistic case. These solutions involve the (four-dimensional) principal quantum number n and are continued analytically to complex angular momentum l via the connection $n=l+1+n_r$, where the "radial quantum number" $n_r=0, 1, 2, \cdots$. The Regge trajectories (analytic functions $l=f(E^2)$ connecting the masses and analytically continued spins of these bound states) are explicitly determined in the weak coupling approximation by making use of the radial wave equation. *R. Ingraham* (University Park, N.M.)

Khuri, N. N. 3173

Extension of the Regge representation.

Phys. Rev. (2) **130** (1963), 429–435.

It is well known that the Watson-Sommerfeld transform of the partial wave expansion of the scattering amplitude is incorrect in the forward direction [A. Bottino, A. M. Longoni and T. Regge, Nuovo Cimento (10) **23** (1962), 954–1004; MR **26** #1090] unless the contribution of the background integral is taken into account. In this paper the author proposes a new definition of the contribution of each pole in the complex angular momentum plane to the scattering amplitude. This definition has the advantage of yielding a representation where the contribution of each pole has the correct analytic behaviour. A relatively simple bootstrap procedure is established for constructing the scattering amplitude from unitarity and analyticity. *T. Regge* (Turin)

McMillan, M. 3174

Regge cuts in a nonrelativistic three-body problem involving separable potentials. (Italian summary)

Nuovo Cimento (10) **32** (1964), 919–938.

The presence of Regge cuts in the angular momentum variable is discussed here in the non-relativistic three-body problem. Two particles are supposed to interact with a fixed centre represented by non-local separable potentials, in which case the author is able to write in closed form the free two-particle Green's function (the two incoming particles interact with the fixed centre but not with each other). This function shows energy-dependent cuts in the angular momentum variable. This is so because such a Green's function involves integration of the full Green's function of one of the systems whose energy-dependent poles become therefore energy-dependent cuts. As an illustration the case when one particle collides against a bound state of the second one and the fixed centre is considered. Allowing again for a separable interparticle potential, the three-body wave function is written in closed form, and the partial wave elastic scattering amplitude is shown to possess an energy-dependent branch point beside poles. *E. Predazzi* (Turin)

Warnock, R. L. 3175

Motion of poles as a broken resonance multiplet approaches degeneracy.

Nuovo Cimento (10) **32** (1964), 255–257.

Although the Gell-Mann-Okubo mass formula is well satisfied by the masses of the $j=3/2^+$ meson-baryon decuplet, Oakes and Yang [Phys. Rev. Lett. **11** (1963), 174–178] have recently attacked its significance. Concerned with the behavior of the scattering amplitude poles corresponding to these states as the symmetry-breaking interactions are removed, they concluded that the motion of these poles must be exceedingly complicated and that the assumption of linear behavior (required to derive the mass formula) is violated. In this work, the motion of the poles is studied with the use of the N/D formalism. It is found that the complicated pole motion does not occur; instead, several poles are associated with a resonance when it is near threshold. One pole causes the resonance peak when it is above threshold, another when it is below. *S. L. Glashow* (Berkeley, Calif.)

Marr, R. B.; Shimamoto, Y. 3176

On a Lee model with a four-point coupling. (Italian summary)

Nuovo Cimento (10) **31** (1964), 599–614.

Authors' summary: "It is shown that a modified Lee model in which a four-point interaction involving the N and θ particles is appended to the original trilinear interaction is soluble and provides a convenient example for the discussion of CDD poles. The parameters of the model may be selected in such a way as to allow the existence of (a) two stable particles, or (b) one stable and one unstable particle with the quantum numbers of a V-particle. The N-θ scattering amplitude is compared with that of a Lee model with two V-particles."

Nakamura, Koichi; Muta, Taizo; Ezawa, Hiroshi 3177
Inversion problem in the model field theory.
Progr. Theoret. Phys. **31** (1964), 488–508.

An attempt is made to give a criterion for deciding, from a study of phase shifts and scattering amplitudes, whether a given "particle" is elementary or composite. The difficulty in principle is that the scattering matrix elements which are measured directly ("on the energy shell") are insufficient for this task. The authors get around this by making an ad hoc assumption of a "factorizable" scattering amplitude, which then allows them to deduce the off-the-energy shell matrix elements from the measured quantities. Application of the formalism to the deuteron leads to the expected result that the deuteron is a composite particle. However, this reviewer cannot accept the assumption of a factorizable amplitude as satisfactory, at least not without a much more thorough physical discussion and motivation than given here.
J. M. Blatt (Kensington)

Sankaranarayanan, A. 3178
On the transformation of relativistic single-particle Hamiltonians to odd and even forms.
Nuovo Cimento (10) **32** (1964), 1715–1720.

Author's summary: "The reduction of the relativistic single-particle equations to even and odd forms is discussed and questions of uniqueness are considered."
G. Barton (Brighton)

Coleman, Sidney; Schnitzer, Howard J. 3179
Mixing of elementary particles.
Phys. Rev. (2) **134** (1964), B863–B872.

Authors' summary: "Particle mixing is studied in a field-theoretic context, as a further approximation to the pole approximation. Although particle mixing is well suited for treating spinless particles, another approximation, also a further approximation to the pole approximation, called vector mixing, is better for treating particles of spin one. Vector mixing is applied to several processes involving the mixing of the ω and the ϕ by the interaction that breaks unitary symmetry."
S. L. Glashow (Berkeley, Calif.)

Ladányi, K. 3180
Broken symmetries in nonlinear spinor theories. (Italian summary)
Nuovo Cimento (10) **32** (1964), 1629–1644.

Im Rahmen einer einheitlichen Feldtheorie der Elementarteilchen vom Heisenberg-Dürrschen Typ werden verschiedene Möglichkeiten untersucht, aus den Grundgleichungen hoher Symmetrie spezielle Lösungen mit geringerer Symmetrie abzuleiten. Dabei werden einige Annahmen gemacht und diskutiert, welche von den ursprünglichen Heisenbergschen abweichen. Z.B., macht Verfasser die Annahme, daß das Photon als stark gebundener Zustand eines Λ-Anti-Λ Paares zu verstehen sei. Die Konstanz der Baryonen- und Leptonenzahl soll nicht aus einer Symmetrieeigenschaft der Grundgleichung, sondern aus der Dynamik der speziellen Wechselwirkungen (genähert) folgen. Verfasser deutet die Möglichkeit an, daß Prozesse mit Nichterhaltung der Baryonenzahl z.B. im kosmischen Geschehen eine Rolle spielen könnten.
G. Heber (Leipzig)

Gilbert, Walter 3181
Broken symmetries and massless particles.
Phys. Rev. Lett. **12** (1964), 713–714.

This letter points out that the arguments used by Klein and Lee [same Lett. **12** (1964), 266–268; MR **28** #4856] to cast doubts on the validity of the Goldstone theorem [J. Goldstone, Nuovo Cimento (10) **19** (1961), 154–164; MR **23** #B1417; J. Goldstone, A. Salam and S. Weinberg, Phys. Rev. (2) **127** (1962), 965–970; MR **28** #3712] are inapplicable to a relativistic theory. In essence this is because the contribution to the commutator $i\int dx e^{ikx} \times \langle[j_\mu(x), \phi(0)]\rangle$ of a conserved current j with a scalar field quantity ϕ arising from the "spurion" states introduced by Klein and Lee can only have the form $Ck_\mu\delta^4(k)$, there being no other vector than k_μ available in a relativistic theory. This contribution is then identically zero.
J. M. Charap (London)

Shpiz, J. M. 3182
On Pais's charge correlation coefficients.
J. Mathematical Phys. **5** (1964), 1052.

Author's summary: "An alternative proof of a theorem of Pais which states that the charge correlation coefficients for an n-π system are independent of the row of the representation of the symmetric group is presented in which no detailed knowledge of the representations is used."
Y. Ne'eman (Pasadena, Calif.)

Levinson, C. A.; Lipkin, H. J.; Meshkov, S. 3183
A unitary symmetry selection rule and its application to new resonances. (Italian summary)
Nuovo Cimento (10) **32** (1964), 1376–1379.

Authors' summary: "A selection rule is formulated, which forbids the decay into two pseudoscalar mesons of all members of a unitary multiplet containing a neutral boson which is odd under CP. It is applied to the classification of resonances like the x and B which decay into pseudoscalar vector mesons. An additional rule forbids the decay into two pseudoscalar mesons of all members of a tenfold multiplet which has even parity."
Y. Ne'eman (Pasadena, Calif.)

Lurié, D.; Macfarlane, A. J. 3184
Matrix elements of the octet operator of SU_3.
J. Mathematical Phys. **5** (1964), 565–575.

From the authors' summary: "All the nonvanishing

matrix elements of all the components of the tensor operator which belongs to the regular representation (the octet) of SU_3 have been evaluated."
S. L. Glashow (Berkeley, Calif.)

Ravatin, Jacques; Mesnard, Guy 3185
Extension des possibilités d'application pratique de la méthode de la mésomérie par l'emploi de graphes.
Cahiers de Phys. **18** (1964), 38–44.
The application of graphical methods to valence bond theory is considered. Two types of graphs, spin-graphs and space-graphs, are introduced, and it is shown how these graphs may be used in the calculation of matrix elements. It is suggested that these new graphical methods may be useful not only in the case of orthogonal orbitals, which is easily treated by other methods, but also in certain cases of limited overlap between neighbouring orbitals. *A. C. Hurley* (Melbourne)

Szépfalusy, P. 3186
Pairing correlation in the nuclear surface layer. (Russian summary)
Acta Phys. Acad. Sci. Hungar. **17** (1964), 229–239.
Author's summary: "A nonuniform system of fermions in superfluid state is treated starting from the equations for the one-particle Green's function in the framework of the generalized Hartree-Fock method taking into account pairing correlation. Equations for determining the self-consistent pairing potential and expressions for the phase space density of tractable form are deduced with the aid of an approximation method which, to the lowest order, gives formulas of similar form to those of infinite systems, but the one-particle effective potential, and therefore all the other quantities too depend on the position. In that approximation the phase space density goes over into that of the Thomas-Fermi model if the pairing correlation is neglected. By taking into account first- and second-order terms we derive inhomogeneity corrections both to the phase space density and the equations determining the pair correlation. The presence of the pair correlation plays an intrinsic role in the evaluation of the correction terms to the phase space density from the Green's function. Thus the pairing correlation may have considerable effect on the density distribution."

STATISTICAL PHYSICS, STRUCTURE OF MATTER
See also 2507, 3097, 3099, 3127, 3186.

Jackson, Julius L.; Coriell, Sam R. 3187
On trapped trajectories in Brownian motion.
J. Mathematical Phys. **5** (1964), 1075–1078.
The Brownian motion of an ion in the attractive field of an infinite line charge is studied. The nature of the motion as a function of the line charge density shows a discontinuity when this density equals $\varepsilon kT/e$, where ε is the dielectric constant of the medium. The discontinuity is investigated and proves to be associated with the fact that, for charge densities above the critical value, trajectories have a non-zero probability of becoming trapped by the attractive field and never leaving its immediate vicinity.
The methods used are elementary and elegant, and are based on the study of the standard partial differential equation for the distribution. *J. Gillis* (Rehovot)

Zalewski, Kacper 3188
Evolution of a simple system towards equilibrium.
Acta Phys. Polon. **24** (1963), 485–492.
A simple model, consisting of a linear oscillator in interaction with harmonic oscillators, is studied in order to investigate different approximate solutions of the master equation. The author concludes that the weak coupling approximation, corresponding to the limit $\lambda \to 0$ for t fixed, may fail for sufficiently large t if λ is fixed. The pole approximation fails for long times in the same way as the weak coupling approximation. The strong coupling approximation is found to remain valid in the limit $t \to \infty$.
E. J. Verboven (Nijmegen)

Takimoto, Noboru 3189
Excitation of density waves in a condensed boson system.
Progr. Theoret. Phys. **31** (1964), 399–411.
A self-consistent method, similar to the Thomas-Fermi method, is applied to a condensed boson system in order to examine the energy loss of incident particles by irradiating phonons. By assuming the Landau energy spectrum, the author derives Landau's critical condition, above which a strong resistance appears. This theory does not, however, explain a recent experiment by Careri on liquid helium. *R. Kubo* (Tokyo)

Abe, Ryuzo 3190
Cluster expansion theory for a system of random Ising spins.
Progr. Theoret. Phys. **31** (1964), 412–423.
Statistical theory of random Ising spins is treated by the cumulant expansion method as formulated by the reviewer [J. Phys. Soc. Japan **17** (1962), 1100–1120; MR **29** #5364]. It is shown that a cluster expansion is possible for this problem. In particular, the magnetic susceptibility is expanded in powers of the spin concentration p. As an application the critical concentration p_c for ferromagnetism is calculated using the Padé approximation for bcc and fcc. *R. Kubo* (Tokyo)

Lewis, M. B. 3191
Time-reversed motion in kinetic theory.
Phys. Rev. (2) **134** (1964), A1410–A1416.
Author's summary: "A set of reversible equations for F_1, the first distribution function and g, the correlation function, are derived for the weak force case. The 'forward motion', i.e., development in time from uncorrelated initial conditions, and the corresponding reverse motion are examined. In the 'forward motion' the equation for F_1 evolves into the Fokker-Planck equation while in the corresponding reverse motion F_1 is described by an anti-Fokker-Planck equation." *E. J. Verboven* (Nijmegen)

Ludwig, Günther 3192

Zur Begründung der Thermodynamik auf Grund der Quantenmechanik.

Z. Physik **171** (1962), 476–486.

This article concerns itself with two central problems relevant to the foundations of the thermodynamics of states of equilibrium: (a) It is to be inferred why a system tends towards a state of equilibrium and remains almost constantly in this state; (b) The validity of the Second Law for such states is to be established. The author begins by pointing out that before (a) can be tackled at all one first has to say clearly what exactly one means by "system" and "state of equilibrium", and by the conceptions of "tending towards" on the one hand and "inferring" on the other. Difficulties which arise here are discussed at length, and it may be worth emphasising the noteworthy conclusion that thermodynamics cannot be founded deductively on an atomic theory, whether classical or quantum mechanical, since one cannot infer logically what the "macroscopic properties" of a system are. The author then proceeds to establish two sufficient conditions which, when they are satisfied, will ensure that almost all systems are almost always in equilibrium. The paper concludes essentially with comments on the conventional treatment of problem (b) mentioned initially. (The paper represents a lecture given in 1961 and is therefore largely discursive, except for a short appendix.)

H. A. Buchdahl (Canberra)

Ludwig, Günther 3193

Zur Begründung der Thermodynamik auf Grund der Quantenmechanik. II. Masterequation.

Z. Physik **173** (1963), 232–240.

It is shown that one can consider in quantum mechanics from a single point of view both the general ergodic theory of macroscopic bodies and the master equation theory.

D. ter Haar (Oxford)

Fuliński, Andrzej 3194

Statistical and thermodynamical description of correlated systems. I. General considerations.

Acta Phys. Polon. **23** (1963), 93–99.

General considerations on the master equation of the isolated system which consists of two subsystems are given. The aim of the present and following papers of this sequence [#3195—#3198 below] is to deal with the thermodynamical and statistical description of the case when the correlation between the two subsystems cannot be neglected.

A quantum state of the isolated system $[U]$ is specified by a combination of the quantum states of the two subsystems $[U^1]$ and $[U^2]$, which are expressed by $\alpha\lambda$ and $\iota\rho$, respectively. The symbols λ, ρ denote the shells of the phase-spaces, and α, ι denote the quantized states of the shells (λ) and (ρ), respectively. The shell implies a part of the phase-space such that the phase-point runs only over the states belonging to the same shell when the system is isolated. Thus the system $[U]$ remains in the same shell since it is isolated, but $[U^1]$ and $[U^2]$ do not since they are not isolated.

The temporal variations of the probability are assumed to obey the Pauli master equation. The relation of the transition probability in $[U]$ to the transition probabilities in $[U^1]$ and $[U^2]$ is derived. It is, however, tacitly assumed in the above-mentioned derivation of the relation that the probability distributions $f^1(\alpha\lambda)$ of $[U^1]$ and $f^2(\iota\rho)$ of $[U^2]$, as well as the probability distribution $f(\alpha\lambda, \iota\rho)$ of $[U]$, obey their respective master equations at the same time. This assumption seems to be important for the reviewer, though the author of this paper pays little attention to it.

According to the derivation of the Pauli master equation from the von Neumann equation for the temporal development of the density matrix of the quantum-statistical ensemble of the systems, the assumption of random a priori phases plays an essential role. It seems to be difficult to justify that, in the cases of strong interaction, the distribution of the phase-factors of the wave functions which describe a mixed state of $[U]$ are random when the phase-factors for the subsystems are distributed randomly. Thus it is the reviewer's opinion that the master equation for $[U]$ is not compatible with the simultaneous master equations for $[U^1]$ and $[U^2]$ unless the interaction and correlation between two subsystems are sufficiently weak. In other words, it is impossible to diagonalize the density matrix for $[U]$ and the density matrices for $[U^1]$ and $[U^2]$ at the same time when the interactions between these two subsystems are not weak.

As pointed out by the author, such correlations between two macroscopic systems have not been dealt with in quantum-statistical mechanics, but it seems to be necessary to start with the quantum-statistical equation of change for the density matrix of the ensemble of the correlated systems for making further progress in this direction.

S. Ono (Tokyo)

Fuliński, Andrzej 3195

Statistical and thermodynamical description of correlated systems. II. Entropy.

Acta Phys. Polon. **23** (1963), 159–165.

The entropy and the entropy production of the system composed of two correlated subsystems are treated in this part on the basis of the general considerations developed in Part I [#3194].

The total entropy S of the system is shown to consist of the entropies S^1 and S^2 of the constituent subsystems, and S^{12}, the entropy of interaction between these two subsystems. The total entropy production is also divided into three parts $\sigma^1, \sigma^2, \sigma^{12}$ in a similar manner. The entropy production between the subsystems σ^{12} can be negative. However, the derivations of the formulae in this paper do not seem to be justified for the cases of strong interaction between the subsystems. The density matrix ρ for the coupled system $[U]$ and the density matrices ρ^1, ρ^2 for $[U^1]$ and $[U^2]$ cannot be diagonalized at the same time, and the expression for the entropy, $-k$ trace $[\rho \log \rho]$, is not necessarily equal to

$$-k \sum_r f_r \log f_r$$

unless the ρ is diagonal, where f_r is the diagonal element of ρ for the state r.

In addition, the same critique as that given to Part I is applicable to the treatment of the temporal change of the entropies given in this part.

S. Ono (Tokyo)

Fuliński, Andrzej 3196
Statistical and thermodynamical description of correlated systems. III. Equilibrium and stationary distributions.
Acta Phys. Polon. **23** (1963), 773–781.
The stationary distribution of the system consisting of two correlated subsystems is derived from the equation of change for the entropies S^1, S^2, S^{12} given in Part II [#3195]. *S. Ono* (Tokyo)

Fuliński, Andrzej 3197
Statistical and thermodynamical description of correlated systems. IV. Time-dependent distributions.
Acta Phys. Polon. **24** (1963), 143–151.
The solution of the master equation derived in Part I [#3194] for describing the time evolution of the correlated two subsystems which form an isolated system is discussed. It is shown that in nearby equilibrium or stationary states the probability of finding the subsystems in their quantized states, and also the correlations, are the exponential functions of momentary values of the observables and do not depend on the initial state of the system. *S. Ono* (Tokyo)

Fuliński, Andrzej 3198
Statistical and thermodynamical description of correlated systems. V. Phenomenological equations near by equilibrium.
Acta Phys. Polon. **24** (1963), 153–164.
With the use of the time-dependent statistical probability distributions derived in Part III [#3196] the phenomenological equations for time evolution of entropies and observables are obtained. It is found and emphasized by the author that the phenomenological description of irreversible processes derived from the point of view of the isolated system as a whole is not equivalent to that derived from the point of view of the single non-isolated subsystem by the use of the entropy productions σ^1, σ^2, σ^{12} introduced in Part II [#3195].

The reviewer believes, however, that this discrepancy arises from the incompatibility of the description of the strongly correlated subsystems from the exact quantum-statistical point of view as mentioned in the reviews of Parts I and II above. *S. Ono* (Tokyo)

Morse, Philip M. 3199
★Thermal physics.
Revised edition.
W. A. Benjamin, Inc., New York-Amsterdam, 1964. xiii+455 *pp.* $9.75.
This is a revised and enlarged second edition of the author's earlier book of the same title [W. A. Benjamin, New York, 1962], and based on his M.I.T. lecture notes for a half-year senior course in thermodynamics and statistical mechanics. The revision has eliminated many misprints and errors that had plagued the first edition, while retaining the informal, colloquial style of presentation.

As one might expect from an experienced teacher and scholar of the author's stature, the present volume contains a thorough, competent and readable account of a difficult subject. The first part of the book deals with thermodynamics, which is presented in a straightforward manner. Thermodynamics differs from most of physical theory in its basic structure and point of view; it does not seem to this reviewer that the author's book imparts a "feeling" for the subject to any greater extent than a number of standard textbooks. However, his presentation is enlivened by many interesting and illuminating physical examples.

The next part of the book, dealing with kinetic theory, is particularly good and includes a number of subjects not frequently presented at this level, namely, chapters on transport phenomena and on the theory of fluctuations. The last part on statistical mechanics is extensive and readable; it also includes a number of advanced subjects that are not often seen in books for college seniors.

The book concludes with a collection of well-chosen problems and a good index. The colloquial style referred to earlier cannot be called elegant or finished, but it is solid and unambiguous. Some of the quality of a tape-recorded lecture, rather than a carefully edited book, remains. This is a characteristic of the Benjamin series. However, this book seems to suggest that a revised second edition of a Benjamin lecture-based book is much better than a first edition. On the whole, the author has undoubtedly succeeded in conveying to the reader the feeling that physics is a living and developing, rather than a static, subject, as stated in the preface.
W. Franzen (Boston, Mass.)

Chernikov, N. A. [Černikov, N. A.] 3200
Derivation of the equations of relativistic hydrodynamics from the relativistic transport equation.
Phys. Lett. **5** (1963), 115–117.
Starting from the relativistic Boltzmann equation and the relativistic form of the conservation laws, the author uses the generalization of Grad's thirteen-moments distribution function in order to obtain relativistic transport equations. The transport coefficients are approximately evaluated, and to this order the bulk viscosity vanishes.
H. Weitzner (New York)

Israel, Werner 3201
Relativistic kinetic theory of a simple gas.
J. Mathematical Phys. **4** (1963), 1163–1181.
Author's summary: "A consistent relativistic theory of transport processes in a simple gas is developed. The approach is the four-dimensional geometric one due to Synge. Scalar and vector eigenfunctions of the linearized collision operator are derived when the scattering cross-section is a simple separable function of scattering angle and relative velocity (Maxwellian particles). The bulk viscosity and thermal conductivity are computed explicitly for this case." *H. Weitzner* (New York)

Hastie, R. J.; Taylor, J. B. 3202
Stability of magnetic wells at finite plasma pressure.
Phys. Rev. Lett. **13** (1964), 123–124.
This note refers to a special class of equilibria which is only possible in magnetic fields possessing a non-zero minimum in field strength. Stability conditions are derived which apply for any value of β (ratio of plasma

pressure to magnetic pressure), thus extending a previous result which was valid only in the low β limit.
T. J. M. Boyd (Culham)

Knorr, G. 3203
Zur Lösung der nicht-linearen Vlasov-Gleichung. (English summary)
Z. Naturforsch. **18a** (1963), 1304–1315.
Author's summary: "Wave-like equations of the nonlinear Vlasov equation for a one-dimensional electron plasma are discussed analytically and numerically. The quasilinear theory is derived in a simple way and some conclusions are drawn from it. It is found that, if the initial conditions of the harmonics of the electrical field differ by orders of magnitude the type of the solution varies appreciably. The numerical solution is obtained by integrating the characteristic function of the distribution function f, i.e., its Fourier transform in space and velocity. It is shown that this method is, in principle, superior to the direct integration of f. Stable and instable cases are discussed. Good agreement between theory and numerical computations is found."

McQuarrie, Donald A. 3204
Classical theory of fluids and the square-well potential.
J. Chem. Phys. **41** (1964), 1197–1198.
Author's summary: "The term E_3, proportional to the cube of the density in an expansion of the internal energy in powers of the density, has been evaluated for the Percus-Yevick, hyperchain, Born-Green-Yvon, Kirkwood, and Meeron equations. The potential used is a square well having a hard-core diameter a and an attractive diameter $2a$. A certain thermodynamic requirement of the E_3 is investigated and it is found that the hyperchain, Born-Green-Yvon, and Meeron equations give the best result, in contrast to fourth virial coefficient studies, in which the Percus-Yevick equation gives the best result. On the basis of these calculations, it seems that the use of virial expansions are of little use in the elucidation of the relative merits of liquid theories."

Berencz, F. 3205
The development of the configuration-interaction method by the aid of the spin-operator method. (Russian summary)
Acta Phys. Acad. Sci. Hungar. **16** (1963), 249–259.
The energies of the spin-singlet states of the π-electrons in benzene are calculated with a variational wavefunction consisting of all the spin-singlets which can be formed from product wave functions having one and only one π electron on each C atom. *A. Herzenberg* (Manchester)

Tietz, T. 3206
Über Eigenwerte und Eigenfunktionen in der statistischen Theorie des Atoms.
Ann. Physik (7) **12** (1963/64), 373–376.
An analytic method is described for determining approximate eigenvalues and eigenvectors of the Schrödinger equation for a neutral atom with a Thomas-Fermi potential. This is a special application of the author's more general approximation [Acta Phys. Acad. Sci. Hungar. **11** (1960), 391–400; MR **22** #7713].
C. Froese (Cambridge, Mass.)

Ziman, J. M. 3207
★Principles of the theory of solids.
Cambridge University Press, New York, 1964. xiii + 360 *pp.* $8.50.
Considered as an introductory text in the basic applications of wave-mechanical theory to problems of the solid state, this book should be found exceptionally useful by both teachers and students. The choice of topics, ranging from the formal theory of periodic lattices to superconductivity, is much the same as that to be found in other texts on the same subject. But the author is exceptionally knowledgeable in the details of applications, and has good judgment in the pedagogical formulation of his material. Technically difficult or laborious calculations have been wisely left to the original literature. This book is recommended for use at the graduate level by students who have had a formal course in atomistics and intermediate wave mechanics.

{The title of the book suggests the necessity of a cautionary remark for mathematicians. Considered as a text for study of principles by mathematicians it is as bad as, though no worse than, most others on the subject. The theory of solids as developed by mathematical physicists is particularly fruitful in examples of the mathematical difficulties underlying formal quantum mechanics from the vantage point of modern functional analysis. For a mathematician who might wish to study theoretical physics in a critical sense this book is recommended on the grounds that the author's clear exposition reduces to a minimum the overlay of undigested ideas which usually presents an insuperable barrier to analysis. From the point of view of a critical study of principles the author's statement that better proofs are to be found in the literature is to be disregarded as at least misleading.} *E. L. Hill* (Minneapolis, Minn.)

Bhagavantam, S.; Pantulu, P. V. 3208
Crystal symmetry and physical properties: Application of group theory.
Proc. Indian Acad. Sci. Sect. A **60** (1964), 1–10.
Authors' summary: "A group-theoretical method given by one of us in an earlier publication [S. Bhagavantam, same Proc. **16** (1942), 359–365] for studying the effect of crystal symmetry on physical properties is now extended to cover the galvanomagnetic, thermomagnetic and piezo-galvanomagnetic effects in single crystals. These effects have gained importance in semi-conductor physics and such studies are therefore of current interest and may be expected to yield valuable new information."

Gabriel, Helmut; Klein, Rudolf 3209
Zur Lösung der Boltzmann-Gleichung bei Anwesenheit eines äusseren Magnetfeldes. (English summary)
Z. Naturforsch. **19a** (1964), 524–532.
Authors' summary: "A new method for the solution of the Boltzmann transport equation in the presence of an external magnetic field is proposed. Applying this procedure to the conduction electrons in a solid a general

expression for the conductivity tensor is derived, which can be simplified by using the symmetry properties of the crystal. Conductivity tensor and magnetoresistance are given for the case of cubic solids and are compared with experiments on tungsten. Our results disagree with the theoretical expressions given by Garcia-Moliner. The method does not rest upon a power series expansion in the magnetic field and is therefore applicable to the intermediate field region."

Wood, Herbert T.; Curtiss, C. F. 3210

Quantum corrections to the transport cross sections.

J. Chem. Phys. **41** (1964), 1167–1173.

Authors' summary: "By a method developed earlier, the second quantum correction to the phase shift in a collision between particles with spherically symmetric potentials is obtained. This expression, along with the classical limit and the first quantum correction developed earlier, is used to obtain the classical limit and the first and second quantum corrections to the transport cross-sections $Q^{(1)}$ and $Q^{(2)}$. In these expressions the potential function is not restricted to monotonic functions; the results apply to a potential with an attractive minimum."

Bennett, John H. 3211

Integral equation methods for transport problems. Some numerical results.

Numer. Math. **6** (1964), 49–54.

Numerical methods and results are presented and analysed for the critical number c_0 of secondary neutrons per collision for the mono-energetic transport equation with isotropic scattering in slab and (x, y)-geometry.

The methods compared are: (A) A variational method, due to Schlesinger [LA-1908 (1955)], and Francis, Stewart, Bohl and Krieger [Proc. Second U.N. Internat. Conf. Peaceful Uses Atomic Energy (Geneva, 1958), Vol. 16, pp. 517–529, United Nations, Geneva, 1958]; (B) a collocation method, due to Bennett [Ph.D. Dissertation, Harvard Univ., Cambridge, Mass., 1962], both based on the integral equation for slab geometry; and (C) a highly accurate discrete ordinate method due to the reviewer [Proc. Second U.N. Internat. Conf. Peaceful Uses Atomic Energy (Geneva, 1958), Vol. 16, pp. 503–516, United Nations, Geneva, 1958]. Method (C) is more accurate than (A) and (B), and requires less storage and less time per iteration for the same number of equidistant spatial mesh points.

Method (B) is then generalized to a "non-uniform mesh, integral-equation method" for (x, y)-geometry, which yields highly accurate results in c_0.

{Note: To generalize method (C) to higher space dimensions, one takes the characteristics of the discrete equations to construct the basic network.}

E. H. Bareiss (Argonne, Ill.)

Ueno, Sueo 3212

The invariant imbedding method for transport problems. II. Resolvent in photon diffusion equation.

J. Math. Anal. Appl. **3** (1961), 361–372.

Part I appeared in same J. **2** (1961), 217–222 [MR **23** #B2460]. The author derives various interesting functional equations for the probability of photon emergence, $p(\mu; \tau)$, and related functions for the case of semi-infinite slabs. *R. Kalaba* (Santa Monica, Calif.)

RELATIVITY

See also 2149.

Anderson, James L. 3213

Twins, clocks and geometry.

Trans. New York Acad. Sci. (2) **26** (1963/64), 934–946.

Expository paper.

Lovelock, David 3214

A new formulation of the relativistic wave equations.

Ann. Physik (7) **12** (1963/64), 361–372.

There have been suggestions of various theories of particles with internal spin which purport to furnish a classical counterpart to quantum mechanical spin phenomena (cf. the comprehensive survey article by H. Hönl [Ergebnisse Exakten Natur. **26** (1952), 291–382]). Many of these theories are based on variational principles whose Lagrangians depend on second-order derivatives of the position vectors. An alternative theory involving only a simple variational principle which leads to slightly more general equations of motion was developed by the reviewer [Ann. Physik (7) **7** (1961), 17–27; MR **22** #10725]. The present article is devoted to the quantization of the latter theory, it being shown that the well-known wave equations of Dirac and Kemmer can thus be obtained. The transition scheme is of the usual (Poisson bracket)→(commutator) type, special commutation rules for the angular momentum operators being assumed. The Klein-Gordon equation, which is also postulated, is used to select suitable commutation rules to be satisfied by the operators defining the canonical structure of the wave equations.

H. Rund (Pretoria)

Eriksson, K.-E.; Yngström, S. 3215

A simple approach to the Schwarzschild solution.

Phys. Lett. **5** (1963), 327–328.

In a previous paper the authors gave a simple derivation of Schwarzschild's external solution based on the equivalence principles, special relativity and Newton's law of gravitation [same Lett. **5** (1963), 119; MR **27** #5588]. Full details are in the first author's paper [Ark. Fys. **25** (1964), 167–173; MR **28** #2893]. It was pointed out to them that essentially the same idea was used by Sommerfeld for the same purpose. Here the authors claim that Sommerfeld's treatment is different from theirs and is incorrect, making no clear distinction between coordinate differentials and measured length and time differentials. {In the reviewer's opinion, the paper of the first author quoted is appreciably clearer and more detailed than Sommerfeld's treatment, but Sommerfeld's does not seem to be essentially different.}

A. S. Wightman (Princeton, N.J.)

Bel, Louis; Montserrat, August 3216

Ondes planes à l'infini dans l'espace-temps de Schwarzschild.

C. R. Acad. Sci. Paris **258** (1964), 4659–4661.

Authors' summary: "Nous esquissons la construction d'une famille à trois paramètres d'hypersurfaces

$S(x^\rho, E, P_t) = \text{const}$, solutions de $\Delta_1 S = m^2c^4$, et dont la partie d'espace à l'infini sont des plans. Nous obtenons explicitement les deux premiers termes du développement en série de S suivant le paramètre kM/c^2. A cette approximation S est aussi solution de $\Delta_2 S = 0$, ce qui nous permet d'obtenir une famille à trois paramètres de fonctions propres de l'opérateur Δ_2." *W. Rindler* (Dallas, Tex.)

Florides, P. S.; Synge, J. L. 3217

Stationary gravitational fields due to single bodies.

Proc. Roy. Soc. Ser. A **280** (1964), 459–465.

In a series of earlier papers a method for the evaluation of stationary gravitational fields due to single bodies by successive approximations is described in detail (see J. L. Synge [same Proc. **270** (1962), 315–326; MR **26** #2294] and the references cited there). This method is now improved and made more compact by the use of the contravariant (instead of covariant) components of the energy tensor. The argument, which applies only to single bodies but does not involve any symmetry conditions, depends essentially on the concept of a compact stationary universe of order $(N-1)$, briefly CSU$(N-1)$, which is defined as follows. A 4-space with coordinates x_r (where $x_4 = it$) is divided into two domains, an interior I and an exterior E, by means of a 3-cylinder B with equation $f(x_\sigma) = 0$ $(\sigma = 1, 2, 3)$. Given a small dimensionless constant k, a CSU$(N-1)$ is a 4-space with metric tensor g_{rs} and contravariant energy tensor T^{rs} with the following properties: (i) $g_{rs} = \delta_{rs} + \underset{1}{g_{rs}} + \cdots + \underset{N-1}{g_{rs}}$, these being functions of x_σ of class C^3 in $E+I$ and C^1 across B; each is a product of a function of x_σ and a power of k as indicated by the numerical subscript; each vanishes at infinity like $(x_\sigma x_\sigma)^{-1/2}$; (ii) $T^{rs} = \underset{1}{T^{rs}} + \underset{2}{T^{rs}} + \cdots + \underset{N-1}{T^{rs}}$, these being functions of x_σ of class C^1 in I and vanishing in E, numerical subscripts again indicating powers of k; (iii) $\underset{1}{G^{rs}} = -8\pi \underset{1}{T^{rs}}, \cdots, \underset{N-1}{G^{rs}} = -8\pi \underset{N-1}{T^{rs}}$ in $E+I$, where $G^{rs} = \underset{1}{G^{rs}} + \underset{2}{G^{rs}} + \cdots + \underset{N-1}{G^{rs}} + \underset{N}{G^{rs}} + \cdots$ (this series going on to infinity) is the Einstein tensor. It is shown that for any CSU$(N-1)$ a so-called critical condition is satisfied, and the passage from CSU$(N-1)$ to CSU(N) is discussed in detail with special reference to the cases $N = 1, 2, 3$. An Einstein universe is obtained by solving (for T^{rs}) the field equations $G^{rs} = -8\pi T^{rs}$, and it is found that the residual energy tensor (which represents the error in the sense that a zero energy tensor outside the body is required) tends to zero as the inverse fourth power of the distance.

H. Rund (Pretoria)

Bialas, Andrzej 3218

Conservation laws for the spin-particle in the gravitational field.

Acta Phys. Polon. **25** (1964), 71–74.

It is shown that if a gravitational field admits an r-parameter group of motions, then the motion of a particle with spin moving in this field satisfies r conservation laws.

H. A. Buchdahl (Canberra)

Schmutzer, Ernst 3219

Entwicklung einer physikalischen Geometrie der Raum-Zeit zum Zwecke der Interpretation der allgemeinen Relativitätstheorie. (English summary)

Z. Naturforsch. **19a** (1964), 665–675.

The paper gives a detailed discussion of the decomposition of the 4-dimensional space of general relativity into a 3-dimensional "spatial" subspace and the orthogonal "time" axis, together with the resulting theory of the affine connection, the object being the development of a physical geometry of space-time which may serve as a basis for physical interpretations of the theory. The author's approach is based on the concept of invariance under transformations of the type $x^{m'} = x^{m'}(x^m)$, $x^{4'} = x^{4'}(x^m, x^4)$ $(m, m' = 1, 2, 3)$. The (generally non-integrable) differential coordinate transformation $d\bar{x}^m = dx^m$, $d\bar{x}^4 = dx^4 + (g_{4m}/g_{44})\,dx^m$, is of fundamental importance when applied to the basis vectors. As regards the "spatial" subspace, many results found previously by C. Cattaneo [Nuovo Cimento (10) **10** (1958), 318–337; MR **21** #1889; Rend. Mat. e Appl. (5) **20** (1961), 18–37; MR **25** #3774] are thus re-derived. The projections of the so-called mathematical tensors lead to physical components for which the processes of projective partial and projective covariant differentiation are introduced. The connection parameters defining the latter are determined by means of certain natural postulates concerning these covariant derivatives. This gives rise to a somewhat complicated theory of curvature, analogues of the Ricci identities being used for this purpose. The equations of motion of general relativity are decomposed on the basis of the author's method, certain terms being singled out and interpreted as Coriolis acceleration and centrifugal-gravitational acceleration, respectively. Finally the problem of energy conservation is analyzed. *H. Rund* (Pretoria)

Brill, Dieter R.; Hartle, James B. 3220

Method of the self-consistent field in general relativity and its application to the gravitational geon.

Phys. Rev. (2) **135** (1964), B271–B278.

Authors' summary: "Concentrations of radiation held together for a long time by their own gravitational attraction ('geons') have been studied for nearly a decade. We extend the previous analyses to the case where gravitational waves are the source of the geon's mass energy. To analyze these solutions of the free-space Einstein equations with persistent features, we develop an approximation method to treat small ripples on a strongly curved background metric. The background metric describes the large-scale persistent features of the geon and is taken to be spherically symmetric. The waves superimposed on this background have an amplitude small enough so that their dynamics can be analyzed in the linear approximation; however, their wavelength is so short, and their time dependence so rapid, that their energy is appreciable and produces the strongly curved background metric in which they move. The Einstein equations are investigated in this limit of short wavelength. It is found that the large-scale features of thin-shell spherical gravitational geons—in fact, of thin-shell spherical geons constructed from any field of zero rest mass—are identical to those of the spherical electromagnetic geons analyzed previously."

H. A. Buchdahl (Canberra)

Kondo, Kazuo 3221

A statistical approach to the principle of general relativity.

Tensor (N.S.) **15** (1964), 168–174.

The opinion is expressed that the commonly accepted approach to the general theory of relativity is too formal and axiomatic in nature. The author suggests a new statistical version based on the idea "... that our universe is an aggregation of an infinite number of space-times where an analytical connexion is defined, ...", while "... what we can know of such an ensemble ... cannot but be a coarse statistical summary ...". In essence, the author's approach appears to be as follows: A contracted form of the equation of geodesic deviation of Riemannian geometry is supposed to give rise, via some statistical reasoning, to the well-known variational principle from which Einstein's field equations are usually derived.

H. Rund (Pretoria)

Misra, R. M. 3222

On stationary systems with spherical symmetry. (Italian summary)

Nuovo Cimento (10) **32** (1964), 939–942.

Einstein [Ann. of Math. (2) **40** (1939), 922–936; MR **1**, 61] has investigated the Schwarzschild singularity by considering the field produced by a large number of particles moving in circular paths under the influence of the gravitational field produced by them. The field equations for such a system obtained by Einstein are somewhat erroneous, and the purpose of the present paper is to obtain the correct field equations. Einstein's final results remain unaltered.

P. S. Florides (Dublin)

Levy, H.; Robinson, W. J. 3223

The rotating body problem.

Proc. Cambridge Philos. Soc. **60** (1964), 279–285.

By symmetry arguments and the use of the vacuum field equations the authors obtain a metric appropriate for an axisymmetric stationary system which is a generalization of Weyl's static metric. Approximate solutions of the vacuum field equations are then obtained by means of multipole expansions of the metric tensor.

The authors then obtain a metric appropriate for a stationary axially symmetric rotating system. By assuming the metric to be Minkowskian at infinity, and by suitable multipole expansions of the metric, approximate solutions of the vacuum field equations are obtained. The physical components of the Riemann tensor of these solutions are calculated, and it is shown that the largest components are quadrupole terms corresponding to the Schwarzschild mass terms. The physical components which depend on the rotation of the system are octopole terms. {The paper contains a number of mistakes which are obviously misprints.}

P. S. Florides (Dublin)

Møller, C. 3224

Momentum and energy in general relativity and gravitational radiation.

Mat.-Fys. Medd. Danske Vid. Selsk. **34**, *no.* 3, 67 *pp.* (1964).

It is well known that the Einstein energy momentum complex $\theta_i^{\ k}$ $(i, k=1, \cdots, 4)$ satisfies only the first two of the following three conditions, which a satisfactory energy momentum complex $T_i^{\ k}$ should satisfy: (1) $T_i^{\ k}$ is an affine tensor density depending algebraically on the gravitational field variables and their derivatives of the first and second orders such that $\partial T_i^{\ k}/\partial x^k=0$; (2) In a matter system for which the metric at large spatial distances from the system is asymptotically of the Schwarzschild type and in which one can thus use coordinates which are asymptotically rectilinear, the integrals

$$P_i = \iiint_{x^4=\mathrm{const}} T_i^{\ 4}\, dx^1 dx^2 dx^3$$

are constant in time and transform as the covariant components of a free vector under linear space-time transformations; (3) The components $T_4^{\ k}$ transform like a 4-vector density under the group of purely spatial transformations (this being the condition of localizability of the energy in a gravitational field). The resulting difficulties inherent in Einstein's complex $\theta_i^{\ k}$ are avoided in the theory of a new energy-momentum complex $T_i^{\ k}$ which satisfies conditions (1)–(3), in which, however, the gravitational field is represented by a tetrad field and not directly by the metric [the author, Mat.-Fys. Skr. Danske Vid. Selsk **1** (1961), no. 10; MR **25** #3772]. This complex is applied to the results of a paper by H. Bondi, M. G. J. van der Burg and A. W. K. Metzner [Proc. Roy. Soc. Ser. A **269** (1962), 21–52; MR **26** #4793] in which the exact form of the metric at large spatial distances from an axi-symmetric, but otherwise arbitrary, insular system of matter emitting gravitational waves is given. After a detailed discussion of the properties of $T_i^{\ k}$ and a resumé of the results of Bondi et al., it is shown that, in the coordinate system adopted by the latter, Einstein's $\theta_i^{\ k}$ gives unreasonable results for the energy radiation and the total energy, while the author's $T_i^{\ k}$ yields a value for the energy radiation which is in agreement with the corresponding relation of Bondi. A conjecture by Bondi regarding the total energy flux from an axi-symmetric system expressed by this relation is thus corroborated. In addition, the author's method yields the angular distribution of the energy flux. The evaluation of the total energy and momentum at any time leads to the verification of a conjecture by Bondi to the effect that the mean value of the so-called mass aspect represents the total mass or energy of the system. It is also found that the radiated gravitational energy administers a recoil to the system of the same amount as in the case of electromagnetic radiation.

The complex $T_i^{\ k}$ is expressed in terms of tetrad fields which are determined by the metric only up to arbitrary Lorentz rotations of the tetrads, and $T_i^{\ k}$ is not in general invariant under the latter. It is shown, however, that the total 4-momentum P_i as well as the asymptotic form of $T_i^{\ k}$ are invariant under such tetrad rotations, provided these satisfy certain boundary conditions. A detailed discussion is given also of the transformation properties of P_i and of the emitted radiation under transformations which, at large spatial distances from the system, have the form of a Lorentz transformation, and it is finally shown that the gravitational radiation at large distances has the form of approximate plane waves with an everywhere positive energy density, the momentum density being the energy current density divided by c^2 as in the case of a plane electromagnetic wave.

H. Rund (Pretoria)

Beiglböck, Wolf 3225

Zur Theorie der infinitesimalen Holonomiegruppe in der Allgemeinen Relativitätstheorie. (English summary)

Z. Physik **179** (1964), 148–160.

Author's summary: "Starting from the infinitesimal holonomy group H_t of a V^4, $(+++-)$ the spin-holonomy group $\tilde{H}_t \equiv \tilde{\sigma}^1(H_t)$ defined by the covering isomorphism $\sigma: G \to L_+^\uparrow$ is introduced. In Einstein-spaces we may replace its real Lie-algebra by a complex one. With the complex calculus we may reproduce the results of Schell [J. Mathematical Phys. **2** (1961), 202–206; MR **23** #A2834], Goldberg and Kerr [ibid. **2** (1961), 327–332; MR **23** #A2836] with very much simplified proofs. A theorem on non-empty Einstein-spaces is given. In part 4 we prove a theorem on the connection between the H_t-behaviour of a vector (spinor) and its covariant derivative in a V^4. With its help we get in a simple manner the metrics of a V^4 with given H_t and $\mathrm{Dim}(H_t) < 6$; our results agree with those given by Goldberg and Kerr, Cahen and Debever. Finally we make some new statements on imperfect holonomy groups." *S. Ide* (Sapporo)

Stakvilevičus, M. I. 3226

Particle-like solutions with gravitation included. (Russian)

Vestnik Moskov. Univ. Ser. III Fiz. Astronom. **1964**, *no.* 3, 86–90.

This is an investigation to see whether gravitational forces can be significant for the description of a particle as a region where the energy momentum density is large but not infinite. As a model the general covariant form of the Klein-Gordon equation with an additional non-linear term is solved in association with Einstein's equations. It is found that if r_0 is the radius of the particle, determined by the distances at which the matter field ϕ decreases rapidly (only the spherically symmetric case is considered), r_g is the gravitational radius $2\mu/c^2$, and μ the mass, then $r_g/r_0 \sim 10^{-42}$ if $r \sim 10^{-13}$ cm and $\mu \sim 10^{-27}$ gm. This can be interpreted as saying that gravitational forces must be negligible compared with non-gravitational forces. However, the author feels that there may be circumstances in which this is not so. *C. A. Hurst* (Adelaide)

Jordan, Pascual; Ehlers, Jürgen; Kundt, Wolfgang; Ozsváth, István; Sachs, Rainer K.; Trümper, Manfred 3227

Lösungen der Einsteinschen Feldgleichungen mit einfach transitiver Bewegungsgruppe. Strenge Lösungen der Feldgleichungen der allgemeinen Relativitätstheorie. VI.

Akad. Wiss. Lit. Mainz Abh. Math.-Natur. Kl. **1962**, 1001–1021.

Previous parts of this excellent compendium of new and known results appeared as follows. Part I (computational aids, axially symmetric vacuum fields and plane-fronted gravitational waves, by Jordan, Ehlers and Kundt), same Abh. **1960**, 21–105 [MR **23** #B982]; Part II (gravitational ray-optics and radiation, by Jordan, Ehlers and Sachs), ibid. **1961**, 1–62 [MR **27** #6601]; Part III (electromagnetic null fields in geometrodynamics, by Jordan and Kundt), ibid. **1961**, 85–97 [MR **25** #1001]; Part IV (on mechanics of continua, by Jordan, Ehlers, Kundt and Sachs), ibid. **1961**, 791–837 [MR **27** #6602]; Part V (Einstein-Maxwell and interior Einstein fields of Petrov type N, by Kundt and Trümper), ibid. **1962**, 965–1000 [MR **28** #5804].

This part, by Ozsváth, treats space-times admitting simply transitive groups of motions. First, a general procedure is developed for solving the field equations in the case where space-time is homogeneous and admits a 4-parameter simply transitive group. Five explicit solutions are presented: three vacuum solutions, including the anti-Mach metric of Ozsváth and Schücking [*Recent developments in general relativity*, pp. 339–350, Pergamon, Oxford, 1962; MR **29** #2015] and two solutions corresponding to pressure-free dust and non-vanishing cosmological constant, one of which generalizes Einstein's static model and represents a closed rotating universe.

Finally, a method is outlined for solving the pressure-free interior field equations in the case where the spatial sections (in co-moving co-ordinates) admit a 3-parameter simply transitive group [cf. A. Z. Petrov: *Einstein spaces* (Russian), Fizmatgiz, Moscow, 1961; MR **25** #4897; German translation (edited by H. J. Treder) *Einstein-Räume*, Chaps. IV, V, Akademie-Verlag, Berlin, 1964; MR **28** #5792]. *W. Israel* (Edmonton, Alta.)

Mirianašvili, M. M.; Gobedžišvili, M. S. 3228

Solution of the gravitational field equations by the "falling box" method. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **33** (1964), 543–548.

The authors claim to have generated several exact solutions of the field equations, extending a method attributed to W. Lenz [A. Sommerfeld, *Electrodynamics*, p. 313 Academic Press, New York, 1952; MR **14**, 433]. They obtain (1) exterior and (2) interior Schwarzschild solutions; (3) the Reissner-Nordström solution; and (4) the cylindrically symmetrical field of a line source. The method uses the principle of equivalence and the Newtonian formula for the potential energy of a falling box {It seems to the reviewer that, quite apart from certain openly stated approximations, the method cannot, in general, lead to exact solutions. Two separate errors have been compounded: It is assumed first that there exist coordinates tied to the falling box in which the metric assumes Minkowskian form throughout a finite region then a non-integrable coordinate transformation is used to produce a non-flat metric. Moreover, metrics (2) and (4) above appear to be incorrectly given.}

R. H. Boyer (Austin, Tex.)

Dautcourt, G. 3229

Zum charakteristischen Anfangswertproblem der Einsteinschen Feldgleichungen.

Ann. Physik (7) **12** (1963), 302–324.

The author discusses the initial-value problem of Einstein's field equations for data given on an arbitrary null hypersurface. Unless this is the light cone, these data do not determine the neighboring field uniquely. The author consequently considers two null hypersurfaces U and V and the "wave surface" Σ in which they intersect. It is necessary to specify the complex shear (or two equivalent parameters) of the null geodesics in U and V as well as four scalars in Σ in order to determine the field in a four-dimensional neighborhood of Σ uniquely. In the case of the light cone, the future and past sheets serve as U and V. The discussion is purely local, and no estimate can be given of the convergence of the approximating series, which are in terms of coordinates leading out of U and V. As an example, the formalism is applied to a discussion of gravitational shock waves. *W. Rindler* (Dallas, Tex.)

Günther, H. 3230

Die elektrische Kraft in der unitären Feldtheorie von Einstein-Schrödinger.

Monatsb. Deutsch. Akad. Wiss. Berlin **6** (1964), 204–210.

Nimmt man in der unitären Feldtheorie Schrödingers die "kosmologische Konstante λ" als groß an, so liefert die EIH-Methode für einfache Polteilchen eine Kraftwirkung gemäss dem Kraftgesetz von Bopp und Podolsky. Für Teilchen komplizierterer Struktur resultiert u.a. das Coulombsche Gesetz. Wegen der vorausgesetzten Größe von λ erhält man jedoch für die Gravitationskraft einen sinnlosen Ausdruck. Wird in der Schrödingerschen Theorie λ hingegen als klein eingeführt, so folgt für die elektrische Kraftwirkung zwischen Singularitäten wieder der von Treder [Ann. Physik (6) **19** (1956), 369–380; MR **19**, 816] für die Einsteinsche Theorie hergeleitete Ausdruck.

H. Treder (Berlin)

Feynman, R. P. 3231

Quantum theory of gravitation.

Acta Phys. Polon. **24** (1963), 697–722.

The aim of the work that is discussed here is a formalism which describes the gravitational interaction in terms of the exchange of a virtual graviton between the interacting matter. The technique follows the standard calculational methods which were developed in quantum electrodynamics.

The Lagrangian is first expressed as the sum of the standard Einstein form and a part that relates to a scalar matter field. The procedure, then, is to express the metric tensor solution of Einstein's equations as $g_{\mu\nu}=\delta_{\mu\nu}+\kappa h_{\mu\nu}$, where $\delta_{\mu\nu}$ is the flat-space metric, $h_{\mu\nu}$ is a tensor field that is to be treated as a perturbation on the flat-space approximation and κ is the square root of the gravitational constant. By treating the perturbing field $h_{\mu\nu}$ and the scalar matter field as the independent variables, the variational procedure yields the corresponding field equations. It is then shown that with the selection of a particular gauge, the field equation in $h_{\mu\nu}$ reduces to $\bar{h}_{\mu\nu,\sigma\sigma}=\bar{S}(h,\varphi)$. This is, of course, the tensor analogue to the vector field equation of electrodynamics, $A_{\mu,\sigma\sigma}=j_\mu(A)$. The matter field solves the scalar equation, $\varphi_{,\sigma\sigma}-m^2\varphi=\bar{\chi}(h,\varphi)$.

The ensuing discussion develops an approach to solving the resulting coupled field equations with the usual propagator techniques of quantum field theory. In addition to the standard difficulties that are encountered in quantum electrodynamics, the author points out that other difficulties occur here. He then concentrates his attention on these difficulties, which have to do with the evaluation of the closed loop diagrams. The author remarks further that analogous difficulties are encountered in the Yang-Mills theory, applied to massless mesic fields. To resolve the difficulty, the latter theory was considered in detail, with the hope of application to the proposed quantum theory of gravitation. The attempted resolution is based on a theorem (which the author discovered) that asserts a relation between the evaluation of the closed-loop diagrams and diagrams without closed loops (which he calls "trees"). From the theorem, the author concludes that "if every closed diagram can be expressed as trees, and if the trees produce no trouble and can be computed, then all you have to do is to say that the closed-loop diagram is the sum of the corresponding tree diagrams, that it should be. Finally, in each tree diagram for which the graviton line has been opened, take only the real transverse graviton to represent that term".

The author discusses a theorem which leads to a computational technique for handling the perturbation expansion of solutions of a particular type of field equation. It seems, however, to this reviewer that neither a rigorous derivation nor a concise expression of the theorem is made clear in the paper. Suppose, however, that these requirements were to be met. Several questions in regard to this approach still persist. Do the resulting calculations relate, in fact, to the solutions of Einstein's equations (in their original or generalized forms)? It is clear that they would relate to the solutions of the coupled equations from which he started. However, is $\delta_{\mu\nu}+\kappa h_{\mu\nu}$ (with $h_{\mu\nu}$ mapped onto a flat space-time!) a valid solution of Einstein's equations? Are the calculational techniques of quantum electrodynamics (whose non-linear part can be "switched off" at will, and in any case, is linearized through second quantization) compatible with a formalism whose structure is intrinsically nonlinear?

The author gives a refreshingly objective account of his own investigation. He points out that a similarity exists between his equations and the Yang-Mills formalism for massless mesons. Thus, his results could be applicable to the latter case. However, he also admits that this approach does not lead to a formulation of quantum gravidynamics; it only implies a possible method of calculation. Thus, the computations give numbers that cannot be identified with a physical law that has to do with the quanta of "gravitation". Further, the author admits that the same mathematical inconsistencies which reside in the scheme of quantum electrodynamics are carried over here (in addition to others!). Thus, it is difficult, at this stage, to call the scheme that is discussed a "theory". Consequently, until these difficulties can be removed, this reviewer feels that the only valid remainder in the title of his paper is "Quantum".

Needless to say, this reviewer feels equally strongly that if it is possible that these difficulties can be resolved within the framework of the quantum theory, it may very likely be done one day by the author.

M. Sachs (Boston, Mass.)

ASTRONOMY

Pringle, Ralph, Jr. 3232

Bounds on the librations of a symmetrical satellite.

AIAA J. **2** (1964), 908–912.

The problem discussed in this paper concerns that of the motion of a symmetrical rigid body in a circular orbit about a central attracting body, the rigid body being free to rotate about its axis of symmetry. This is basically the problem of precession and nutation that has been exhaustively treated for the case of high spin. The author is interested in the use of the theory to predict phenomena occurring in artificial satellites. This requires exploration of the complete range of motions instead of restriction to the high-spin case.

The study proceeds by obtaining the equilibrium points (in which the axis of rotation of the rigid body may be at

rest in a coordinate system rotating with angular velocity equal to the orbital mean motion), and by a discussion of the motions in the vicinity of these equilibrium points. Particular attention is given to the curves of zero velocity which define regions of bounded motions and to curves of bifurcation on which the nature of the stability of equilibrium points changes.

D. Brouwer (New Haven, Conn.)

Novoselov, V. S. 3233

Optimal two-impulse transfer between orbits with small inclinations and eccentricities. (Russian. English summary)

Bjull. Inst. Teoret. Astronom. **9** (1963), 295–309.

D. F. Lawden [Advances in Space Science, Vol. 1, pp. 1–53, Academic Press, New York, 1959] gave a general variational method for the determination of optimal transfer trajectories with an impulsive thrust in the gravitational field of a spherically symmetric central body. Restricting himself to terms of first-order smallness, he determined the optimal transfer trajectory between coplanar orbits with small eccentricities. Under the same restriction R. S. Long [Astronaut. Acta **6** (1960), 167–178] determined the optimal transfer trajectory between non-coplanar orbits with finite mutual inclination and small eccentricities. In the present paper the author is concerned with optimal transfer trajectories between orbits with small inclinations and eccentricities. While in the theories of Lawden and Long the initial and final orbits are described in terms of undetermined multipliers and both orbits must be extremals, the author starts from the theory of optimal impulsive transfers, not requiring the examination of the undetermined multipliers along the terminal orbits.

In constructing the optimal transfer orbits spherical coordinates are used and small terms up to second order are taken into account. As a possible application of the theory optimal transfers between Earth's orbit and those of Venus and Mars are mentioned.

E. Leimanis (Vancouver, B.C.)

Nikolenko, L. D. 3234

On the effect of a non-spherical earth on the motion of a satellite. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 745–748.

Author's summary: "The results presented are of an analytical investigation of the motion of an artificial satellite in a non-central field of gravity of the earth. An approximate solution of the problem is obtained by the averaging method of N. M. Krylov and N.N. Bogolyubov."

Včerašnjuk, P. P. 3235

On the motion of an artificial earth satellite about its center of mass. (Russian)

Ukrain. Mat. Ž. **15** (1963), 305–309.

The paper is concerned with the motion of an artificial satellite relative to its centre of mass, which, by assumption, describes a circular orbit in the gravitational field of the Earth, assumed to be a homogeneous sphere. Let the mass distribution of the satellite be such that $A=B$ and $(C-A)/C=\varepsilon$ is small, where A, B, C are the principal inertia moments of the satellite. Further, disregard the aerodynamical forces. In order to integrate the corresponding canonical system of equations, an averaging method developed by the author in an earlier paper is applied [Trudy Konferencii Aspirantov i Sotrudnikov Inst. Matem. Akad. Nauk Ukrain. SSR, Izdat. Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev, 1963]. Since the ellipsoid of inertia of the satellite is not a sphere ($A\neq C$), the motion of the artificial satellite about its mass centre reduces to a perturbed regular precession.

E. Leimanis (Vancouver, B.C.)

Codina, Sayd 3236

Some astronomical applications of the virial theorem. (Spanish. English summary)

Univ. Nac. Tucumán Rev. Ser. A **14** (1962), 197–219.

Danby, J. M. A. 3237

The stability of the triangular Lagrangian points in the general problem of three bodies.

Astronom. J. **69** (1964), 294–296.

The author's summary: "If three particles of masses m_1, m_2, and m_3 move subject to their mutual attractions, it is possible for them to maintain a relative configuration of an equilateral triangle, while each moves in a Keplerian orbit about the centre of mass of the system. It is shown here that the configuration is stable against infinitesimal displacements if $(m_2m_3+m_3m_1+m_1m_2)/(m_1+m_2+m_3)^2$ is less than some quantity that depends only on the eccentricity of the Keplerian orbits."

P. C. Chaudhuri (Nsukka)

Brumberg, V. A. 3238

Series of polynomials in the three-body problem. (Russian. English summary)

Bjull. Inst. Teoret. Astronom. **9** (1963), 234–256.

If $f(w)$ is an analytic function given by its local Taylor series (1) $f(w)=a_0+a_1w+a_2w^2+\cdots$ which converges in a certain finite circle with center at the origin, then, by a well-known theorem of Mittag-Leffler, $f(w)$ can be represented in any region inside the star region of this function by a uniformly convergent series of polynomials, the coefficients of which are linear combinations of the coefficients of series (1). In other words, at any point inside the star region there exists a sequence of polynomials $f_n(w)$ $(n=1, 2, \cdots)$, where (2) $f_n(w)=c_0^{(n)}a_0+c_1^{(n)}a_1w+\cdots+c_{m_n}^{(n)}a_{m_n}w^{m_n}$, which converges uniformly to $f(w)$. The convergence coefficients $c_k^{(n)}$ do not depend on the form of $f(w)$ but are given by the coefficients of polynomials (3) $g_n(w)=c_0^{(n)}+c_1^{(n)}w+\cdots+c_{m_n}^{(n)}w^{m_n}$ which converge uniformly to the value of the function (4) $g(w)=1/(1-w)$ provided that w does not assume any real values between 1 and ∞. The more rapidly the sequence of polynomials (3) converges to the function $g(w)$, the more rapidly, in general, the sequence of polynomials (2) converges to the function $f(w)$. Several authors such as G. Mittag-Leffler, E. Le Roy, E. Lindelöf and O. Perron have constructed different sets of coefficients $c_k^{(n)}$. The ideal set of $c_k^{(n)}$ would be that which implies the maximal rapidity of convergence of the polynomials (3) to $g(w)$ for the minimal order of these polynomials.

E. Picard and P. Painlevé have pointed out the close connection between the problem of expanding analytic

functions into polynomial series and the problem of integrating differential equations by the method of Cauchy and Lipschitz. One of the most suitable expansions for $g(w)$ is that given by E. Goursat, which is the result of applying the Cauchy-Lipschitz method to the differential equation $dg/dw = g^2$ which for the initial condition $g(0) = 1$ determines the function (4). The convergence of the sequence of polynomials obtained by the Cauchy-Lipschitz method was improved by a modification of Picone.

In Section 1 such more effective convergence coefficients $c_k^{(n)}$ are calculated by application of the modified Cauchy-Lipschitz method. In celestial mechanics the simplest example for the application of polynomial series is the two-body problem. Therefore in Section 2 series of polynomials in terms of the mean anomaly are constructed for the elliptic two-body problem. The coefficients of the Sundman series for this problem can be calculated in finite form. The corresponding Sundman series are constructed for reasons of comparison in Section 3. Finally in Section 4 are constructed polynomial series in terms of the variable which regularizes double collisions in the three-body problem. The method is illustrated by four examples due to J. L. Lagrange (1772), J. Zumkley (1941), E. Strömgren (1909) and C. Burrau (1913).

The results of this paper show that in principle polynomial series converging for any real instant of the time can be used with advantage for solving the three-body problem by numerical methods.

E. Leimanis (Vancouver, B.C.)

Szebehely, Victor 3239

Perturbations of the regularized equations of the restricted problem of three bodies.

Astronom. J. **69** (1964), 309–315.

The author's summary: "The differential equations of the restricted problem of three bodies are regularized using Birkhoff's transformation. The terms occurring in the regularized equations are expanded according to the powers of the dimensionless parameter μ which is associated with the mass ratio of the primaries. The nth-order terms are presented and details are given for the first-order theory. The general results are valid for collision with either one or both primaries."

P. C. Chaudhuri (Nsukka)

Zatepjakin, M. M. 3240

On stability of motion in one case of the three-body problem. (Russian)

Approximate methods of solving differential equations, pp. 27–32. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, 1963.

Laplace showed that in the case of an arbitrary law of attraction, $mm'f(r)$, between two particles of masses m and m' at the distance r, where f is an arbitrary function of r, the problem of three bodies admits a particular solution for which the attracting particles remain at the vertices of an equilateral triangle, the plane of which has an invariable position in space. Such a Laplacian motion is said to be stable in the sense of Lyapunov if for arbitrary infinitesimal perturbations of the motion the triangle, at the vertices of which are located the three particles, differs only slightly from an equilateral triangle and the limits of variability for the sides differ only slightly from those in the unperturbed motion during the whole course of motion.

Lyapunov [Soobšč. Har'kov. Mat. Obšč. (2) **2** (1889), 1–94; *Collected works* (Russian), Vol. I, pp. 327–401, Izdat. Akad. Nauk SSSR, Moscow, 1954; MR **19**, 109] solved the stability problem of the Laplacian motions for an arbitrary law of attraction. However, the method used by him for expanding the invariants of the characteristic equation in powers of a small parameter ε leads to a complicated discussion of the stability problem.

The author of the present paper investigates the stability of the Lagrangian equilateral solutions which are sufficiently close to circular for the Newtonian inverse square law of attraction. Lyapunov showed that if λ is not a root of one of the equations

$$\text{(i) } 9\lambda(1-\lambda) = \tfrac{3}{16}, \quad \text{(ii) } 9\lambda(1-\lambda) = \tfrac{1}{4}, \quad \text{(iii) } \lambda(1-\lambda) = 0,$$

then for sufficiently small $\varepsilon \neq 0$ the Lagrangian solutions are stable. Hence cases are of interest where λ is a root of one of the above equations. Case (iii), however, must be disregarded since for $\lambda = 0$ and $\lambda = 1$ the system of equations for the perturbations is integrable and the unperturbed motion is unstable.

In cases (i) and (ii) the Lyapunov result is obtained by a simpler method based on investigations of M. G. Kreĭn [*In memory of Aleksandr Aleksandrovič Andronov* (Russian), pp. 413–498, Izdat. Akad. Nauk SSSR, Moscow, 1955; MR **17**, 738]. In addition, in case (i) the λ-zone of instability

$$\lambda_0 - \varepsilon|\lambda_1| + \cdots < \lambda < \lambda_0 + \varepsilon|\lambda_1| + \cdots$$

is determined, where λ_0 is a root of equation (i) and $\lambda_1 = \pm[\lambda_0 - (1-\lambda_0)(\tfrac{1}{4} + 3\lambda_0)^2]/2[1 - (\tfrac{1}{4} + 3\lambda_0)^2]$. In case (ii) the Lagrangian motions are stable.

E. Leimanis (Vancouver, B.C.)

Surdin, Maurice 3241

Étude de la distribution des masses des poussières interplanétaires et de celles des étoiles.

C. R. Acad. Sci. Paris **258** (1964), 82–85.

This article consists of two essentially independent parts, both dealing with distribution laws of cosmic masses. In the first part the author notes that empirically obtained laws for the distribution of interplanetary masses, ranging from 10^{-10} to 10^{10} g, are all of the form

$$df = \frac{C}{m^a}\, dm,$$

df being the number of particles with masses between m and $m + dm$, C a constant, and a a number close to 2. The author then derives such a distribution law by considering a process of ablation.

The second part of the paper is concerned with the state of equilibrium of a system of interacting bodies with exchange of mass. The distribution law obtained is

$$df = \frac{1}{m_0} \exp\left(-\frac{m}{m_0}\right) dm,$$

m_0 being the average mass. The empirical distribution law obtained for stars in the vicinity of the sun is in good agreement with this law if $m_0 = 0.5 m_\odot$.

D. Brouwer (New Haven, Conn.)

ECONOMICS, OPERATIONS RESEARCH, GAMES

See also 2171, 2711, 2796, 2935, 2937, 2945, 2946, 3015, 3312.

Brans, J. Pierre 3242

Quelques aspects du problème de la ruine en théorie collective du risque.

Cahiers Centre Études Rech. Opér. 5 (1963), 139–159.

The author deals with the ruin problem of collective risk theory. He considers two types of risk processes: $Z_1(t) = Z_1(0) + X(t) - t$, $0 \leq t < \infty$, and $Z_2(t) = Z_2(0) + t - X(t)$, $0 \leq t < \infty$, where $\{X(t), 0 \leq t < \infty\}$ is a compound Poisson process with positive increments. Denote by $T_1(u)$ the smallest t such that $Z_1(t) = 0$, given that $Z_1(0) = u$, and by $T_2(u)$ the smallest t such that $Z_2(t) = 0$, given that $Z_2(0) = u$. The author finds explicit expressions for the distributions of $T_1(u)$ and $T_2(u)$. The problems discussed are equivalent to finding the distribution of the length of the initial busy period and the distribution of the virtual waiting time for a single-server queue with Poisson input and general service times. *L. Takács* (New York)

Barra, J. R.; Brodeau, F. 3243

Les problèmes d'extrema de la programmation dynamique de type discret déterministe.

Rev. Française Traitement Information [*Chiffres*] 7 (1964), 49–80.

A direct (forward) method is presented for solving deterministic dynamic programming problems of discrete nature. It is shown that this method leads to the same "optimal policy" as obtained by the backward solution of Bellman. In order to obtain an algorithm, it has been necessary to impose one of two restrictions on the values which the "decision" variable x_i can assume in period i. Either x_i is bounded from below, i.e., $x_i \geq a_i$, where a_i is a real number, or $x_{i+1} \geq x_i$ (it is claimed that a large number of practical problems, especially inventory control problems, falls in one of these two categories). The first approach leads to type A solutions for which a computer program, written in ALGOL, is presented; the second one to type B solutions which are of a more complicated nature. Finally, an application of both types of solutions is given for a classic inventory control problem.

H. F. Karreman (Madison, Wis.)

Fukao, Takeshi; Nureki, Reiko 3244

Applications of dynamic programming.

Information Processing in Japan 2 (1962), 4–9.

This paper contains a sketchy account of the application of the dynamic programming idea to a stochastic multi-period minimization problem arising from the control of a system of reservoirs used in power generation.

M. L. Balinski (Princeton, N.J.)

Kubát, Libor 3245

Note on the reliability of parallel redundant systems.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 453–460. *Publ. House Czech. Acad. Sci., Prague,* 1964.

The author considers a system composed of components working simultaneously. A failure of one component causes a failure of the entire system. The lifetimes of individual components are assumed to be mutually independent random variables, not necessarily identically distributed. The reliability of the system is defined as the probability that the system will not fail within a prescribed time interval, with a similar definition of the reliability of individual components. To increase the reliability of the system, some parts of it are duplicated, triplicated, etc. Parallel activity of the redundant parts is assumed. It is shown in the paper that optimum redundancy under the conditions stated above consists in multiplying individual components, i.e., in including parallel elements and not whole chains of them. To compare individual component redundancy with entire part redundancy, the author introduces an "advantage factor" and shows that the advantage of the former type of redundancy over the latter is great if the reliabilities of individual components are small, and may be negligible if they are near 1.

J. Machek (Prague)

Klee, Victor 3246

A 'string algorithm' for shortest path in directed networks.

Operations Res. 12 (1964), 428–432.

It is shown that any shortest-path algorithm A for undirected networks can be combined with a simple cutting procedure C to form a shortest-path algorithm for directed networks having positive arc lengths. The general scheme of the combined A-C algorithm is to ignore arc directions and apply A, thereby locating a shortest path from the starting node p to the finishing node q in the associated undirected network. If this path is also a (directed) path in the directed network, terminate the procedure. If not, apply C as follows. Locate first and last arcs α and β in the path which are traversed against their orientations. Delete these arcs together with all non-path arcs which are co-terminal with α and all non-path arcs which are co-initial with β. Then ignore directions, apply A again to the new network, and so on. If the starting network has v nodes and e arcs, it is shown that a shortest path from p to q can be found by $\min[v, \frac{1}{2}(e+2)]$ or fewer applications of A, each followed by an application of C.

For digital computation it would not be worthwhile to use such a procedure, since locating shortest paths in directed networks is no harder than doing the same for undirected networks. But the method does show how certain analogue devices, which work only for undirected networks (for example, the Minty "string computer"), can be applied to directed networks.

D. R. Fulkerson (Santa Monica, Calif.)

Ross, John; Cliff, Norman 3247

A generalization of the interpoint distance model.

Psychometrika 29 (1964), 167–176.

Authors' summary: "Coombs and Kao have conjectured that, given the distances between a set of points representing persons and another representing objects, factor analytic methods can be used to recover the locations of the persons and objects. In the present paper it is proved

that the conjecture applies, with some qualifications, to squared distances rather than distances themselves. Possible applications to scaling and to inventory responses are discussed."

Savage, I. Richard 3248

Surveillance problems: Poisson models with noise.

Naval Res. Logist. Quart. **11** (1964), 1–13.

The author generalises his earlier contribution [same Quart. **9** (1962), 187–209] by the introduction of noise. He considers two cases: (1) continuous surveillance and (2) costly surveillance. If production and noise are independent Poisson processes, the strategies which maximize average income per time unit depend only on the last observation. Rules are derived for the problem of when to make repairs and (in the case of costly observations) when to make the next observation. These rules are (a) stationary, (b) Markov in the sense that they depend on known observations, (c) non-randomized.

G. Tintner (Los Angeles, Calif.)

Gluss, Brian 3249

An alternative method for continuous line segment curve-fitting.

Information and Control **7** (1964), 200–206.

Author's summary: "In a recent paper [Information and Control **5** (1962), 261–267; MR **26** #4851], the author described an algorithm for determining the best least-squares fit of $N+1$ continuous line segments to a curve $g(x)$ in an interval (α, β). The method involved the solution of functional equations in which the functional was defined over a two-dimensional space. In this paper is formulated an alternative algorithm, using a different criterion function, that reduces to consideration of a functional of one variable. The criterion function gives the best least-squares fit of (constant) slopes of line segments to the derivative $g'(x)$ of the function $g(x)$; this leads to a class of solutions, from which is selected that which minimizes the original criterion function. An analogy to control theory along lines recently pointed out by Aoki is discussed."

M. Aoki (Los Angeles, Calif.)

Gordon, W. J.; Newell, G. F. 3250

Equilibrium analysis of a stochastic model of traffic flow.

Proc. Cambridge Philos. Soc. **60** (1964), 227–236.

On a highway vehicles are traveling in one direction in accordance with a Poisson stream. The vehicles are traveling at independently chosen constant speeds $v_1, v_2, \cdots, v_n$ unless they are prevented by slower moving vehicles ahead. If a fast moving group of vehicles catches up with a slow moving group of vehicles, then the former overtakes the latter and the time differences between successive overtakings are exponentially distributed random variables whose expectations depend only on the speed of the slow moving group. For a stationary process the authors deduce a system of equations for the determination of the expected number of groups of vehicles of speed v_i $(i=1, 2, \cdots, n)$ per unit distance. The cases $n=2$ and $n=3$ are investigated in detail.

L. Takács (New York)

Pietrzykowski, T. 3251

Application of Monte-Carlo method to linear programming.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 297–300.

Author's introduction: "The present paper deals with the description of an iteration method of solving the problem of linear programming. This method is a modification of the one described by the author in [Prace ZAM Ser. A No. 13 (1961); MR **25** #4422]. The disadvantage of the latter is that the number of computations required for separate iteration steps is not always the same: when a successive term of a sequence is 'in the neighbourhood' of a k-dimensional face of an n-dimensional set of solutions $(0 \leq k \leq n-2)$ then for finding the next term it is necessary to solve the system of $n-k+1$ linear inequalities. In this paper such a situation is avoided by admitting no term of the sequence to be in the neighbourhood of the boundary; it is prevented by applying pseudo-random modifications."

Bartkus, A. T. 3252

An approximate solution of certain combinatorial problems of linear programming by a dichotomy method. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 776–780.

The following problem is studied: Given the real numbers a_{ij} $(i=1, \cdots, m;\ j=1, \cdots, n)$, b_i', b_i'' $(i=1, \cdots, m)$, c_j $(j=1, \cdots, n)$ such that $b_i' > b_i''$ and $c_j \geq 0$, determine non-negative numbers x_j $(j=1, \cdots, n)$ satisfying, for any i, either

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i'$$

or

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i'',$$

and maximizing under these conditions the linear form

$$L(X) = \sum_{j=1}^{n} c_j x_j.$$

The basic idea of the proposed method for solving this problem consists in successively dividing the set of solutions in two subsets, and eliminating one of these subsets which cannot contain the optimal solution. A simple criterion for selecting the desired subsets, and on this basis, an algorithm for solving the problem, are given.

P. L. Ivănescu (Bucharest)

Heinhold, J.; Kuntze, K. 3253

Ein Verfahren der linearen Optimierung.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2**, 255–272 (1964).

For the linear program Max $z=\mathfrak{c}\mathfrak{x}$ under the restrictions $\mathfrak{a}_\mu\mathfrak{x}+b_\mu \geq 0$, where $\mathfrak{c}$, $\mathfrak{a}_\mu \neq 0$ and $\mathfrak{x}$ are vectors, a solution method to be used for analog computers is given, which is based on the solution of the differential equation $d\mathfrak{x}/dt = \mathfrak{v}(\mathfrak{x})$. Here the parameter t represents the time and $\mathfrak{v}(\mathfrak{x})$ is either written explicitly for points which do not lie in any hyperplane $\mathfrak{a}_\mu\mathfrak{x}+b_\mu=0$, or if $\mathfrak{x}$ is contained in a hyperplane, is determined recursively. It is proved that the procedure converges towards the optimum, if such a one exists; on the other hand, one may also determine by this procedure whether there is no solution.

H. Künzi (Zürich)

Hoffman, A. J. 3254

On abstract dual linear programs.

Naval Res. Logist. Quart. **10** (1963), 369–373.

Let S be a set which is a commutative semigroup under an operation (addition) and a semigroup under another operation (multiplication), distributive with respect to addition. It is assumed that S is partially ordered by a relation ($\leqq$) such that $a \leqq b$ implies $a+x \leqq b+x$ for all $x \in S$. Finally, S admits a subset P such that $a \leqq b$, $x \in P$ imply $ax \leqq bx$ and $xa \leqq xb$.

For any choice of a_{ij}, b_i, c_j in S ($i=1, 2, \cdots, m$; $j=1, 2, \cdots, n$), define the sets of tuples $F=\{(x_1, \cdots, x_n) | x_j \in P, \sum_{j=1}^n a_{ij}x_j \leqq b_i\}$ and $G=\{(y_1, \cdots, y_m) | y_i \in P, \sum_{i=1}^m a_{ij}y_i \geqq c_j\}$; then if $F \neq \varnothing$ and $G \neq \varnothing$, the trivial part of the duality theorem of linear programming holds, i.e.,

$$(*) \qquad \max_{(x_1,\cdots,x_n)\in F} \sum_{j=1}^n c_j x_j \leqq \min_{(y_1,\cdots,y_m)\in G} \sum_{i=1}^m y_i b_i.$$

In addition to the proof of this result, several ingenious examples of sets S are given.

Finally, the following two problems are stated: (1) Characterize the subfamilies of sets S for which equality holds in (*); (2) For any given S find all matrices $A=(a_{ij})$, such that for any choice of b_i and c_j defining non-empty sets F and G, equality holds in (*).

A. G. Azpeitia (Amherst, Mass.)

Jaeger, Arno; Mond, Bertram 3255

On direct sums and tensor products of linear programs.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **3**, 19–31 (1964).

The authors deduce, in the language of abstract algebra, that the feasible vectors, optimal vectors, dual space and value of the direct sum of a family of linear programming problems (of the same type) are respectively the direct sum (or sum) of the feasible vectors, optimal vectors, dual spaces and values of the summand programs. For tensor products, the situation is more complicated and partial results are supplied for cases in which equality is required in the component programs. For example, assuming further that the components require non-negative solutions and have non-negative objective functions, the tensor product of optimal solutions will be optimal for the product problem. *J. J. Stone* (Cambridge, Mass.)

Bahtin, I. O. [Bahtin, I. A.] 3256

An approximate method of finding the optimal values of a non-linear problem. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 730–734.

Author's summary: "An approximate method is described for finding the least value and the points at which this value is attained of the function

$$(1) \qquad \Phi = \sum_{i=1}^L a_i^{x_i} b_i^{y_i} \qquad (0 < a_i, b_i \leqq 1)$$

on the polyhedron

$$(2) \quad \sum_{i=1}^L x_i = m_1, \quad \sum_{i=1}^L y_i = m_2 \qquad (x_i, y_i \geqq 0; m_1, m_2 > 0).$$

As compared with the exact method of M. A. Krasnoselsky, A. Y. Levin and I. A. Bahtin, this method considerably reduces the number of operations required for finding the optimal magnitudes of function (1)–(2)."

Evstigneev, V. A. 3257

A time transportation problem in graph theory. (Russian)

Dokl. Akad. Nauk SSSR **157** (1964), 814–815.

Direct formulas are established for the determination of the minimal time needed for the transportation of a given amount of a commodity through a loop-free network with given lengths and capacities of arcs.

P. L. Ivănescu (Bucharest)

Pšeničnii, B. M. [Pšeničnyĭ, B. N.] 3258

The connection between the theory of graphs and the transportation problem. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 427–430.

Author's summary: "An algorithm for the solution of the transportation problem, based on the application of the theory of graphs and the theory of Kirchhoff's equations, is presented in this paper." *P. L. Ivănescu* (Bucharest)

Verhovskiĭ, B. S. 3259

Distribution of a non-homogeneous production allowing for conversion at intermediate points. (Russian)

Dokl. Akad. Nauk SSSR **158** (1964), 17–20.

The problem of producing in different centres several non-homogeneous products which are converted at some intermediate points with limited capacities is viewed as a four-index transportation problem: Find x_{ijkl} minimizing

$$L = \sum_{i,j,k,l} x_{ijkl} p_{ijkl}$$

and subject to

$$\sum_{k,l} x_{ijkl} \leqq a_{ij},$$

$$\sum_{i,k} x_{ijkl} = c_{jl},$$

$$\sum_{i,j,l} x_{ijkl} \leqq b_k, \qquad x_{ijkl} \geqq 0.$$

Properties of the solutions of the above problem are examined, and an algorithm for its solution is outlined.

P. L. Ivănescu (Bucharest)

Richter, Reinhold 3260

Über eine Erweiterung des Transportproblems.

Wiss. Z. Hochsch. Verkehrswesen "Friedrich List" Dresden **11** (1964), 73–77.

Expository paper on the capacitated transportation problem. Examples are given.

P. L. Ivănescu (Bucharest)

Fan, Liang-tseng; Wang, Chiu-sen 3261

The application of the discrete maximum principle to a transportation problem.

J. Math. and Phys. **43** (1964), 255–260.

The authors consider a non-linear version of the Hitchcock-Koopmans transportation problem. It is shown how the discrete form of the Pontryagin maximum principle can be used to solve the problem. Numerical solution is obtained by hand-calculation for the case where the transportation costs involve a fixed charge plus a cost that is a quadratic function of the amount shipped. It is

seen that in this case the method presented is more efficient than the general dynamic programming algorithm previously used by Bellman and the reviewer to solve the same problem. *S. E. Dreyfus* (Santa Monica, Calif.)

Goldstein, A. A. 3262
Convex programming in Hilbert space.
Bull. Amer. Math. Soc. **70** (1964), 709–710.
Let **C** be a closed convex subset of a Hilbert space **H**. The author gives an iterative algorithm for minimizing certain twice-differentiable functions $f(\cdot)$ on **C**. The iteration is $x_{k+1} = \mathbf{P}(x_k - \rho_k \nabla f(x_k))$, where **P** denotes the projection operator for **C** and $\nabla f(\cdot)$ is the Fréchet derivative and ρ_k is a suitably chosen scalar. The convergence proof involves minimal assumptions on $f(\cdot)$. The algorithm is more general than the familiar Arrow-Hurwicz algorithm of convex programming in that it is not required that **C** be characterized by inequalities. Of course, it does require knowledge of **P**. The result has important application to control problems; for instance, see the reviewer's paper [J. Soc. Indust. Appl. Math. Ser. A Control **1** (1963), 109–127; MR **27** #5645].
A. V. Balakrishnan (Los Angeles, Calif.)

Warga, J. 3263
A convergent procedure for solving the thermochemical equilibrium problem.
J. Soc. Indust. Appl. Math. **11** (1963), 594–606.
The author suggests that the problem of minimizing the free energy of a mixture of chemical elements and compounds might be solved by using the procedures described in the two previous papers [same J. **11** (1963), 579–587; MR **28** #5835; ibid. **11** (1963), 588–593; MR **28** #5834]. No reference is made to other work on this problem.
E. M. L. Beale (London)

Ash, Milton; Jones, Wayne 3264
Optimal strategies for maximum-number games.
J. Math. Anal. Appl. **9** (1964), 138–140.
Consider the following problem: Given N different real numbers on cards face down on a table, what is the stopping rule that maximizes the probability of choosing the largest of the N numbers? Additional rules of the game are that previously chosen cards remain face up, and that only the immediate choice can be declared to be largest. Using dynamic programming, the authors obtain the solution to the game. The nature of the solution is to continue until $N-k_0$ cards have been chosen, where $(1-k_0/N)(1-\sum_{j=1}^{k_0-1} 1/(N-j)) = 1/N$, and then stop on the first occasion on which a number is chosen that is larger than the first $N-k_0$ numbers. Similar solutions are obtained when trial costs and prizes are considered and it is required to maximize the expectation of winning.
B. Gluss (Berkeley, Calif.)

Genin, M. A. 3265
Solution of a discrete two-person game with bluffing. (Russian. English summary)
Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **19** (1964), *no.* 1, 26–35.
Solution of a simplification of a card game resembling "I Doubt It"; however, the denomination to be announced does not cycle. Strategies involving more than two successive false calls do not occur in the solution; the value for m-card hands is asymptotic to cr^m, where r is an algebraic number near .36 and c seems to be close to r.
J. R. Isbell (New Orleans, La.)

BIOLOGY AND BEHAVIORAL SCIENCES

Simon, Herbert A. 3266
A note on mathematical models for learning.
Psychometrika **27** (1962), 417–418.
Author's summary: "It is shown that a rational learning curve developed by Estes for paired associate learning is a special case of a more general function. The latter is the product of two functions and assumes that the discovery and fixation aspects of learning are independent. The indications are that the form of one of the functions has not been tested sharply by Estes in the paired associate learning setting."

INFORMATION, COMMUNICATION, CONTROL

See also 2171, 2173, 2464, 2474, 2839, 2930, 3011, 3013, 3014, 3019, 3020, 3021, 3022, 3115, 3116, 3262.

★Mathematische und physikalisch-technische Probleme der Kybernetik. 3267
Vorträge, gehalten auf der Konferenz der Forschungsgemeinschaft der Deutschen Akademie der Wissenschaften zu Berlin vom 20. bis 23. März 1962 in Berlin-Adlershof. Deutsche Akademie der Wissenschaften zu Berlin, Reihe A: Tagungen, Band V. Über Wissenschaftliche Grundlagen der Modernen Technik.
Akademie-Verlag, Berlin, 1963. 557 *pp.* (2 *inserts* + 2 *plates*) *DM* 55.00.
The papers of mathematical interest in the conference described above will be reviewed individually.

Erdős, Paul; Rényi, Alfréd 3268
On two problems of information theory. (Russian summary)
Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 229–243.
The first problem considered by the authors is the behavior of the minimal number $A(n)$ of weighings required to determine the number of false coins in a set of n coins. This problem was proposed for $n=5$ by H. S. Shapiro [Amer. Math. Monthly **67** (1960), 82] and for arbitrary n by N. J. Fine [ibid. **67** (1960), 697–698]. The second problem is the behavior of the minimal number $B(n)$ of testing sequences (which report the number of coincidences) required to determine an unknown binary sequence of length n. The authors state several reformulations of both problems and show that

$$2 \leq \liminf_{n\to\infty} [n^{-1}A(n)\log_2 n] \leq \limsup_{n\to\infty} [n^{-1}A(n)\log_2 n] \leq \log_2 9,$$

$$2 \leq \liminf_{n\to\infty} [n^{-1}B(n) \log_2 n] \leq \limsup_{n\to\infty} [n^{-1}B(n) \log_2 n] \leq \log_2 9.$$

In a remark, they acknowledge the following results (some unpublished) concerning asymptotic behavior, constructive proofs, and sharper bounds for $A(n)$: The result $\lim_{n\to\infty} n^{-1}A(n) = 0$ was conjectured by N. J. Fine and proved by D. G. Cantor, who also obtained $A(n) = O[(\log n)^{-1} n \log\log n]$. L. Moser and H. L. Abbott, and H. S. Shapiro and S. Söderberg obtained

$$\limsup_{n\to\infty} [n^{-1}A(n) \log_2 n] \leq \log_2 27.$$

E. R. Berlekamp obtained $A(n) \leq (\log_2 n)^{-1}(n \log_2 9)$. B. Gordon and L. Moser proved

$$\liminf_{n\to\infty} [n^{-1}A(n) \log_2 n] \geq 2.$$

E. Mills proved $A(n) = O[n(\log n)^{-1}]$. B. Lindström has proved $\lim_{n\to\infty} [n^{-1}A(n) \log_2 n] = 2$.

H. P. Edmundson (Pacific Palisades, Calif.)

Dobrušin, R. L. 3269

An asymptotic bound for the probability error of information transmission through a channel without memory using the feedback. (Russian)

Problemy Kibernet. No. 8 (1962), 161–168.

Consider a discrete memoryless channel and a rate of transmission R such that $A < R < C$, where C is the capacity of the channel and A is a certain "critical" value. Let e_n [respectively, g_n] be the minimum error for codes of word length n when there is no feedback [when there is feedback] in the channel. Then there exist d, D, $0 < d < D < \infty$, such that for all sufficiently large n, $d < e_n/g_n < D$. For all $R < C$ a bound on the order of g_n for n large is given. *J. Wolfowitz* (Ithaca, N.Y.)

Kendall, David G. 3270

Functional equations in information theory.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2**, 225–229 (1964).

D. K. Faddeev [Uspehi Mat. Nauk **11** (1956), no. 1 (67), 227-231; MR **17**, 1098] has characterized Shannon's entropy functions

$$(*)\quad H_k(p_1, \cdots, p_k) = \sum_{j=1}^{k} p_j \log_2 \frac{1}{p_j} \qquad (k = 2, 3, \cdots)$$

by the following three postulates. (1) H_k is a real symmetric function of the variables $p_1, \cdots, p_k$ ($p_j \geq 0$, $\sum p_j = 1$) for $k \geq 2$. (2) $h(t) = H_2(t, 1-t)$ is continuous on the interval $[0, 1]$ and $h(\frac{1}{2}) = 1$. (3) For $k \geq 2$ and $0 \leq t \leq 1$,

$$H_{k+1}(tp_1, (1-t)p_1, \cdots, p_k) = H_k(p_1, \cdots, p_k) + p_1 h(t).$$

In the present paper the author considers a similar system of postulates but for $k=2$ and $k=3$ only; he supposes that (1) holds for $k=2$ and $k=3$, further that (3) holds for $k=2$. Postulate (2) is replaced by the following: (2′) $h(t)$ is nondecreasing for $0 < t \leq \frac{1}{2}$ and $h(\frac{1}{2}) = 1$. It is shown that these postulates imply that H_2 and H_3 are given by (*). *A. Rényi* (Budapest)

Nedoma, Jiří 3271

The synchronization for ergodic channels.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 529–539. *Publ. House Czech. Acad. Sci., Prague*, 1964.

Consider a channel which transmits letters continuously. The letters are encoded in blocks of n. When the receiver does not know which of the sequence of letters received correspond to the initial letters of the various blocks of n, the channel is called unsynchronized. The author proves that for unsynchronized ergodic channels with a finite past history (the exact conditions are not easy to state in a brief review) it is possible to find a method (with a synchronizing signal) which enables the transmission of an ergodic source with entropy rate less than the capacity of the channel. The method involves coding long groups of messages. *J. Wolfowitz* (Ithaca, N.Y.)

Balakrishnan, A. V. 3272

Effect of linear and nonlinear signal processing on signal statistics.

J. Res. Nat. Bur. Standards Sect. D **68D** (1964), 953–965.

Most of this paper is a review of earlier work on some statistical questions involved in signal processing. The problems which are considered include quantization error, errors involved in the use of the sampling theorem, prediction errors and their effects on adaptive data processors, and the distribution of a functional of a Markov process. Some new results are derived or stated in connection with the last two problems.

L. L. Campbell (Kingston, Ont.)

Carlyle, J. W.; Thomas, J. B. 3273

On nonparametric signal detectors.

IEEE Trans. Information Theory **IT-10** (1964), 146–152.

Authors' summary: "This paper is a survey and summary of some of the simpler one-input and two-input detection schemes. Particular emphasis is placed on those systems where less than a complete statistical description of the input is required. Such nonparametric detectors offer the advantage of ease of instrumentation and insensitivity to changes in the input statistics while yielding a fixed maximum false alarm rate. For both the one-input and two-input cases, the paper begins with a review of well-known parametric detectors, proceeds in the direction of increasing departure from the parametric noise model, and introduces corresponding nonparametric techniques."

E. L. Lehmann (Berkeley, Calif.)

Csáki, F. 3274

Simplified derivation of optimum transfer functions in the Wiener-Newton sense.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 55–62. *Publ. House Czech. Acad. Sci., Prague*, 1964.

Integral equations of Wiener-Hopf type have been obtained for optimal filters with semi-free configuration and with semi-free configuration with constraints by Newton. Since it is usually necessary to transform the integral equations and utilize Wiener spectrum factorization theorems, the author presents a frequency-domain

method for obtaining an optimal Wiener filter for systems with semi-free configuration with and without constraint.
M. Aoki (Los Angeles, Calif.)

Kondo, Jiro 3275

A theory of multiple prediction.

J. Operations Res. Soc. Japan **6** (1963/64), 63–81.

This paper considers the problem of prediction for a vector stationary time series, having in mind applications in economics. An attempt is made to obtain the predictor by the method of Wiener-Hopf, but finally a solution is obtained by assuming that the prediction can be accomplished by using a fixed finite number of lags. The paper, which is difficult to read because of its style, suffers from a lack of references to recent work on prediction theory for vector processes, for example, that due to Helson and Lowdenslager [Acta Math. **99** (1958), 165–202; MR **20** #4155] and Wiener and Masani [ibid. **98** (1957), 111–150; MR **20** #4323; ibid. **99** (1958), 93–137; MR **20** #4325].
E. J. Hannan (Baltimore, Md.)

Perez, Albert 3276

Extensions of Shannon-McMillan's limit theorem to more general stochastic processes.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 545–574. *Publ. House Czech. Acad. Sci., Prague,* 1964.

From the author's summary: "The purpose of the present paper is to extend to the case of more general stochastic processes certain fundamental statements concerning the abstract alphabet version of the Shannon-McMillan limit theorem. The versions proved in the present paper represent an extension of the fundamental statement to more general pairs of stochastic processes subject to only very general and natural conditions, applying in particular to all preceding versions." *J. Wolfowitz* (Ithaca, N.Y.)

Strassen, V. 3277

Asymptotische Abschätzungen in Shannons Informationstheorie.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 689–723. *Publ. House Czech. Acad. Sci., Prague,* 1964.

Let $N(n, \varepsilon)$ be the length of the longest code of word length n and probability of error ε for the discrete memoryless channel. It is known that $\log N(n, \varepsilon) = nC + O(\sqrt{n})$, where C is the capacity of the channel. For the $O(\sqrt{n})$ term the author obtains the result $K\sqrt{n} + O(\log n)$, with K given. Other related results are obtained.
J. Wolfowitz (Ithaca, N.Y.)

Hu, Guo Ding [Hu, Kuo-Ting] 3278

On Shannon theorem and its converse for sequences of communication schemes in the case of abstract random variables.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 285–332. *Publ. House Czech. Acad. Sci., Prague,* 1964.

Author's summary: "The transmission schemes that we shall consider are formed by sequences of general message (signal) variables as in the work of Dobrushin, and are not as usual formed by that determined by certain random processes. The Shannon fundamental problem has its different forms when the sense of error function varies. Besides the error function sequences involved in Shannon's first and second theorem (we call them the probability error function sequence and the information error function sequence, respectively), there are frequency error function sequences introduced by Nedoma and general error function sequences by Kolmogorov and Dobrushin. The metric error function sequence and the uniformly bounded non-negative error function sequence also introduced by the author in this paper have the probability and frequency error function sequences as particular cases, while all these sequences are different forms of the general error function sequence.

"In the case of discrete variables capable of taking a finite number of values we have obtained in previous papers all the algebraic relations of information quantities by means of the duality relation between information quantities and a certain additive set function, and a necessary and sufficient condition for the conclusion of Shannon's theorem in the sense of probability and metric error function sequences and also that in the sense of information error function sequence under some slight restriction. This paper is devoted to extending all the above results to the case of abstract variables capable of taking infinitely many values."
J. Wolfowitz (Ithaca, N.Y.)

Smith, George W., Jr.; Clement, Preston R. 3279

A study of the correlation technique of generalized spectral analysis.

J. Franklin Inst. **278** (1964), 133–146.

Authors' summary: "Correlation techniques are used to determine generalized Fourier coefficients of an orthogonalized exponential series corresponding to the impulse response of an unknown linear system whose output is corrupted by additive noise. Useful approximate expressions for the short-term variances of these coefficients under aperiodic and periodic analysis signal excitations are presented. Conditions are derived under which the short-term variances of the spectral coefficients are proportional to the ratio of the power spectral densities of the control signal and the analysis signal and inversely proportional to the short-term averaging time. Simulation studies indicate very close agreement with theoretically expected results."

Winkelbauer, K. 3280

On discrete information sources.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 765–830. *Publ. House Czech. Acad. Sci., Prague,* 1964.

From the author's summary: "In this paper we shall deal with the class of all conceivable discrete information sources each of which produces messages whose letters are taken from a finite or denumerably infinite alphabet. It is shown that such information sources possess a type of

'rates', the values of which, compared with usual capacities of communication channels, play a decisive role in transmission of information. For a stationary (or, more generally, periodic) source, such a uniquely determined 'rate' in general differs from the usual entropy rate and may be expressed as the essential supremum of entropy rates associated with ergodic sources of which the stationary source is composed." *J. Wolfowitz* (Ithaca, N.Y.)

Leskovar, Branko 3281
Probability density function of the envelope of a sine wave superimposed on a narrow-band Gaussian noise for the coherent detection. (Serbo-Croatian summary)
Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II **17** (1962), 243–253 (1963).
Author's summary: "An analysis of the probability density function of the envelope of a sine wave superimposed on the narrow-band Gaussian noise for coherent detection of electrical signals is given. The probability density function of the envelopes is calculated and plotted as a function of both the signal-to-noise and coherent wave-to-noise ratios. Particular attention is given to the probability density functions of the envelopes for various phase angles between the coherent wave and the sine wave plus narrow-band Gaussian noise, which are important in a number of practical applications."

Ogg, Frank C., Jr. 3282
A note on Bayes detection of signals.
IEEE Trans. Information Theory **IT-10** (1964), 57–60.
The Bayes rule for decision among multiple alternatives is reviewed and shown to lead to the choice of the alternative with the least expected cost, when one exists. When signal classes may overlap, the usual definitions of errors and correct decisions are invalid. The paper shows how to select a suitable cost function for overlapping signal classes to obtain a likelihood-ratio detector; the cost function depends on the prior probabilities of the classes and on the distribution of the signal within each class. Generalized Neyman-Pearson detectors are defined as those that fix a relation among certain error probabilities and minimize some combination of the remaining ones. An example is presented to show that such detectors may not correspond to a Bayes rule for any cost assignment. Some cases are exhibited where Neyman-Pearson and Bayes detectors are equivalent.
C. W. Helstrom (Pittsburgh, Pa.)

Fano, Robert M. 3283
A heuristic discussion of probabilistic decoding.
IEEE Trans. Information Theory **IT-9** (1963), 64–74.
An expository paper.

Kasami, Tadao 3284
A decoding procedure for multiple-error-correcting cyclic codes.
IEEE Trans. Information Theory **IT-10** (1964), 134–138.
Author's summary: "A decoding procedure for multiple-error-correcting cyclic codes is described. This method is very simple in principle, and the mechanization is easy for short codes with relatively high redundancy. Applications are made to the cyclic Golay code, the Bose-Chaudhuri (63, 45), (31, 16), (31, 11) codes and the (41, 21) cyclic codes. A block diagram for a decoder for the Golay code is shown." *H. B. Mann* (Madison, Wis.)

Kruševs'kiĭ, A. V. 3285
Mean decoding error for a conjunctive code. (Ukrainian. Russian and English summaries)
Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 737–739.
Author's summary: "The mean error of decoding is considered for a conjunctive code in the case when the equiprobable and stochastic independent numerical communications are transmitted along the coupling channel without memory undergoing the effect of random noises. It is assumed that on transition of the code combination to a forbidden one, a command is carried out in the system which was carried out in it before the new command was sent."

Parthasarathy, K. R. 3286
A note on McMillan's theorem for countable alphabets.
Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 541–543. *Publ. House Czech. Acad. Sci., Prague*, 1964.
The purpose of this note is to give a simple proof of McMillan's theorem for the case of countable alphabets.
J. Wolfowitz (Ithaca, N.Y.)

Belima, O. S. [Belima, A. S.] 3287
Investigation of a discrete centralized information-gathering system with cyclo-sporadic action. (Ukrainian. Russian and English summaries)
Avtomatika **1964**, *no.* 1, 19–28.
Author's summary: "A discrete information-gathering system with random duration of the cycle, serving s processes continuously varying in time and m sporadic (Poisson) streams of information is investigated in this research. Such a system is called a system with cyclo-sporadic action. The method of Markov chains was used for investigation. Formulae are obtained for estimating the basic criteria of a system of cyclo-sporadic action."

Vapnik, V. N.; Červonenkis, A. Ja. 3288
On a class of algorithms for pattern recognition learning. (Russian. English summary)
Avtomat. i Telemeh. **25** (1964), 937–945.
Authors' summary: "Two recognition problems are formulated: the finding of a pattern by a learning system, the synthesis of a learning system. A formal scheme of a learning is described for the first problem. Estimates of learning sequence duration are obtained. An heuristic programme for the solution of problem of learning system synthesis is proposed."

Shen, D. W. C.; Rosenberg, Aron 3289
A nonlinear stochastic learning model.
Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (*Liblice*, 1962), *pp.* 629–636. *Publ. House Czech. Acad. Sci., Prague*, 1964.

The authors consider the estimation of the parameters of the functional relationship between the stochastic series for output and input by means of a system of orthogonal functional polynomials. The set of kernels that characterize the nonlinear system are self-optimized by comparing the trial output with a target function and by establishing a feedback from the error to the variable parameters of the system in such a way as to minimize the mean square error at the next trial.

{The mathematical worth of this paper is obscured by grammatical and mathematical errors. The most likely explanation is that the paper was typeset in Prague and not proofread by the authors. In any case, the authors' statement that "It is conceivable that Wiener's problem can be extended without a logical break to include recognition of characters, recoding and even translation from one language into another" is without foundation in regard to machine translation.}

H. P. Edmundson (Pacific Palisades, Calif.)

Fel'dbaum, A. A. 3290

On a class of self-learning systems with dual control. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 433–444.

Author's summary: "The formulation and the solution of the problem of the optimum learning in a class of closed-loop systems are proposed. The learning is realized by means of the dual control which combines trial actions of a cognitive nature, produced on the plant by the control device, with directive ones bringing the plant to the required state. A simple example illustrates the theory."

Landweber, Peter S. 3291

Three theorems on phrase structure grammars of type 1.

Information and Control **6** (1963), 131–136.

The author shows that the class of languages generated by type 1 phrase structure grammars [Chomsky, Information and Control **2** (1959), 137–167; MR **21** #4107] is not enlarged by allowing end markers, that this class is closed under the operation of intersection, and that those languages representable by linear bounded automata [see Myhill, WADD Tech. Note No. 60-165 (1960)] belong to this class. *H. P. Edmundson* (Pacific Palisades, Calif.)

Al'breht, È. G.; Krasovskiĭ, N. N. 3292

Observation of a non-linear control system in the vicinity of a given motion. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 1047–1057.

Authors' summary: "Problems on observability of a non-linear system with small disturbances and a quasi-linear system are considered. The solution is given in the form of a power series of some functionals and a small parameter."

Balakrishnan, A. V. 3293

A general theory of nonlinear estimation problems in control systems.

J. Math. Anal. Appl. **8** (1964), 4–30.

Certain problems in mathematical physics can be recast as extremum problems of quadratic functionals. In particular, considerable attention has been given to the minimization of a quadratic polynomial associated with a bounded and positive semi-definite linear operator T defined on a Hilbert space H, viz., the quadratic polynomial

$$Q(x) = [Tx, x] - [x, g] - [g, x],$$

where g is a fixed element of H and [,] denotes dot product.

In principle, at least, this problem can be solved completely. Set $x = x_0 + x_1$ and $g = g_0 + g_1$, where x_0 and g_0 belong to H_0, the null-space of T, whilst x_1 and g_1 belong to $H_0^\perp$. If T_1 denotes the restriction of T to $H_0^\perp$, we have

$$\begin{aligned} Q(x) &= [Tx_1, x_1] - [x_1, g_1] - [g_1, x_1] - [x_0, g_0] - [g_0, x_0] \\ &= [\sqrt{T}x_1, \sqrt{T}x_1] - [\sqrt{T}x_1, T_1^{-1/2}g_1] \\ &\qquad - [T_1^{-1/2}g_1, \sqrt{T}x_1] - [x_0, g_0] - [g_0, x_0] \\ &= \|\sqrt{T}x_1 - T_1^{-1/2}g_1\|^2 - \|T_1^{-1/2}g_1\|^2 \\ &\qquad - [x_0, g_0] - [g_0, x_0]. \end{aligned}$$

If $g_0 = 0$, then

$$Q(x) = \|\sqrt{T}x_1 - T_1^{-1/2}g\|^2 - \|T_1^{-1/2}g\|^2 \geqq -\|T_1^{-1/2}g\|^2 = Q(T_1^{-1}g) = Q(T_1^{-1}g + v),$$

where v is any vector in H_0. If $g_0 \neq 0$, then

$$\inf_{x \in H} Q(x) \leqq \inf_{\lambda \text{ real}} Q(\lambda g_0) = -\sup_{\lambda \text{ real}} ([\lambda g_0, g_0] + [g_0, \lambda g_0]) = -\infty.$$

However, this solution is not constructive, for it involves operator T_1^{-1}, which cannot always be calculated explicitly. The method of steepest descent provides an algorithm which determines a minimizing sequence for $Q(x)$. If $g \in H_0^\perp$, so that $Q(x)$ has a finite minimum, the method proceeds as follows.

A first approximation x_0 is chosen. If $Tx_0 = g$, proceed no further, for $Q(x_0) = Q(T_1^{-1}g)$ which is the minimum. Otherwise, take

$$z_0 = Tx_0 - g, \qquad \varepsilon_0 = \frac{[z_0, z_0]}{[Tz_0, z_0]}, \qquad x_1 = x_0 - \varepsilon_0 z_0.$$

If $Tx_1 \neq g$, the process continues. Thus, at the nth step, one takes

$$\begin{aligned} z_n &= Tx_n - g, \\ \varepsilon_n &= \frac{[z_n, z_n]}{[Tz_n, z_n]}, \quad z_n \neq 0, \\ &= 0, \qquad\qquad z_n = 0, \\ x_{n+1} &= x_n - \varepsilon_n z_n. \end{aligned}$$

Thus, the corrections at successive steps are related by the non-linear transformation $z_{n+1} = z_n - \varepsilon_n T z_n$.

The sequence $\{x_n\}$ so defined does not necessarily converge, but it always provides a minimizing sequence, in the sense that

$$\lim_{n \to \infty} Q(x_n) = \min_{x \in H} Q(x).$$

In proving this result, one may assume that no z_n is zero, for otherwise the result is trivial. Since g is in $H_0^\perp$

$$-\infty < \min_{x \in H} Q(x) \leqq Q(x_{n+1}) = Q(x_n) - \varepsilon_n \|z_n\|^2 \leqq Q(x_n),$$

so that $\{Q(x_n)\}$ converges. Thus, $\varepsilon_n\|z_n\|^2 \to 0$. But

$$\varepsilon_n = \frac{[z_n, z_n]}{[Tz_n, z_n]} = \frac{\|z_n\|^2}{\|\sqrt{T}z_n\|^2} \geq \frac{1}{\|\sqrt{T}\|^2} > 0,$$

so that

$$\|z_n\| = \|Tx_n - g\| = \|\sqrt{T}(\sqrt{T}x_n - T_1^{-1/2}g)\| \to 0.$$

Since $\sqrt{T}x_n$ and $T_1^{-1/2}g$ are both in $H_0^{\perp}$, this implies

$$Q(x_n) - \min_{x\in H} Q(x) = Q(x_n) + \|T_1^{-1/2}g\|^2 = \|\sqrt{T}x_n - T_1^{-1/2}g\| \to 0.$$

The author explores the application of the method of steepest descent to the theory of nonlinear estimation, with special regard to problems in stochastic control.
W. Langlois (San Jose, Calif.)

Grišin, V. P. 3294
The minimax problem in the theory of analytic controller design. (Russian. English summary)
Avtomat. i Telemeh. **25** (1964), 868–880.
Author's summary: "A system for the stabilization of the object, with initial data deviation and constant disturbances concerning a certain program trajectory is considered. The best (in the sense of the selected integral criterion) control of the system with the worst nature of constant disturbances is stated. The problem is solved for a finite control time."

Hsieh, H. C. 3295
The least squares estimation of linear and nonlinear system weighting function matrices.
Information and Control **7** (1964), 84–115.
The author develops a least-square solution to two identification problems in control theory. The first is the linear problem:

$$\mathbf{y}(t) = \int_0^\infty \mathbf{W}(\tau)\mathbf{x}(t-\tau)\,d\tau \quad \text{and} \quad \mathbf{z}(t) = \mathbf{y}(t) + \mathbf{n}(t),$$

where $\mathbf{y}(t) = m$-dimensional output in the absence of noise; $\mathbf{x}(t) = n$-dimensional input; $\mathbf{W}(t) = (m\times n)$-dimensional (unknown) weighting function matrix; $\mathbf{n}(t) = m$-dimensional noise; $\mathbf{z}(t) = m$-dimensional observed output. The least-squares estimate of $\mathbf{W}(t)$ amounts to the minimization of a quadratic form, and the author applies the Hilbert-space version of the method of steepest descent to this problem. He extends the treatment to a related problem when the computation is being done in real time and the information is continually being up-dated.

The second problem is the time-invariant non-linear problem:

$$y(t) = \sum_{i=1}^{n} F_i(x_i(t)),$$

$$F_i(x_i(t)) = \sum_{j=1}^{N} \int_0^T \cdots \int_0^T \omega_{ij}(\tau_1, \tau_2, \cdots, \tau_j) \times x_i(t-\tau_1)x_i(t-\tau_2)\cdots x_i(t-\tau_j)\,d\tau_1 d\tau_2 \cdots d\tau_j,$$

where a least-squares estimate is required for the ω_{ij}. The problem is treated by setting it in a product Hilbert space and then using the same procedures as for the linear problem.
W. L. Root (Ann Arbor, Mich.)

Jazwinski, Andrew H. 3296
Optimal trajectories and linear control of nonlinear systems.
AIAA J. **2** (1964), 1371–1379.
The variational differential equations that are necessary conditions for a relative minimum of a control problem generally have boundary conditions specified at two points. This paper shows that an extension of the adjoint method for the numerical integration of two-point boundary-value problems presented by Goodman and Lance is applicable to the variational problem even in the case when one boundary is implicitly, rather than explicitly, specified. The resulting algorithm is similar to several referenced methods, and the unreferenced technique, called quasilinearization, proposed by Bellman and Kalaba. It is also shown how the proposed algorithm yields a feedback control scheme for the correction of deviations from an optimal trajectory. Equivalent feedback schemes, differing in computational details, have been derived by several authors.
S. E. Dreyfus (Santa Monica, Calif.)

Krasovskiĭ, A. A. 3297
Quasi-stationary processes of continuous extremal control with coordinate restrictions. (Russian. English summary)
Avtomat. i Telemeh. **24** (1963), 1633–1642.
Author's summary: "Dynamics of quasi-stationary processes in continuous extremal systems with limitations of the type of inequalities for the plant inputs or their functions is considered. The convergence of the controls for such conditions is proved."

Kelley, Henry J. 3298
A second variation test for singular extremals.
AIAA J. **2** (1964), 1380–1382.
Author's summary: "The second variation for the Mayer problem is examined with reference to singular test extremals and evaluated for the case of a special variation in a control variable. By means of a limiting process, a necessary condition for a minimum is obtained from the requirement of positive semidefiniteness of the second variation. Some applications of the test to singular extremals arising in flight mechanics problems are presented."
U. D'Ambrosio (Providence, R.I.)

Lazarěv, G. B. 3299
Determination of the stability region of automatic control systems with random parameters. (Ukrainian. Russian and English summaries)
Avtomatika **1964**, *no.* 4, 10–14.
Author's summary: "A method for determining the stability region is proposed for structurally stable systems with parameters gradually varying in accordance with a random function of time. The method is based on finding the most unfavourable combinations of variable parameters."

Naumov, B. N.; Cypkin, Ja. Z. 3300
Frequency criteria for absolute stability of processes in non-linear automatic control systems. (Russian. English summary)
Avtomat. i Telemeh. **25** (1964), 852–867.

Authors' summary: "There is proposed a frequency criterion for the process absolute stability in nonlinear automatic control systems which is similar to the common frequency criterion of the linear automatic control systems. The proposed criterion is based on the generalization of the equilibrium state absolute stability of V. M. Popov to the case of absolute stability of the process in nonlinear control systems. To study the absolute stability of the process, phase-amplitude and logarithmic frequency characteristics are used. Methods for the synthesis of stabilizing devices providing the process with absolute stability and the stability degree not lower than the required one are described."

Roĭtenberg, Ja. N. [Ройтенберг, Я. Н.] 3301

★Some problems in the control of motion [Некоторые задачи управления движением].

Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1963. 140 *pp.* 0.38 *r.*

The author gives an account in this monograph of his work on two problems. The first is the determination of a control to achieve a certain output. His method is an approximate one for a special class of linear and a special class of nonlinear problems. In the examples given (the reduction of error in a gyroscopic compass) it gives satisfactory results. How well it might work in general is not discussed. The second problem is that of the identification of a state of a system at a given time from the observation of a sequence of subsequent changes of state. Chapter I is devoted to the discussion of these problems for continuous motions (ordinary differential equations) and Chapter II is for sampled-data systems (difference equations). *J. P. LaSalle* (Providence, R.I.)

Stancil, R. T. 3302

A new approach to steepest-ascent trajectory optimization.

AIAA J. **2** (1964), 1365–1370.

A modification of the steepest-ascent technique of Bryson et al. [J. Aerospace Sci. **29** (1962), 420–430] is proposed wherein the need to guess the increment of the coordinate to be optimized is removed. Some computational experience is discussed. *L. D. Berkovitz* (Lafayette, Ind.)

Sobolev, O. S. 3303

Analysis of the structure of a higher-dimensional automatic system with a non-linear connection. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 347–355.

Author's summary: "A particular problem of the theory of coordinated identical automatic control systems, a case of the degeneracy of the interconnection matrix, is considered. The degeneracy of interconnections involves additional requirements for the automatic control system and the introduction of essential nonlinearities. A problem of structure division of a complicated system into more simple ones is solved."

Gorban', A. V. 3304

On the problem of simulation of control systems. (Russian. English summary)

Avtomat. i Telemeh. **24** (1963), 1297–1302.

A constructive approach is proposed to the problem of determining the solution of degenerate variational problems in non-parametric form. The allowable control functions are assumed to assume values for an open region, and the minimized functional has the property of convexity. *F. D. Faulkner* (Monterey, Calif.)

Jakubovič, V. A. 3305

Frequency conditions for absolute stability of control systems with hysteresis-like non-linearities. (Russian)

Dokl. Akad. Nauk SSSR **149** (1963), 288–291.

Consider a system

$$(1) \qquad \dot z = P_z + q\varphi(\sigma, \varphi_0)_t, \qquad \sigma = (z, r),$$

where P is a constant $n \times n$ matrix and z, q, r are n-vectors, while φ is a hysteresis or "relay-hysteresis" scalar function defined (for the present) as follows: Let $\mathscr{M}_{\sigma_0}(0, t_1)$ denote the collection of functions $\sigma(z)$ continuous on $[0, t_1]$ and such that $\sigma(0) = \sigma_0$. Then (a) assume that to each σ_0 there corresponds a certain collection $\mathscr{E}(\sigma_0)$ of initial values φ_0; (b) to each $t_1 \geqq 0$ and $\varphi_0 \in \mathscr{E}(\sigma_0)$ there corresponds an operator $\varphi(\cdot|_{t_0}^{t_1}, \varphi_0)$ mapping $\mathscr{M}_{\sigma_0}$ on $C(0, t_1)$; (c) $0 \leqq \sigma\varphi \leqq \mu_0\sigma^2$. The system (1) is absolutely stable whenever every solution $z(t)$ exists on $(0, \infty)$, and moreover, there exists a continuous and increasing function $\psi(\rho)$ depending solely upon P, q, r, μ_0, defined on $[0, \infty)$ with $\psi(0) = 0$, such that whatever $\varphi_0 \in \mathscr{E}(\sigma_0)$, $\sigma_0 = (z(0), r)$ we have $\|z(t)\| \leqq \psi(\|z(0)\|)$.

Several sufficient conditions for absolute stability are given which resemble somewhat the well-known Popov conditions. [References: V. M. Popov, Avtomat. i Telemeh. **22** (1961), 857–875; MR **24** #A3394; Jakubovič, ibid. **24** (1963), 293–303; MR **28** #311a; ibid. **24** (1963), 717–731; MR **28** #311b; Aĭzerman and Gantmaher, *Absolute stability of controlled systems* (Russian), Izdat. Akad. Nauk SSSR, Moscow, 1963].

{A translation of this article appears in Soviet Physics Dokl. **8** (1963), 235–237 with a carelessly translated title.} *S. Lefschetz* (Princeton, N.J.)

Weiss, Leonard 3306

Weighting patterns and the controllability and observability of linear systems.

Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 1122–1127.

The author generalises Kalman's work on the decomposition of solutions of linear dynamical systems, and discusses the relation between the internal (or free motion) and the external (or forced motion) system characteristics in terms of a weighting pattern. The paper is a useful resumé of the author's work in this field. *A. W. Babister* (Glasgow)

Kelendzheridze, D. L. [Kelendžeridze, D. L.] 3307

On a problem of optimum tracking.

Avtomat. i Telemeh. **23** (1962), 1008–1013 (*Russian. English summary*); *translated as Automat. Remote Control* **23** (1963), 942–947.

A minimax solution is considered for a minimum time problem. The positions of two points x, y in R^n are determined by two differential equations $\dot x = f(x, u)$, $\dot y = g(y, v)$ with control functions u, v, respectively, piecewise continuous vectors of dimension r, s lying in open regions U^r, V^s. f, g are continuous in u, v and of

class C' in x, y. Initial values x_0, y_0 are given. Let u, v be any allowed controls such that, at some time T_{uv}, $x=y$, and let $T=\max_{v\in V'}\min_{u\in U'} T_{uv}$; the problem is to determine T and the associated paths. Solutions ψ, χ to the adjoint differential equations are introduced, a Hamiltonian is defined, and canonical equations with a minimax principle are introduced to define conditions for the solution. The existence proof makes use of a "Bellman" function that is assumed to be twice differentiable, a condition which the author observes to be stronger than necessary. The problem is called tracking; it seems more a model of a pursuit problem.

F. D. Faulkner (Monterey, Calif.)

Aoki, Masanao 3308

On the approximation of trajectories and its applications to control systems optimization problems.

J. Math. Anal. Appl. **9** (1964), 23–41.

The author considers a control system given by $dx/dt=Ax+Bu$, $x(t_0)=x^0$, where x and u are the state- and control-vectors. It is desired to follow as closely as possible a prescribed trajectory r, $r(t_0)=r^0$, in a time-interval (t_0, t_0+T) by determining $u(t)$ such that $\int_{t_0}^{t_0+T} \|x(t)-r(t)\|\,dt$ is minimized, where the norm $\|x\|$ is either $\max|x_i|$ or $\sum_1^n |x_i|$, and where the x_i are the n components of x.

Methods are shown for determining suboptimal policies, and in some special cases optimal policies are found. Considered in some detail is the case of independent control in which each component of state can be controlled independently.

It might be noted that the problem is closely related to Bellman's theory of differential approximation.

B. Gluss (Berkeley, Calif.)

Dem'janov, V. F.; Homenjuk, V. V. 3309

Solution of a linear optimal control problem. (Russian. English summary)

Avtomat. i Telemeh. **24** (1963), 1174–1177.

The problem of minimizing a function of one variable at a specified terminal time is discussed. The variables satisfy a linear differential equation $\dot{x}^i=\sum a_j{}^i x^j+\sum b_j{}^i u^j+f^i$. The input or control variables u^j satisfy the constraint $|u^i|\le 1$, $i=1,\cdots,r$. This problem is easily solved on a computer.

F. D. Faulkner (Monterey, Calif.)

Beĭko, I. V.; Karpenko, M. F. 3310

On a method of finding optimal controls. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 1418–1424.

Authors' summary: "A method is outlined for finding the control which is optimal as regards quick response. The method is based on reducing the problem of optimal quick response to a sequence of variational problems of the classical type with a quadratic functional."

Beĭko, I. V. 3311

Synthesis of controls which are close to high-speed optimal controls. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1964**, 842–845.

Berezovs'kiĭ, A. I.; Ivanov, V. V. 3312

Some algorithms for optimal high-speed controls. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 580–583.

Authors' summary: "A numerical solution is given for the problem of optimal quick-response controls. The problem is reduced to that of quadratic programming."

P. L. Ivănescu (Bucharest)

Kolemaev, V. A. 3313

An optimal control problem for a Wiener process. (Russian. Lithuanian and English summaries)

Litovsk. Mat. Sb. **3** (1963), *no.* 1, 169–174.

Let $\xi=\{\xi_t\}$, $\xi_0=0$ be a random process with unit local diffusion and with local shocks $\{\theta(t,x_t), t\ge 0, -a<x_t<a\}$, where the functions $\theta(t,x_t)$ take values from a given set Θ. Denote by ν_a the moment of first exit of the path x_t of the process ξ from the interval $(-a,a)$ and consider the functional

$$(1)\qquad R(\xi)=\int_0^{\nu_a} g(t,x_t,\theta(t,x_t))\,dt.$$

The following optimal control problem is investigated: At every moment t, by means of the observed data $\{\xi_\tau, \tau<t, |\xi_\tau|<a\}$, to vary the parameter $\theta\in\Theta$ in such a way that the mathematical expectation of (1) be extreme. It is shown that if the function $g(t,x,\theta)$ depends explicitly upon θ, an optimal rule must, generally, employ all values $\theta\in\Theta$. A detailed discussion of the case $g(t,x,\theta)=1-c(\theta)$ is given.

R. Theodorescu (Bucharest)

Kolemaev, V. A. 3314

On the optimal control of a Wiener process. (Russian. Lithuanian and English summaries)

Litovsk. Mat. Sb. **3** (1963), *no.* 1, 175–180.

Let $\xi(t)$ be a random process verifying the stochastic differential equation

$$d\xi=\theta dt+\sigma d\eta,\qquad \xi(0)=0,$$

where $\eta(t)$ is a Wiener process. Optimal control problems of $\xi(t)$ are discussed under various assumptions about the parameter θ.

R. Theodorescu (Bucharest)

Orford, Richard J. 3315

Optimal stochastic control systems.

J. Math. Anal. Appl. **6** (1963), 419–429.

The optimal control of a scalar system governed by a differential equation $\dot{X}=aX+bu$ is discussed; a, b are given functions of time t, the input u is a set of impulses u_j at times t_j, $j=1,\cdots,N$. Measurements of X are made at times t_j; these have random errors and the input u also has random errors. The inputs individually satisfy $|u_j|\le c_j$ and $\sum|u_j|\le c$, where the c's are given. The conditional expectation of a given function $L(X)$ is to be minimized with respect to the data available and allowed future inputs. It is shown that X is a stochastic variable; the concept of a state is derived for the system; a functional equation is given for the optimal control; and a dynamic programming relation is set up for optimal control.

F. D. Faulkner (Monterey, Calif.)

Pontrjagin, L. S. [Понтрягин, Л. С.]; 3316a
Boltjanskiĭ, V. G. [Болтянский, В. Г.];
Gamkrelidze, R. V. [Гамкрелидзе, Р. В.];
Miščenko, E. F. [Мищенко, Е. Ф.]
★The mathematical theory of optimal processes [Математическая теория оптимальных процессов].
Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1961. 391 *pp.* 1.13 *r.*

Pontryagin, L. S. [Pontrjagin, L. S.]; 3316b
Boltyanskii, V. G. [Boltjanskiĭ, V. G.];
Gamkrelidze, R. V.;
Mishchenko, E. F. [Miščenko, E. F.]
★The mathematical theory of optimal processes.
Translated from the Russian by K. N. Trirogoff; edited by L. W. Neustadt.
Interscience Publishers [*John Wiley & Sons, Inc.*], *New York-London*, 1962. viii + 360 *pp.* $11.95.

Pontrjagin, L. S.; 3316c
Boltjanskij, V. G. [Boltjanskiĭ, V. G.];
Gamkrelidze, R. V.;
Misčenko, E. F. [Miščenko, E. F.]
★Mathematische Theorie optimaler Prozesse.
R. Oldenbourg, Munich-Vienna, 1964. 340 *pp.* *DM* 68.00.

In this book the authors give a systematic development of a theory of optimal control processes, the central result of which is a "maximum principle". This theory has a fundamental practical significance for various automatic control processes. It is also of significant interest from a purely mathematical point of view, since it is a direct generalization of the classical calculus of variations to problems to which, up to now, the known methods were not applicable. The book consists of an introduction and seven chapters (42 sections); at the end of the book there is a bibliography arranged by chapters.

Those control processes are considered which can be described by the system

$$(1) \qquad \frac{dx}{dt} = f(x, u),$$

where x is an n-dimensional vector which characterizes the process, u is an r-dimensional control vector, and t is the time. Corresponding to the system the functional

$$(2) \qquad J = \int_{t_0}^{t} f^0(x, u)\, dt$$

is considered, where f^0 is a known function and the initial and terminal states (but not the initial and terminal times!) of the process are fixed (where it is assumed that there exist controls $u(t)$ which take the process from one state to another). In Chapters 1 and 2 the authors formulate and solve the problem of determining a control $u^*(t)$ which realizes the passage of the control process from the initial to the final state in such a way that the functional J has a least value. A feature of the presentation is the proposition that the vector U can be chosen from a closed set U; it is this property that makes the problem under consideration non-classical. Necessary conditions for optimality are formulated in view of the maximum principle (§ 3). The application of this principle is illustrated by a series of examples. In addition, Chapter 1 contains a series of modifications of this fundamental problem: The case when the initial and terminal times are fixed; the problem of optimal transfer from a certain initial manifold onto a certain terminal manifold; the problem of the optimal hitting of a moving point (the equivalent case when the system (1) is non-autonomous). The authors also formulate a synthesis problem and solve it in a series of particular cases. In § 9 the authors discuss at length the relation between the maximum principle and the method of dynamic programming; it is shown that even in the very simplest examples (which can be easily solved by means of the maximum principle) the hypotheses which are necessary for the derivation of Bellman's equation are not fulfilled. Chapter 3 is concerned with a solution of a minimum-time optimal problem (i.e., under the hypothesis that $f^0(x, u) \equiv 1$ in the functional equation (2)) in the case when all the right-hand sides of the system (1) are linear forms of their arguments (with constant or variable coefficients), and the set U is a closed convex polyhedron. This problem, which has important technological applications, and which is completely inaccessible by classical methods, admits a complete solution within the framework of the maximum principle. Under well-defined hypotheses the authors prove existence and uniqueness theorems for an optimal control and also solve a synthesis problem for a series of examples. § 22 is concerned with the synthesis of linear optimal minimum-time problems (i.e., the search for an optimal trajectory) by means of relay circuits. Chapter 4 includes a solution of a series of related problems together with several applications: the case when the functional J is given by an improper integral (i.e., the upper limit of integration in (2) is equal to ∞); a consideration of optimal processes with parameters which are also allowed to vary (up to the beginning of the process); consideration of optimal processes with time lag; applications of the theory to problems of approximation of functions. In § 28 the authors give the solution to a (sufficiently special) tracking problem. It is remarked that the general problem of tracking an object controlled by the system

$$\frac{dz}{dt} = g(z, v)$$

by another object controlled by system (1) has not been solved.

In Chapter 7 another problem is solved, that of tracking an object whose motion is assumed to be a Markov process. In this connection one seeks a control by the tracking object (1) for which one has the largest probability of intersecting a certain small neighborhood of this object with the evader. The short Chapter 5 is devoted to clarifying the relation between theories of optimal processes and the classical calculus of variations. It is shown that an optimal problem is a generalization of the problem of Lagrange, and it is equivalent to it whenever the domain U is an open set of an r-dimensional vector space. In this latter case one obtains from the maximum principle all the fundamental necessary conditions of the classical theory (including the Weierstrass criterion). In Chapter 6 the authors solve the problem of optimal control by an object under the condition that the points describing the state always lie in a certain closed region in the phase space (i.e., the theory of optimal processes with restrictions on the phase coordinates).

They also obtain a certain generalization of the maximum principle.

{The second and third listings above are competent translations into English and German, respectively.}

N. Rozov (RŽMat **1963** #8 B280)

Salukvadze, M. E. 3317

On the invariance of optimal controllers. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 650–652.

Author's summary: "It is proved that there are no conditions under which the solution of the synthesis problem in the optimum linear control system [the author, Avtomat. i Telemeh. **23** (1962), 721–731] is invariant in relation to constant external disturbances."

Senin, A. G. 3318

A statistical design problem for an object with distributed parameters. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 653–660.

The author derives an integral equation whose solution determines an optimal sampled-data function for a distributed control system for extrapolating a random field. He also gives a practical method of constructing such a measuring system.

Parhomenko, P. P. 3319

Design of relay structures on various functionally complete systems of logical elements. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 963–979.

Author's summary: "The algorithm of logical synthesis of switching circuits for different functionally complete logical element systems is described. The algorithm permits one to get different type functional diagrams, it yields approximate results good enough for practical use and increases the possibilities of the machine aid by reducing the number of tries. Structural diagrams obtained with the help of the described method are given."

Bobrovnik, G. A.; Lazarev, V. M. 3320

On the design of optimal measuring systems containing digital computers. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 696–701.

Authors' summary: "Design of measuring predictor systems with digital computers obtaining an optimum mean-square error is described. Structural circuits are proposed for certain cases of input function realization. The advantage of the system compared with the system without prediction, or with a simple law of prediction, is determined."

Nečiporuk, È. I. 3321

On self-correcting gating circuits. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 1045–1048.

A "gate" is a two-poled element conducting current only in one direction. O. B. Lupanov [same Dokl. **111** (1956), 1171–1174; MR **19**, 807] has defined the rank of a gating circuit as the length of a maximal oriented chain connecting an input pole to an output pole. In gating circuits two types of errors are possible: a gate may be shorted closed or it may be broken and open. Under broad conditions, the author shows that, for circuits of rank 2, the complexity of a circuit correcting an m-fold break is not less than $C(m+1)$ times that of an ordinary circuit having the same admittance matrix, where C is a certain constant. Under certain narrower conditions, as the size of the matrix increases, the constant C asymptotically approaches 1.

G. N. Raney (Storrs, Conn.)

Andreoli, Giulio; Fadini, Angelo 3322

Nuova teoria delle reti di comunicazione. I. I circuiti ad interruttore.

Giorn. Mat. Battaglini (5) **10** **(90)** (1962), 5–25.

In questa nota, un circuito è un simplesso di n vertici $V_1, \cdots, V_n$, collegati due a due da $r=\binom{n}{2}$ rami su ciascuno dei quali è inserito un interruttore con due role possibili stati: passa e non passa. Se ogni interruttore può assumere l'una o l'altra delle due posizioni, indipendentemente dagli stati degli altri interruttori, il circuito si dice libero, in caso contrario, vincolato.

Lo stato del circuito, nel suo complesso, è la totalità degli stati dei singoli interruttori, cioè la totalità ordinata di r-ple $\sigma=(\varepsilon_1, \cdots, \varepsilon_r)$ con $\varepsilon=0$ (o ν, booleanamente) oppure $\varepsilon=1$ (o τ) rispettivamente per non passa o passa. Se il circuito è libero allora le σ formano gli elementi di un'algebra di Boole a 2^r elementi, in caso contrario le σ formano una varietà immersa in quell'algebra.

Se certi interruttori sono bloccati nella posizione passa ed altri nella posizione non passa vengono a mancare le communicazioni dirette fra certe coppie di nodi e restano in comunicazione diretta certe altre. Gli interruttori effettivi che possono assumere i due valori ν, τ sono allora $\rho<r$ e gli stati del circuito ridotto formano una sottoalgebra booleana di quella totale.

Una centrale di comando ad interruttori liberi, dalla cui manovra dipendono gli stati possibili per il circuito, è le totalità degli stati di tutti gli interruttori.

In questa nota i due problemi tratatti sono: Il problema diretto: Dato un simplesso di n nodi ed $\binom{n}{2}$ rami, su ciascuno dei quali sia posto un interruttore bloccato o no, manovrato da una centrale di comando, assegnate le m variabili della centrale ed i vincoli ad esse eventualmente importi fissate la funzioni di queste variabili per ogni interruttore; determinare le partizioni dei nodi per ogni manovra della centrale, la struttura poliedrale di ciascuna partizione.

Il problema inverso è: Dato il circuito da n vertici $V_1, \cdots, V_n$ ed $r=\binom{n}{2}$ rami l_{rs} ciascuno dotato di un interruttore I_{rs}, determinare le m variabili indipendenti della centrale di comando e le r funzioni di queste, atte a realizzare assegnate partizioni dei vertici; o addirittura assegnata strutture poliedrali.

M. Coroi-Nedelcu (Bucharest)

Kel'mans, A. K. 3323

Some optimal problems in the theory of reliability of information networks. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 661–667.

Author's summary: "An optimum reservation problem is solved for an arbitrary structure of information circuits without loops considering weights of points yielding information. Formulae which permit one to determine the optimum distribution of the reserve and achieved loss minimum are found."

Tal', A. A. 3324

Questionnaire language and abstract design of minimal sequential machines. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 946–962.

Author's summary: "An algorithm for the abstract synthesis of sequential machines is described. The algorithm differs from the known algorithm by the peculiarity that it does not require to formulate beforehand the synthesized machine set by means of any language. The whole information on synthesized machine properties the 'designer' gets from 'the customer' during the algorithm development consisting in answering the two types of questions about the sequences of external situations. The algorithm use results in the recovery of the state diagram for the minimum sequential machine required by 'the customer'."

Böhling, Karl-Heinz 3325

★Zur Strukturtheorie sequentieller Automaten.

Forschungsberichte des Landes Nordrhein-Westfalen, Nr. 1279 (Nr. 4 der Schriften des IIM· Serie A).

Westdeutscher Verlag, Cologne, 1964. 73 *pp.* *DM* 45.00.

This is a monograph whose purpose is to define complete or incomplete, deterministic or non-deterministic sequential machines; and complete or incomplete, deterministic or non-deterministic automata (essentially those of Rabin and Scott) by relational systems. For the devices the author treats, three-place relations suffice. [Pushdown automata are not mentioned, but they can be described by 5-place relations.] No theorems are proved, but the author asserts that consequences of his treatment will appear elsewhere. *S. Ginsburg* (Van Nuys, Calif.)

Svečinskiĭ, V. B. 3326

On self-correcting constructions of finite automata. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 685–691.

Author's summary: "A method is proposed for the synthesis of reliable automata with memory out of elements of limited reliability. The method is based on using the theory of self-correcting codes. The reliability of the automata is estimated."

Kämmerer, Wilhelm 3327

Ein Algorithmus zur Verarbeitung logischer Operatoren im Rahmen einer automatischen Programmierung mittels programmierenden Programms.

Z. Math. Logik Grundlagen Math. **10** (1964), 139–146.

Eine Beschreibung einer Methode für die Konstruktion der Klammerstruktur wird gegeben, und ergibt sich eine Darstellung des logischen Operators, da eine dynamischen Umkodierung und der Aufbau des Maschinenprogramms entstehen zu lassen. *P. Constantinescu* (Bucharest)

Zarovnyĭ, V. P. 3328

On the representation of machines by semigroup-theoretical formations (by m-semigroups). (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 716–731.

Let $\mathfrak{A}$ be a set, $\mathfrak{X}$ a semigroup and δ a mapping $\mathfrak{A}\times\mathfrak{X}\to\mathfrak{A}$ such that $\delta(a, x_1\cdot x_2)=\delta(\delta(a, x_1), x_2)$; then $\langle\mathfrak{A}, \mathfrak{X}, \delta\rangle$ is called an automaton. An m-semigroup is a triplet $\langle\bar{A}, \bar{\mathfrak{A}}, \bar{\mathfrak{X}}\rangle$ such that $\bar{A}$ is a semigroup, $\bar{\mathfrak{A}}\subseteq\bar{A}$ and $\bar{\mathfrak{X}}$ is a sub-semigroup and $\bar{\mathfrak{A}}\cdot\bar{\mathfrak{X}}\subseteq\bar{\mathfrak{A}}$, $\bar{\mathfrak{A}}$ and $\bar{\mathfrak{X}}$ generate $\bar{A}$. A representation of the automaton by this m-semigroup is a couple $\langle\varphi, \psi\rangle$ such that φ is a one-to-one mapping of $\mathfrak{A}$ onto $\bar{\mathfrak{A}}$ and ψ is an isomorphism between $\mathfrak{X}$ and $\bar{\mathfrak{X}}$ such that $\delta(a, x)=\varphi^{-1}(\varphi(a)\psi(x))$. First it is established that the general problem of representability can be reduced to the case of the so-called p-automata, in which case a necessary and sufficient condition is given. Two further results solve the similar problem for two special types of m-semigroups (the second type is commutative).

G. Grätzer (University Park, Pa.)

AUTHOR INDEX